Algorithms for Programming Contests - Week 03

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Graphs

Graphs

A *graph* is a tuple $G = (V, E)$, where V is a non-empty set of *vertices* and E is a set of edges.

A *directed* graph is a graph with $E \subseteq V \times V = \{(u, v) \mid u, v \in V\}$.

An *undirected* graphs is a graph with $E \subseteq \{\{u, v\} \mid u, v \in V\}$.

For a vertex v, we denote the successors of v by
$vE := \{u \mid (v, u) \in E\}$ for directed graphs;
$vE := \{u \mid \{v, u\} \in E\}$ for undirected graphs.

A *path* from v_1 to v_n is a sequence $p = v_1 v_2 \dots v_n$ such that $v_{i+1} \in v_i E$ for all $i \in [1, n-1]$, and $v_i \neq v_j$ for all $i \neq j$.

- A graph is *cyclic* if there is a path $p = v_1 \dots v_n$ with $v_1 \in v_n E$, otherwise it is *acyclic*.
- An undirected graph is *connected* if for every pair of vertices $u, v \in V$, there is path from u to v.
- For an undirected graph, a *connected component* is a maximal set $V' \subseteq V$ where for all $u, v \in V'$, there is a path from u to v.
- An undirected graph is a *tree* if it is acyclic and connected. For any tree (V, E), we have $|V| = |E| + 1$.
- An undirected acyclic graph is a *forest*, and each connected component is a tree.
- A directed acyclic graph is also called a *DAG*.

Graphs as an abstract data type

Graph representation

- Adjacency list: For each vertex v, store a list of successors vE.
- Adjacency matrix: For each pair of vertices u, v, store existence of an edge $(u, v) \in E$.

Graph operations

- Make graph: build a graph from a list of vertices and edges.
- Get vertices: Iterate over all vertices $v \in V$.
- Get edges: Iterate over all edges $e \in E$.
- Test edge: Test existence of an edge $(u, v) \in E$.
- Get successors: For a vertex v, iterate over all successors $u \in vE$.

Graph traversal

Graph traversal

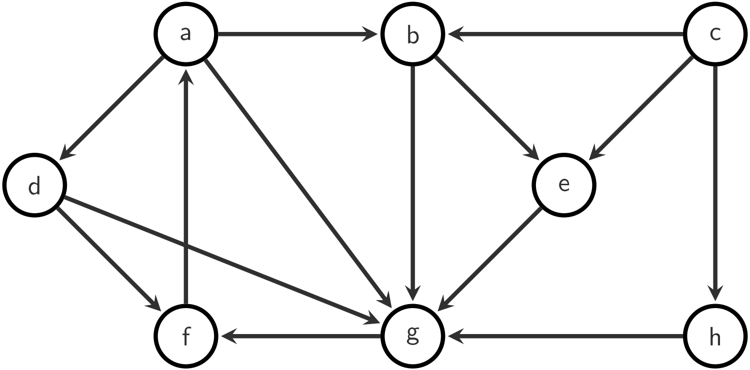
Graph traversal

- Visit vertices in certain order.
- Assign vertices an order $o : V \to \mathbb{N} \cup \{\infty\}$ of discovery time.
- Possibly keep track of other information such as finishing time, predecessor, etc.

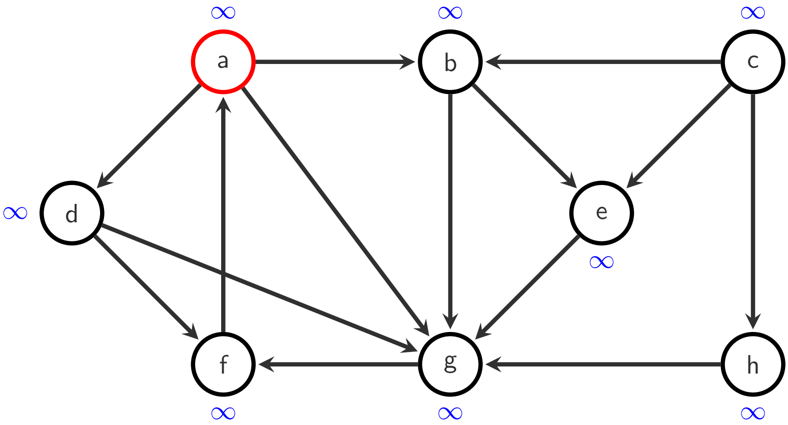
Usages

- Find vertex with certain properties.
- Check property for all vertices.
- Find connected components.
- Check for cycles.
- ...

Depth First Search (DFS)

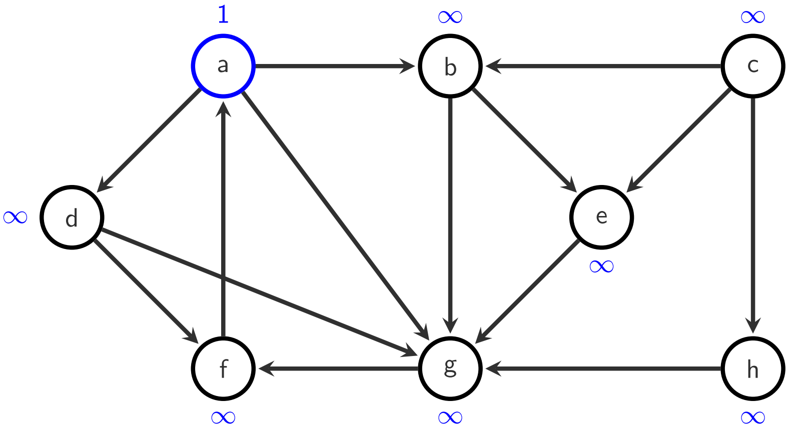


Depth First Search (DFS)



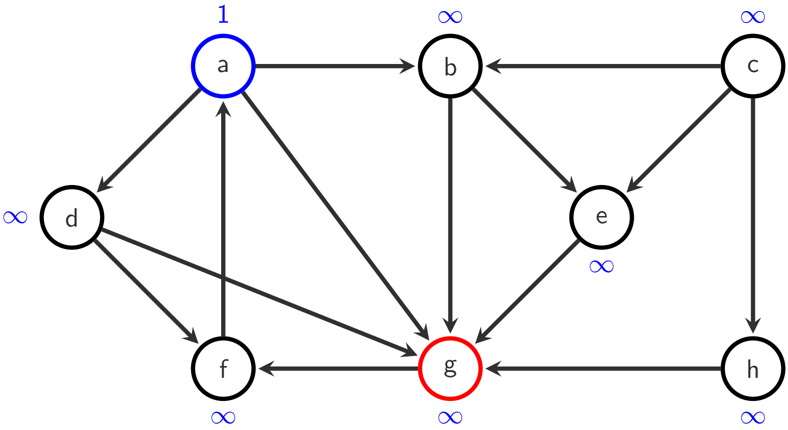
$S = [a]$

Depth First Search (DFS)



