Audio Processing
Assignment N.1

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Abstract

Note: All insights and explanations in this section are primarily drawn from [1], with specific references to pages in the beginning of an answer.

1 Continuous Signals

1.1 Fourier transform

Annotations: The Fourier transform of x(t) is X(w)

a. Time convolution property:

Given: $X_1(w) = F(x_1(t)), X_2(w) = F(x_2(t))$

Prove: $F(x_1(t)*x_2(t)) = X_1(w)X_2(w)$ where * is the continuous convolution operator

Ans: Page 263

$F(x_1(t) * x_2(t)) = F(\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau)$ (Convolution Def.)

$= \int_{-\infty}^{\infty} e^{-jwt}[\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau]dt$ (FT Def.)

$= \int_{-\infty}^{\infty} e^{-jwt}[\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)dt]d\tau$ (Fubini's Theorem)

$= \int_{-\infty}^{\infty} x_1(\tau)[\int_{-\infty}^{\infty} e^{-jwt}x_2(t-\tau)dt]d\tau$ ($x_1(\tau)$ is constant for t)

$= \int_{-\infty}^{\infty} x_1(\tau)F(x_2(t-\tau))d\tau$ ($x_2(t-\tau)$ FT Def.)

$= \int_{-\infty}^{\infty} x_1(\tau)F(x_2(t))e^{-jw\tau}d\tau$ (Time shifting property)

$= F(x_2(t)) \int_{-\infty}^{\infty} e^{-jw\tau}x_1(\tau)d\tau$ ($F(x_2(t))$ is constant for τ)

$= F(x_2(t))F(x_1(t)) = X_1(w)X_2(w)$ ($x_1(\tau)$ FT Def.)

b. Linearity property:

Given: $X_1(w) = F(x_1(t)), X_2(w) = F(x_2(t))$

Prove: $F(ax_1(t) + bx_2(t)) = aX_1(w) + bX_2(w)$

Ans:

$F(ax_1(t) + bx_2(t)) = \int_{-\infty}^{\infty} e^{-jwt}[ax_1(t) + bx_2(t)]dt$ (FT Def.)

$= \int_{-\infty}^{\infty} e^{-jwt}ax_1(t) + e^{-jwt}bx_2(t)]dt$ (Multiplication factorization)

$= a\int_{-\infty}^{\infty} e^{-jwt}x_1(t)dt + b\int_{-\infty}^{\infty} e^{-jwt}x_2(t)]dt$ (Linearity of Integration)

$= aF(x_1(t)) + bF(x_2(t)) = aX_1(w) + bX_2(w)$ (FT Def.)

c. Scaling property:

Given: $X(w) = F(x(t))$

Prove: For $a > 0$ $F(x(at)) = \frac{1}{a}X(\frac{w}{a})$

Ans: Page 255

$F(x(at)) = \int_{-\infty}^{\infty} e^{-jwt}x(at)dt$ (FT Def.)

$= \frac{1}{a}\int_{-\infty}^{\infty} e^{-jw\frac{\tau}{a}}x(\tau)d\tau$ (Var change)

$= \frac{1}{a}X(\frac{w}{a})$ (FT Def.)

d. Time shifting property:

Given: $X(w) = F(x(t))$

Prove:

1. For a given $t_0, F(x(t - t_0)) = X(w)e^{-jwt_0}$
2. What is the effect of time shifting on the amplitude spectrum?
3. What is the effect of time shifting on the phase spectrum?

Ans: Page 257

1. $F(x(t - t_0)) = \int_{-\infty}^{\infty} e^{-jwt}x(t - t_0)dt$ (FT Def.)

$= \int_{-\infty}^{\infty} e^{-jw(\tau+t_0)}x(\tau)d\tau$ (Var change)

$= \int_{-\infty}^{\infty} e^{-jw\tau-jwt_0}x(\tau)d\tau = \int_{-\infty}^{\infty} e^{-jwt_0}e^{-jw\tau}x(\tau)d\tau$ (Exponential properties)

$= e^{-jwt_0}\int_{-\infty}^{\infty} e^{-jw\tau}x(\tau)d\tau$ (e^{-jwt_0} is constant for τ)

$= e^{-jwt_0}F(x(t)) = X(w)e^{-jwt_0}$ (FT Def.)

