Algorithms for Programming Contests - Week 04

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Graphs

A *weighted graph* is a tuple $G = (V, E, c)$, where

- V is a non-empty set of *vertices*,
- E is a set of edges,
- $c : E \to \mathbb{R}$ is the weight function.

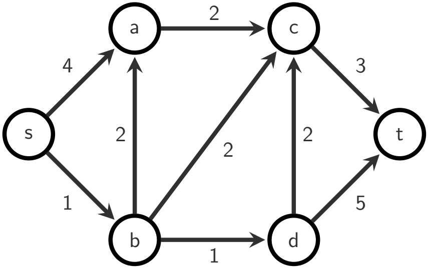
A *directed* graph is a graph with $E \subseteq V \times V = \{(u, v) \mid u, v \in V\}$.

An *undirected* graph is a graph with $E \subseteq \{\{u, v\} \mid u, v \in V\}$.

A *path* from v_1 to v_n is a sequence $p = v_1 v_2 \dots v_n$ such that $(v_i, v_{i+1}) \in E$ for all $i \in [1, n-1]$, and $v_i \neq v_j$ for all $i \neq j$.

The *length of a path* is the sum of its edge weights.

Shortest Path Problem - Classification



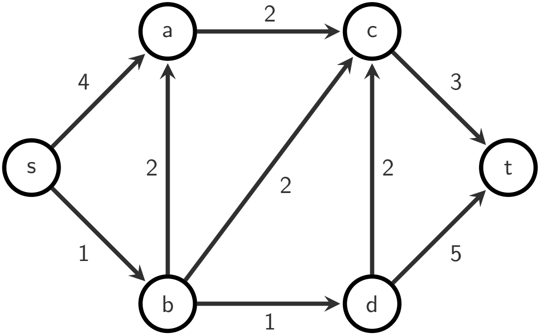
- **Single Pair Shortest Path (SPSP):**
  Find the shortest path between s and t.
- **Single Source Shortest Path (SSSP):**
  Find the shortest path between s and all the other nodes.
- **All Pairs Shortest Path (APSP):**
  Find the shortest path between any pair of nodes.

Shortest Path Problem - Applications

- transportation
- networking and telecommunication
- six degrees of separation
- plant and facility layout
- ...

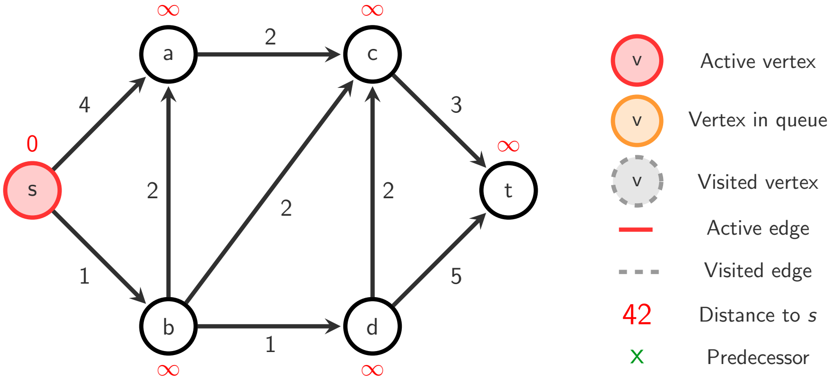
Dijkstra's Algorithm

- Published by Edsger W. Dijkstra in 1959
- Dijkstra's Algorithm solves the SSSP.

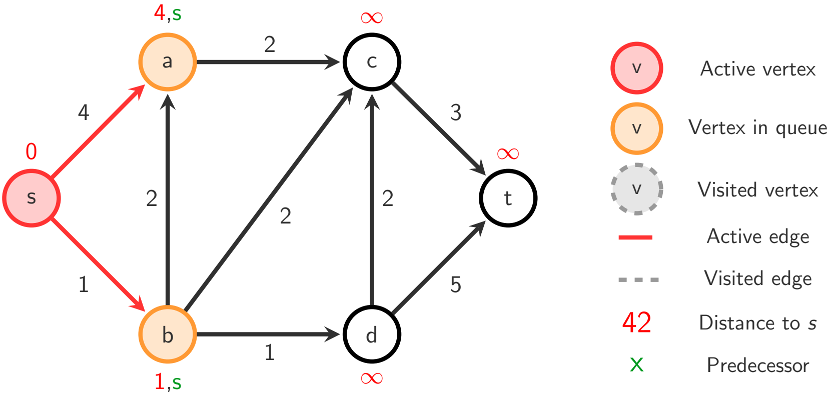


Dijkstra's Algorithm

Find the shortest path between s and t!



Dijkstra's Algorithm



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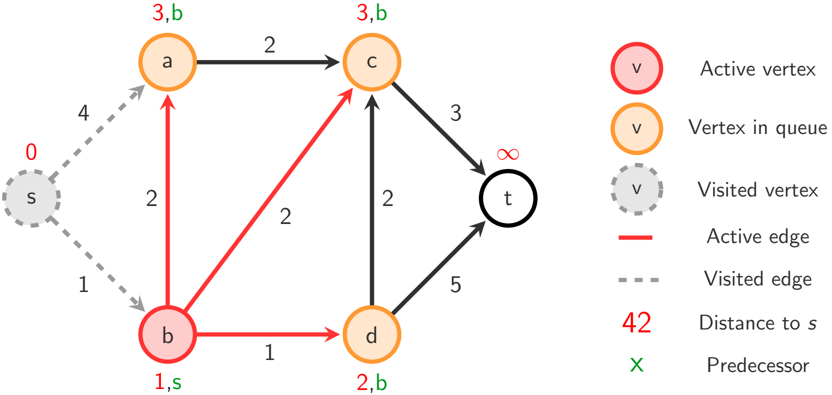
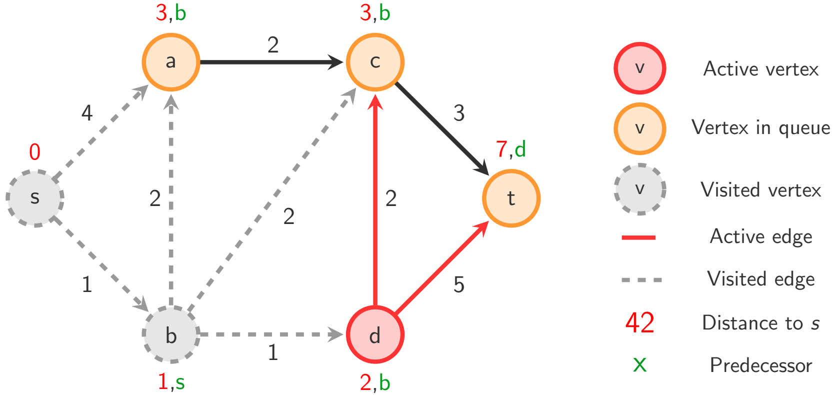
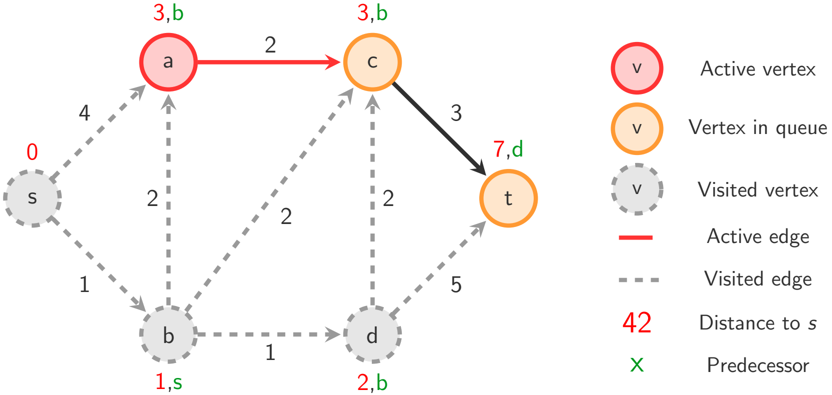


