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ON THE ORIGIN OF COMMENSURABILITIES IN THE SOLAR SYSTEM—I

THE TIDAL HYPOTHESIS

S. F. Dermott

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Summary

A method of statistically analysing the distribution of mean motions in the solar system is developed. The results of the analysis suggest that the preference for near-commensurability among pairs of mean motions is a condition of formation rather than the result of evolution. Tidal evolution in the satellite systems of Jupiter, Saturn and Uranus is discussed and it is shown that whilst, in general, no limits can be placed on the dissipative function, Q^{-1}, it is probable that tides have produced only small changes in the mean motions of the satellites.

1. *Introduction.* The satellite system of Jupiter provides the best-known examples of pairs of satellites with near-commensurate mean motions. If n_1, n_2 and n_3 are, respectively, the mean motions of Io, Europa and Ganymede then we have,

$$\frac{n_2}{n_1} = \frac{1}{2} - 0 \cdot 001817$$

$$\frac{n_3}{n_2} = \frac{1}{2} - 0 \cdot 003647.$$

Also, within observational accuracy (nine significant figures) the mean motions of these satellites obey the Laplace relation $n_1 - 3n_2 + 2n_3 = 0$, viz.

$$n_1 - 2n_2 = 0 \cdot 739469091$$

$$n_2 - 2n_3 = 0 \cdot 739469092.$$

Indeed, according to Laplace's analysis (**1**), this relation is exact. There are other examples of near-commensurability in the solar system but none quite so striking as the latter examples. The question of whether the observed number of near-commensurate pairs of mean motions in the solar system is too great to have arisen from a random distribution of mean motions has been considered by Roy & Ovenden (**2**), (**3**) and more recently by Goldreich (**4**). In their first paper, Roy & Ovenden were able to answer the latter question in the affirmative. A summary of their proof, as modified by Goldreich, is given below.

Let n_1 and $n_2 (n_1 > n_2)$ be the mean motions of two bodies about a common centre of force. If two integers, A_1 and A_2, exist such that

$$\left| \frac{n_2}{n_1} - \frac{A_2}{A_1} \right| = \epsilon$$

where ϵ is a small positive number, then these mean motions are said to be nearly commensurate in the ratio A_2/A_1. (As the orbital period, T of a body is given by $T = 2\pi/n$, an entirely equivalent definition of near-commensurability can be given in terms of orbital periods. In later work it is found more convenient to use the latter rather than mean motions.) Since the ratio n_2/n_1 can always be approximated with arbitrary accuracy by the ratio of two integers, it is necessary to limit the size of the integers considered. In the papers of Roy & Ovenden and that of Goldreich the limit for A_1 was set arbitrarily at 7. This confines the ratio n_2/n_1 to the interval 1/7 to 6/7 (inclusive). The number of pairs of mean motions in the solar system which comply with the latter restriction is 46. Of these 46 pairs, seven pairs can be singled out as being special in that they possess what I refer to as configurational regularity. Not only are the mean motions of these satellites near-commensurate, the motions also show a relation between the conjunctions of the satellites and one or more of their orbital elements. For example, for Titan and Hyperion we have,

$$\frac{n_2}{n_1} = \frac{3}{4} - 0\cdot000566$$

and conjunctions always occur near the aposaturnium of Hyperion. The seven near-commensurabilities possessing configurational regularity are: Io and Europa, Europa and Ganymede, Ganymede and Callisto, Io and Ganymede, Mimas and Tethys, Enceladus and Dione, and Titan and Hyperion. Full details of the orbital characteristics of these pairs have been given by Goldreich (4).

The preference for near-commensurability among the observed mean motions was determined by Goldreich as follows. For each ratio, n_2/n_1 the difference of the two fractions, A_2/A_1 and A_2'/A_1' which bound it most closely from above and below is found. Then the probability, P_{ϵ_0} that n_2/n_1 should be within ϵ_0 of one of these fractions is taken to be

$$P_{\epsilon_0} = \frac{2\epsilon_0}{\dfrac{A_2}{A_1} - \dfrac{A_2'}{A_1'}}.$$

This procedure is repeated for all possible pairs (46 in all) and for all possible pairs less those with configurational regularity (39 in all). If, aside from their particular distribution between the fractions 1/7 and 6/7, the ratios n_2/n_1 showed no preference for commensurability then the expectation value, E_{ϵ_0} for the number of near-commensurabilities with $\epsilon \leqslant \epsilon_0$ is found to be 2092 ϵ_0 (46 pairs) and 1856 ϵ_0 (39 pairs). Table I, which includes Goldreich's results (with a few corrections), shows the results of this analysis. The distribution of mean motions definitely deviates from randomness when all 46 pairs are considered but not

TABLE I

ϵ_0	0·0119	0·0089	0·0059	0·0030	0·0015
E_{ϵ_0} (46 pairs)	24·9	18·6	12·3	6·3	3·1
Observed number of pairs of mean motions with $\epsilon \leqslant \epsilon_0$	32	26	20	12	5
E_{ϵ_0} (39 pairs)	22·1	16·5	10·9	5·6	2·8
Observed number of pairs of mean motions with $\epsilon \leqslant \epsilon_0$	25	19	13	6	1

significantly so when the seven pairs with configurational regularity are not included.

