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Interior Point algorithm for Parcimonious problem of Mars spectral observion.

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Nantes, 2025

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1 Introduction

Satellite have been send to Mars last few decades in order to get data of the red planet (Images, Videos, Spectroscopy, ... ect). We focus here on the analysis of Mars Spectrum ; our goal is here to find the chemical and mineral composition of the Mars soil.

In order to achieve such objectif, we raise an optimisation problem. In fact, we know the spectrum of every mineral ; we can thus create what we call a dictionnary D of size p ; we know that our observation y is the linear combinaison of k element of D (for the most part) ; we then consider that in reality our observation have been noised and therefore have an error e. Moreover, we consider a parcimonious problem as we only want k coefficient in x to non-zero values.

Therefore, our minimisation problem can be raise as :

$$\min ||y - Dx||^2 \tag{1}$$

where, $y = y_{gt} + e = Dx_{gt} + e$ (gt for ground truth).

Then, we ask for some constraint on the x_i coefficient ; as we want each coefficient to be positif or null (Negative proportion will not make sens here), and we should have the sum of every coefficient to be close to 1. The new problem is :

$$\begin{aligned} \min_x \quad & ||y - Dx||^2 \\ \text{s.t.} \quad & x_i \geq 0 \\ & \sum x_i = 1 \end{aligned} \tag{2}$$

To finish, we know that the problem can be rewritten as a quadratic program, as our objectif function is quadratic, and our constraints are linear :

$$\begin{aligned} \min_x \quad & \frac{1}{2} x^T G x + d^T x \\ \text{s.t.} \quad & x_i \geq 0 \\ & \sum x_i = 1 \end{aligned} \tag{3}$$

where, G here is the a square positif-definite matrix such that $G = D^T D$, and d is a column vector with dimension equal to the state-space dimension p, and such that $d = -D^T y$.

Therefore, to solve such problem, we will create a **Interior Point method algorithm**, oftenly use for his good time computation performance on quadratic problems.

2 Interior Point method

2.1 Theory

2.1.1 KKT

This section is inspired by the book 'Numerical Optimisation' from UCI mathematics.

Before showing results of our algorithm, lets dive into the mathematics of it, to better understand how such algorithm should help us to win computation time for our specific problem.

Lets first consider a general quadratic program with only inegality constraint :

$$\begin{aligned} \min_x \quad & \frac{1}{2}x^T G x + d^T x \\ \text{s.t.} \quad & Ax \geq b \end{aligned} \tag{4}$$

G is symetric and positive-semidefinite, A is a $m \times p$ matrix and b a m column vector.

We know that we can rewrite this constraint problem with KKT conditions :

$$\begin{aligned} \frac{1}{2}x^T G x + d^T x \\ Ax - b \geq 0 \\ (Ax - b)_i \lambda_i = 0 \\ \lambda \geq 0 \end{aligned} \tag{5}$$

We can introduce a slack vector $s = Ax - b$ with $s \geq 0$, and thus rewrite KKT conditions as :

$$\begin{aligned} \frac{1}{2}x^T G x + d^T x \\ Ax - b = s \\ (Ax - b)_i \lambda_i = 0 \\ \lambda \geq 0 \end{aligned} \tag{6}$$

NB : As G is positive-definite, note that KKT conditions are not only necessary but also sufficent, so we know that the solution found would be the global minimum of our problem.

Next, we want to solve the pertubed KKT condition :

$$\begin{bmatrix} Gx - A^T\lambda + d \\ Ax - b - s \\ S\Lambda e - \sigma\mu e \end{bmatrix} = 0 \tag{7}$$

with $S = \mathrm{diag}(s_1, s_2, ..., s_m)$, Λ=diag$(\lambda_1, \lambda_2, ..., \lambda_m)$, $e = [1, 1, ..., 1]^T$, $\sigma \in [0, 1]$, and $\mu = \frac{s^T\lambda}{m}$.

2.1.2 Central Path

The solution for (7) give the **Central Path**, leading to the solution of the quadratic program as $\sigma\mu$ tends to zero ; we can then fix μ and applied a Newton Method to the following linear system :

$$\begin{bmatrix} G & 0 & -A^T \\ A & -I & 0 \\ 0 & \Lambda & Y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta s \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} -r_d \\ -r_p \\ -\Lambda Y e + \sigma\mu e \end{bmatrix} \tag{8}$$

where, $r_d = Gx - A^T\lambda + d$ and $r_p = Ax - b - s$ are the 2 residus.

Therefore, we can find the next iterates by setting ;

$$(x^+, s^+, \lambda^+) = (x, s, \lambda) - \alpha(\Delta x, \Delta s, \Delta\lambda) \tag{9}$$

Here α need to be choose such that $(s^+, \lambda^+) > 0$, so that the solution stay in the feasible zone.

To choose alpha, we solve the equation : $(s, \lambda) - \alpha(\Delta s, \Delta\lambda) = 0$, so that $\alpha = \min((s/\Delta s, \lambda/\Delta\lambda))$ should sastify feasibility for every s_i and λ_i.

2.2 ISTA.m