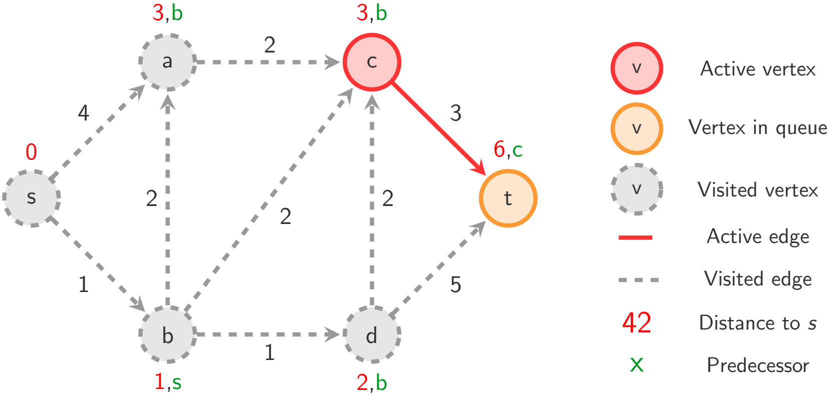
Dijkstra's Algorithm



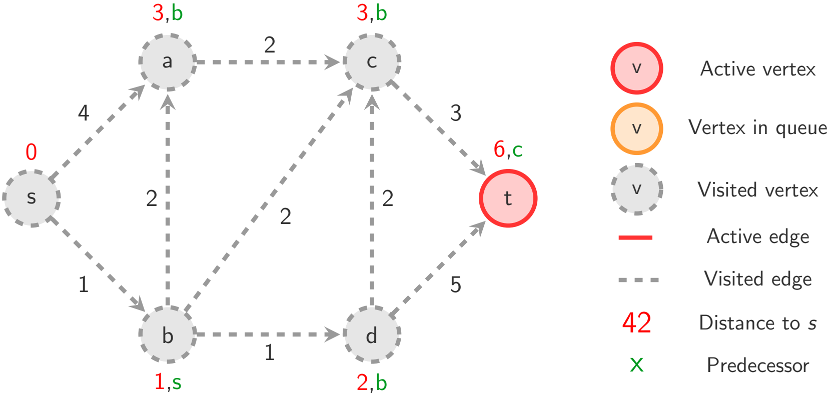
Dijkstra's Algorithm



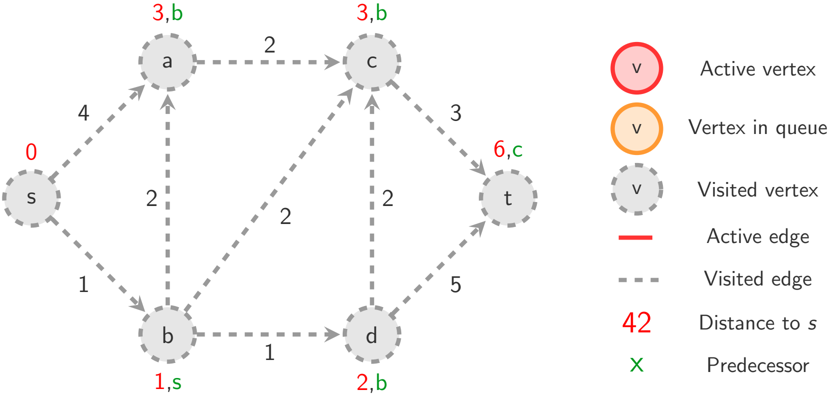
Dijkstra's Algorithm



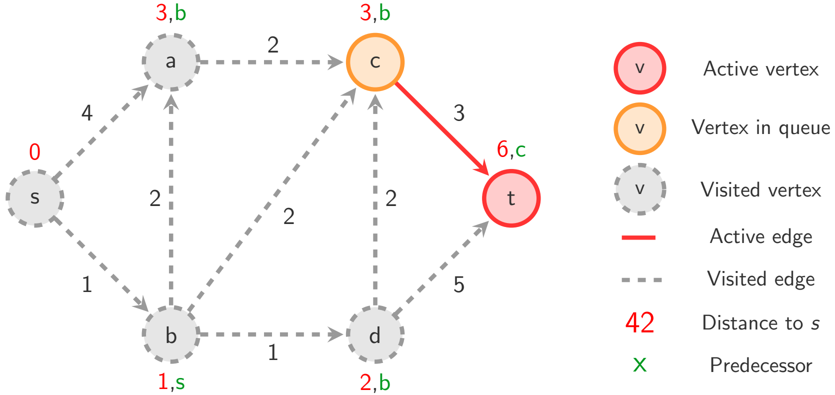
Dijkstra's Algorithm



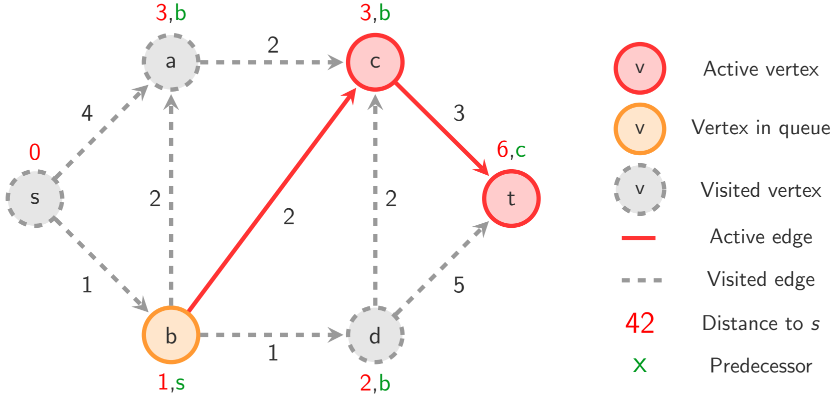
Dijkstra's Algorithm



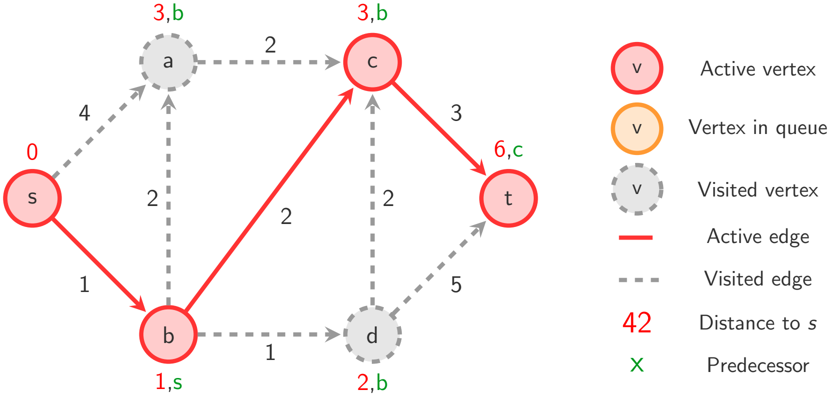
Shortest Path Tree



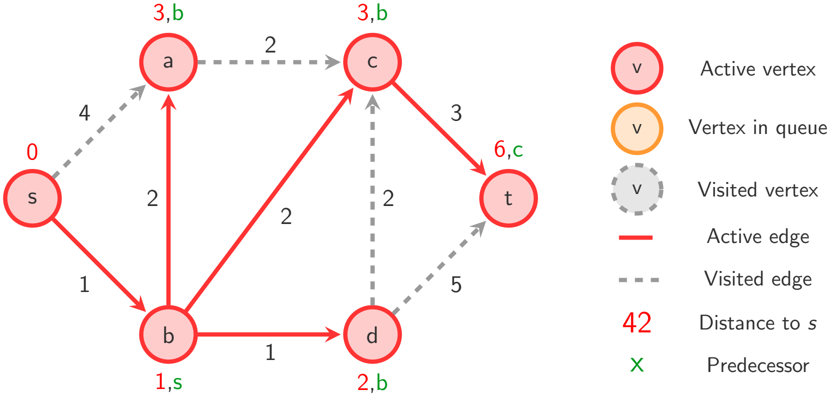
Shortest Path Tree



Shortest Path Tree



Shortest Path Tree



Algorithm 1 Dijkstra's Algorithm

```
Input: Graph G = (V, E, c)
  procedure DIJKSTRA(G, src)
      for each vertex v ∈ V do
          dist[v] ← ∞, prev[v] ← null
      end for
      dist[src] ← 0
      PQ ← PriorityQueue over V
      for each vertex v ∈ V do
          PQ.insert(v, dist[v])
      end for
      while PQ is not empty do
          v ← PQ.deleteMin()
          for each neighbor w of v do
              if dist[v] + c(v, w) < dist[w] then
                  dist[w] ← dist[v] + c(v, w)
                  PQ.decreaseKey(w, dist[w])
                  prev[w] ← v
              end if
          end for
      end while
  end procedure
```

Analysis of Dijkstra's Algorithm

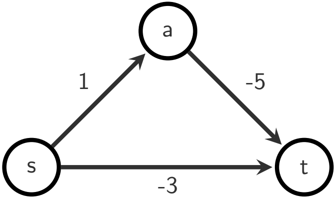
Running time

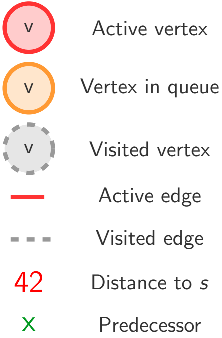
- With Fibonacci heap as priority queue:
- $|V|$ insert operations: $\mathcal{O}(|V|)$
- $|E|$ decreaseKey operations: $\mathcal{O}(|E|)$
- $|V|$ deleteMin operations: $\mathcal{O}(|V| \log |V|)$
- In total: $\mathcal{O}(|E| + |V| \log |V|)$

Note, that the running time is the same as for Prim's Algorithm.

Limitations of Dijkstra's Algorithm

Dijkstra's Algorithm may not work for graphs with negative edge weights!





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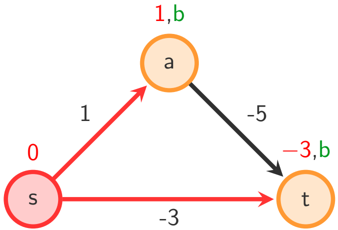

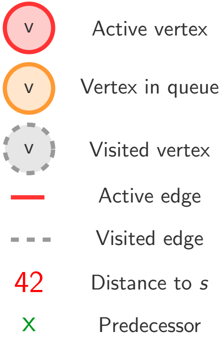

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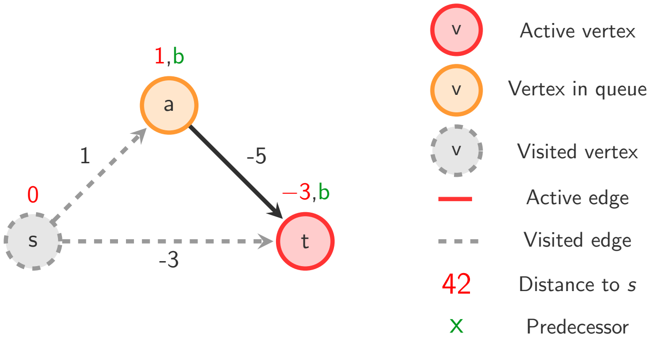

Limitations of Dijkstra's Algorithm

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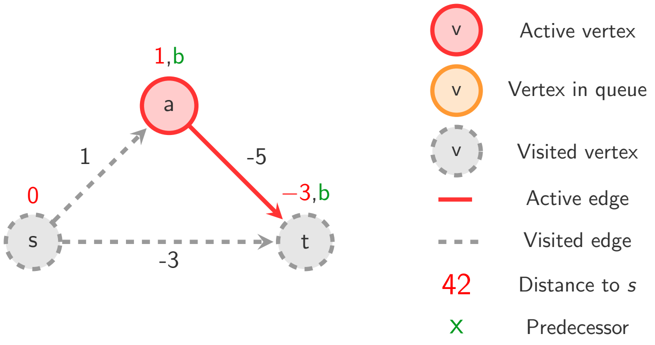
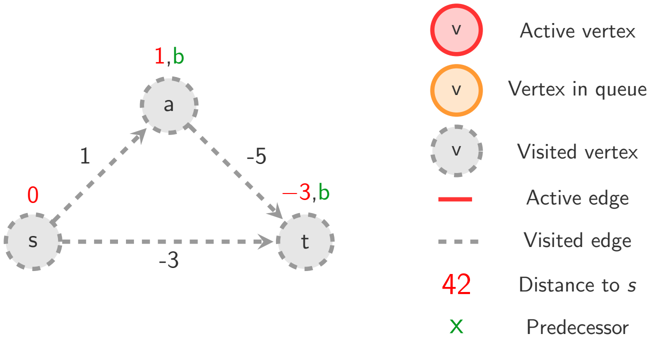


Vertex t is not updated because it was already visited.

Limitations of Dijkstra's Algorithm

Dijkstra's Algorithm may not work for graphs with negative edge weights!

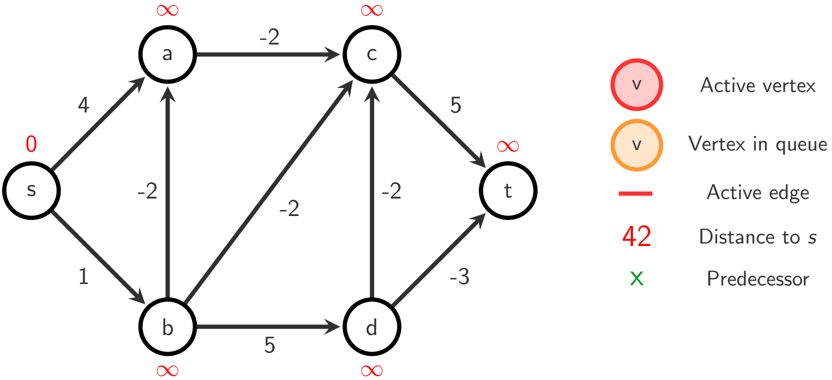


Bellman-Ford Algorithm

- Published by Richard Bellman and Lester Ford in 1958 and 1956 respectively.
- Solves SSSP even if the graph has negative edge weights.
- Idea: Start with shortest paths of length 1 and then successively construct all shortest paths of length 2, 3, $\ldots$, $|V| - 1$.