Roy & Ovenden (2) have suggested that the observed distribution of orbits may be explained by supposing either that the mechanism of formation of the planets and satellites was such as to favour orbits with commensurate mean motions, or that such orbits are stable and the present distribution is the result of the evolution of the mean motions since formation. The discussion of the stability of near-commensurabilities, initiated by Roy & Ovenden in their second paper (3), has been extended by Goldreich (4). Both discussions favour the hypothesis that there has been considerable evolution in the solar system, either as the result of direct gravitational interactions (Roy & Ovenden) or as the result of tides raised on the planets by the satellites (Goldreich).

The assumptions and requirements of the tidal hypothesis are quite specific and thus amenable to criticism. Firstly, it is assumed that the distribution of orbits was initially random, the preference for commensurability being consonant with this assumption. Secondly, in those satellite systems which now contain near-commensurabilities with configurational regularity (the Jovian and Saturnian systems) it is assumed that tidal torques have produced considerable evolution of the mean motions of the individual satellites over a period comparable to the age of the solar system. If the masses of two satellites which are part of a near-commensurability with configurational regularity are such that their mutual gravitational interaction is appreciable, then Goldreich (4) has proved that such a near-commensurability is stable. Under the action of perturbing forces, tidal or otherwise, angular momentum is exchanged between the satellites in just the correct proportion to keep their motions near-commensurate. His theory finds application to the two-body systems of Mimas and Tethys, Enceladus and Dione, and Titan and Hyperion. The four-body problem of Io, Europa, Ganymede and Callisto is too complex for direct analysis but the configurational regularities possessed by pairs of these satellites suggest that they have the same stability as the former examples. Thus, according to Goldreich, the mean motions of the individual satellites decreased, independently, as the tides fed angular momentum from the planet's spin into their orbits until, by chance, pairs of satellites attained commensurability and consequently stability. It follows from the discussion that when the pairs of satellites (seven in all) which now possess this stability are discounted then the remaining 39 pairs of satellites and planets considered should show a preference for commensurability consonant with a random distribution of mean motions. Table I would seem to indicate that this is so and thus lends support to Goldreich's two hypotheses. One of the aims of this paper is to show that this problem has not been studied in sufficient depth. If, initially, there was a preference for commensurability but not all of the near-commensurabilities were stable then tides and other perturbing forces could have acted to radically change the distribution. The results of Goldreich's analysis would be equally consonant with such initial conditions and thus the question of the origin of the observed preference for commensurability must be considered to be as yet unanswered.

The height of the tidal protuberance raised on a planet by some satellite is proportional to r^{-3}, r being the orbital radius of that satellite. The gravitational force between the protuberance and the satellite is proportional to r^{-2}. Thus all other factors being constant the tidal torque experienced by that satellite falls off rapidly with increasing r and in a satellite system such as that of Saturn it is to be expected that tides have had the greatest effect on the evolution of the inner satellites rather than the outer ones. Indeed, Goldreich (4) has suggested that in the Saturnian system tides have played no appreciable part in the evolution of the Titan–Hyperion commensurability. This may not invalidate his thesis but it is to be noted that this commensurability is one of the three which have been proved to be stable. It is shown below that Goldreich's conclusions with respect to this commensurability are not necessarily correct.

2. *Tidal theory*. The mechanism of tidal evolution has been described in detail by, amongst others, Goldreich & Soter (5), and Jeffreys (6). In what follows I draw freely upon both of these sources but the notation employed is that of Goldreich & Soter—see below.