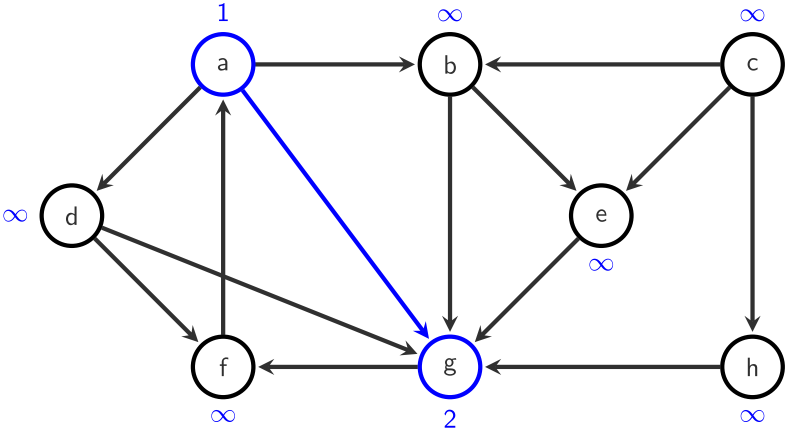
$S = [\text{b}, \text{d}, \text{g}]$

Depth First Search (DFS)



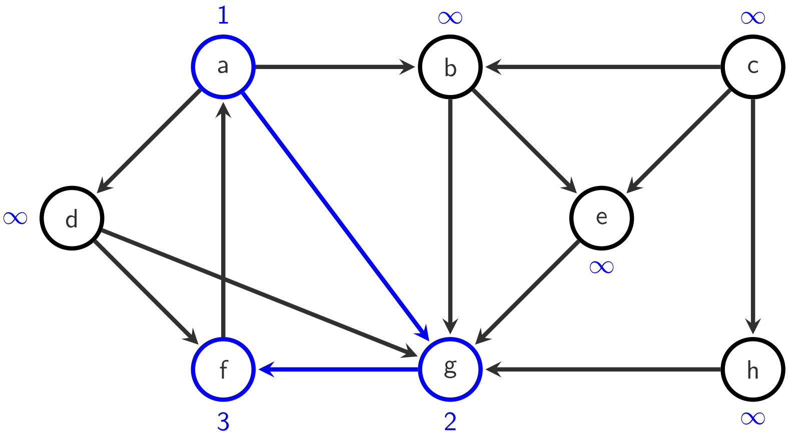
$S = [\mathrm{b}, \mathrm{d}, \mathrm{g}]$

Depth First Search (DFS)



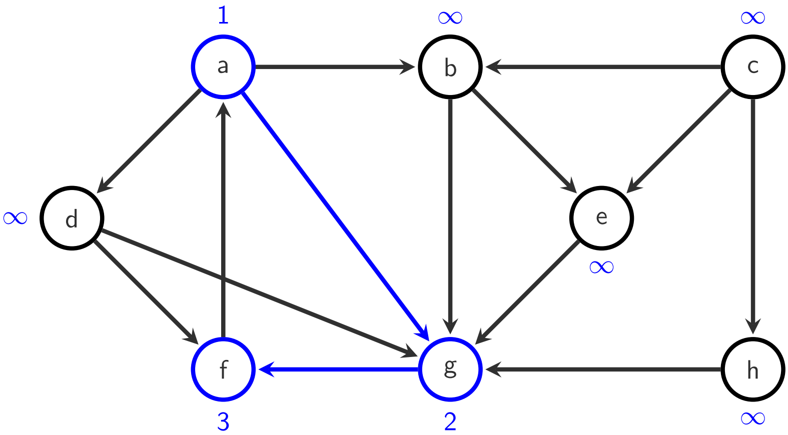
$S = [\text{b}, \text{d}, \text{f}]$

Depth First Search (DFS)



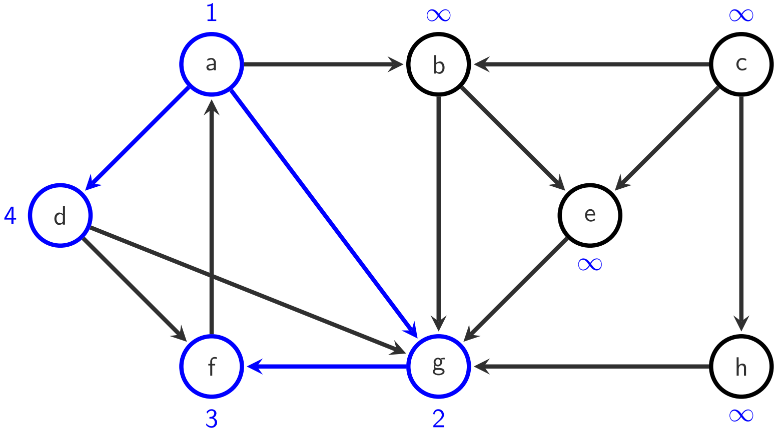
$S = [\mathrm{b}, \mathrm{d}, \mathrm{a}]$

Depth First Search (DFS)



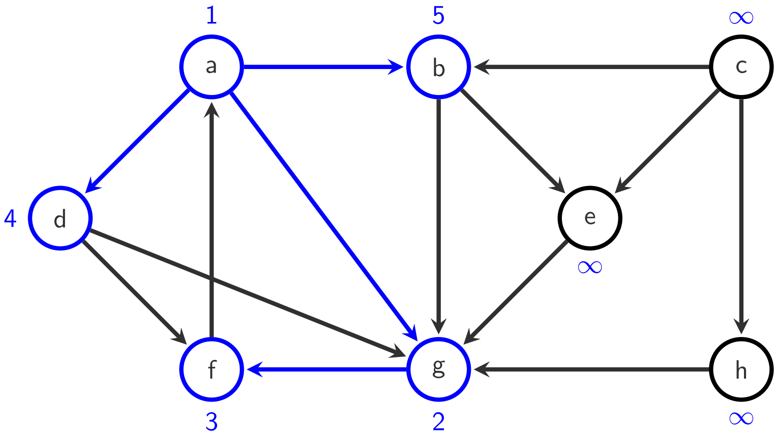
$S = [\mathrm{b}, \mathrm{d}]$

Depth First Search (DFS)



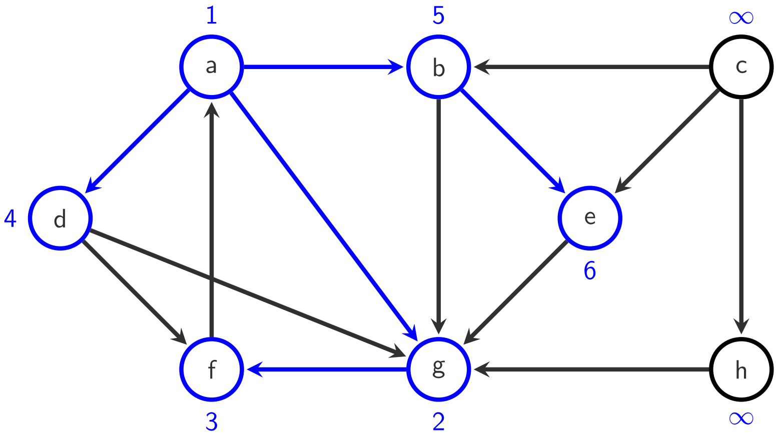
$S = [\mathrm{b}, \mathrm{f}, \mathrm{g}]$