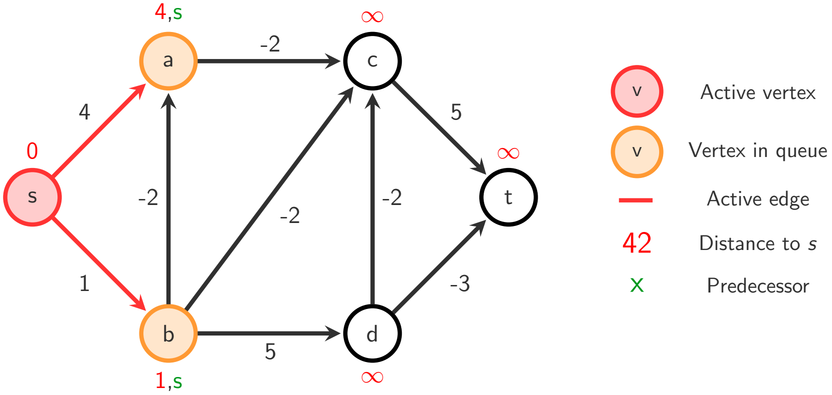
Bellman-Ford Algorithm

$Q = (s)$



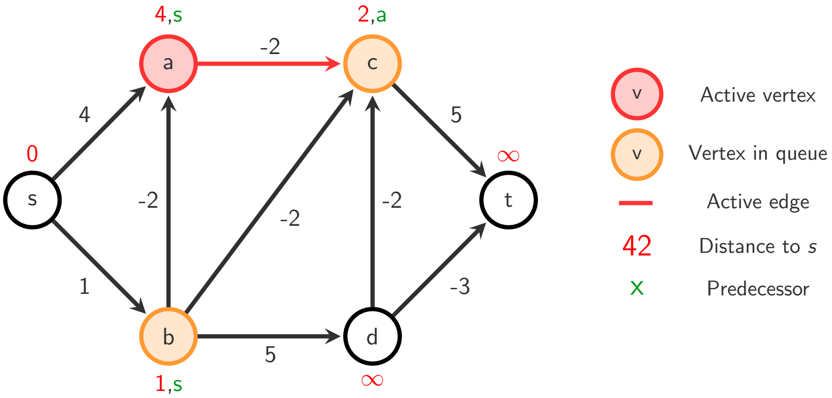
Bellman-Ford Algorithm

$Q = (a,b)$



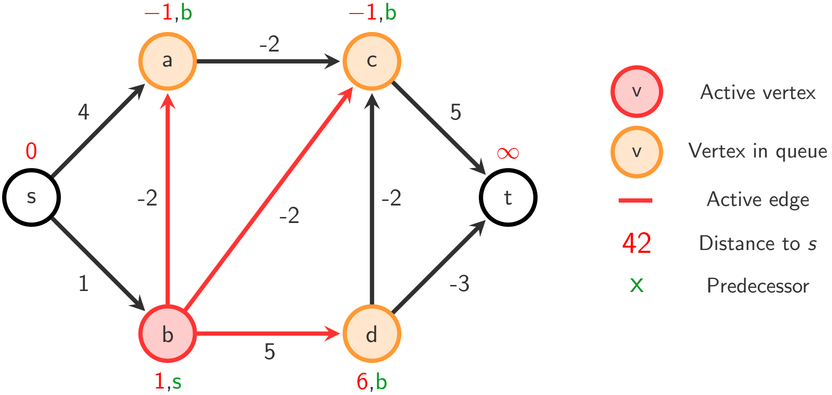
Bellman-Ford Algorithm

$Q = (b, c)$



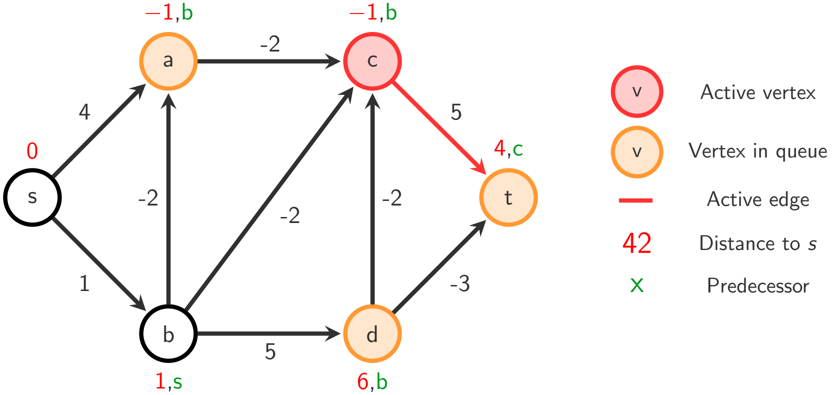
Bellman-Ford Algorithm

$Q = (c, a, d)$



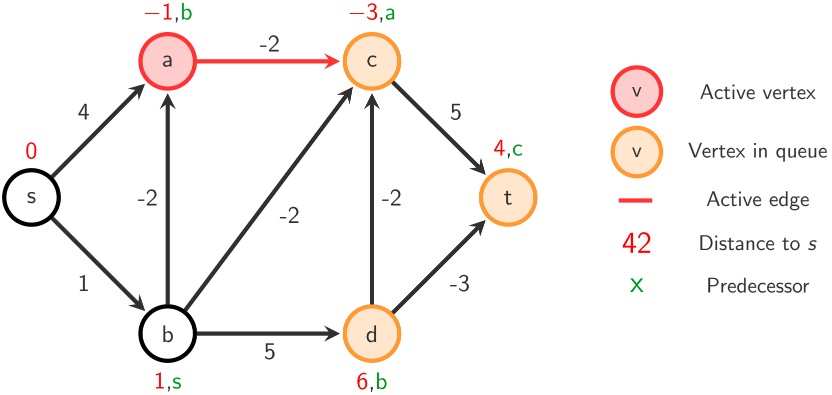
Bellman-Ford Algorithm

$Q = (a, d, t)$



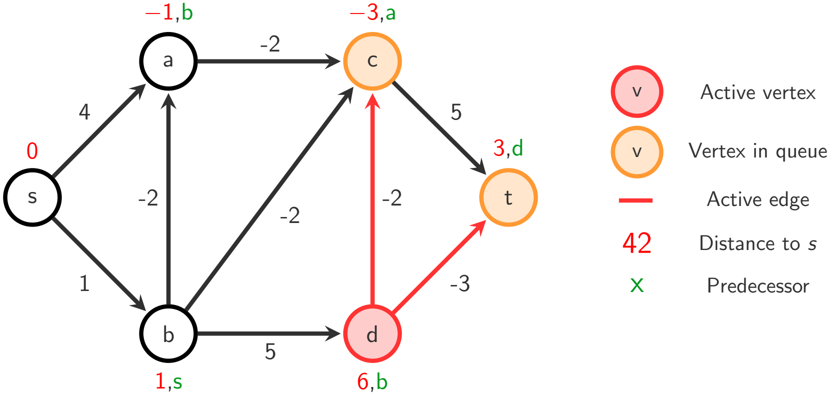
Bellman-Ford Algorithm

$Q = (d, t, c)$



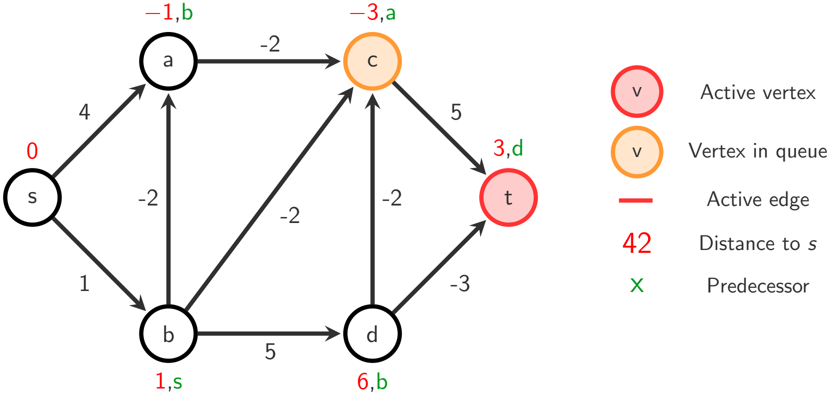
Bellman-Ford Algorithm

$Q = (t, c)$



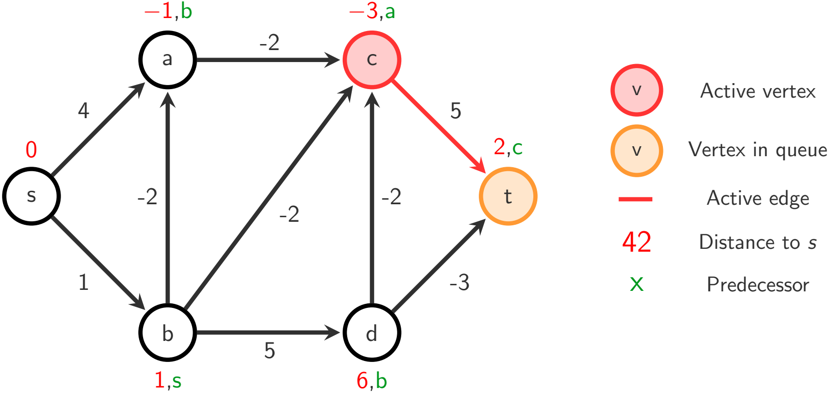
Bellman-Ford Algorithm

$Q = (c)$



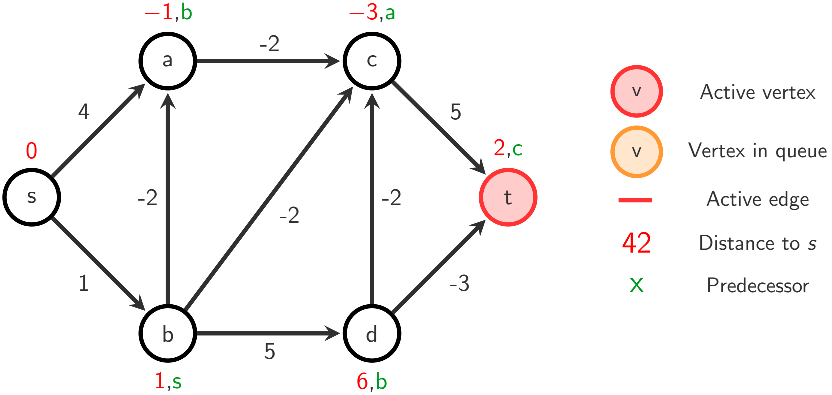
Bellman-Ford Algorithm

$Q = (t)$



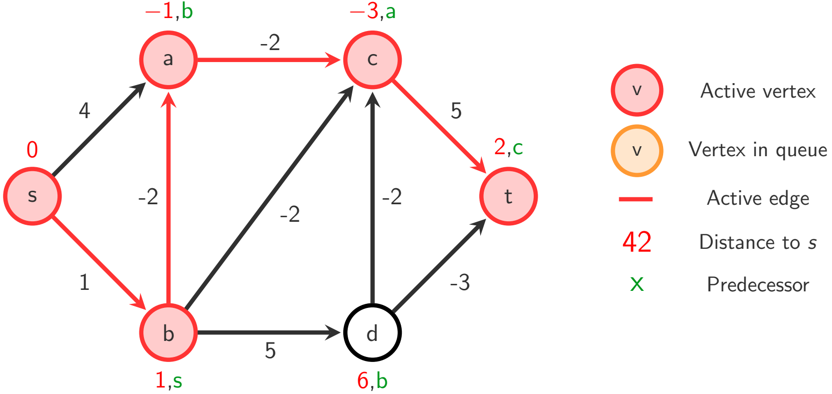
Bellman-Ford Algorithm

$Q = ()$



Bellman-Ford Algorithm

$Q = ()$



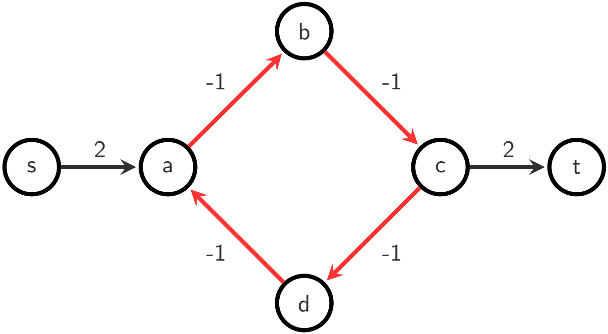
Algorithm 2 Bellman-Ford Algorithm (no negative cycles)

Input: Graph $G = (V, E, c)$ with no negative cycles

```
procedure BELLMAN-FORD(G, src)
    for each vertex v ∈ V do
        dist[v] ← ∞, prev[v] ← null
    end for
