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ON THE ORIGIN OF COMMENSURABILITIES IN THE SOLAR SYSTEM—III

The Resonant Structure of the Solar System

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Summary

A method of estimating the statistical significance of resonance relations found among the mean motions of planets and satellites is described. The resonance relations found recently by Molchanov are shown to be consonant with a random distribution of mean motions and thus have no significance.

1. *Introduction.* It has recently been claimed by Molchanov (**1**) that the structure of the solar system is determined by sets of resonance relations, i.e. that the solar system as a whole is in a resonant state. Representing the orbital frequency (or mean motion) of a body by ω_i various sums

$$\sum_{i=1}^{s} \alpha_i \omega_i = \Sigma$$

can be formed, α_i (i = 1, 2, ..., s) being either a small integer or zero and s the number of bodies in the system. If $\Sigma \simeq 0$ for some set of α_i then the set of n ($\leqslant s$) frequencies for which $\alpha_i \neq 0$ are said to be in resonance, the resonant state being characterized by the set of α_i. The number of independent resonance relations possible in any one system is smaller by one than the number of frequencies, all frequencies being expressed in terms of one particular frequency (arbitrarily chosen). As a result, the configuration of a system with maximum resonance is uniquely determined by a set of whole numbers (the various sets of α_i)—a table of resonance vectors. Molchanov has shown that under certain circumstances the appearance of such a resonant structure in a given oscillating system is inevitable. Any random configuration of planets and satellites will evolve under the action of small dissipative perturbations until a resonant state is achieved. Thus Molchanov's theory is capable of not only accounting for the observed preference for near-commensurability among pairs of mean motions in the solar system (**2**) (a large percentage of the resonances cited involve only two bodies) but also for the complete structure of the solar system, excepting the question of scale.

The ' resonances ' among the nine planets found by Molchanov are:

$$\begin{array}{lllllllllr}
\omega_1 & -\omega_2 & -2\omega_3 & -\omega_4 & & & & & & = 0 \\
 & \omega_2 & & -3\omega_4 & & -\omega_6 & & & & = 0 \\
 & & \omega_3 & -2\omega_4 & +\omega_5 & -\omega_6 & +\omega_7 & & & = 0 \\
 & & & \omega_4 & -6\omega_5 & & -2\omega_7 & & & = 0 \\
 & & & & 2\omega_5 & -5\omega_6 & & & & = 0 \\
 & & & & \omega_5 & & -7\omega_7 & & & = 0 \\
 & & & & & & \omega_7 & -2\omega_8 & & = 0 \\
 & & & & & & \omega_7 & & -3\omega_9 & = 0
\end{array}$$

Following Molchanov, Jupiter's frequency (ω_5) is chosen as the unit of measurement and the other frequencies, ω_{theor}, determined from the above relations. The discrepancies, $\Delta\omega$ between these frequencies and the observed frequencies, ω_{obs}, (along with the resonance vectors of the system in the form of a coefficient matrix) are given in Table I. All data used in this paper, unless otherwise stated, are from Allen (3).

TABLE I

Frequencies and resonance vectors of the solar system

Planet	ω_{obs}	ω_{theor}	$\lvert\Delta\omega/\omega_{\text{obs}}\rvert$	Coefficient matrix									$\left\lvert\dfrac{\Sigma}{\omega_{\min}}\right\rvert$
Mercury	49·24	49·20	0·0008	1	−1	−2	−1	0	0	0	0	0	0·0110
Venus	19·28	19·26	0·0010	0	1	0	−3	0	−1	0	0	0	0·1022
Earth	11·862	11·828	0·0031	0	0	1	−2	1	−1	1	0	0	0·0933
Mars	6·307	6·287	0·0032	0	0	0	1	−6	0	−2	0	0	0·1731
Jupiter	1·000	1·000	0·0000	0	0	0	0	2	−5	0	0	0	0·0339
Saturn	0·4027	0·400	0·0068	0	0	0	0	1	0	−7	0	0	0·0840
Uranus	0·14116	0·14286	0·0120	0	0	0	0	0	0	1	−2	0	0·0395
Neptune	0·07199	0·07143	0·0078	0	0	0	0	0	0	1	0	−3	0·0471
Pluto	0·04775	0·04762	0·0027										