	Mass	Radius	Orbital semi-major axis	Angular velocity
Satellite, *s*	m	a	r	n (orbital)
Primary, *p*	M	A	R	Ω (axial)

Tidal evolution is a phenomena associated with any departure of a body's material from perfect elasticity and fluidity. The gravitational field of the satellite has the effect of raising tidal protuberances on the planet but without the presence of friction in the motion there would be no secular transfer of angular momentum between the planet and the satellite. The problem is analogous to that of a driven harmonic oscillator, the driving force being provided by the satellite, the frictional forces present having the effect of making the tidal oscillations of the planet lag in phase. For a solution of the problem a knowledge of the dissipative function, Q^{-1} is essential, this being defined by

$$Q^{-1} = \frac{1}{2\Pi E_0} \oint \left(-\frac{dE}{dt}\right) dt \tag{1}$$

where E_0 is the maximum energy stored in the tidal distortion and the integral over $-dE/dt$, the rate of dissipation, is the energy lost during one complete cycle. If the satellite's orbital period is longer than the planet's rotational period (as it is for all cases of interest here), the lagging tide is carried ahead of the satellite by an angle θ. If the system is far from resonance then the phase lag, θ is related to Q by $Q^{-1} = \tan 2\theta$ (7) or, since Q is generally large, $Q^{-1} \simeq 2\theta \simeq \sin 2\theta$.

From an integration of the satellite-produced tidal couple throughout the tidal distortion of the planet, the net effective torque, N is found to be (6)

$$N = \frac{8}{5} \Pi \left(\frac{GmA^4}{r^3}\right) (\rho H \sin 2\theta)_P \tag{2}$$

where G is the gravitational constant. Strictly, the density ρ should refer to the tidal bulge, but as other uncertainties in the discussion are large, this is set equal to the mean density of the planet. The actual height of the tide, H is given by ζh, where

$$\zeta = \frac{3}{4} \left(\frac{m}{M}\right) \left(\frac{A^4}{r^3}\right) \tag{3}$$

is the equilibrium tidal height (disturbing potential divided by undisturbed surface gravity) and

$$h = \frac{5/2}{1+19\mu/2g\rho A}$$

is a correction factor for the rigidity of the planet, μ and for a second degree disturbance of the tidal potential arising from the deformation itself. For large planets (e.g. Jupiter, Saturn and Uranus) self-gravity far exceeds rigidity and h can be assumed to be equal to 5/2 (5). On substituting these equations into equation (2) we obtain

$$N = \frac{9}{4} G \left(\frac{A^5}{Q}\right)\left(\frac{m^2}{r^6}\right).$$

The rate of change of the satellite's mean motion caused by the above torque can then be shown to be (5)

$$\frac{dn}{dt} = -\frac{27}{4} . G \left(\frac{A^5}{Q}\right) \frac{m}{r^8}. \tag{4}$$

Further progress cannot be made without knowledge of Q. This function is peculiar to each satellite-raised tide and is in general dependent on the amplitude and frequency of the latter. For small changes in the mean motion of a satellite it is, perhaps, permissible to assume that Q is a constant, different for each satellite. In no sense, though, can Q be considered to be a parameter of the planet alone. If the interval of time in equation (4) is set equal to the age of the solar system ($4\cdot5 . 10^9$ years) then the various values of Q that would be necessary to bring about various percentage changes in the mean motion of a given satellite during such a period can be calculated—see Table II.

Table II

Primary	Secondary	Q necessary for changes in mean motion during the lifetime of the system ($4\cdot5 . 10^9$ y) of 1%, 10% and 50%					
		1%		10%		50%	
Jupiter	V	1·54	10^5	1·5	10^4	3	10^3
	Io	2·65	10^7	2·6	10^6	5	10^5
	Europa	7·69	10^5	7·7	10^4	1	10^4
	Ganymere	1·20	10^5	1·2	10^4	2	10^3
	Callisto	1·80	10^3	1·8	10^2	4	10
Saturn	Janus	<1		<1		<1	
	Mimas	1·93	10^6	1·9	10^5	4	10^4
	Enceladus	6·83	10^5	6·8	10^4	1	10^4
	Tethys	1·57	10^6	1·6	10^5	3	10^5
	Dione	5·13	10^4	5·1	10^3	1	10^3
	Rhea	1·30	10^4	1·3	10^3	2	10^2
	Titan	3·29	10^4	3·3	10^3	7	10^2
	Hyperion	<1		<1		<1	
Uranus	Miranda	1·22	10^6	1·2	10^5	2	10^4
	Ariel	1·55	10^6	1·5	10^5	3	10^4
	Umbriel	7·42	10^4	7·4	10^3	1	10^3
	Titania	2·69	10^4	2·7	10^3	5	10^2
	Oberon	2·48	10^3	2·5	10^2	5	10

The minimum value of Q ($= \sin 2\theta^{-1}$) is unity and so changes in the mean motions of Janus and Hyperion, caused by tidal action, can be assumed to have been negligible. Strictly, no other conclusions can be drawn from the above table but, all other factors being equal, the results do suggest that changes in the mean motion of Mimas have been $\sim 10^2$ those of Titan, suggesting further that tides have played no appreciable part in the formation of the Titan–Hyperion commensurability.