Depth First Search (DFS)



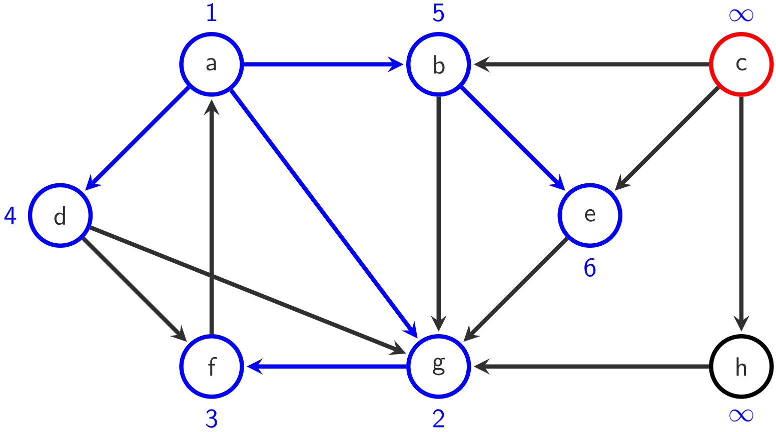
$S = [\text{e}, \text{g}]$

Depth First Search (DFS)



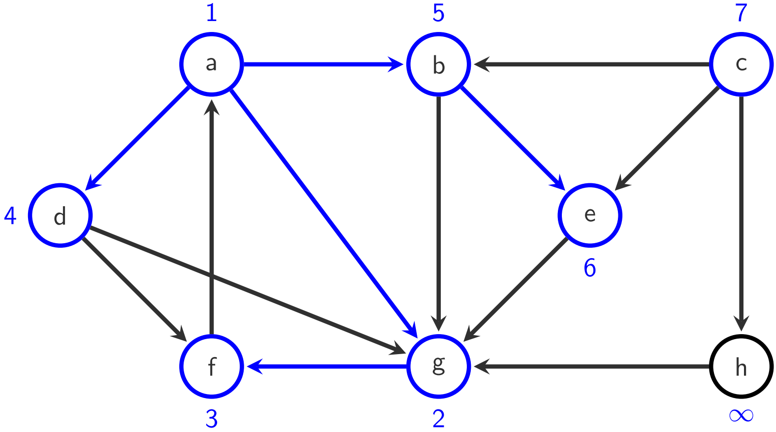
$S = [\text{g}]$

Depth First Search (DFS)



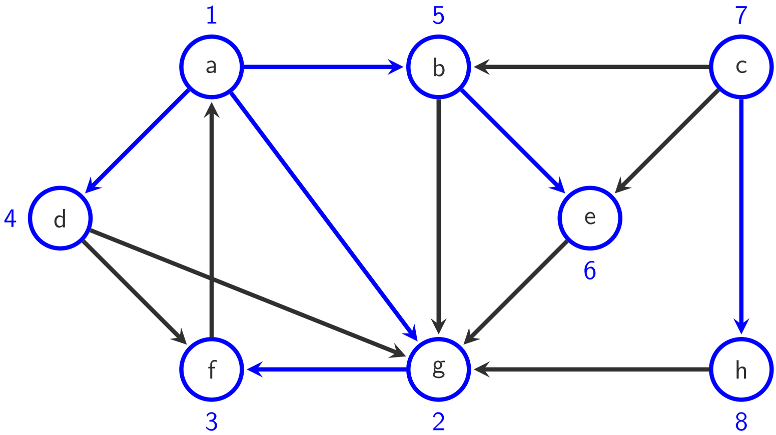
$S = [c]$

Depth First Search (DFS)



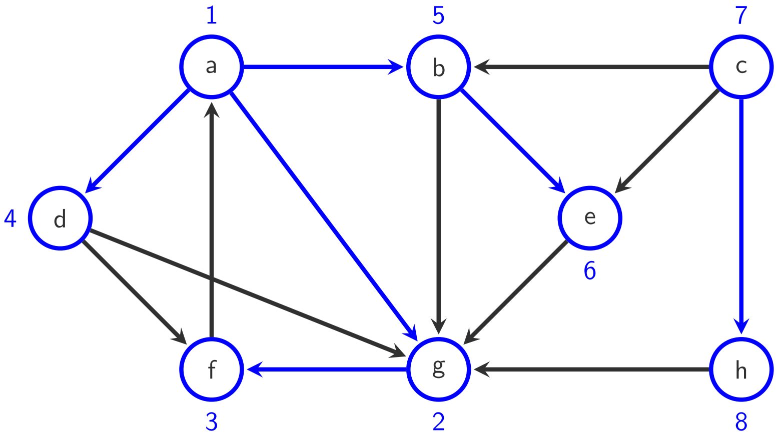
$S = [\mathrm{e}, \mathrm{h}]$

Depth First Search (DFS)



$S = [\mathrm{e}, \mathrm{g}]$

Depth First Search (DFS)



$S = []$

DFS Algorithm

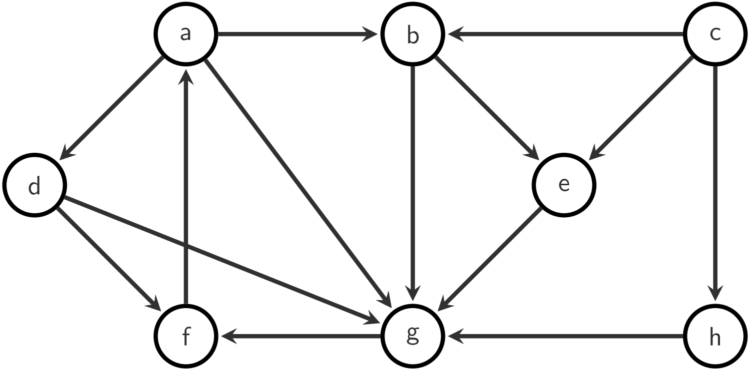
Algorithm 1 Depth-first search

Input: Graph $G = (V, E)$

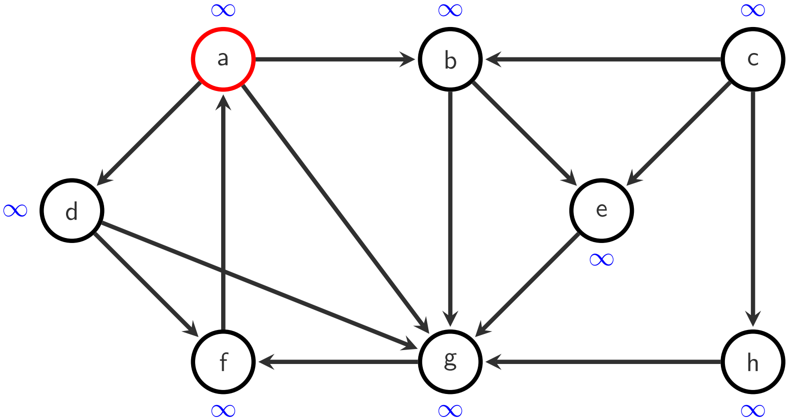
```
procedure DFS(G)
    for each vertex v ∈ V do
        o(v) ← ∞
    S ← EmptyStack()
    i ← 1
    for each vertex v ∈ V do
        if o(v) = ∞ then
            DFSExplore(G, v)

procedure DFSExplore(G, v)
    S.push(v)
    while S is not empty do
        v = S.pop()
        if o(v) = ∞ then
            o(v) ← i;
            i ← i + 1
            for each u ∈ vE do
                S.push(u)
```