For all planets $\Delta\omega/\omega$ is remarkably small. It must be noted, however, that in forming the sums, Σ, the coefficients, α_i, are arbitrary and so it is suggested here that a better indication of whether a resonance relation corresponds to an actual resonance in the dynamical sense is afforded by comparing Σ with $\omega_{\min}$, the smallest frequency involved in the relation. Accordingly, the values of $\Sigma/\omega_{\min}$ for the relations found by Molchanov are included in Table I. The mean value of the latter quantities is $\sim 0\cdot 07$. The aim of this paper is to show that contrary to Molchanov's conjecture the occurrence of such relations is simply due to chance. The various resonance relations found by Molchanov among the satellites of the major planets and the corresponding values of $\Sigma/\omega_{\min}$ for these relations are given in Table II. In Molchanov's paper ω_{obs} for Miranda (Uranus 5) is mistakenly given as 6·529, the correct value being 6·157. In effect the value 6·529 corresponds to a fictitious satellite and the fact that it can be fitted into a scheme of resonance relations is in itself an indication that these relations have no significance.

2. *Statistical analysis.* Consider a system of s planets or satellites of frequencies $\omega_1, \omega_2, \ldots, \omega_s$ where $\omega_1 > \omega_2 > \ldots > \omega_s$. Let $\omega_1 = 1$ then $0 < \omega_i \leqslant 1 (1 \leqslant i \leqslant s)$. Assume that any ratio of frequencies, ω_i/ω_j, is never an exact integer and that all ω_i are known with such accuracy that all Σ are different. These assumptions serve to simplify the discussion.

Resonance relations can exist in any set of n frequencies ($n \leqslant s$), the number of different sets of n which can be formed from a total of s being given by ϕ_n where

$$\phi_n = \frac{s(s-1) \ldots (s-n+1)}{n(n-1) \ldots 1}.$$

For a resonance relation to be considered significant the magnitudes of the coefficients, α_i, must be small, say, $\leqslant 6$. This limit, $\alpha_{\max}$, on α_i is, of course, arbitrary. It is, however, found convenient in this paper to assume that $\alpha_{\max}$ is even so that $\frac{1}{2}\alpha_{\max}$ is also an integer. As $\alpha_{\max}$ is arbitrary this restriction is unimportant.

TABLE II

Frequencies and resonance vectors of satellite systems

Satellite	ω_{obs}	ω_{theor}	$\lvert\Delta\omega/\omega_{\mathrm{obs}}\rvert$	Coefficient matrix								$\left\lvert\frac{\Sigma}{\omega_{\min}}\right\rvert$
				Saturn's satellites								
1. Mimas	16·920	16·800	0·0071	−1	0	2	0	0	0	0	0	0·0031
2. Enceladus	11·637	11·600	0·0032	0	−1	0	2	0	0	0	0	0·0026
3. Tethys	8·446	8·400	0·0055	0	0	−1	0	2	1	0	2	0·0738
4. Dione	5·826	5·800	0·0045	0	0	0	−1	2	−1	0	−1	0·1609
5. Rhea	3·530	3·500	0·0086	0	0	0	0	−1	2	2	0	0·0411
6. Titan	1·000	1·000	0·0000	0	0	0	0	0	−3	4	0	0·0030
7. Hyperion	0·7494	0·7500	0·0008	0	0	0	0	0	−1	0	5	0·0247
8. Iapetus	0·2010	0·2000	0·0050									
				Jupiter's satellites								
1. Io	4·044	4·000	0·0110	1	−2	0	0					0·0073
2. Europa	2·015	2·000	0·0075	0	1	−2	0					0·0147
3. Ganymede	1·000	1·000	0·0000	0	0	−3	7					0·0021
4. Callisto	0·4287	0·4285	0·0004									
				Uranus' satellites								
5. Miranda	6·157	6·545	0·0630	−1	1	1	1	0				0·3980
1. Ariel	3·454	3·454	0·0000	0	−1	1	2	−1				0·0001
2. Umbriel	2·101	2·091	0·0048	0	0	−2	1	5				0·0490
3. Titania	1·000	1·000	0·0000	0	0	1	−4	3				0·0629
4. Oberon	0·6466	0·6364	0·0160									