Goldreich (4) has claimed that a lower bound can be given to Q by consideration of the fact that a satellite in the past could not have had an orbital radius smaller than the planetary radius. His reasoning, however, is incorrect. No matter what source of tidal dissipation is considered, the tidal dissipative function, Q^{-1} associated with such dissipation is always dependent in some way on the frequency of the tide, this being related to $\Omega - n$. Going back in time, as n tends to Ω, $\Omega - n$ tends to zero, Q tends to infinity and all tidal action ceases. The orbital radius, r_b for which $n = \Omega$ is the locus of a barrier through which no satellite can pass if acted on by purely tidal forces. For Jupiter, Saturn and Uranus $r_b \sim 2A$ (see Table III) and is to all intents a parameter of the planet, the orbital angular momentum of the satellites being so much less ($\sim 10^{-2}$) than the rotational angular momentum of the planet.

An alternative method of giving a lower bound to the Q associated with some satellite is afforded if the satellite was formed with an initial orbital radius $r_i > r_b$. A lower limit can be set to r_i by reference to the Roche limit (8) of the satellite. Assume that the satellite was formed by accretion and that the initial density of the agglomeration of particles first formed was ρ_i, this being, in all probability, less than the present density of the satellite ρ_s; also assume that the agglomeration rotated so as to keep the same face to the planet. Then by consideration of the stability of such a body, held together by gravitational forces alone (another assumption!) and acted on by the tidal shear exerted by the planet, we obtain (8)

$$r_i/A > 2\cdot 52 \left(\frac{\rho_P}{\rho_i}\right)^{1/3} = r_L/A.$$

The 'Roche limit', r_L is peculiar to each satellite, being dependent on the initial density of the latter. Satellites of interest here are Jupiter 1 (Io), Saturn 1 (Mimas), and Uranus 5 (Miranda). The densities of these satellites are from *J.B.A.A. Handbook* (9) and all other necessary data from Allen (10)—see Table III.

TABLE III

Primary	r_b/A	r/A	ρ_P	ρ_s
Jupiter	2·22	5·91 (J1)	1·33	4·1 (J1)
Saturn	1·87	3·07 (S1)	0·69	1·0 (S1)
Uranus	2·55	8·05 (U5)	1·56	5 (U5)

As $\rho_i \leqslant \rho_s$, then $r_i > r_b$ only if $r_L(\rho_i = \rho_s) > r_b$. Plots of r_L/A against ρ_i for the three planets are given in Fig. 1. For the Jovian and Uranian systems, $r_L(\rho_i = \rho_s) < r_b$ for the satellites considered and actually for all other satellites (except, perhaps, Callisto) in these systems as well, the densities of these satellites being particularly high. Thus no bounds can be given to Q for the satellites in these systems without additional hypotheses concerning ρ_i. It may be possible to bound Q by considering collisions or near-collisions of the satellites themselves but this is not done here. For Saturn 1 (Mimas), $r_L(\rho_i = \rho_s) > r_b$. Also as $r_i < r = 3{\cdot}07$ Å and $\rho_i \leqslant \rho_s$, then it follows that Mimas must have been formed in the shaded sector of Fig. 1. Even if $\rho_i = \rho_s$ (which is most unlikely, the initial agglomeration must have had a more loosely-packed structure than the present satellite) then the orbital radius of Mimas could not have changed, under the action of tides alone, by more than 45 per cent of the initial value. Hence, with