Breadth First Search (BFS)

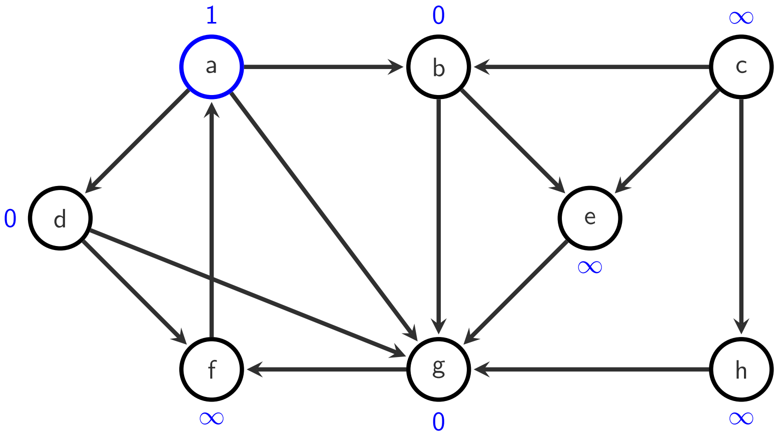


Breadth First Search (BFS)



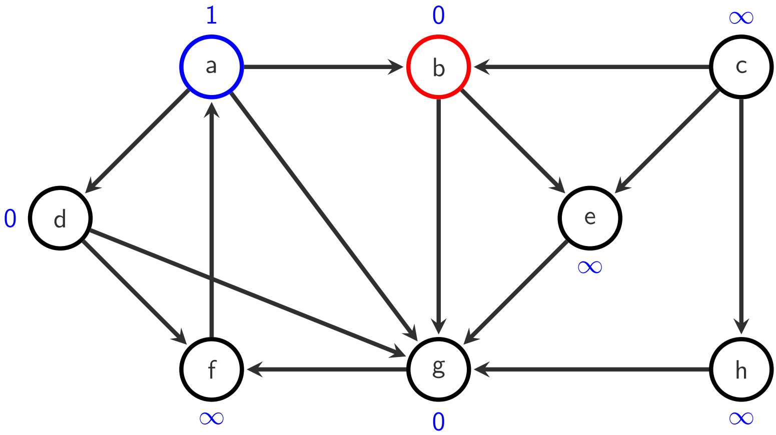
$S = [a]$

Breadth First Search (BFS)



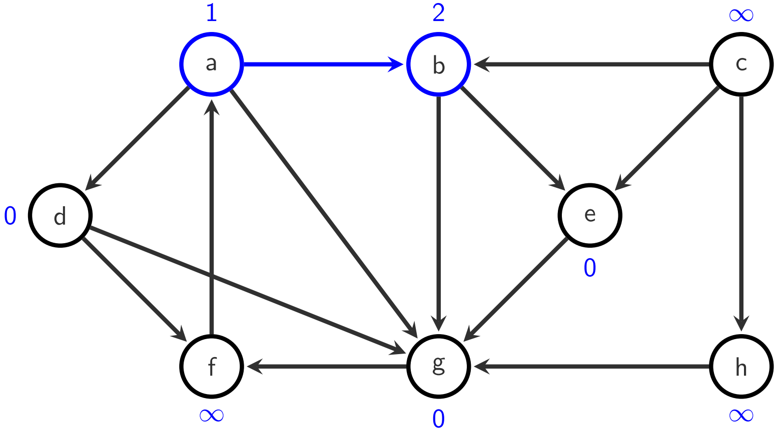
$S = [\text{b}, \text{d}, \text{g}]$

Breadth First Search (BFS)



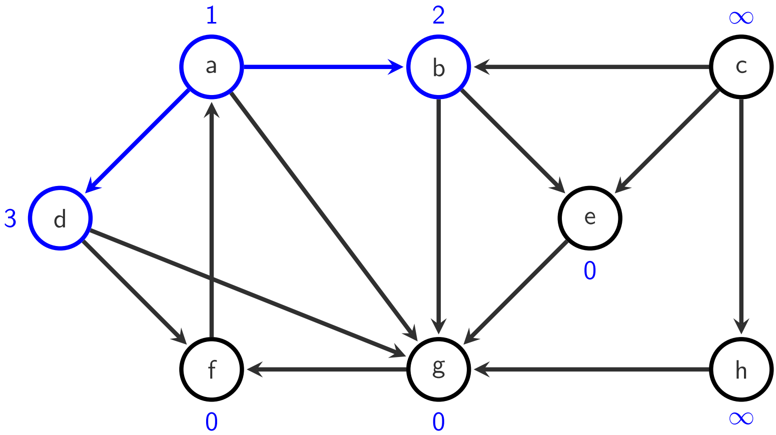
$S = [\text{b}, \text{d}, \text{g}]$

Breadth First Search (BFS)



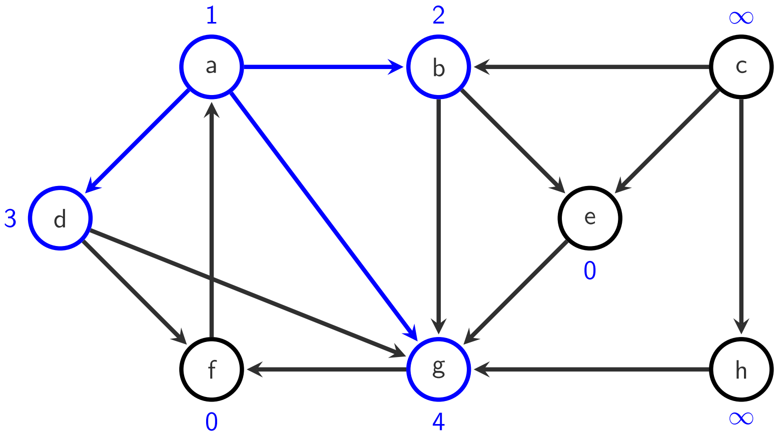
$S = [\mathrm{d}, \mathrm{g}, \mathrm{e}]$

Breadth First Search (BFS)



