On Arbitrage and Replication for Fractal Models

Albert N. Shiryaev

Steklov Mathematical Institute, Moscow, and MaPhySto, Aarhus[1]

1. On a filtered probability space

$$\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\le T}, P\big)$$

we consider a standard Brownian motion $B = (B_t)_{t\le T}$ and a fractal (or fractional) Brownian motion $B^H = \big(B_t^H\big)_{t\le T}$ with the Hurst exponent $H, 0 < H \le 1$, i.e. a Gaussian process with continuous trajectories and with $B_0^H = 0$, $EB_t^H = 0$, $E(B_t^H)^2 = t^{2H}$, $\operatorname{cov}\big(B_t^H, B_s^H\big) = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big)$.

For case $H = \frac{1}{2}$ a fractal Brownian motion is a Brownian motion. For $H = 1$ we have

$$\mathrm{Law}\big(B_t^1; t \le T\big) = \mathrm{Law}(\xi \cdot \mathrm{t}; \mathrm{t} \le \mathrm{T})$$

where $\xi \sim N(0,1)$.

2. Now we shall consider a *fractal version of the Bachelier model* (see [1; Chapter VIII]) of the $(B(r), S(\mu))-$market with a bank account $B(r) = (B_t(r))_{t\le T}$ and a risk asset $S(\mu) = (S_t(\mu))_{t\le T}$, where for simplicity we assume $r = 0$ and $B_t(0) \equiv 1$ and

$$S_t(\mu) = S_0 + \mu t + B_t^H. \tag{1}$$

For $H \in (0, 1/2) \cup (1/2, 1)$ a fractal Brownian motion B^H is a *non-semimartingale* [2]. So, in these cases martingale measures *do not exist* and the "First fundamental asset pricing theorem" ("No arbitrage is equivalent, essentially, to existence of a martingale measure") does not "work". This creates difficulties for finding a rational price for *hedging* of the contingent claim f_T by martingale methods.

Recall that by a standard definition a price $\mathbb{C}_T$ is *rational* if

$$\mathbb{C}_T = \inf\{x \ge 0 : \exists \pi \text{ with } X_0^\pi = x, X_T^\pi \ge f_T\}$$

where $\pi = (\beta, \gamma)$ is a portfolio and the corresponding capital is

$$X_t^\pi = \beta_t B_t(r) + \gamma_t S_t(\mu), t \le T.$$

We assume here, as usual, that π is a *self-financing* portfolio, i.e.

$$X_t^\pi = X_0^\pi + \int_0^t \gamma_u dS_u(\mu) \tag{2}$$

---

[1] MaPhySto - Centre for Mathematical Physics and Stochastics, funded by a grant from The Danish National Research Foundation

(for the case $B_t(r) \equiv \mathrm{Const.}$)

The corresponding stochastic integral in (2) we understand in the following sense.

Suppose that $H \in \left(\frac{1}{2}, 1\right)$ and let $f = f(x), x \in R$, be a function that belongs to the class C^1.

For

$$F(x) = F(0) + \int_0^x f(y)dy$$

by Taylor's formula with reminder in the integral form we have

$$F(x) = F(y) + f(y)(x-y) + \int_y^x f'(u)(x-u)du. \tag{3}$$

Hence, as in [3] and [4], for each sequence $T^n \equiv \left\{t^{(n)}(m), m \geq 1\right\}$, $n \geq 1$, of times $t^{(n)}(m)\left(0 = t^{(n)}(1) \leq t^{(n)}_{(2)} \leq \cdots\right)$ we have

$$\begin{aligned} F\left(B_t^H\right) - F\left(B_0^H\right) &= \sum_m \left[F\left(B^H_{t\wedge t^{(n)}(m+1)}\right) - F\left(B^H_{t\wedge t^{(n)}(m)}\right)\right] \\ &= \sum_m f\left(B^H_{t\wedge t^{(n)}(m+1)}\right)\left(B^H_{t\wedge t^{(n)}(m+1)} - B^H_{t\wedge t^{(n)}(m)}\right) + R_t^{(n)}, \end{aligned} \tag{4}$$

where

$$R_t^{(n)} = \sum_m \int_{B^H_{t\wedge t^{(n)}(m)}}^{B^H_{t\wedge t^{(n)}(m+1)}} f'(u)\left(B^H_{t\wedge t^{(n)}(m+1)} - u\right)du.$$

