

Audio Processing

Assignment N.1

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Abstract

Note: All insights and explanations in this section are primarily drawn from [1], with specific references to pages in the beginning of an answer.

1 Continuous Signals

1.1 Fourier transform

Annotations: The Fourier transform of $x(t)$ is $X(w)$

a. Time convolution property:

Given: $X_1(w) = F(x_1(t))$, $X_2(w) = F(x_2(t))$

Prove: $F(x_1(t) * x_2(t)) = X_1(w)X_2(w)$ where $*$ is the continuous convolution operator

Ans: Page 263

$$\begin{aligned} F(x_1(t) * x_2(t)) &= F\left(\int_{-\infty}^{\infty} x_1(\tau)x_2(t - \tau)d\tau\right) && \text{(Convolution Def.)} \\ &= \int_{-\infty}^{\infty} e^{-j\omega t} \left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t - \tau)d\tau\right] dt && \text{(FT Def.)} \\ &= \int_{-\infty}^{\infty} e^{-j\omega t} \left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t - \tau)dt\right] d\tau && \text{(Fubini's Theorem)} \\ &= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} e^{-j\omega t} x_2(t - \tau)dt\right] d\tau && (x_1(\tau) \text{ is constant for } t) \\ &= \int_{-\infty}^{\infty} x_1(\tau) F(x_2(t - \tau)) d\tau && (x_2(t - \tau) \text{ FT Def.}) \\ &= \int_{-\infty}^{\infty} x_1(\tau) F(x_2(t)) e^{-j\omega\tau} d\tau && \text{(Time shifting property)} \\ &= F(x_2(t)) \int_{-\infty}^{\infty} e^{-j\omega\tau} x_1(\tau) d\tau && (F(x_2(t)) \text{ is constant for } \tau) \\ &= F(x_2(t)) F(x_1(t)) = X_1(w)X_2(w) && (x_1(\tau) \text{ FT Def.}) \end{aligned}$$

b. Linearity property:

Given: $X_1(w) = F(x_1(t))$, $X_2(w) = F(x_2(t))$

Prove: $F(ax_1(t) + bx_2(t)) = aX_1(w) + bX_2(w)$

Ans:

$$\begin{aligned} F(ax_1(t) + bx_2(t)) &= \int_{-\infty}^{\infty} e^{-j\omega t} [ax_1(t) + bx_2(t)] dt && \text{(FT Def.)} \\ &= \int_{-\infty}^{\infty} e^{-j\omega t} ax_1(t) + e^{-j\omega t} bx_2(t) dt && \text{(Multiplication factorization)} \\ &= a \int_{-\infty}^{\infty} e^{-j\omega t} x_1(t) dt + b \int_{-\infty}^{\infty} e^{-j\omega t} x_2(t) dt && \text{(Linearity of Integration)} \\ &= aF(x_1(t)) + bF(x_2(t)) = aX_1(\omega) + bX_2(\omega) && \text{(FT Def.)} \end{aligned}$$

c. Scaling property:

Given: $X(\omega) = F(x(t))$

Prove: For $a > 0$ $F(x(at)) = \frac{1}{a}X(\frac{\omega}{a})$

Ans: Page 255

$$\begin{aligned} F(x(at)) &= \int_{-\infty}^{\infty} e^{-j\omega t} x(at) dt && \text{(FT Def.)} \\ &= \frac{1}{a} \int_{-\infty}^{\infty} e^{-j\omega \frac{\tau}{a}} x(\tau) d\tau && \text{(Var change)} \\ &= \frac{1}{a} X(\frac{\omega}{a}) && \text{(FT Def.)} \end{aligned}$$

d. Time shifting property:

Given: $X(\omega) = F(x(t))$

Prove:

1. For a given t_0 , $F(x(t - t_0)) = X(\omega)e^{-j\omega t_0}$
2. What is the effect of time shifting on the amplitude spectrum?
3. What is the effect of time shifting on the phase spectrum?

Ans: Page 257

$$\begin{aligned} 1. \quad F(x(t - t_0)) &= \int_{-\infty}^{\infty} e^{-j\omega t} x(t - t_0) dt && \text{(FT Def.)} \\ &= \int_{-\infty}^{\infty} e^{-j\omega(\tau + t_0)} x(\tau) d\tau && \text{(Var change)} \\ &= \int_{-\infty}^{\infty} e^{-j\omega\tau - j\omega t_0} x(\tau) d\tau = \int_{-\infty}^{\infty} e^{-j\omega t_0} e^{-j\omega\tau} x(\tau) d\tau && \text{(Exponential properties)} \\ &= e^{-j\omega t_0} \int_{-\infty}^{\infty} e^{-j\omega\tau} x(\tau) d\tau && (e^{-j\omega t_0} \text{ is constant for } \tau) \\ &= e^{-j\omega t_0} F(x(t)) = X(\omega)e^{-j\omega t_0} && \text{(FT Def.)} \\ 2. \quad &\text{Amplitude spectrum won't change } (|e^{-j\omega t_0}| = 1) \\ 3. \quad &\text{Phase spectrum will change by } -\omega t_0 \end{aligned}$$

e. Time shifting property:

Given:

$$\text{rect}(t) = \begin{cases} 0 & \text{if } |t| > 0.5, \\ 0.5 & \text{if } |t| = 0.5, \\ 1 & \text{if } |t| < 0.5. \end{cases}$$

Prove:

1. $F(\text{rect}(\frac{t}{\tau})) = \tau \text{sinc}(\frac{w\tau}{2})$
2. Draw $x(t) = \text{rect}(t)$
3. Draw $X(w)$, $|X(w)|$, $\angle X(w)$

Ans: Page 247

$$\begin{aligned} 1. F(\text{rect}(\frac{t}{\tau})) &= \int_{-\infty}^{\infty} e^{-j\omega t} \text{rect}(\frac{t}{\tau}) dt && \text{(FT Def.)} \\ &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j\omega t} \text{rect}(\frac{t}{\tau}) dt && \text{(rect = 0 for other values)} \\ &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j\omega t} dt && \text{(rect = 1 for all values)} \\ &= -\frac{1}{j\omega} [e^{-j\omega \frac{\tau}{2}} - e^{j\omega \frac{\tau}{2}}] && \text{(Simple integration)} \\ &= -\frac{1}{j\omega} [\cos(-\frac{\omega\tau}{2}) + j\sin(-\frac{\omega\tau}{2}) - \cos(\frac{\omega\tau}{2}) - j\sin(\frac{\omega\tau}{2})] && \text{(Euler's formula)} \\ &= -\frac{1}{j\omega} [\cos(\frac{\omega\tau}{2}) - j\sin(\frac{\omega\tau}{2}) - \cos(\frac{\omega\tau}{2}) - j\sin(\frac{\omega\tau}{2})] && \text{(cos(-x)=cos(x), sin(-x)=-sin(x))} \\ &= -\frac{1}{j\omega} [-2j\sin(\frac{\omega\tau}{2})] = \frac{2\sin(\frac{\omega\tau}{2})}{\omega} && \text{(Simple math)} \\ &= \tau \frac{\sin(\frac{\omega\tau}{2})}{\frac{\omega\tau}{2}} = \tau \text{sinc}(\frac{\omega\tau}{2}) && \text{(sinc Def.)} \end{aligned}$$

2+3. Appears at Figure 1

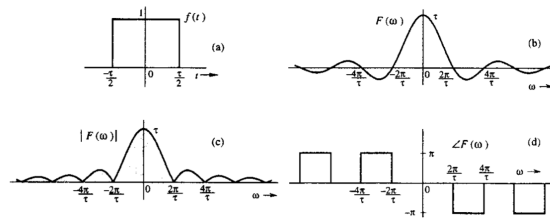


Figure 1: Draw $x(t) = \text{rect}(t)$, $X(w)$, $|X(w)|$, $\angle X(w)$

1.2 Fourier Series

a. Fourier series of delta function

- i. Given the unit impulse train function $\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$
- ii. Find the values of D_n of the exponential form

Ans: Page 212

$$\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \text{ where } D_n = \frac{1}{T_0} \int_{T_0} f(t) e^{-jn\omega_0 t} dt \quad (\text{by Def.})$$

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta_{T_0}(t) e^{-jn\omega_0 t} dt \quad (\text{Changing the interval of integration})$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} e^0 = \frac{1}{T_0}$$

(Sampling property of unit impulse function)

- iii. Draw $x(t) = \delta_{T_0}(t)$ and $X(\omega)$

Ans: Page 213

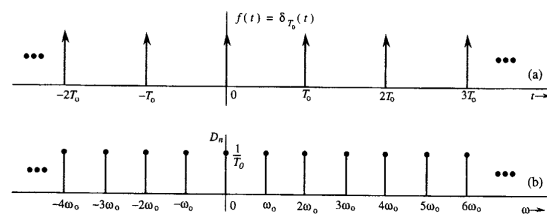


Figure 2: Top: $x(t)$. Bottom: $X(\omega)$

- iv. What is T_0

Ans:

The period of the impulse train, time interval between impulses in the train.