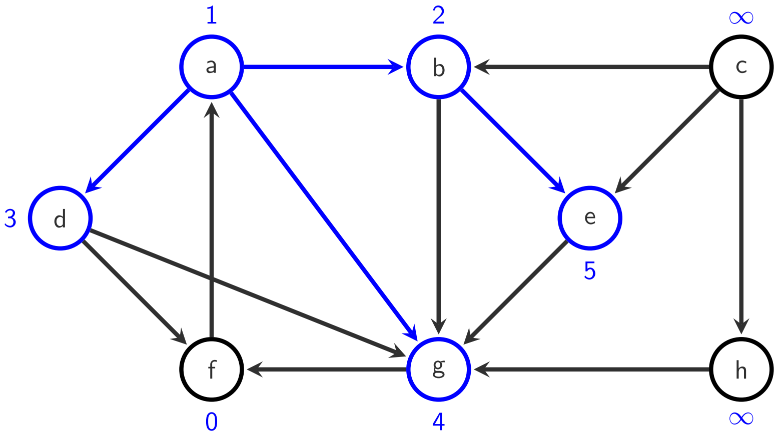
$S = [\mathrm{g}, \mathrm{e}, \mathrm{f}]$

Breadth First Search (BFS)



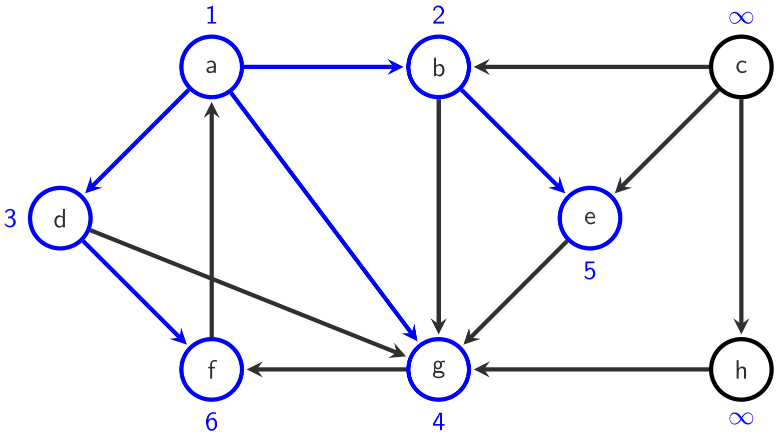
$S = [\mathrm{e}, \mathrm{f}]$

Breadth First Search (BFS)



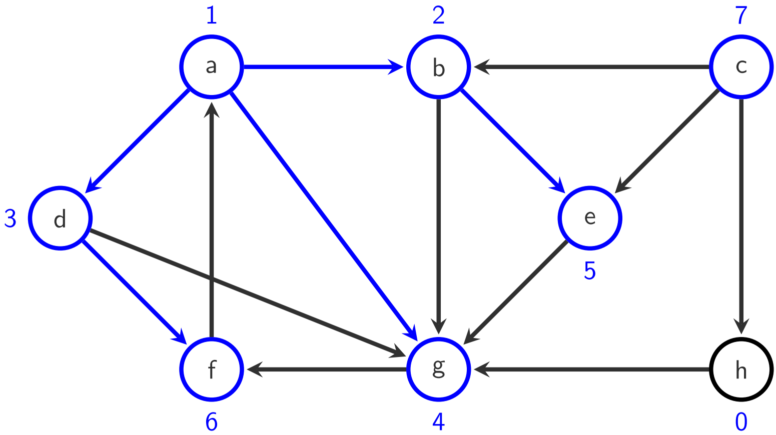
$S = [\mathrm{f}]$

Breadth First Search (BFS)



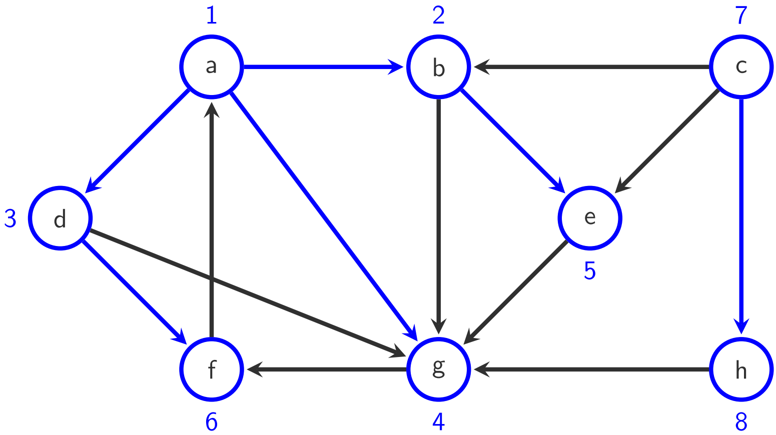
$S = []$

Breadth First Search (BFS)



$S = [\mathrm{h}]$

Breadth First Search (BFS)



$S = []$

BFS Algorithm

Algorithm 2 Breadth-first search

Input: Graph $G = (V, E)$

procedure $\mathrm{BFS}(G)$
    for each vertex $v \in V$ **do**
        $o(v) \leftarrow \infty$
    $S \leftarrow$ EmptyQueue()
    $i \leftarrow 1$
    for each vertex $v \in V$ **do**
        if $o(v) = \infty$ **then**
            $\mathrm{BFS}\textsc{Explore}(G, v)$

procedure $\mathrm{BFS}\textsc{Explore}(G, v)$
    S.enqueue(v)
    while S is not empty **do**
        $v = S$.dequeue()
        $o(v) \leftarrow i$;
        $i \leftarrow i + 1$
        for each $u \in vE$ **do**
            if $o(u) = \infty$ **then**
                $o(u) \leftarrow 0$;
                S.enqueue(u)

Topological sort (TS)

Topological order

For a directed graph $G = (V, E)$, a *topological order* is an assignment $o : V \to \mathbb{N}$ such that for all $(u, v) \in E$, we have $o(u) < o(v)$.

- Topological order exists *if and only if* graph is acyclic (i.e. a DAG).
- Topological order may not be unique.
- Topological sort: Problem of finding a topological order.

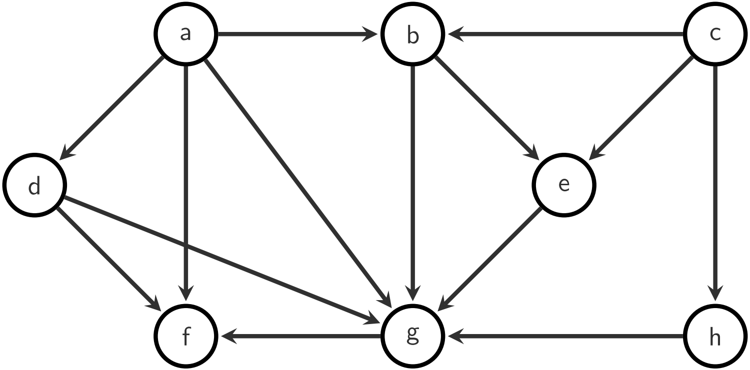
Usages

- Resolving dependencies.
- Instruction scheduling.
- Determine order for compilation multi-source programs.
- Detecting cycles.