Clearly, $P\left(\sup\limits_{0\leq u\leq t}\left|f'(B_u^H)\right| < \infty\right) = 1$ and because for $H \in \left(\frac{1}{2}, 1\right)$

$$P - \lim_n \sum_m \left|B^H_{t^{(n)}(m+1)\wedge t} - B^H_{t^{(n)}(m)}\right|^2 = 0,$$

we obtain

$$\left|R_t^{(n)}\right| \leq \frac{1}{2}\sup_{0\leq u\leq t}\left|f'\left(B_u^H\right)\right| \cdot \sum_m \left|B^H_{t\wedge t^{(n)}(m+1)} - B^H_{t\wedge t^{(n)}(m+1)}\right|^2 \xrightarrow{P} 0.$$

The left-hand side of (4) is independent of n and $R_t^{(n)} \xrightarrow{P} 0$. So,

$$P - \lim_n \sum_m f\left(B^H_{t\wedge t^{(n)}(m)}\right)\left(B^H_{t\wedge t^{(n)}(m+1)} - B^H_{t\wedge t^{(n)}(m)}\right)$$

exists and we denote it by

$$\int_0^t f\left(B_u^H\right)dB_u^H \tag{5}$$

and call it the *stochastic integral with respect to the fractal Brownian motion* $B^H = \left(B_u^H\right)_{u \le t}$, $H \in \left(\frac{1}{2}, 1\right)$, $f \in C^1$.

The arguments given also prove that (P-a.s.)

$$F\left(B_t^H\right) - F\left(B_0^H\right) = \int_0^t f\left(B_u^H\right) dB_u^H, \tag{6}$$

which can be regarded as an analogue of *Itô's formula*, for a fractal Brownian motion.

Having the definition of the stochastic integral (5) we see that the stochastic integral in (2) is well defined at least for functions $\gamma = (\gamma_u)$ of the type of $\gamma_u = \gamma(B_u^H)$ (with $\int_0^t \gamma_u ds_u(\mu) \equiv \int_0^t \gamma_u \mu du + \int_0^t \gamma_u dB_u^H$).

Now we show that for (non-semimartingale) fractal Brownian motion B^H, $\frac{1}{2} < H < 1$, the corresponding $(B(0), S(\mu))-$market has an *arbitrage* property in the sense that there exists a portfolio π with $X_0^\pi = 0$, $X_T^\pi \ge 0$ (P-a.s.) and $P(X_T^T > 0) > 0$.

Indeed, let us define the (Markov) portfolio $\pi = (\beta, \gamma)$ with

$$\begin{aligned} \beta_t &= -\left(B_t^H\right)^2 - 2B_t^H, \\ \gamma_t &= 2B_t^H. \end{aligned} \tag{7}$$

then its value (with $S_0 = 1, \mu = 1$) is

$$X_r^\pi = \beta_t + \gamma_t S_s = \left(B_t^H\right)^2.$$

Using Itô's formula (6) we obtain

$$dX_t^\pi = d\left(B_t^H\right)^2 = 2B_t^H dB_t^H = \gamma_t dS_t.$$

This means that the constructed portfolio $\pi = (\beta, \gamma)$ is self-financing. Since $X_0^\pi = 0$ and $X_t^\pi = (B_t^H)^2 > 0$ for $t > 0$ it follows that arbitrage occurs in the $(B(0), S(1))-$market at each instant $t > 0$.

3. From the financial point of view the $(B(0), S(\mu))-$model described above is fairly artificial because the prices $S_t(\mu)$ can take negative values.

The next example of a *fractal version of the Black-Merton-Scholes model* does not have this deficiency.