- v. Find the interval between D_n and D_{n+1} , and its relation to T_0

Ans:

The interval is $\frac{2\pi}{T_0}$ the shorter the period T_0 in the time domain, the wider the spacing $\Delta\omega$ in the frequency domain, and vice versa.

3. Using the properties of the transform and the results from 2.a and 1.d, draw the spectrum of the following continuous and periodic function:

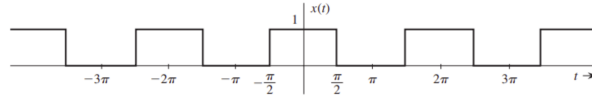


Figure 3: Given function

Ans: Page 196

$$x(t) = \begin{cases} 1 & -\frac{\pi}{2} + kT_0 \leq t \leq \frac{\pi}{2} + kT_0, \quad k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \quad \text{Where } T_0 = 2\pi$$

$x(t)$ is continuous and periodic function \rightarrow CTFS

$$F(x(t)) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}, \quad \text{where } D_n = \frac{1}{T_0} \int_{-\pi}^{\pi} x(t) e^{-jn\omega_0 t} dt$$

$$D_n = \frac{1}{T_0} \int_{-\pi/2}^{\pi/2} e^{-jn\omega_0 t} dt \quad (x(t) = 1 \text{ when } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \text{ else } 0)$$

$$\text{when } n = 0, D_n = \frac{1}{T_0} \pi = \frac{1}{2} \text{ else } \frac{1}{2\pi} \text{sinc}\left(\frac{n\pi}{2}\right)$$

$$|D_n| = \frac{1}{2\pi} \left| \text{sinc}\left(\frac{n\pi}{2}\right) \right|, \quad \angle D_n = \begin{cases} 0, & \text{if } \text{sinc}\left(\frac{n\pi}{2}\right) > 0, \\ \pi, & \text{if } \text{sinc}\left(\frac{n\pi}{2}\right) < 0. \end{cases}$$

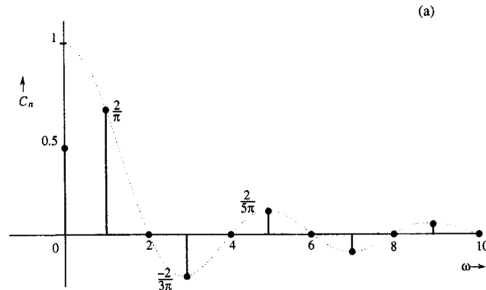


Figure 4: Spectrum of F(x(t))

4. Given $x(t) = e^{-at}u(t)$, where $a > 0$, and $u(t)$ is the unit step function.

a. Find $F(x(t))$

Ans: $F(x(t)) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$ (FT Def.)
 $= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t} dt$ (x(t) Def.)
 $= \int_0^{\infty} e^{-at}e^{-j\omega t} dt$ (u(t) Def.)
 $= \int_0^{\infty} e^{-(a+j\omega)t} dt$ (Combine the Exponential)
 $= 0 - \frac{-1}{a+j\omega} = \frac{1}{a+j\omega}$ ($a > 0$ so converge)

b. Draw it's magnitude and phase

Ans:

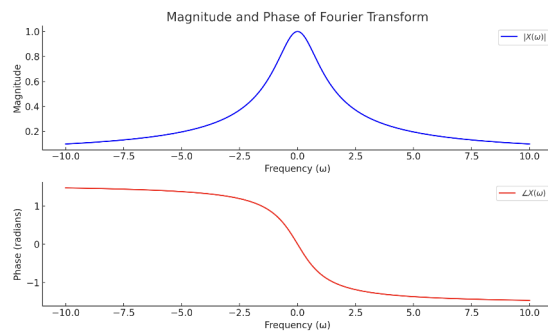


Figure 5: $|X(w)| = \frac{1}{\sqrt{a^2 + w^2}}$, $\angle X(w) = -\tan^{-1}(\frac{w}{a})$

c. What kind of filter can it be used for?

Ans: Low Pass Filter

2 Discrete Signals

2.1 Given $F_s = 8000Hz(8KHz)$

a. To which frequency 10KHz will be aliased to?

Ans: 2KHz

- b. How could you prevent the aliasing if we had the analogue signal? Explain shortly in words

Ans:

Set Low pass filter to ensure the filter allows only frequencies below the Nyquist frequency to pass.

If i must have the higher frequencies, i could increase the sample rate to capture them also.

2.2 Stereo hearing

- a. Record yourself counting till 10 using your mobile device/phone - save it under the name 'audio_r.wav'
- b. Make a copy of the file under 'audio_l.wav'
- c. Open both files in Audacity / any other audio editing app that enables playing audio in stereo
- d. Wear headphones:
 - i. Play both channels
 - ii. Shift 'audio_l.wav' 2ms to the right w.r.t 'audio_r.wav' and play both channels
 - iii. Shift 'audio_r.wav' 2ms to the right w.r.t 'audio_l.wav' and play both channels
- e. What do you hear? And why?

