Introduction to Partial Differential Equations

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September 20, 2020

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I. DEFINITION AND CLASSIFICATION OF PDE

A. Notation and definitions

Definition 1. Given a domain $\Omega \subset \mathbb{R}^n$ and a smooth function $u : \bar{\Omega} \to \mathbb{R}$, denote the tensor of all partial derivatives of order k by:

$$D^k u = \left(\frac{\partial^k u}{\partial x_{i_1} ... \partial x_{i_k}} \right)_{(i_1,...,i_k) \in 1,...,n^k} \tag{1}$$

Definition 2. A k-th order PDE on a domain Ω is an expression of the form:

$$F(D^k u(x), ..., u(x), x) = 0), \forall x \in \Omega \tag{2}$$

where $F : \mathbb{R}^{n^k} \times ... \times \mathbb{R} \times \Omega \mapsto \mathbb{R}$ is given and $u : \Omega \mathbb{R}$ is the unknown.

Definition 3. We call a solution to 2 a classical solution.

Definition 4. A partial differential equation is called linear if 2 is of the form:

$$a_{i_1 ... i_s}(x) \frac{\partial^s u(x)}{\partial x_{i_1} ... \partial x_{i_s}} = f(x) \tag{3}$$

It is called semi-linear if:

$$F(D^k u(x), ..., u(x), x) = a_{i_1 ... i_k}(x) \frac{\partial^k u(x)}{\partial x_{i_1} ... \partial x_{i_k}} + \tilde{F}(D^{k-1} u(x), ..., u(x), x) \tag{4}$$

It is called quasi-linear if:

$$F(D^k u(x), ..., u(x), x) = a_{i_1 ... i_k}(D^{k-1} u(x), ..., u(x), x) \frac{\partial^k u(x)}{\partial x_{i_1} ... \partial x_{i_k}} + \tilde{F}(D^{k-1} u(x), ..., u(x), x) \tag{5}$$

If none of these cases happens, the PDE is called fully non linear.

B. Linear Second Order PDE

Since for a classical solution to this problem, partial derivatives commute, one can assume the matrix A of highest order coefficient is symmetrical.

Definition 5. A linear second order PDE is called elliptic at $x \in \Omega$ if $A(x)$ is positive (or negative) definite. It is parabolic if it has one eigenvalue zero and all other positive (or negative). It is hyperbolic if it has one negative (resp. positive) eigenvalue and all other positive (resp. negative). If the property is satisfied on the whole domain Ω, we say it is satisfied uniformly.

Appendix A: Gauss-Green formulae and integration by parts

Those formulae follow directly from vector identities and the divergence theorem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary.

Proposition 1. Green-Gauss formula

$$\forall u \in C^1(\overline{\Omega}) : \int_\Omega \partial_{x_i} u dx = \int_{\partial\Omega} u\nu_i dS, i = 1, ..., n \tag{A1}$$

Proposition 2. Integration by parts formula

$$\forall u, v \in C^1(\overline{\Omega}) : \int_\Omega \partial_{x_i} uvdx = \int_\Omega \partial_{x_i} vudx + \int_{\partial\Omega} uv\nu_i dS, i = 1, ..., n \tag{A2}$$

Proposition 3. First Greens' identity

$$\forall u \in C^2(\overline{\Omega}) : \int_\Omega \nabla^2 udx = \int_{\partial\Omega} \partial_\nu udS \tag{A3}$$

Proposition 4. Second Greens' identity

$$\forall u, v \in C^2(\overline{\Omega}) : \int_\Omega \nabla^2 uvdx = -\int_\Omega \nabla u \cdot \nabla vdx + \int_{\partial\Omega} \partial_\nu uvdS \tag{A4}$$

$$= \int_\Omega u\nabla^2 vdx + \int_{\partial\Omega} \partial_\nu uv - u\partial_\nu vdS \tag{A5}$$