Namely, consider a $(B(r), S(r))-$market with

$$\begin{aligned} B_t(r) &= e^{rt}, \\ S_t(r) &= e^{rt + B_t^H}. \end{aligned} \tag{8}$$

In view of Itô's formula (6) we have

$$dS_t(r) = S_t(r)\left(rdt + dB_t^H\right) \tag{9}$$

and evidently

$$dB_t(r) = rB_t(r)dt.$$

We now consider the portfolio $\pi = (\beta, \gamma)$ with

$$\begin{aligned} \beta_t &= 1 - e^{2B_t^H}, \\ \gamma_t &= 2\Big(e^{B_t^H} - 1\Big). \end{aligned} \tag{10}$$

For this portfolio we have

$$X_t^\pi = \beta_t B_t(r) + \gamma_t S_t(r) = e^{rt}\Big(e^{B_t^H} - 1\Big)^2.$$

Using again a simple extension of Itô's formula (6) (for $e^{rt}f(B_t^H)$) we obtain

$$dX_t^\pi = re^{rt}\Big(e^{B_t^H} - 1\Big)^2 dt + 2e^{rt+B_t^H}\Big(e^{B_t^H} - 1\Big)dB_t^H$$

and it is easy to see that the expression on the right-hand side is just the expression for $\beta_t dB_t(r) + \gamma_t dS_t(r)$ which means that the portfolio (10) is self-financing.

Since for this portfolio we also have $X_0^\pi = 0$ and $X_t^\pi > 0$ for $t > 0$, this model of a $(B(r), S(r))-$market leaves space for arbitrage for each $t > 0$. (Compare with another approach in [6]).

4. As we see for the above models there are arbitrage opportunities. However, it does not mean that we cannot discuss the problem of existence of opportunities for *replications* (i.e. completeness).

Indeed, if for example $S_t = S_0 + B_t^H$ then for the function $F = F(x)$ with $F'(x) \in C'$ we have by (6)

$$F(S_t) = F(S_0) + \int_0^t F'(Su)dS_u. \tag{11}$$

Thus, if we need to replicate the function $f_T = F(S_T)$ on the $(B(0), S(0))-$market (with $B_t(0) \equiv 1$, $S_t(0) = S_o + B_t^H$) then it is sufficient to define the corresponding portfolio $\pi = (\beta, \gamma)$ with a property of replication in the following way:

$$\begin{aligned} \gamma_t &= F'(S_t), \\ B_t &= F(S_t) - S_t \cdot F'(S_t). \end{aligned}$$

From (11) it follows that the initial capital of the replication portfolio π is $X_0^\pi = F(S_o)$.

5. Suppose now that a stock process $S = (S_t)$ has the following structure

$$S_t = S_o + \mu t + \sigma B_t + \sigma_H B_t^H, \tag{12}$$

where $B = (B_t)$ is a Brownian motion and $B^H = (B_t^H)$ is a fractal Brownian motion with $\frac{1}{2} < H < 1$.

If $\sigma_H > 0$ then the process $S = (S_t)$ is again a non-semimartingale and as a result there are no martingale measures. So, we cannot use here the standard martingale approach for pricing of the contingent claims.

However, it turns out that in the case $\sigma > 0$ the Brownian motion in (12) plays a kind of a "regularisation" role (cf. [5]) prevailing on a fractal Brownian motion.

Concretely, suppose that on a $(B(0), S_t(\mu, \sigma, \sigma_H))-$market with $B_t(0) \equiv 1$, $S_t(\mu, \sigma, \sigma_H) = S_t$ we consider the problem of a replication of the contingent claim $f_T = (S_T - K)^+$ of the standard call-option.

If $\sigma_H = 0$ then we have the usual Bachelier model with a unique martingale measure $\tilde{P}_T$ such that

$$d\tilde{P}_T = e^{-\frac{\mu}{\sigma}B_T - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T} dP_{T.}$$

From the general theory of pricing on arbitrage-free markets, [1], it follows that the rational price is

$$\mathbb{C}_T = E_{\tilde{P}_T} f_T. \tag{13}$$

For $f_T = (S_T - K)^+$ we find easily that

$$\mathbb{C}_T = (S_0 - K)\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\varphi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right), \tag{14}$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $\Phi(x) = \int\limits_{-\infty}^{x} \varphi(y)dy$. For an optimal portfolio $\tilde{\pi} = (\tilde{\beta}, \tilde{\gamma})$ the corresponding capital is

$$X_t^{\tilde{\pi}} = E_{\tilde{P}_T}(f_T | S_t) = C(t, S_t),$$

where

$$C(t, s) = (S - K)\Phi\left(\frac{S - K}{\sigma\sqrt{T - t}}\right) + \sigma\sqrt{T - t}\varphi\left(\frac{S - K}{\sigma\sqrt{T - t}}\right)$$