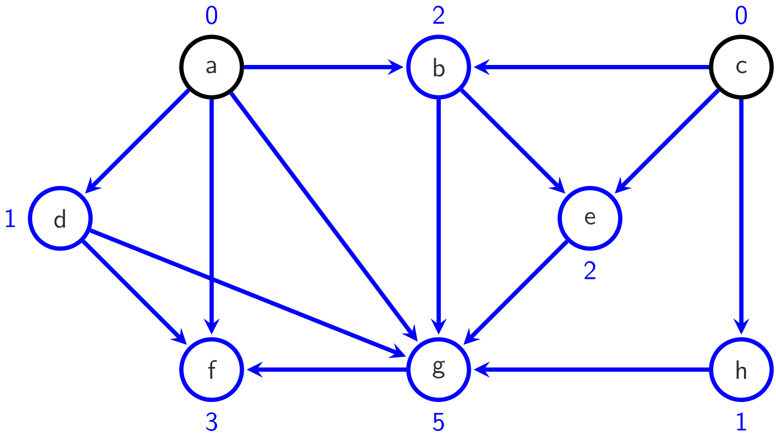
Topological sort (Implementation)

1. For every node, store the number of predecessors.
2. Choose a node with 0 predecessors and remove it from the graph.
3. Repeat until no nodes with 0 predecessors left.
4. $\Rightarrow$ The order in which the nodes are removed is *topological*

TS (example)

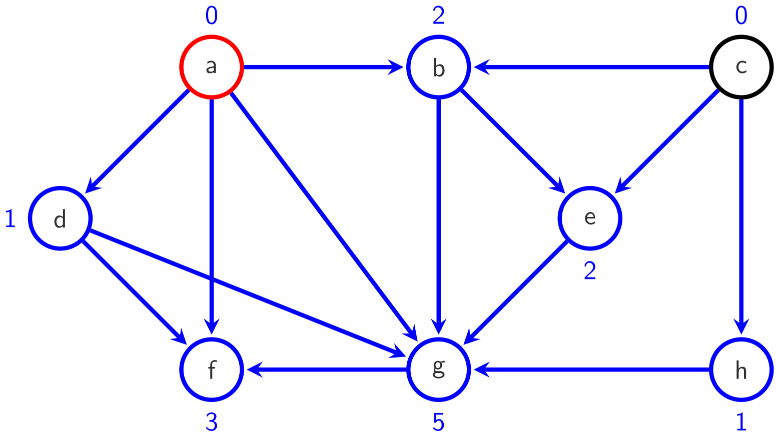


TS (example)



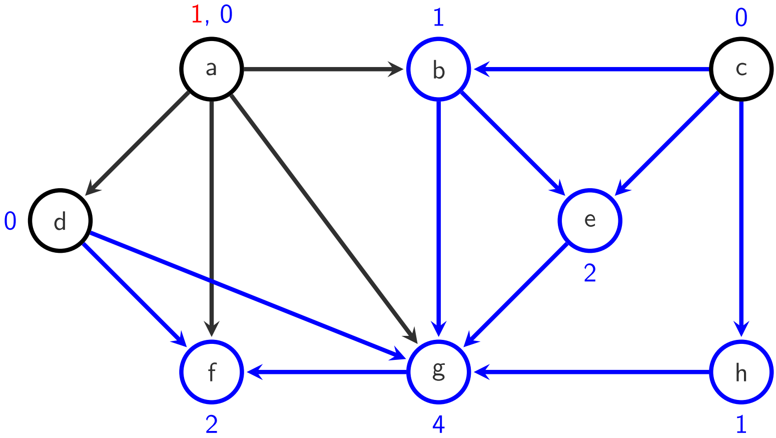
$S = [\mathrm{a}, \mathrm{c}]$

TS (example)



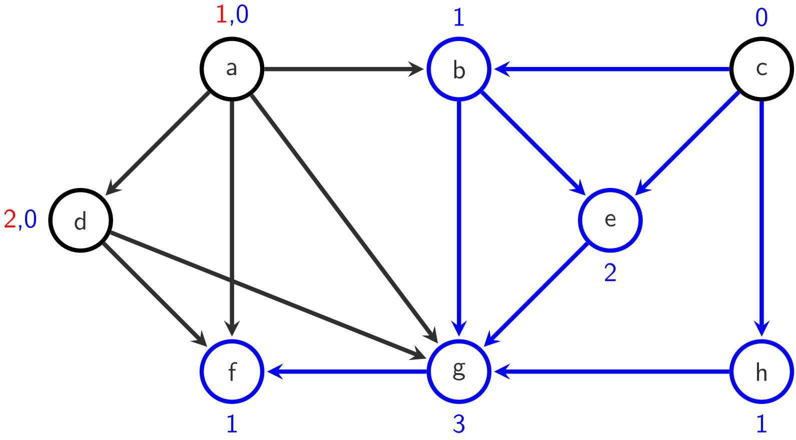
$S = [\textcolor{red}{\text{a}}, \text{c}]$

TS (example)



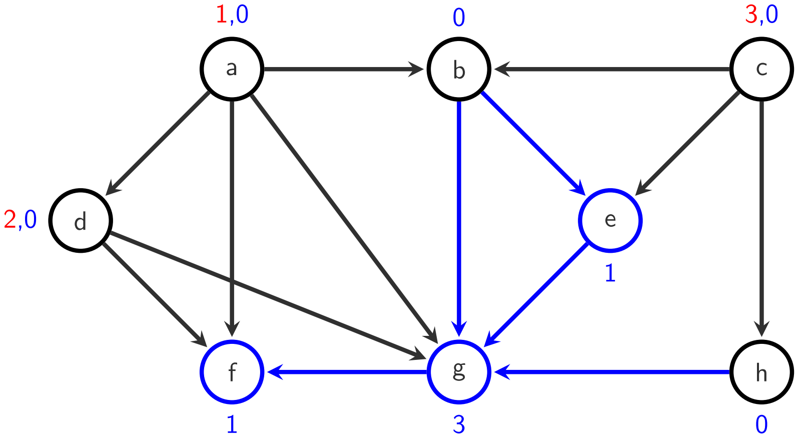
$S = [\mathrm{c}, \mathrm{d}]$

TS (example)



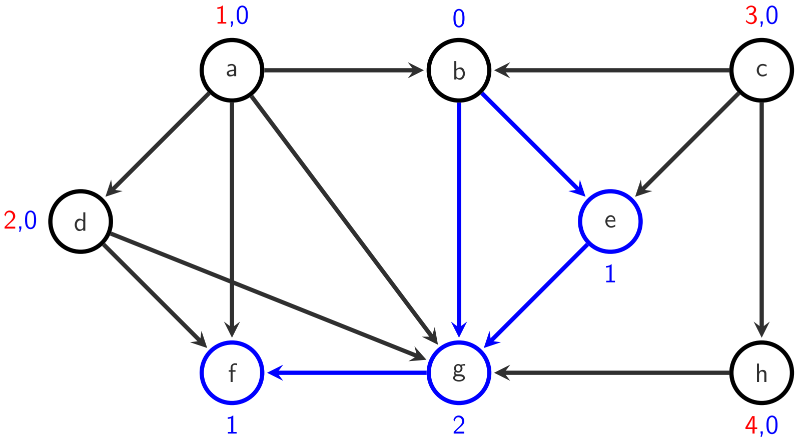
$S = [\text{c}]$

TS (example)