2. Amplitude spectrum won't change ($|e^{-jwt_0}| = 1$)
3. Phase spectrum will change by $-wt_0$

e. Time shifting property:

Given:

$$\text{rect}(t) = \begin{cases} 0 & \text{if } |t| > 0.5, \\ 0.5 & \text{if } |t| = 0.5, \\ 1 & \text{if } |t| < 0.5. \end{cases}$$

Prove:

1. $F(rect(\frac{t}{\tau})) = \tau sinc(\frac{w\tau}{2})$
2. Draw x(t) = rect(t)
3. Draw X(w), $|X(w)|, \angle$X(w)

Ans: Page 247

1. $F(rect(\frac{t}{\tau})) = \int_{-\infty}^{\infty} e^{-jwt} rect(\frac{t}{\tau})dt$ (FT Def.)

$= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-jwt} rect(\frac{t}{\tau})dt$ (rect = 0 for other values)

$= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-jwt} dt$ (rect = 1 for all values)

$= -\frac{1}{jw}[e^{-jw\frac{\tau}{2}} - e^{jw\frac{\tau}{2}}]$ (Simple integration)

$= -\frac{1}{jw}[cos(-\frac{w\tau}{2}) + jsin(-\frac{w\tau}{2}) - cos(\frac{w\tau}{2}) - jsin(\frac{w\tau}{2})]$ (Euler's formula)

$= -\frac{1}{jw}[cos(\frac{w\tau}{2}) - jsin(\frac{w\tau}{2}) - cos(\frac{w\tau}{2}) - jsin(\frac{w\tau}{2})]$ (cos(-x)=cos(x), sin(-x)=-sin(x))

$= -\frac{1}{jw}[-2jsin(\frac{w\tau}{2})] = \frac{2sin(\frac{w\tau}{2})}{w}$ (Simple math)

$= \tau \frac{sin(\frac{w\tau}{2})}{\frac{w\tau}{2}} = \tau sinc(\frac{w\tau}{2})$ (sinc Def.)

2+3. Appears at Figure 1

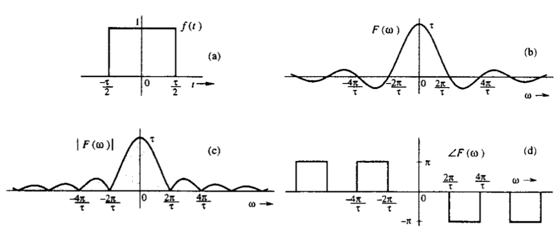

Figure 1: Draw x(t) = rect(t), X(w), $|X(w)|, \angle$X(w)

1.2 Fourier Series

a. Fourier series of delta function

i. Given the unit impulse train function $\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$

ii. Find the values of D_n of the exponential form

Ans: Page 212

$\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jnw_0 t}$ where $D_n = \frac{1}{T_0} \int_{T_0} f(t) e^{-jnw_0 t} dt$ (by Def.)

$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta_{T_0}(t) e^{-jnw_0 t} dt$ (Changing the interval of integration)

$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jnw_0 t} dt = \frac{1}{T_0} e^0 = \frac{1}{T_0}$

(Sampling property of unit impulse function)

iii. Draw $x(t) = \delta_{T_0}(t)$ and X(w)

Ans: Page 213

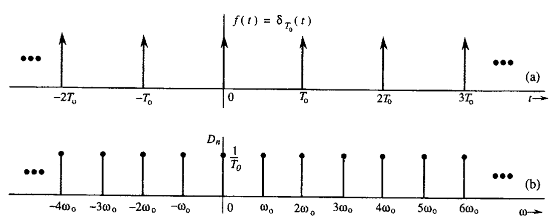

Figure 2: Top: x(t). Bottom: X(w)

iv. What is T_0

Ans:

The period of the impulse train, time interval between impulses in the train.

v. Find the interval between D_n and D_{n+1}, and its relation to T_0

Ans:

The interval is $\frac{2\pi}{T_0}$ the shorter the period T_0 in the time domain, the wider the spacing $\Delta\omega$ in the frequency domain, and vice versa.