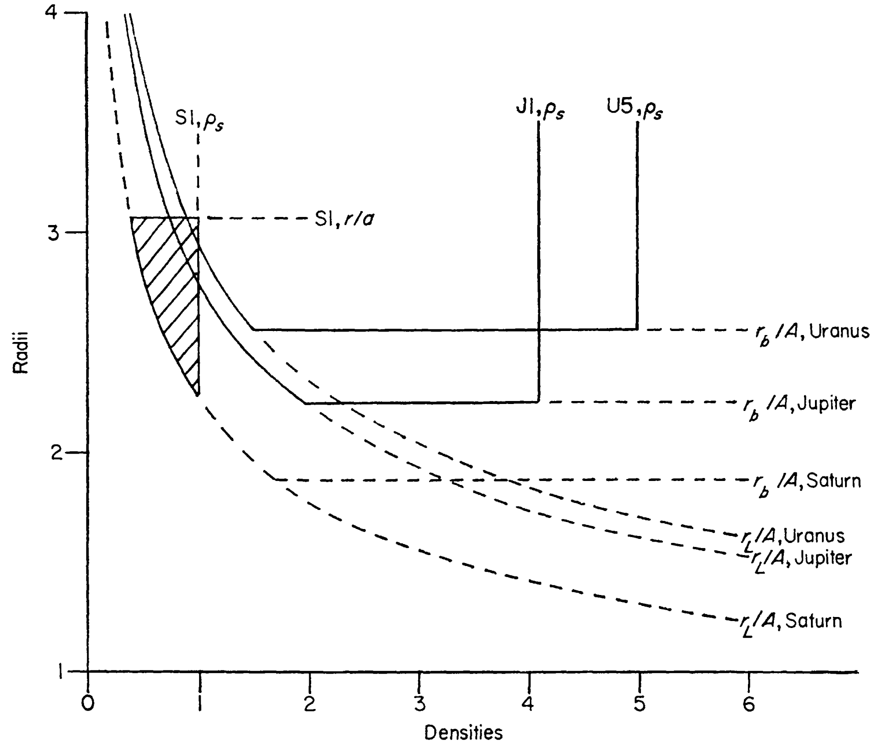

FIG. 1. *Plot of Roche limits r_L as functions of the initial densities ρ_i of the satellites concerned, these being: J1 (Io), S1 (Mimas), and U5 (Miranda). These satellites must have been formed (if they have only been acted on by tidal forces) in the sector bounded by r_L, r_b and r. For Mimas only, this sector has been shaded.*

the assumption that Q remained constant during the change and neglecting the fact that Mimas and Tethys form a stable commensurability, we have, on subtituting r for n (by means of Kepler's third law) in equation (4) and integrating,

$$\frac{2}{13}\left(r^{13/2} - r_i^{13/2}\right) = \frac{9}{2}\left(\frac{G}{M}\right)^{1/2}\frac{A^5}{Q}\, m\, t,$$

where t is the age of the solar system. Hence $Q(\text{Mimas}) > 5 . 10^4$. This fact alone, however, cannot be used to bound Q for any other satellite, say, Q (Titan) until the source of tidal dissipation has been specified and the amplitude and frequency dependence of Q determined.

Without detailed knowledge of the internal structures of the major planets it is impossible to calculate the Q's associated with the various tides raised on these planets. However, the rate of turbulent dissipation in the atmosphere can be estimated and it is not unreasonable to assume that the latter may account for a substantial fraction of the total tidal dissipation. The mechanism of this source of dissipation has already been described by Goldreich & Soter (5). Their model is followed here with the exception of the approximation that the frequency of the semi-diurnal tide is independent of the mean motion of the tide-raising satellite, i.e. that $\Omega - n \simeq \Omega$. This change is not related to any of the other assumptions of the theory and therefore it is only necessary, here, to describe the results of such a modification.

According to Goldreich & Soter the protuberance racing along the surface of the planet under the action of the satellite gives rise to tidal currents in the depth of the atmosphere. The motion of these currents in the boundary layer of the solid–fluid interface is turbulent if

$$D \leqslant \frac{3}{4} A^5 \left(\frac{\Omega - n}{2\nu}\right)^{1/2} \frac{m}{M} \cdot \frac{r^{-3}}{Re},$$

D being the depth of the atmosphere, ν the kinematic viscosity of the fluid which is assumed to be $< 10^{-3}$ c.g.s. and Re the Reynolds number. With the assumption that the Reynolds number is 300 the turbulence criterion becomes

$$D \leqslant 5 \cdot 6 \; 10^{-3} \frac{A^5}{r^3} \cdot \frac{m}{M} (\Omega - n)^{1/2} \text{ cm.}$$

The maximum depth of atmosphere, corresponding to each satellite, such that the motion associated with the satellite in the boundary layer is turbulent, is given in Table IV.

TABLE IV

Turbulence criteria: maximum depth of atmosphere, D

Primary	Secondary	$R_e D/A$	$D(R_e = 300)$ (km)
Jupiter	V	407	$>A$
	Io	290	69 000
	Europa	46	11 000
	Ganymede	37	9000
	Callisto	4	1000
Saturn	Mimas	22	4000
	Enceladus	2	400
	Tethys	10	2000
	Dione	8	2000
	Rhea	8	2000
	Titan	37	7000
	Hyperion	0·02	4
Uranus	Miranda	260	$>A$
	Ariel	130	11 000
	Umbriel	20	2000
	Titania	40	3000
	Oberon	11	900

An upper bound to the Jovian atmosphere of ~2800 km has recently been suggested by Hide (**11**) from hydrodynamic considerations concerning the Great Red Spot. This would suggest that for this planet the motions are indeed turbulent for all tides except the tide raised by the satellite Callisto. No independent estimate of the depth of the Saturnian atmosphere is available but it is, perhaps, reasonable to assume that it is about that of Jupiter; the two planets have similar radii and the lower surface gravity of Saturn must compensate, to some extent, for its cooler temperature. This being so, it is not at all obvious that the turbulence criteria are satisfied except, perhaps, for the tides raised by the satellites Mimas and Titan. Calculations are made, however, on the assumptions that all motions are turbulent. Thus the rate of dissipation calculated is an upper rate but note that this does not imply that bounds are placed on the total dissipation rate.