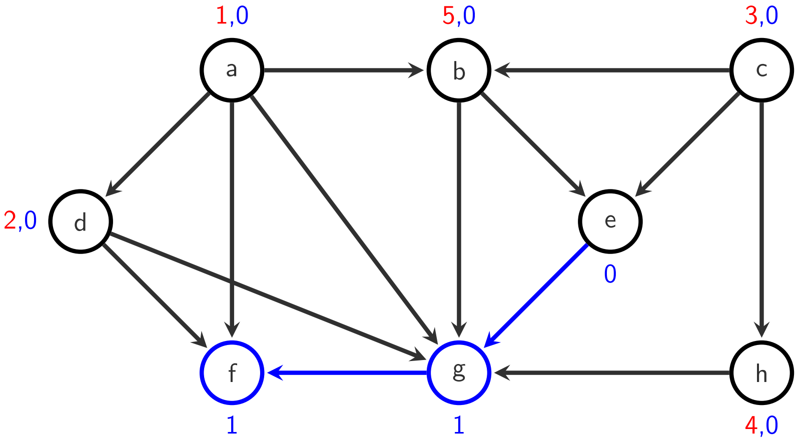
$S = [\mathrm{b}, \mathrm{h}]$

TS (example)



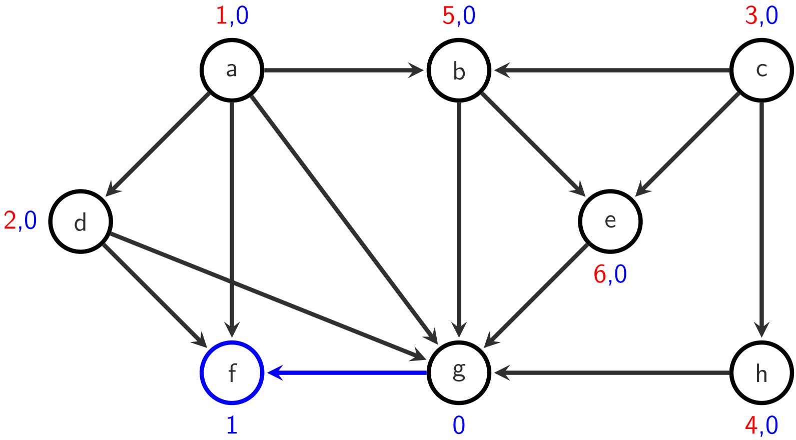
$S = [\mathrm{b}]$

TS (example)



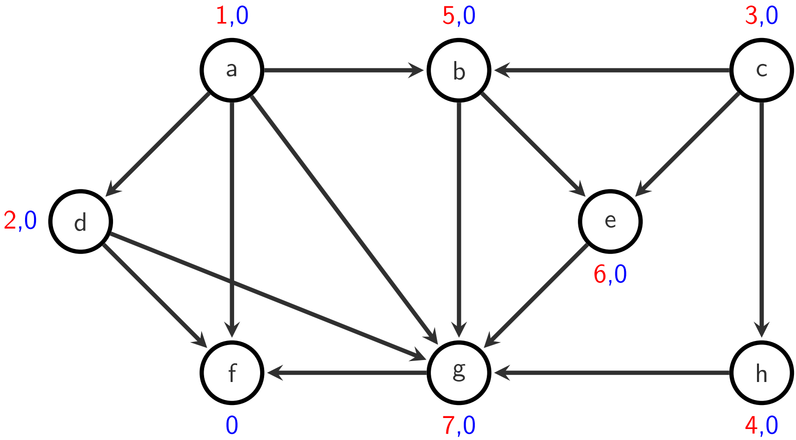
$S = [\text{e}]$

TS (example)



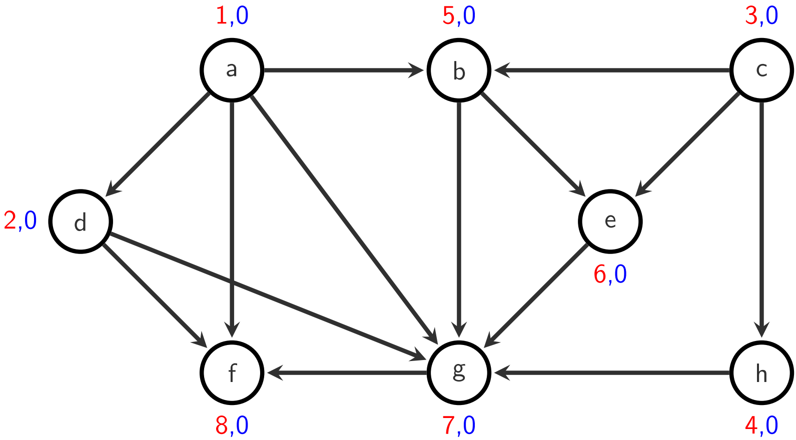
$S = [\text{g}]$

TS (example)



$S = [\text{f}]$

TS (example)



$S = []$

Algorithm 3 Topological sort

```
Input: Directed graph G = (V, E)
  procedure TS(G)
      for each vertex v ∈ V do
          o(v) ← ∞
          ▷ count predecessors
          pre(v) ← |{u | v ∈ uE}|
      S ← EmptyQueue()
      i ← 1
      for each vertex v ∈ V do
          if pre(v) = 0 then
              TSExplore(G, v)

procedure TSExplore(G, v)
    if o(v) = ∞ then
        S.push(v)
    while S is not empty do
        v = S.pop()
        o(v) ← i; i ← i + 1
        for each u ∈ vE do
            pre(u) ← pre(u) − 1
            if pre(u) = 0 then
                S.push(u)
```

If unvisited vertices with $o(v) = \infty$ remain, then the graph is cyclic.

Analysis of DFS, BFS and TS

Running time

- Each vertex is visited at most once: $\mathcal{O}(|V|)$
- For each vertex, each successor considered at most once: $\mathcal{O}\left(\sum_{v \in V} |vE|\right) = \mathcal{O}(|E|)$
- In total: $\mathcal{O}(|V| + |E|)$
- For topological sort, count number of predecessors in linear time.

Minimum spanning trees

Minimum spanning trees (MST)

Spanning tree

For an undirected graph $G = (V, E)$, a *spanning tree* of G is a subset of edges $T \subseteq E$ such that (V, T) forms a tree, i.e. is connected and acyclic.

Weighted graphs

We now consider graphs with a *weight function* $w : E \to \mathbb{R}$ on the edges. For a subset of edges $E' \subseteq E$, we define $w(E') := \sum_{e \in E'} w(e)$.