3. Using the properties of the transform and the results from 2.a and 1.d, draw the spectrum of the following continuous and periodic function:

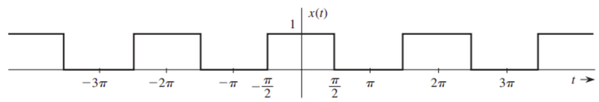

Figure 3: Given function

Ans: Page 196

$$x(t) = \begin{cases} 1 & -\frac{\pi}{2} + kT_0 \leq t \leq \frac{\pi}{2} + kT_0, \quad k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \text{Where } T_0 = 2\pi$$

x(t) is continuous and periodic function $\rightarrow$ CTFS

$F(x(t)) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$, where $D_n = \frac{1}{T_0} \int_{-\pi}^{\pi} x(t) e^{-jn\omega_0 t} dt$

$D_n = \frac{1}{T_0} \int_{-\pi/2}^{\pi/2} e^{-jn\omega_0 t} dt$ $\quad$ ($x(t) = 1$ when $\frac{-\pi}{2} \leq t \leq \frac{\pi}{2}$ else 0)

when $n = 0$, $D_n = \frac{1}{T_0}\pi = \frac{1}{2}$ else $\frac{1}{2\pi} sinc(\frac{n\pi}{2})$

$$|D_n| = \frac{1}{2\pi} \left|\text{sinc}\left(\frac{n\pi}{2}\right)\right|, \angle D_n = \begin{cases} 0, & \text{if sinc}\left(\frac{n\pi}{2}\right) > 0, \\ \pi, & \text{if sinc}\left(\frac{n\pi}{2}\right) < 0. \end{cases}$$

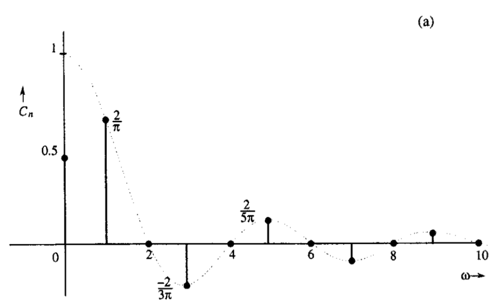

Figure 4: Spectrum of F(x(t))

4. Given $x(t) = e^{-at}u(t)$, where $a > 0$, and $u(t)$ is the unit step function.

   a. Find F(x(t))

Ans: $F(x(t)) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$ (FT Def.)

$= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-jwt}dt$ (x(t) Def.)

$= \int_{0}^{\infty} e^{-at}e^{-jwt}dt$ (u(t) Def.)

$= \int_{0}^{\infty} e^{-(a+jw)t}dt$ (Combine the Exponential)

$= 0 - \frac{-1}{a+jw} = \frac{1}{a+jw}$ ($a > 0$ so converge)

b. Draw it's magnitude and phase

Ans:

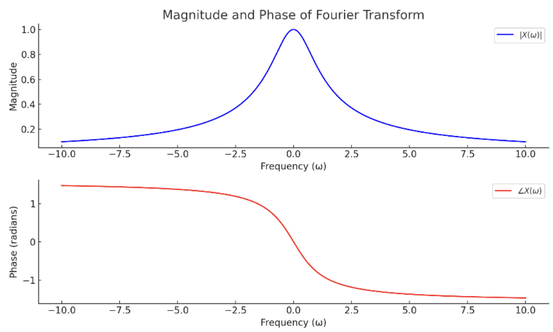

Figure 5: $|X(w)| = \frac{1}{\sqrt{a^2+w^2}}, \angle X(w) = -tan^{-1}(\frac{w}{a})$

c. What kind of filter can it be used for?

Ans: Low Pass Filter

2 Discrete Signals

2.1 Given $F_s = 8000Hz(8KHz))$

a. To which frequency 10KHz will be aliased to?

Ans: 2KHz

b. How could you prevent the aliasing if we had the analogue signal? Explain shortly in words

Ans: Set Low pass filter to ensure the filter allows only frequencies below the Nyquist frequency to pass.

If i must have the higher frequencies i could add more samples to capture them also.

2.2 Stereo hearing

a. Record yourself counting till 10 using your mobile device/phone - save it under the name 'audio_r.wav'

b. Make a copy of the file under 'audio_l.wav'

c. Open both files in Audacity / any other audio editing app that enables playing audio in stereo

d. Wear headphones:

   i. Play both channels

   ii. Shift 'audio_l.wav' 2ms to the right w.r.t 'audio_r.wav' and play both channels

   iii. Shift 'audio_r.wav' 2ms to the right w.r.t 'audio_l.wav' and play both channels

e. What do you hear? And why?