    dist[src] ← 0
    Q ← FIFO-Queue
    Q.insert(src)
    while Q is not empty do
        v ← Q.pop()
        for each neighbor w of v do
            if dist[v] + c(v, w) < dist[w] then
                dist[w] ← dist[v] + c(v, w)
                prev[w] ← v
                if w not in Q then
                    Q.push(w)
                end if
            end if
        end for
    end while
end procedure
```

Negative Cycles

- If there are negative cycles in the graph, the distance between s and t can become arbitrarily short.
- Detection of negative cycles becomes necessary.



Negative Cycle Detection

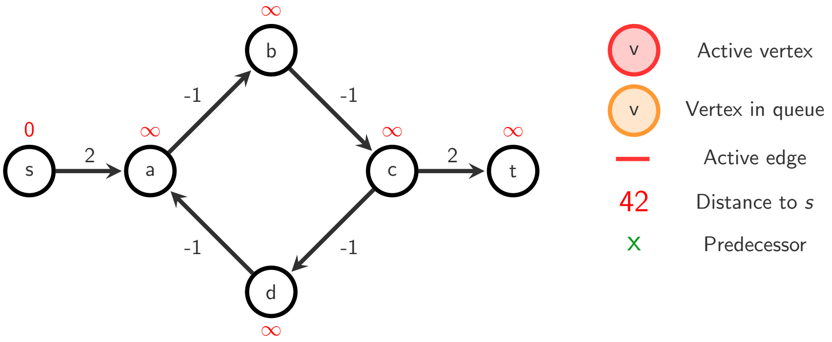
- Idea: Process FIFO-Queue in phases.
- One phase = processing all nodes currently in the queue.
- After phase i, all shortest paths of length i were detected.
- Longest shortest path contains at most $n - 1$ edges if there is no negative cycle.
- If there are nodes left in the queue after phase n, then there is a negative cycle.
- Cycle can be constructed by recursively visiting the predecessors of a node that is left in the queue after phase n.