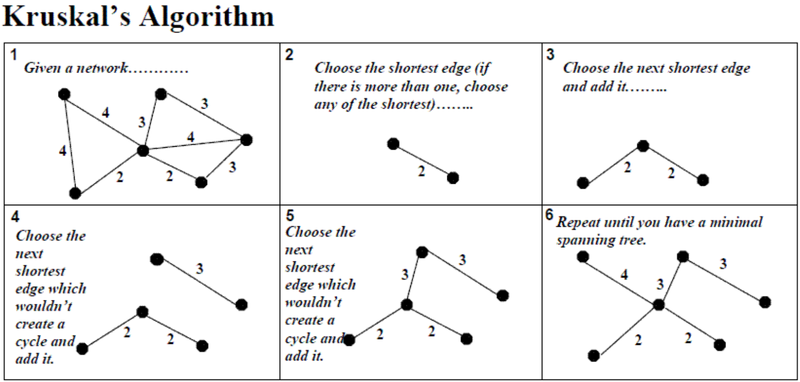
Minimum (weight) spanning tree

For an undirected graph $G = (V, E)$ with a weight function $w : E \to R$, a *minimum spanning tree (MST)* is a spanning tree S of G such that for all spanning trees T of G, we have $w(S) \leq w(T)$.

Minimum spanning trees (MST)

- Spanning trees only exist for connected graphs.
- Otherwise, a spanning tree exists for each connected component.
- All spanning trees of a graph have the same number of edges.
- Negative weights can be avoided by adding a constant to all weights.
- Maximum spanning tree can be obtained with $w'(e) = -w(e)$.

Kruskal and Prim



Algorithm 4 Kruskal's algorithm

Input: Undirected graph $G = (V, E)$

procedure $\textsc{Kruskal}(G)$

  $S \leftarrow \emptyset$ ▷ Current set of edges

  $L \leftarrow$ List of edges $e \in E$ sorted in increasing order by $w(e)$

  $U \leftarrow$ Union-Find structure initialized over set V

  for each edge (u, v) in L in order **do**

    ▷ Test if vertices are in different components

    if $U.\text{find}(u) \neq U.\text{find}(v)$ **then**

      ▷ If yes, add edge to MST and merge components

      $U.\text{union}(u, v)$

      $S \leftarrow S \cup \{e\}$

If vertices in different components remain, the graph is not connected.

Analysis of Kruskal's algorithm

Running time

- Sorting of edges: $\mathcal{O}(|E| \log |E|)$
- With α as the inverse Ackermann function, i.e. $\alpha = f^{-1}$ with $f(n) = A(n, n)$:
- $2|E|$ find operations: $\mathcal{O}(|E| \alpha(|V|))$
- $|V|$ union operations: $\mathcal{O}(|V| \alpha(|V|))$
- In total: $\mathcal{O}(|E| \log |E|)$

Proof of correctness for Kruskal's algorithm

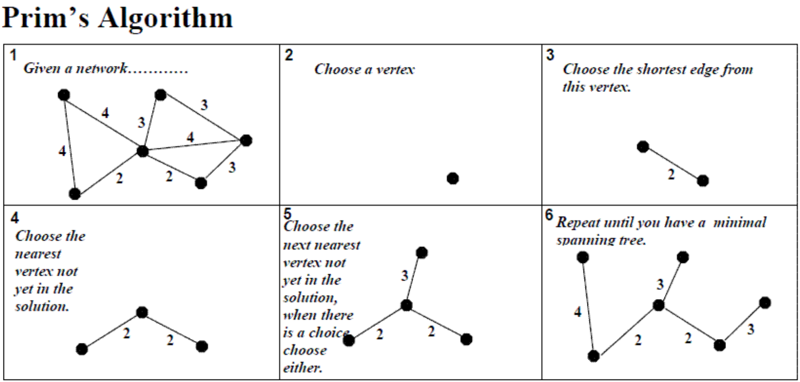
Lemma

Let $T \subseteq E$ be a set of edges such that there is a minimum spanning tree S of G with $T \subseteq S$.
Let $e \in E \setminus T$ be an edge such that $T \cup \{e\}$ does not create a cycle, with e having minimal weight among all of these edges.
Then, there is a minimum spanning tree S' of G such that $T \cup \{e\} \subseteq S'$.

Proof.

When $e \in S$, then $S' := S$ fulfills the requirement.
When $e \notin S$, then $S \cup \{e\}$ has a cycle c, and there is an edge $f \neq e$ in c that is not in T (otherwise adding e to T would create a cycle). Then $S' := S \setminus \{f\} \cup \{e\}$ is a also a spanning tree, and $w(S') \leq w(S)$, as $w(e) \leq w(f)$. As S is a minimum spanning tree, we have $w(S') = w(S)$, and therefore S' is also a minimium spanning tree.

Kruskal and Prim



Prim (Implementation)

- Use 3 colors for the vertices:
  - *black* (finished node — already part of the MST)
  - *grey* (discovered node – at least one connection to a black node)
  - *white* (unknown node — no connection to initial node found yet)

1. Start at a single node, keep track of encountered nodes.
2. Choose a grey node that is closest to any black node, color it black and all its neighbors grey.
3. Repeat until no grey nodes are left.

Algorithm 5 Prim's algorithm

Input: Graph $G = (V, E)$

procedure $\textsc{Prim}(G)$
- $S \leftarrow \emptyset$
- **for** each vertex $v \in V$ **do**
  - $\mathit{visited}(v) \leftarrow$ false
  - $c(v) \leftarrow \infty$
- $PQ \leftarrow$ PriorityQueue over V
- $s \leftarrow$ any $v \in V$
- $\textsc{PrimVisit}(s)$
- **while** PQ is not empty **do**
  - $v \leftarrow PQ.\text{deleteMin}()$
  - $S \leftarrow S \cup \{\{\mathit{pre}(v), v\}\}$
  - $\textsc{PrimVisit}(v)$

procedure $\textsc{PrimVisit}(v)$
- $\mathit{visited}(v) \leftarrow$ true
- **for** each $u \in vE$ **do**
  - **if** not $\mathit{visited}(u)$ **then**
    - **if** $w(v, u) < c(u)$ **then**
      - $\mathit{pre}(u) \leftarrow v$
      - $c(u) \leftarrow w(v, u)$
      - **if** u in PQ **then**
        - $PQ.\text{decreaseKey}(u, c(u))$
      - **else**
        - $PQ.\text{insert}(u, c(u))$

If not all vertices were visited, the graph is not connected.

Analysis of Prim's algorithm

Running time

- Graph exploration without priority queue: $\mathcal{O}(|V| + |E|)$
- With Fibonacci heap as priority queue:
- $|V|$ insert operations: $\mathcal{O}(|V|)$
- $|E|$ decreaseKey operations: $\mathcal{O}(|E|)$
- $|V|$ deleteMin operations: $\mathcal{O}(|V| \log |V|)$
- In total: $\mathcal{O}(|E| + |V| \log |V|)$

Note: In Java and C++, there is no decreaseKey operation, instead delete and insert again.