With this nomenclature, the number of different relations which can be formed involving n frequencies for which Σ is positive (each relation has a negative counterpart) is

$$\tfrac{1}{2}\phi_n(2\,\alpha_{\max})^n.$$

The maxima, $\Sigma_{\max}$, of the sums of these relations must be $\leqslant n\,\alpha_{\max}$ and on average is $\sim\tfrac{1}{2}n\,\alpha_{\max}$. If we assume that the sums are distributed uniformly between 0 and $\Sigma_{\max}$ then the expected number of relations involving n frequencies for which $\delta\Sigma > \Sigma > 0$, where $\delta\Sigma$ is some arbitrary small limit, is simply

$$\tfrac{1}{2}\phi_n(2\alpha_{\max})^n\,\frac{\delta\Sigma}{\tfrac{1}{2}n\alpha_{\max}}.$$

Of course, there is no reason to suppose that the distribution of Σ is uniform and therefore the above estimate of the expectation value, which I denote by $(\mathscr{E}_n)_{\max}$, is to be regarded as the maximum number of relations which can be expected. The minimum number of relations to be expected, $(\mathscr{E}_n)_{\min}$, is estimated as follows.

With any particular set of n frequencies form sums $\sum\limits_i \alpha_i\omega_i$ (i being summed over the n products, $\alpha_i\omega_i$) where, this time, the coefficients, α_i, take any integral value between $-\tfrac{1}{2}\alpha_{\max}$ and $+\tfrac{1}{2}\alpha_{\max}$ including zero. There is a choice of $\alpha_{\max}+1$ coefficients and therefore the number, D, of different relations for which $\Sigma > 0$ which can be formed with this particular set of n frequencies is given by

$$D = \tfrac{1}{2}[(\alpha_{\max}+1)^n - 1].$$

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As $0 < \omega_i \leqslant 1$ then $\Sigma_{\max} < \frac{1}{2} n\alpha_{\max}$. Construct a histogram the interval width of which is such that if the sums of the D relations were uniformly distributed between 0 and $\Sigma_{\max}$ then each interval would contain two different sums. The interval width which satisfies this requirement is

$$\sim \frac{n\alpha_{\max}}{D}.$$

Combine any two relations found together in such an interval and having sums of, say, Σ and Σ' to form a single relation the magnitude of the sum, Σ'', of which is given by

$$|\Sigma''| = |\Sigma - \Sigma'| = |\sum_i (\alpha_i - \alpha_i')| \leqslant \frac{n\alpha_{\max}}{D}.$$

The coefficients in relation formed in this way are never greater than $\alpha_{\max}$, as $|\alpha_i| \leqslant \frac{1}{2}\alpha_{\max}$ and therefore $|\alpha_i - \alpha_i'| \leqslant \alpha_{\max}$. The number of intervals in the histogram is $\frac{1}{2}D$ and this therefore is the maximum number of such relations which can be formed. Of more interest here, though, is that the minimum number is $\frac{1}{4}D$, the minimum occurring when each interval contains either one or three sums. This argument is completely independent of the distribution of Σ. A minimum of $\frac{1}{4}D$ relations can be found the sums of which equal 0 to within $(n\alpha_{\max})/D$ and it is estimated therefore that a minimum of E_n can be found the sums of which equal 0 to within some limit $\delta\Sigma$, E_n being given by

$$E_n = \frac{D^2}{4} \cdot \frac{\delta\Sigma}{n\alpha_{\max}}.$$

Not all of these E_n relations, though, involve n frequencies as the coefficients, $\alpha_i - \alpha_i'$ are occasionally 0. Let E_n' be the number specifically involving n frequencies. As

$$E_n - E_n' \leqslant E_{n-1}$$

i.e.

$$E_n' \geqslant E_n - E_{n-1},$$

$E_n - E_{n-1}$ can be taken to be a minimum estimate of E_n'. (E_n is usually much greater than E_{n-1} and therefore the error introduced by this approximation is small.) Corrections must also be made for multiple counts. In forming relations by combination it is obvious that some relations are formed more than once. This can be seen by considering the relation

$$5\omega_1 - 3\omega_2 = \Sigma$$

which can be resolved to form pairs of relations such as $3\omega_1 - 2\omega_2$ and $-2\omega_1 + \omega_2$ or $4\omega_1 - 2\omega_2$ and $-\omega_1 + \omega_2$, etc. The number of such pairs of relations into which any one relation can be resolved depends simply on the coefficients involved in that relation. E_n' must be reduced by some factor θ_n to allow for this. By determining the relative frequency with which a given coefficient can be expected to occur in a relation it can be shown that $(\mathscr{E}_n)_{\min}$ is given by

$$(\mathscr{E}_n)_{\min} = \frac{(E_n - E_{n-1})}{\theta_n} \cdot \phi_n$$

where

$$\theta_n = \left\{\frac{(\alpha_{\max})^2 + (\alpha_{\max} - 1)^2 + \ldots + 1}{\alpha_{\max} + \alpha_{\max} - 1 + \ldots + 1}\right\}^n.$$