The energy dissipated per unit area in the turbulent boundary layer is $k\rho v^3$ where $k = 0 \cdot 002$ is the coefficient of skin friction, ρ is the density of the atmosphere, and v is the velocity of the tidal current. The latter being estimated to be

$$v = (\Omega - n)\, A . \frac{\zeta}{D}.$$

The total energy loss per tidal period is

$$\Delta E = \left(\frac{\pi}{\Omega - n}\right) k\rho v^3 \text{ erg cm}^{-2}.$$

The potential energy stored per unit area in the tidal distortion is

$$E_0 = \tfrac{1}{2} g \rho \zeta^2 \text{ erg cm}^{-2}$$

where $g = GM/A^2$ is the surface gravity. Thus, from the definition of Q in equation (1) we have,

$$Q \simeq \frac{2\pi E_0}{\Delta E} = \frac{GMD^3}{kA^5} \cdot \frac{1}{(\Omega - n)^2\, \zeta}$$

$$= \frac{(\text{constant})_P}{(\text{tidal frequency})^2\, (\text{tidal amplitude})}.$$

Substituting for ζ from equation (3)

$$Q = \frac{4}{3} \cdot \frac{GM^2 r^3 D^3}{k\Omega^2 A^9 m} \cdot \frac{1}{(1 - n/\Omega)^2}.$$

The result obtained is that of Goldreich & Soter multiplied by the factor $(1 - n/\Omega)^{-2}$. For most satellites this correction factor is negligible compared with the other uncertainties in the argument, but note that as n tends to Ω, Q tends to infinity, which is to be expected for when the motions are synchronous all oscillations of the planet cease. The above formula for Q can now be substituted in equation (4) and the relative changes in the mean motions of the satellites since their formation $4 \cdot 5 . 10^9$ years ago calculated—see Table V. No account is taken in these calculations of the affects of stable commensurabilities.

The relative changes indicated in Table V only depend on Q being inversely proportional to the square of the tidal frequency and to the amplitude of the tide. They will hold for any source of dissipation with this frequency and amplitude dependence, turbulent dissipation being just an example. If such a source of dissipation accounts for a substantial fraction of the total dissipation associated with any one of the Saturnian satellites then it can be concluded that changes in the mean motions of the satellites (except Hyperion and, of course, Janus) have all been of the same order. Thus, if tides have been responsible for the formation of any of the commensurabilities in this system then they could be responsible for all of them including that of Titan and Hyperion. If, however, the major source of dissipation is specifically atmospheric turbulence then estimates of the

TABLE V

Relative tidal changes in orbital periods

Primary	Secondary	$\left(1-\frac{n}{\Omega}\right)^2$	$-\frac{\delta n}{n}\Big/\left(\frac{A}{D}\right)^3$	$-\delta n/n$ for various D (km)	
				2856	1428
Jupiter	V	0·03	$4{\cdot}33.10^{-13}$	7.10^{-9}	5.10^{-8}
	Io	0·60	$5{\cdot}01.10^{-6}$	8.10^{-2}	6.10^{-1}
	Europa	0·78	$2{\cdot}80.10^{-8}$	4.10^{-4}	4.10^{-3}
	Ganymede	0·89	$5{\cdot}40.10^{-9}$	8.10^{-5}	7.10^{-4}
	Callisto	0·95	$6{\cdot}97.10^{-12}$	1.10^{-7}	9.10^{-7}
				964	482
Saturn	Mimas	0·28	$1{\cdot}19.10^{-8}$	3.10^{-3}	2.10^{-2}
	Enceladus	0·46	$5{\cdot}89.10^{-9}$	1.10^{-3}	1.10^{-2}
	Tethys	0·58	$8{\cdot}17.10^{-8}$	2.10^{-2}	2.10^{-1}
	Dione	0·70	$2{\cdot}53.10^{-8}$	6.10^{-3}	5.10^{-2}
	Rhea	0·81	$6{\cdot}03.10^{-9}$	1.10^{-3}	1.10^{-2}
	Titan	0·94	$8{\cdot}66.10^{-9}$	2.10^{-3}	2.10^{-2}
	Hyperion	0·96	$9{\cdot}90.10^{-16}$	2.10^{-10}	2.10^{-9}
				238	119
Uranus	Miranda	0·47	$6{\cdot}26.10^{-9}$	6.10^{-3}	5.10^{-2}
	Ariel	0·67	$5{\cdot}44.10^{-8}$	5.10^{-2}	4.10^{-1}
	Umbriel	0·79	$4{\cdot}52.10^{-10}$	5.10^{-4}	4.10^{-3}
	Titania	0·90	$3{\cdot}70.10^{-10}$	4.10^{-4}	3.10^{-3}
	Oberon	0·93	$8{\cdot}76.10^{-12}$	9.10^{-6}	7.10^{-5}