Algorithm 3 Bellman-Ford Algorithm (negative cycle detection)

```
Input: Graph G = (V, E, c)
  procedure BELLMAN-FORD(G, src)
      for each vertex v ∈ V do
          dist[v] ← ∞, prev[v] ← null
      end for
      dist[src] ← 0
      Q, Q' ← FIFO-Queue
      Q.insert(src)
      for phase 1 to |V| do
          while Q is not empty do
              v ← Q.pop()
              for each neighbor w of v do
                  if dist[v] + c(v, w) < dist[w] then
                      dist[w] ← dist[v] + c(v, w)
                      prev[w] ← v
                      if w not in Q' then
                          Q'.push(w)
                      end if
                  end if
              end for
          end while
          swap(Q,Q')
      end for
      if Q is not empty then
          return there exists a negative cycle
      end if
  end procedure
```

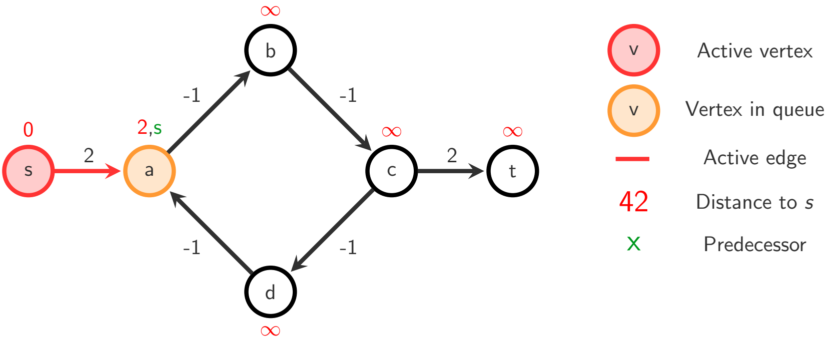
Bellman-Ford Algorithm with Negative Cycles

Initialization



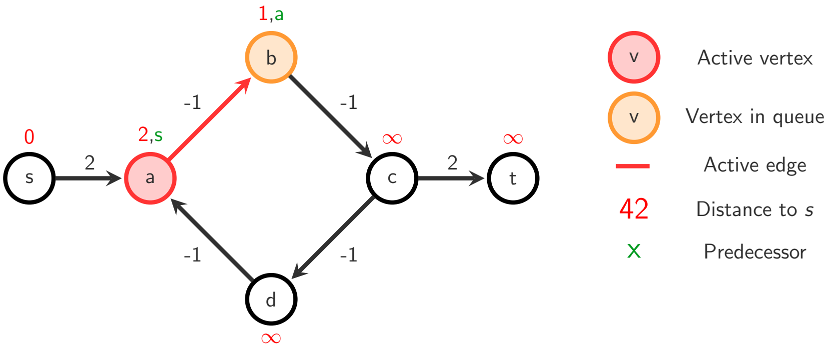
Bellman-Ford Algorithm with Negative Cycles

Phase 1: $Q = (s) \longrightarrow Q' = (a)$



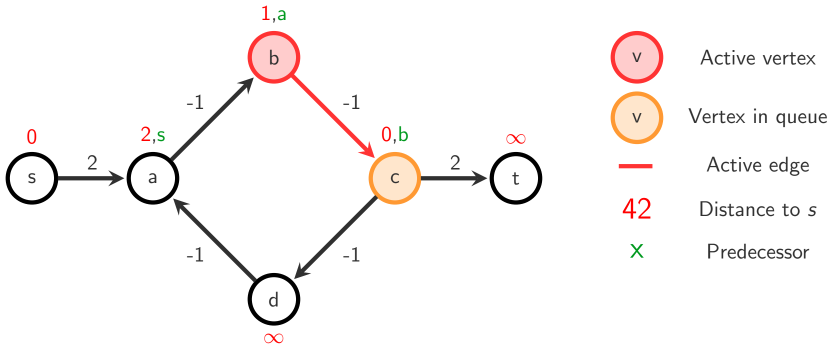
Bellman-Ford Algorithm with Negative Cycles

Phase 2: $Q = (a) \longrightarrow Q' = (b)$



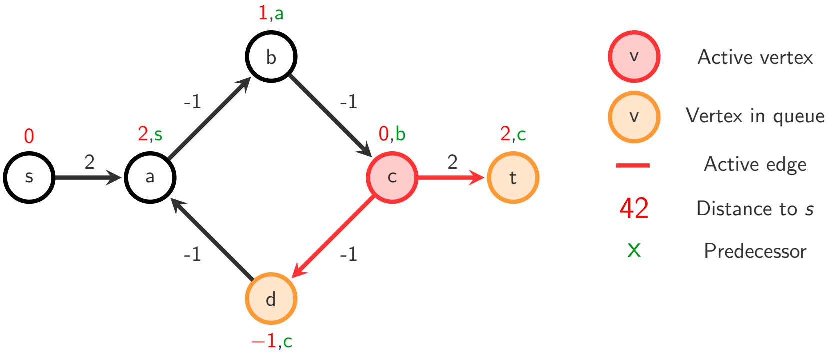
Bellman-Ford Algorithm with Negative Cycles

Phase 3: $Q = (b) \longrightarrow Q' = (c)$



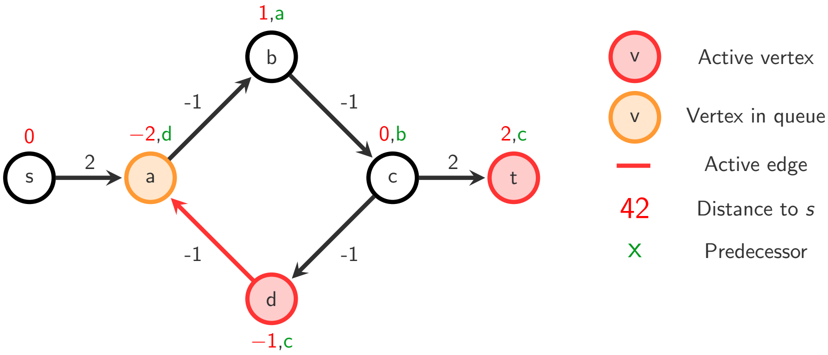
Bellman-Ford Algorithm with Negative Cycles

Phase 4: $Q = (c) \longrightarrow Q' = (d, t)$



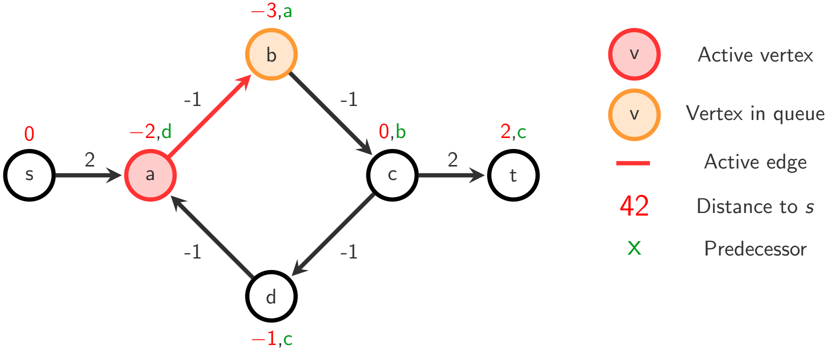
Bellman-Ford Algorithm with Negative Cycles