Formulae have been derived for the maximum, $(\mathscr{E}_n)_{\max}$, and the minimum, $(\mathscr{E}_n)_{\min}$, number of relations to be expected involving n frequencies for which $\delta\Sigma > \Sigma > 0$. Evaluations of these formulae for $\delta\Sigma = 0{\cdot}002$ and $s = 5$, i.e. for a satellite system analogous to that of Uranus, are given in Table III. The ratio $\delta\Sigma/\omega_{\min}$ is of more significance than $\delta\Sigma$ itself but as in a system of five satellites $\omega_{\min}$ is, on average, $\sim 0{\cdot}2$ then for ease of assessment of the various relations the fact that $\delta\Sigma \sim 0{\cdot}002$ can be taken to imply that $\delta\Sigma/\omega_{\min} \sim 0{\cdot}01$. Also shown in Table III are values of $(\mathscr{E}_n)_{\text{mean}}$, the geometric mean of $(\mathscr{E}_n)_{\max}$ and $(\mathscr{E}_n)_{\min}$. As this mean value never differs from the latter values by more than a factor of ~ 5 and as we are only interested in an estimate of the expectation values then this mean value will be used to assess the various relations found by Molchanov.

TABLE III

Expectation values, $\delta\Sigma/\omega_{\min} \sim 0{\cdot}01$

n	$(\mathscr{E}_n)_{\max}$	$(\mathscr{E}_n)_{\min}$	$(\mathscr{E}_n)_{\text{mean}}$
2	0·2	0·01	0·06
3	2	0·1	0·4
4	9	0·4	2
5	17	0·7	3·5

For the Uranian system Molchanov cites two relations involving three frequencies for which $\Sigma/\omega_{\min} \sim 0{\cdot}05$. From Table III it can be seen that the number of such relations to be expected, given a random distribution of frequencies, is $\sim 2 (\mathscr{E}_n \propto \delta\Sigma)$. Thus the cited relations have no particular significance. Similar criticisms can be made of the other relations found by Molchanov. The values of $\Sigma/\omega_{\min}$ for these relations are much too high for the relations to be considered at

TABLE IV

Frequencies and resonance vectors of random systems

Satellite	ω_{obs}	ω_{theor}	$\lvert\Delta\omega/\omega_{\text{obs}}\rvert$	Coefficient matrix					$\lvert\Sigma/\omega_{\min}\rvert$
			System A						
1	16·7125	16·7097	0·0002	−1	3	6	2	−1	0·0005
2	1·8040	1·8036	0·0002	−1	4	4	0	3	0·0005
3	1·6240	1·6235	0·0003	0	4	−4	1	−2	0·0013
4	1·2785	1·2783	0·0002	0	0	5	−4	−3	0·0059
5	1·0000	1·0000	0·0000						
			System C						
1	4·1384	4·1429	0·0011	4	−6	−1	4	2	0·0002
2	3·3518	3·3571	0·0016	0	1	−1	−1	−1	0·0033
3	1·7141	1·7143	0·0001	2	−2	2	−5	0	0·0015
4	1·0000	1·0000	0·0000	1	0	−3	1	0	0·0037
5	0·6355	0·6428	0·0115						
			System D						
1	17·1062	17·0909	0·0009	1	−2	−1	2	−4	0·0011
2	6·9608	6·9545	0·0009	1	−2	−4	3	0	0·0088
3	1·5439	1·5455	0·0010	0	0	2	−4	1	0·0019
4	1·0000	1·0000	0·0000	−2	6	0	−3	−5	0·0006
5	0·9104	0·9091	0·0014						

all interesting. To corroborate this conclusion similar ' resonance relations ' have been found for various fictitious satellite systems (analogous to that of Uranus)—see Table IV. The construction of these systems from random number tables has been described in a previous paper (**4**). The relations found for these random systems involve only small coefficients ($\alpha_i \leqslant 6$) and yet have values of $\Sigma/\omega_{\min}$ considerably smaller than the relations found by Molchanov. In fact, the latter small values of $\Sigma/\omega_{\min}$ indicate, perhaps, that $(\mathscr{E}_n)_{\max}$ is a better estimate of the expectation value than $(\mathscr{E}_n)_{\text{mean}}$.