depth of the Saturnian atmosphere would suggest that tides have been responsible for only small changes in the mean motions of the satellites. In the Jovian system it is only necessary that Io's mean motion should have changed considerably for the existing commensurabilities to be accounted for. Estimates of the depth of the Jovian atmosphere would indicate that the tidal hypothesis is tenable for this system but it is certainly not proved. Until more detailed knowledge of the constitutions of these planets is available none of the above arguments can be considered to be conclusive.

3. *Statistical analysis.* The statistical method of Goldreich (4) only gives a measure of the preference for commensurability among pairs of mean motions. The distribution of mean motions could be non-random whilst not showing the latter preference. Such a distribution would be equally as interesting as the former. A more detailed knowledge of the distribution of mean motions is obtained as follows.

If A_2/A_1 ($A_2 < A_1$) and A_2'/A_1' ($A_2' < A_1'$) are the two fractions which bound n_2/n_1 ($n_2 < n_1$) most closely from above and below, then we define

$$a = \frac{\dfrac{n_2}{n_1} - \dfrac{A_2'}{B_1'}}{\dfrac{A_2}{A_1} - \dfrac{A_2'}{A_1'}}.$$

The number a varies from 0 to 1, the preference for commensurability increasing both as a tends to 0 and as a tends to 1. For ease of presentation, we define the deviation δ of n_2/n_1 from A_2'/A_1' to be

$$\delta = a-b,$$

where

$$b = 0 \text{ if } a \leqslant 0 \cdot 5,$$

and

$$b = 1 \text{ if } a > 0 \cdot 5.$$

Thus, $-0\cdot5 \leqslant \delta \leqslant +0\cdot5$ and the preference for commensurability increases as δ tends to 0. The deviations of all possible pairs of mean motions are found and a histogram constructed. The number of pairs is not large ($\leqslant 46$) but is, perhaps, just large enough for a simple χ^2-test for non-randomness to be applied. The range of δ is divided into five intervals of 0·2 and the observed frequency, f in each interval found. (In applying the χ^2-test, 0·5 is subtracted from $|f-\phi|$, ϕ being the expected frequency in a given interval. (This is Yates's correction for

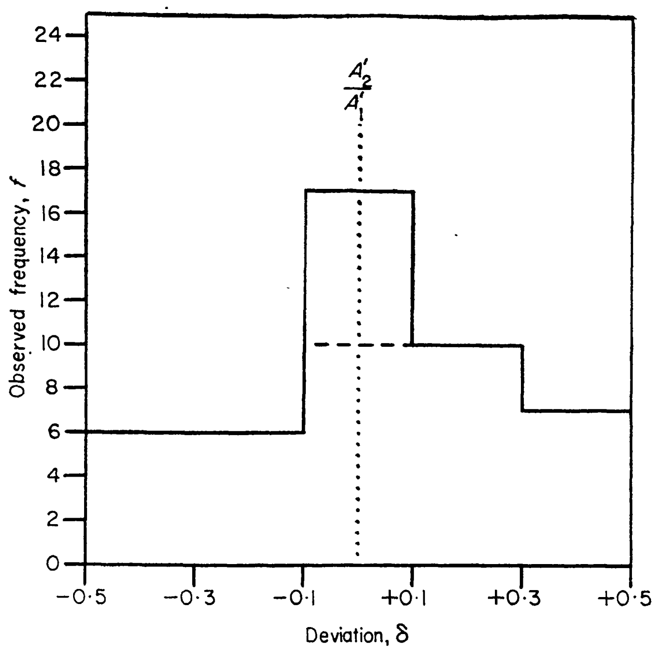

FIG. 2. *Histogram showing the preference for near-commensurability among pairs of mean motions; A_1, $A_1' \leqslant 7$. All possible pairs are considered, i.e.* 46 *pairs (solid line) and all possible pairs less those near-commensurabilities with configurational regularity, i.e.* 39 *pairs (broken line).*

discontinuity.) Also intervals with small frequencies ($\leqslant 5$) are amalgamated with neighbouring intervals.) The advantage of this statistical method is that not only can it indicate a preference for commensurability, i.e. $\delta = 0$, but also a preference for any other deviation, say, $\delta = -0\cdot2$ or $\delta = +0\cdot3$.

