Solutions to Probability Theory by Klenke

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I. MARTINGALES

9.2.1 Let $s \leq t$. Using the tower property (for $\mathcal{F}_s \subseteq \mathcal{F}_t$):

$$\mathbb{E}(X_t|\mathcal{F}_s) = \mathbb{E}((Y|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}(Y|\mathcal{F}_s)..$$

Also using the triangle inequality and the tower property:

$$\mathbb{E}(|X_t|) \leq \mathbb{E}(\mathbb{E}(|Y||\mathcal{F}_t)) = \mathbb{E}(\mathbb{E}(|Y|)|\mathcal{F}_t) < \infty.$$

9.2.2 Notice that: $\mathbb{E}(X_1^2) = \mathbb{E}(\mathbb{E}(X_1^2|\mathcal{F}_0)) = \mathbb{E}(X_1\mathbb{E}(X_1^2|\mathcal{F}_0)) = \mathbb{E}(X_1X_0)$ (using that X_1 is $\mathcal{F}_0$-measurable. Also, for the same reason, $\mathbb{E}(X_0^2) = \mathbb{E}(\mathbb{E}(X_1|\mathcal{F}_0)^2) = E(X_1^2) = E(X_1X_0)$. As a consequence, $\mathbb{E}((X_1 - X_0)^2) = 0$. By induction, the martingale is constant.

9.2.3 Only (i) requires a modified proof. Regarding the integrability condition, there remains to show that $\mathbb{E}(\varphi(X_t)^-) < \infty$, for every $t \in I$. By convexity of φ, corollary 7.8 implies the existence of $a, b \in \mathbb{R}$ such that:

$$\mathbb{E}(\varphi(X_t)^-) \leq \mathbb{E}((aX_t + b)^-) \leq |a|\mathbb{E}(|X_t|) + |b| < \infty.$$

For the submartingale property, take $s \leq t \in I$. Using the Jensen's inequality and then the submartingale property of $(X_t)_t$ and the increasing property of φ:

$$\mathbb{E}(\varphi(X_t)|\mathcal{F}_s) \geq \varphi(\mathbb{E}(X_t|\mathcal{F}_s)) \geq \varphi(X_s).$$

For a counter-example, take $I = \{0, 1\}$, $X_0 = -2$, $X_1 = 1$. It is clearly a submartingale. Define $\varphi(x) \equiv x^2$. Then, $\varphi(X_0) = 4 > \varphi(X_1) = 1$. So $(\varphi(X_t))_{t\in I}$ is not a submartingale.

9.2.4 (i) Take $Y = 2\mathcal{B}_{1,(1+X)/2} - 1$. (ii) Compute:

$$\begin{aligned}
\mathbb{E}(e^{\lambda X}) &= \mathbb{E}(e^{\lambda\mathbb{E}(Y|X)}) \\
&= \mathbb{E}(\mathbb{E}(\lambda Y|X)) \\
&= \mathbb{E}\left(\frac{1+X}{2}e^{\lambda} + \frac{1-X}{2}e^{-\lambda}\right). \\
&= \cosh(\lambda)
\end{aligned} \tag{1}$$

Comparing term by term the series expansion of $\cosh(\lambda)$ and $e^{\lambda^2/2}$ yields the final inequality. (iii) The inequality is clear if we can show that $M_n - M_{n-1}$ is always independent from $M_k - M_{k-1}$, for $n > k$ without loss of generality.

$$\mathbb{E}((M_n - M_{n-1})(M_k - M_{k-1})) = \mathbb{E}(\mathbb{E}((M_n - M_{n-1})(M_k - M_{k-1})|M_{k-1})) = 0$$

since $(M_k)_k$ is a martingale. (iv) Using Markov's inequality:

$$\begin{aligned}
\mathbb{P}(|M_n| \geq \lambda) &\leq \frac{\mathbb{E}(e^{\gamma|M_n|})}{e^{\gamma\lambda}} \\
&\leq \frac{\mathbb{E}(e^{\gamma M_n}) + \mathbb{E}(e^{-\gamma M_n})}{e^{\gamma\lambda}} \\
&\leq 2\exp\left(\frac{\gamma^2\sum_{k=1}^n c_k^2}{2} - \gamma\lambda\right),
\end{aligned} \tag{2}$$

where we used that $(-M_k)$ is a martingale satisfying similar properties to (M_k). Conclude by taking the optimal γ as $\gamma \equiv \lambda/\sum_{k=1}^n c_k^2$.

II. OPTIONAL SAMPLING THEOREMS

10.2.2 (i) Let $\theta \in]-\delta, \delta[$. Fix an arbitrary integer $n \geq 1$. Using the independence of $(Y_j)_j$ and $Y_n \sim Y_1$:

$$\mathbb{E}(Z_n^\theta|\mathcal{F}_{n-1}) = \mathbb{E}(e^{\theta(X_n - X_{n-1})}e^{\theta X_{n-1}}e^{-n\psi(\theta)}|\mathcal{F}_{n-1}) = e^{\theta X_{n-1}-(n-1)\psi(\theta)}e^{-\psi(\theta)}\mathbb{E}(e^{\theta Y_n}) = Z_{n-1}^\theta.$$

(ii) According to the Cauchy-Schwartz inequality:

$$\mathbb{E}(Y_1 e^{\theta Y_1}) < \mathbb{E}(Y_1^2 e^{\theta Y_1})^{1/2}\mathbb{E}(e^{\theta Y_1})^{1/2}.$$

Therefore, $\psi^{''}(\theta) > 0$ and ψ is strictly convex. (iii) Notice that:

$$\mathbb{E}(\sqrt{Z_n^\theta}) = e^{n(\psi(\theta/2)-\psi(\theta)/2)}.$$

But, by strict convexity:

$$\frac{\psi(\theta)-\psi(0)}{\theta}\frac{\theta}{2} + \psi(\theta) > \psi(\theta/2).$$

Hence, $\mathbb{E}\left(\sqrt{Z_n^\theta}\right) \to 0$ when $\theta \neq 0$. (iv) Fix $\epsilon > 0$. By Markov's inequality:

$$\begin{aligned}\mathbb{P}(Z_n^\theta > \epsilon) &\leq \frac{\mathbb{E}\left(\sqrt{Z_n^\theta}\right)}{\sqrt{\epsilon}} \\ &= \frac{e^{n(\psi(\theta/2)-\psi(\theta)/2)}}{\sqrt{\epsilon}}.\end{aligned} \tag{3}$$

As a consequence, strict convexity implies that $\sum_{n=1}^\infty \mathbb{P}(Z_n^\theta > \epsilon) < \infty$ (geometric series). By the Fast Convergence Theorem, Z_n^θ converges to 0 almost surely.

Using strict convexity, ψ has at most two zeros. But since, its derivative at 0 is strictly positive and one of the solutions must be 0, the two solutions are necessarily at $\theta^* < 0$ and 0. Finally, let $\tau = \inf\{n \in \mathbb{N} : X_n + k_0 < 0\}$. Clearly, it is a stopping time. According to the optional stopping theorem, $(Z_{n\wedge\tau}^{\theta^*})_n$ is a martingale. In particular:

$$1 = \mathbb{E}(Z_{n\wedge\tau}^{\theta^*}) = \mathbb{P}(\tau \leq n)e^{\theta^* X_\tau} + (1 - \mathbb{P}(\tau \leq n))Z_n^{\theta^*} \to p(k_0)e^{\theta^* X_\tau} \geq p(k_0)e^{k_0\theta^*}.$$