Phase 5: $Q = (d, t) \longrightarrow Q' = (a)$



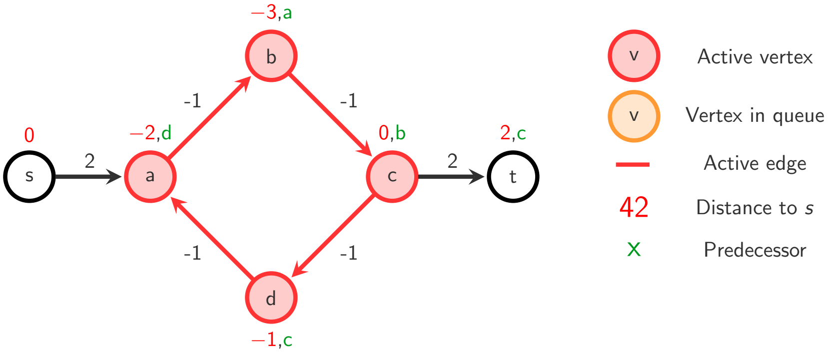
Bellman-Ford Algorithm with Negative Cycles

Phase 6: $Q = (a) \longrightarrow Q' = (b)$



Bellman-Ford Algorithm with Negative Cycles

After phase $6 = |V|$: $Q = (b)$
The queue is not empty $\rightarrow$ negative cycle $\rightarrow$ predecessor backtracking



Analysis of Bellman-Ford Algorithm

Running time

- At most $\mathcal{O}(|V|)$ phases.
- One phase takes at most $\mathcal{O}(|V| + |E|)$ operations.

  Pop all $|V|$ nodes, consider all $|E|$ edges, push all $|V|$ nodes.
- In total: $\mathcal{O}(|V||E|)$

Floyd-Warshall Algorithm

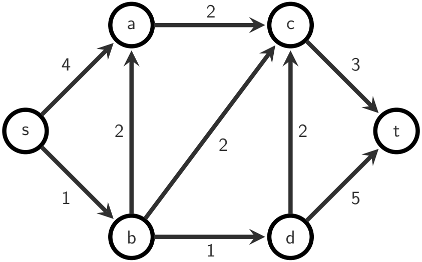
How to solve APSP?

- **Naive approach:** Executing Dijkstra algorithm $|V|$ times
  - Runtime: $\mathcal{O}(|V||E| + |V|^2 \log |V|)$
  - Can neither handle negative edge weights nor negative cycles.
- **Floyd-Warshall Algorithm**:
  - Runtime $\mathcal{O}(|V|^3)$
  - Can handle negative edge weights.
  - Negative cycle detection possible.
  - Easy to code.

$\Rightarrow$ Apply the naive approach if the graph is sparse!

Floyd-Warshall Algorithm

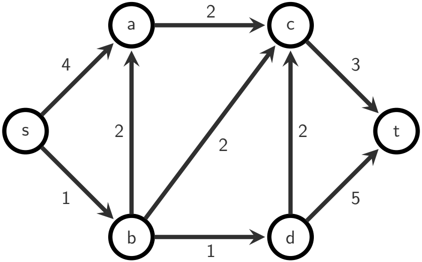
- Represent graph in distance matrix.
- Idea: successively add vertices as intermediate nodes for shortest paths.



$$\text{dist} = \begin{array}{c} \\ s \\ a \\ b \\ c \\ d \\ t \end{array} \begin{array}{c} \begin{array}{cccccc} s & a & b & c & d & t \end{array} \\ \begin{pmatrix} 0 & 4 & 1 & \infty & \infty & \infty \\ \infty & 0 & \infty & 2 & \infty & \infty \\ \infty & 2 & 0 & 2 & 1 & \infty \\ \infty & \infty & \infty & 0 & \infty & 3 \\ \infty & \infty & \infty & 2 & 0 & 5 \\ \infty & \infty & \infty & \infty & \infty & 0 \end{pmatrix} \end{array}$$

Floyd-Warshall Algorithm

- When considering a vertex k as intermediate node, there are two possibilities:
  - Shortest path between i and j does not go over k.
  - Shortest path between i and j uses k as intermediate node.
- Update: $\text{dist}[i][j] = \min\{\text{dist}[i][j], \text{dist}[i][k] + \text{dist}[k][j]\}$



$$\text{dist} = \begin{array}{c c} & \begin{array}{cccccc} s & a & b & c & d & t \end{array} \\ \begin{array}{c} s \\ a \\ b \\ c \\ d \\ t \end{array} & \begin{pmatrix} 0 & 4 & 1 & \infty & \infty & \infty \\ \infty & 0 & \infty & 2 & \infty & \infty \\ \infty & 2 & 0 & 2 & 1 & \infty \\ \infty & \infty & \infty & 0 & \infty & 3 \\ \infty & \infty & \infty & 2 & 0 & 5 \\ \infty & \infty & \infty & \infty & \infty & 0 \end{pmatrix} \end{array}$$

Algorithm 4 Floyd-Warshall Algorithm

Input: Graph $G = (V, E, c)$

```
procedure FLOYD-WARSHALL(G)
    dist[][] ← array of size |V| × |V| initialized to ∞
    for each vertex v ∈ V do
        dist[v][v] ← 0
    end for
    for each edge (u, v) ∈ E do
        dist[u][v] ← c(u, w)
    end for
    for each vertex k ∈ V do
        for each vertex i ∈ V do
            for each vertex j ∈ V do
                if dist[i][k] + dist[k][j] < dist[i][j] then
                    dist[i][j] ← dist[i][k] + dist[k][j]
                end if
            end for
        end for
    end for
end procedure
```

Analysis of Floyd-Warshall Algorithm

Running time

- Consider each of the $\mathcal{O}(|V|)$ vertices as intermediate node.
- Check if the shortest path between all $\mathcal{O}(|V|^2)$ vertex pairs becomes shorter by passing over intermediate node.
- In total: $\mathcal{O}(|V|^3)$

Floyd-Warshall Algorithm

- Order of loops matter: $k \rightarrow i \rightarrow j$
- Negative cycles exists $\Leftrightarrow$ negative entries on diagonal of matrix.
- Shortest path tree can be reconstructed by bookkeeping the update steps in another $|V| \times |V|$ matrix.
- Floyd-Warshall algorithm is an example of Dynamic Programming (discussed later in class).
- Other application: computation of transitive closure.

Longest Path Problem

- **Longest Path Problem:** Find a simple path of maximum length between two nodes in a graph.
- NP-hard for general graphs.
- Polynomial time algorithms exist for directed acyclic graphs.
- Application in DAGs: Finding critical paths in scheduling problems.

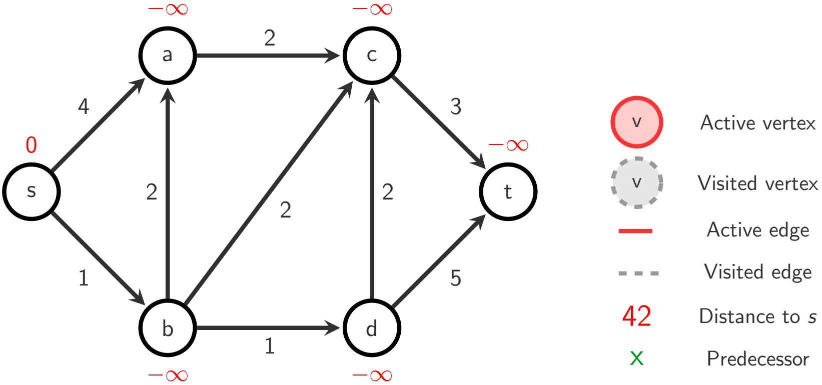
Longest Path Problem

- Approach 1:
  - Negate all edge weights in given DAG.
  - The shortest path in the modified graph is the longest path in the original graph.
  - Use Bellman-Ford to compute shortest path.
  - Complexity: $\mathcal{O}(|V||E|)$

