1 Question 1

$\overline{G}$ is composed of two sets of 10 fully connected vertices, and each of these vertices is also connected to the 20 remaining vertices. To construct a triangle involving one of the 20 vertices, we choose 2 vertices from one of the fully connected sets of size 10, and then choose 1 vertex among the 20 remaining vertices to complete the triangle. To construct a triangle entirely within one of the sets of 10 vertices, we simply choose 3 elements among the 10. This procedure is exhaustive and without redundancy. Therefore, the total number of triangles is:$2 \times \binom{10}{2} \times 20 \ + \ 2 \times \binom{10}{3} = 1800 + 240 = 2040$.

2 Question 2

Define the numerator and denominator

$$g(x) = x^\top Ax, \qquad h(x) = x^\top x,$$

so that

$$R(x) = \frac{g(x)}{h(x)}.$$

Since the adjacency matrix A is symmetric, the gradients are

$$\nabla g(x) = 2Ax, \qquad \nabla h(x) = 2x.$$

Using the quotient rule for gradients, we obtain

$$\nabla R(x) = \frac{h(x)\,\nabla g(x) - g(x)\,\nabla h(x)}{[h(x)]^2} = \frac{2\big((x^\top x)Ax - (x^\top Ax)x\big)}{(x^\top x)^2}.$$

Thus, $\nabla R(x) = 0$ if and only if

$$(x^\top x)\,Ax - (x^\top Ax)\,x = 0.$$

Let

$$\lambda = \frac{x^\top Ax}{x^\top x} = R(x).$$

Then the above equation becomes

$$Ax = \lambda x,$$

which shows that x is an eigenvector of A with eigenvalue λ.

Conversely, if x satisfies $Ax = \lambda x$, then $x^\top Ax = \lambda x^\top x$, and substituting into the gradient expression yields $\nabla R(x) = 0$. Therefore, x is a stationary point of R.

3 Question 3

Case 1: We have two communities: $l_b = 7$ and $l_r = 5$ Total number of edges: $m = 26$ Sum of degrees: $d_b = 15$, $d_r = 11$

$$Q_a = \frac{l_b}{m} - \left(\frac{d_b}{2m}\right)^2 + \frac{l_r}{m} - \left(\frac{d_r}{2m}\right)^2 = \frac{7}{13} - \left(\frac{15}{26}\right)^2 + \frac{5}{13} - \left(\frac{11}{26}\right)^2 \approx 0.41$$

Case 2: We have two communities: $l_b = 2$ and $l_r = 7$ Total number of edges: $m = 26$ Sum of degrees: $d_b = 8$, $d_r = 18$

$$Q_a = \frac{l_b}{m} - \left(\frac{d_b}{2m}\right)^2 + \frac{l_r}{m} - \left(\frac{d_r}{2m}\right)^2 = \frac{2}{13} - \left(\frac{8}{26}\right)^2 + \frac{7}{13} - \left(\frac{18}{26}\right)^2 \approx 0.19$$

4 Question 4

These two graphs have the same shortest path representation $V = [6, 4, 0, 0]$ but are not isomorphic.

5 Question 5

We consider two labeled graphs G and G', each with six nodes. The initial discrete node labels and the labels of their neighbors are given below. For each node we write:

$$v : \ell(v) \,/\, \{\ell(u)\}_{u \in N(v)},$$

where $\ell(v)$ is the node label and $N(v)$ is the set of neighbors.

Graph G:

$$\begin{aligned}
A &: 1 \,/\, \{2, 5\} \\
B &: 5 \,/\, \{1, 2, 3\} \\
C &: 2 \,/\, \{1, 3, 5\} \\
D &: 3 \,/\, \{1, 2, 4, 5\} \\
E &: 1 \,/\, \{3, 4\} \\
F &: 4 \,/\, \{1, 2, 3\}
\end{aligned}$$

Graph G':

$$\begin{aligned}
A &: 1 \,/\, \{2, 5\} \\
B &: 5 \,/\, \{1, 2, 4\} \\
C &: 2 \,/\, \{1, 4, 5\} \\
D &: 4 \,/\, \{3, 5\} \\
E &: 3 \,/\, \{4, 4\} \\
F &: 4 \,/\, \{2, 3\}
\end{aligned}$$

Each unique signature appearing across both graphs is assigned a new compressed label. Identical signatures receive the same new label. The mapping used is as follows:

$$\begin{aligned}
1|2, 5 &\to 1 \\
5|1, 2, 3 &\to 2 \\
2|1, 3, 5 &\to 3 \\
3|1, 2, 4, 5 &\to 4 \\
1|3, 4 &\to 5 \\
4|1, 2, 3 &\to 6 \\
5|1, 2, 4 &\to 7 \\
2|1, 4, 5 &\to 8 \\
4|3, 5 &\to 9 \\
3|4, 4 &\to 10 \\
4|2, 3 &\to 11
\end{aligned}$$

Each graph is represented by a count vector of the frequency of the new labels.

$$V(G) = [1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0]$$

$$V(G') = [1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1]$$

The Weisfeiler–Lehman kernel after one iteration is the inner product of the two histograms:

$$K(G, G') = V(G)^\top V(G').$$

Since only the first component is shared between G and G', we obtain:

$$\boxed{K(G, G') = 1}.$$

After one WL iteration, the two graphs share only one identical subtree pattern (around node A). All other nodes receive distinct new labels, meaning their 1-hop neighborhoods differ. Therefore, the structural similarity between G and G' at this scale is low, and the WL kernel correctly captures this difference.