Other relations of possible interest to which this analysis can be directly applied are:

(i) a relation between the mean motions of Ariel, Umbriel, Titania and Oberon (denoted by n_1, n_2, n_3 and n_4 respectively) found by d'Arrest (**5**), viz.

$$n_1 - n_2 - 2n_3 + n_4 \simeq 0,$$

(ii) the Laplace relation between the mean motions of Io, Europa and Ganymede (denoted by n_1, n_2 and n_3 respectively), viz.

$$n_1 - 3n_2 + 2n_3 = 0,$$

and (iii) a relation analogous to the Laplace relation between the mean motions of Miranda, Ariel and Umbriel (denoted by n_5, n_1 and n_2 respectively) found by Harris (**6**), viz.

$$n_5 - 3n_1 + 2n_2 \simeq 0.$$

The Laplace relation is exact to within observational accuracy and therefore cannot possibly be due to chance. The expectation values of the other relations depend on the limit placed on α_i. The fact that the largest coefficient in the d'Arrest relation is two does not imply that $\alpha_{\max}$ is two as such a relation would, perhaps, still be considered significant even if it contained coefficients much higher than this. Thus there is an unavoidable element of arbitrariness in these estimates. Evaluations of $(\mathscr{E}_n)_{\max}$ and $(\mathscr{E}_n)_{\text{mean}}$ for various values of $\alpha_{\max}$ are given in Table V. When an expectation value is small, i.e. $\ll 1$, it can be interpreted as the probability that a certain relation is due to chance.

TABLE V

Relation	Σ*	$\alpha_{\max}$	$(\mathscr{E}_n)_{\max}$	$(\mathscr{E}_n)_{\text{mean}}$
d'Arrest	$1 \cdot 3 \times 10^{-5}$	6	0·06	0·01
		4	0·02	0·004
		2	0·002	0·0009
Harris	$3 \cdot 2 \times 10^{-4}$	6	0·3	0·07
		4	0·1	0·03

* Data from Kuiper (1956).

3. *Conclusion.* It has been shown that the relations found by Molchanov are consonant with a random distribution of mean motions and thus have no significance. Accurate, or seemingly so, resonance relations can be deceptive. Consider the following relations between the rotational frequency of Jupiter about its own axis, ω_1, and the orbital frequencies of Io, Europa, Ganymede and Callisto these being represented by ω_2, ω_3, ω_4 and ω_5 respectively.

$$\omega_1 = 2\omega_2 + 5\omega_3 - 2\omega_5.$$

The predicted value of ω_1, found by substituting the observed values of ω_2, ω_3 and ω_5 into the above relation, corresponds to a period of 9 h 55·38 m. The rotational period of Jupiter as derived from observations of radio frequency emissions is 9 h 55·49 m (7). The discrepancy, Σ, corresponds to a difference of only 6·6 s, Σ/ω_1 being $\sim 0{\cdot}0002$. The ratio Σ/ω_5, however, is $\sim 0{\cdot}01$ and from Table III it can be seen that a ratio of this magnitude is consonant with a random distribution of frequencies. With $s = 5$ and $n = 4$ the expected number of relations, $(\mathscr{E}_4)_{\text{mean}}$ for which $\Sigma/\omega_{\text{min}} \sim 0{\cdot}01$ is ~ 2. Other relations between these frequencies are:

$$\omega_1 = 4\omega_2 - \omega_3 + 4\omega_4 - 2\omega_5,$$

which predicts a period of 9 h 55·38 m and

$$\omega_1 = 4\omega_2 + \omega_3 - 2\omega_5,$$

which predicts a period of 9 h 54·4 m. This does not imply that all accurate resonance relations are without significance. Those near-commensurabilities with configurational regularity (2) and, of course, the Laplace relation are most definitely in a different category. Rather, it implies that the presence of configurational regularity is, perhaps, a more reliable indication of resonance than mere numerical coincidence. It is noted that no such regularity has been associated with the ‘ resonances ’ found by d’Arrest and Harris.

Finally, following Molchanov, the fact that the solar system as a whole is not in a resonant state indicates that the effects of dissipative mechanisms have been small and thus that the present distribution of the planets and satellites reflects closely the initial distribution. This conclusion is in full accord with previous conclusions (2), (4) of the present author.

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