- Approach 2:
  - Compute topological ordering of nodes in DAG.
  - Process nodes in topological order.
  - For each node v in the DAG check whether the distance to any of its successors can be increased by passing over v.
  - Complexity: $\mathcal{O}(|V| + |E|)$

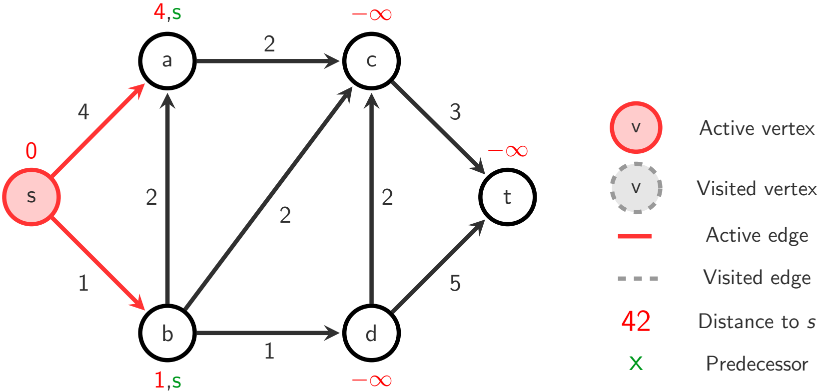
Longest Path in DAG

Topological order: s,b,a,d,c,t



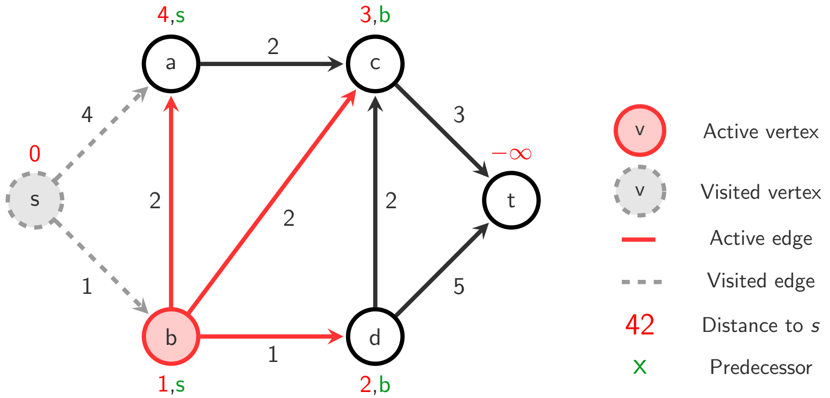
Longest Path in DAG

Topological order: s,b,a,d,c,t



Longest Path in DAG

Topological order: s,b,a,d,c,t



Longest Path in DAG

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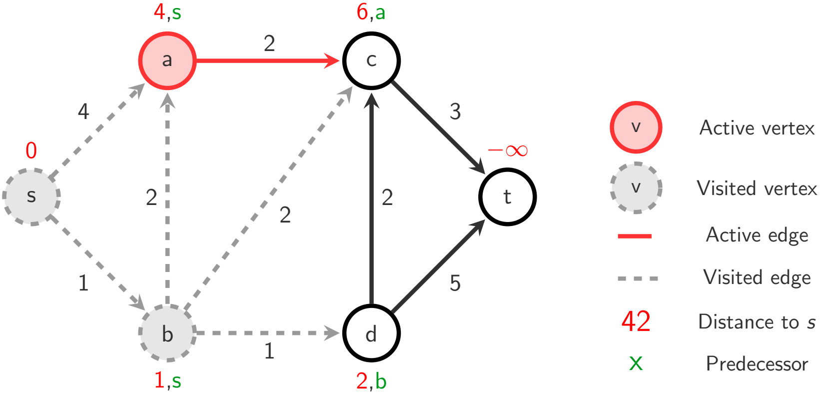

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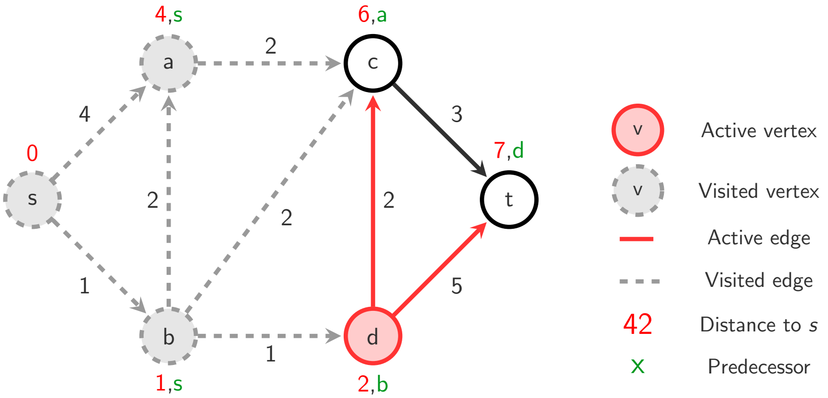

Longest Path in DAG

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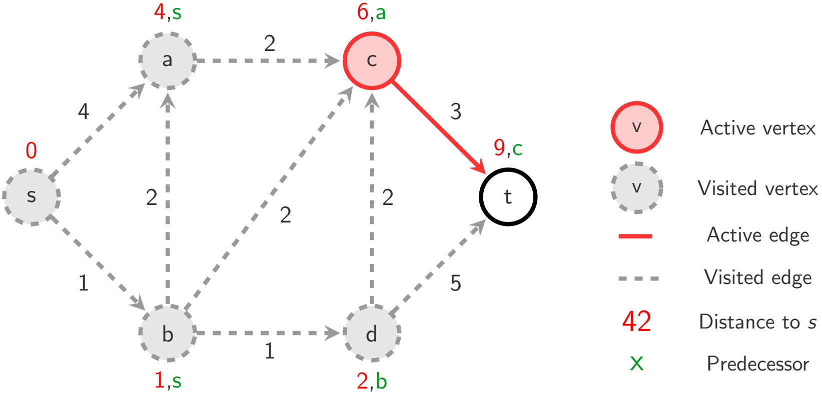

Longest Path in DAG

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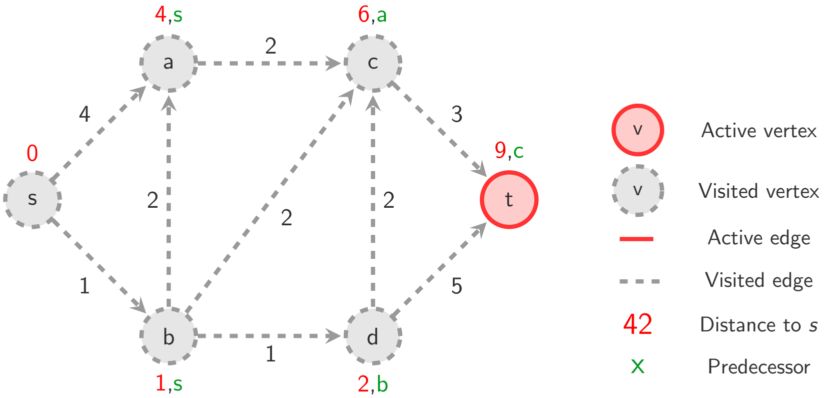

Longest Path in DAG

Topological order: s,b,a,d,c,t