(see details in [1; chapter VIII]). Moreover

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial S^2} = 0, \quad t < T, \tag{15}$$

$$C(T, S) = (S - K)^{\sqrt{t}}, \tag{16}$$

$$dC(t, S_t) = \frac{\partial C}{\partial S} dS_t. \tag{17}$$

So, $dX_t^{\tilde{\pi}} = \tilde{\gamma}_t dS_t$, where $\tilde{\gamma}_t = \frac{dC}{dS}(t, S_t)$ and $\tilde{\beta}_t = C(t, S_t) - \tilde{\gamma}_t S_t$.

The idea of the pricing for the "mixture" model $(B(0), S(\mu, \sigma, \sigma_H))$ with $\sigma_H > 0$ is the following.

Let us take the portfolio $\tilde{\pi} = (\tilde{\beta}, \tilde{\gamma})$ obtained for the "pure" Bachelier model (with $\sigma_H = 0$) and apply it for the "mixture" model.

For this model the portfolio $\tilde{\pi} = (\tilde{\beta}, \tilde{\gamma})$ is self-financing by Itô's formula for $C(t, S_t)$ with S_t from (12):

$$\begin{aligned} dX_t^{\tilde{\pi}} &= dC(t, S_t) \\ &= \frac{\partial C}{\partial S} dS_t + \frac{\partial C}{\partial t} dt + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} (dS_t)^2 \\ &= \frac{\partial C}{\partial S} dS_t + \left[\frac{\partial C}{\partial S} + \frac{\sigma^2}{2} \frac{\partial^2 C}{\partial S^2} \right] dt \\ &= \frac{\partial C}{\partial S} dS_t \end{aligned} \tag{18}$$

where we use the fact that the function $C(t, S)$ satisfies (15).

Remark 1. The function $C(t, S) \in C^{1,2}$ and validity of the "differential" form of Itô's formula (18) is proved by arguments similar to those used for the proof of the formula (6).

For the portfolio $\tilde{\pi} = (\tilde{\beta}, \tilde{\gamma})$ we also have a replication property: $X_T^{\tilde{\pi}} = (S_T - K)^+$.

As a result we see also that for the "mixture" model with $S = (S_t)$, given by (12), the corresponding initial capital $X_0^{\tilde{\pi}}$, which is sufficient for construction a replication portfolio, is equal to $C(0, S_0) \equiv \mathbb{C}_T$.

Remark 2. From the arguments considered *it does not follow* that it is impossible to construct a replicating self-financing portfolio with smaller initial capital than $\mathbb{C}_T$.

Remark 3. A similar approach of replication can be applied to the "mixed" $(B(r), S(r, \sigma, \sigma_H))-$market with

$$S_t(r, \sigma, \sigma_H) = S_0 e \left(r - \frac{\sigma^2}{2} \right) t + \sigma B_t + \sigma_H B_t^H,$$

when $\frac{1}{2} < H < 1$.

References

[1] Shiryaev A.N. *Essentials of Stochastic Finance.* World Scientific Publ. (Dec. 1998). Russian edition: Publ. House FASIS (June 1998), Moscow.

[2] Liptser R.Sh., Shiryaev A.N. *Theory of Martingales.* Kluwer Acad. Publ., Dordrecht, 1989.

[3] Föllmer H., Protter Ph., Shiryaev A.N. Quadratic covariation and an extension of Itô's formula. *Bernoulli* **1** (1995), 149–170.

[4] Lin S.J. Stochastic analysis of fractional Brownian motions. *Stochastics and Stochastic Reports* **55** (1995), 121–140.

[5] Petit F., Yor M. Itô's formula and the martingales of certian submartingales. Preprint. INRIA, 1998.

[6] Rogers L.C.G. Arbitrage with fractional Brownian motion. *Mathematical Finance* **7** (1997), 95–105.