Ans:

a. The sound appears to come from the center of the head.
   The sound reaches both ears at the same time.

b. The sound appears to come from the right side.
   The sound reaches your right ear slightly earlier than your left ear

c. The sound appears to come from the left side.
   The sound reaches your left ear slightly earlier than your rightx ear

2.3 $\mathcal{Z}$ Transform

Annotations: The $\mathcal{Z}$ transform of x[n] is X(z), marked as $\mathcal{Z}(x[n]) = X(z)$

a. Given: $\mathcal{Z}(x_1[n]) = X_1(z), \mathcal{Z}(x_2[n]) = X_2(z)$

Prove: $\mathcal{Z}(x_1[n] * x_2[n]) = X_1(z)X_2(z)$ where $*$ is the discrete convolution operator

Ans: Page 508

$\mathcal{Z}(x_1[n] * x_2[n]) = \mathcal{Z}(\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m])$ (Conv Def.)

$= \sum_{n=-\infty}^{\infty} z^{-n} \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$ ($\mathcal{Z}$ Def.)

$= \sum_{m=-\infty}^{\infty} x_1[m] \sum_{n=-\infty}^{\infty} z^{-n}x_2[n-m]$ (Interchanging order of summation)

$= \sum_{m=-\infty}^{\infty} x_1[m] \sum_{n=-\infty}^{\infty} z^{-m}z^{-r}x_2[r]$ (Variable exchange)

$= \sum_{m=-\infty}^{\infty} x_1[m]z^{-m} \sum_{n=-\infty}^{\infty} z^{-r}x_2[r] = X_1(z)X_2(z)$ ($\mathcal{Z}$ Def.)

b. Given: $\mathcal{Z}(x_1[n]) = X_1(z)$

Prove: $\mathcal{Z}(a^n x_1[n]) = X(\frac{z}{a})$

Ans:

$\mathcal{Z}(a^n x_1[n]) = \sum_{-\infty}^{\infty} a^n x_1[n]z^{-n}$ ($\mathcal{Z}$ Def.)

$= \sum_{-\infty}^{\infty} x_1[n](\frac{z}{a})^{-n} = X(\frac{z}{a})$ ($\mathcal{Z}$ Def.)

2.4 DTFS

a. Given the signal $x[n] = cos(0.1\pi n)$:

i. How many samples are there in one period (what is N_0)?

Ans: Page 846

$\frac{\Omega}{2\pi} = \frac{m}{N_0} \rightarrow N_0 = m\frac{2\pi}{0.1\pi} = 20m = 20$

ii. What is the discrete time fourier series of x[n]?

Ans:

$x[n] = \sum_{r=-10}^{9} D_r e^{-jr0.1\pi n}$ where $D_r = \frac{1}{20}\sum_{n=-10}^{9} x[n]e^{-jr0.1\pi n}$ (DTFS Def.)

$D_r = \frac{1}{20}\sum_{n=-10}^{9} cos(0.1\pi n)e^{-jr0.1\pi n}$

$= \frac{1}{20}\sum_{n=-10}^{9}(\frac{1}{2}e^{j0.1\pi n} + \frac{1}{2}e^{-j0.1\pi n})e^{-jr0.1\pi n}$ ($x[n] = \frac{1}{2}e^{j0.1\pi n} + \frac{1}{2}e^{-j0.1\pi n}$)

$= \frac{1}{40}(\sum_{n=-10}^{9} e^{-(r-1)0.1\pi jn} + \sum_{n=-10}^{9} e^{-(1+r)0.1\pi jn})$ (Simplification)

$D_1 = 0.5, D_{-1} = 0.5$, all other D are 0

$\sum_{n=0}^{N_0-1} e^{j\alpha n} = \begin{cases} N_0, & \text{if } \alpha = 2\pi m \text{ (integer multiple of } 2\pi\text{)}, \\ 0, & \text{otherwise.} \end{cases}$

$x[n] = 0.5e^{-0.1\pi jn} + 0.5e^{0.1\pi jn}$

References

[1] Signal processing and Linear systems - B.P lathi