Ans:

- i. The sound appears to come from the center of the head.
The sound reaches both ears at the same time.
- ii. The sound appears to come from the right side.
The sound reaches your right ear slightly earlier than your left ear
- iii. The sound appears to come from the left side.
The sound reaches your left ear slightly earlier than your right ear
- e. What - written in black, Why written in red.

2.3 Z Transform

Annotations: The Z transform of $x[n]$ is $X(z)$, marked as $\mathcal{Z}(x[n]) = X(z)$

a. Given: $\mathcal{Z}(x_1[n]) = X_1(z)$, $\mathcal{Z}(x_2[n]) = X_2(z)$

Prove: $\mathcal{Z}(x_1[n] * x_2[n]) = X_1(z)X_2(z)$ where $*$ is the discrete convolution operator

Ans: Page 508

$$\begin{aligned}
 \mathcal{Z}(x_1[n] * x_2[n]) &= \mathcal{Z}\left(\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]\right) && \text{(Conv Def.)} \\
 &= \sum_{n=-\infty}^{\infty} z^{-n} \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m] && \text{(\mathcal{Z} Def.)} \\
 &= \sum_{m=-\infty}^{\infty} x_1[m] \sum_{n=-\infty}^{\infty} z^{-n}x_2[n-m] && \text{(Interchanging order of summation)} \\
 &= \sum_{m=-\infty}^{\infty} x_1[m] \sum_{n=-\infty}^{\infty} z^{-m}z^{-r}x_2[r] && \text{(Variable exchange)} \\
 &= \sum_{m=-\infty}^{\infty} x_1[m]z^{-m} \sum_{n=-\infty}^{\infty} z^{-r}x_2[r] = X_1(z)X_2(z) && \text{(\mathcal{Z} Def.)}
 \end{aligned}$$

b. Given: $\mathcal{Z}(x_1[n]) = X_1(z)$

Prove: $\mathcal{Z}(a^n x_1[n]) = X\left(\frac{z}{a}\right)$

Ans:

$$\begin{aligned}
 \mathcal{Z}(a^n x_1[n]) &= \sum_{n=-\infty}^{\infty} a^n x_1[n]z^{-n} && \text{(\mathcal{Z} Def.)} \\
 &= \sum_{n=-\infty}^{\infty} x_1[n]\left(\frac{z}{a}\right)^{-n} = X\left(\frac{z}{a}\right) && \text{(\mathcal{Z} Def.)}
 \end{aligned}$$

2.4 DTFS

a. Given the signal $x[n] = \cos(0.1\pi n)$:

i. How many samples are there in one period (what is N_0)?

Ans: Page 846

$$\frac{\Omega}{2\pi} = \frac{m}{N_0} \rightarrow N_0 = m \frac{2\pi}{0.1\pi} = 20m = 20$$

ii. What is the discrete time fourier series of $x[n]$?

Ans:

$$x[n] = \sum_{r=-10}^9 D_r e^{-jr0.1\pi n} \text{ where } D_r = \frac{1}{20} \sum_{n=-10}^9 x[n] e^{-jr0.1\pi n} \quad (\text{DTFS Def.})$$

$$D_r = \frac{1}{20} \sum_{n=-10}^9 \cos(0.1\pi n) e^{-jr0.1\pi n}$$

$$= \frac{1}{20} \sum_{n=-10}^9 \left(\frac{1}{2} e^{j0.1\pi n} + \frac{1}{2} e^{-j0.1\pi n} \right) e^{-jr0.1\pi n} \quad (x[n] = \frac{1}{2} e^{j0.1\pi n} + \frac{1}{2} e^{-j0.1\pi n})$$

$$= \frac{1}{40} \left(\sum_{n=-10}^9 e^{-(r-1)0.1\pi jn} + \sum_{n=-10}^9 e^{-(1+r)0.1\pi jn} \right) \quad (\text{Simplification})$$

$$D_1 = 0.5, D_{-1} = 0.5, \text{ all other } D \text{ are } 0$$

$$\sum_{n=0}^{N_0-1} e^{j\alpha n} = \begin{cases} N_0, & \text{if } \alpha = 2\pi m \text{ (integer multiple of } 2\pi), \\ 0, & \text{otherwise.} \end{cases}$$

$$x[n] = 0.5e^{-0.1\pi jn} + 0.5e^{0.1\pi jn}$$

References

- [1] Signal processing and Linear systems - B.P lathi