It can easily be shown from classical perturbation theory that the dynamical consequences of near-commensurability decrease as $A_1 - A_2$ increases. In this paper the above procedure is followed for A_1, $A_1' \leqslant 7$ and also for A_1, $A_1' \leqslant 6$. All possible pairs of mean motions are considered and also all possible pairs less those with configuration regularity. The results of the analysis are shown in Figs 2 and 3.

With A_1, $A_1' \leqslant 7$, all possible pairs being considered, the distribution is definitely non-random, the result being significant at the 3 per cent level. The histogram also indicates a preference for commensurability, i.e. $\delta = 0$, but no correlation test is applied. When those pairs with configurational regularity are subtracted from the distribution there is still a slight preference for commensurability but the result of the test for non-randomness is only significant at the 28 per cent level. Note, however, that there is a slight asymmetry in the distribution, δ showing a preference for being greater than 0 rather than less.

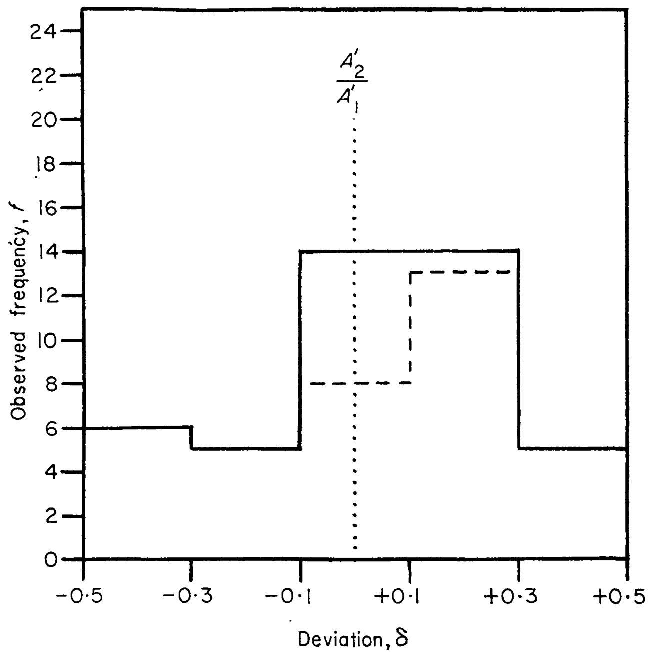

FIG. 3. *Histogram showing the preference for near-commensurability among pairs of mean motions;* A_1, $A_1' \leqslant 6$. *All possible pairs are considered, i.e.* 44 *pairs (solid line) and all possible pairs less those near-commensurabilities with configurational regularity, i.e.* 37 *pairs (broken line).*

With A_1, $A_1' \leqslant 6$, all possible pairs being considered, the distribution is again definitely non-random, the result being significant at the 1 per cent level. Of more interest, however, is the observation that when those pairs with configurational regularity are subtracted from the distribution the latter is still non-random, the result being significant at the 8 per cent level, but there is a preference for $\delta = +0{\cdot}2$ rather than $\delta = 0$. The significance level is too high for the result to be considered at all conclusive but the observation is made striking by the realization that the observed distribution is exactly the opposite to that expected on the basis of Goldreich's tidal origin hypothesis. In general, it is to be expected that n_2/n_1 will increase with time, since dn/dt depends on r^{-8} (see equation (4)). This implies that, in general, δ will also increase with time, the evolutionary path being from 0 to $+0{\cdot}5$ to $-0{\cdot}5$ to 0; stability being attained, presumably, when $\delta = 0$. Thus, if the distribution was initially random and the deviation of each pair increased, some pairs consequently attaining stability, then after appreciable evolution the distribution would show a preference for commensurability (all possible pairs being considered) and a corresponding absence of pairs with deviations slightly greater than 0. This is certainly not observed.

The above arguments cannot, unfortunately, be considered rigorous, there being far too many unknowns in the problem, but it is suggested that the observed distribution is better explained by the hypothesis that initially there was a preference for commensurability among pairs of mean motions but not all of the pairs possessed stability and that tides have since acted to slightly increase the deviations of these pairs. The observed preference for $\delta = +0{\cdot}2$ is consonant with an average increase in n_2/n_1 of the order of 4 per cent.

4. *Conclusion.* It is suggested that the preference for commensurability among pairs of mean motions in the solar system is better explained by the hypothesis that such a preference is original rather than the result of tidal evolution. The arguments are not rigorous but both statistical analysis and tidal theory indicate that changes in the mean motions of the satellites of Jupiter, Saturn, and Uranus have only been small.

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Royal Military College of Science,
Shrivenham,
Wilts.
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