AIFS Lecture 1: Bare Minimum Linear Algebra

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Overview

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Sets

- A *set* is a well-defined collection of distinct objects.
- Two ways to write a set: Explicit or Implicit
- *Explicit*: Write the set by explicitly mentioning its members within curly brackets.
  Eg: $\{Honda, Ferrari, Renault\}$, $\{3, 5, 17\}$, $\emptyset$, $\{1, 3, \{1, 5\}\}$
- *Implicit*: Write in set-builder notation.
  Eg: $A = \{x : x \in H \text{ and } x < 10\}$ where $H = \{94, 324, 1, 9, 50\}$
  $A = \{1, 9\}$
- The set builder notation is read out loud as:
  x such that x in H and x less than 10

Properties of Sets

- All elements of a set are unique.
- The order of the elements in the set does not matter.
- Sets need not be homogeneous.
- *Cardinality*: The number of elements in the set. Eg: Let A={3,4}, then $|A| = 2$
- *Membership*: Given any object x and a set A, we can ask if x is a member of A. If x is a member then we write $x \in A$, if not we write $x \notin A$. Eg: Let $A = \{4, 7\}$, then $4 \in A$ and $9 \notin A$
- *Subset*: Given some sets B and A, B is a subset of A if all the members of B is in A. It's written as $B \subset A$. Eg: Let $A = \{3, 5, 7\}$ and $B = \{3, 7\}$, then $B \subset A$
- *Power Set*: Given a set S, the power-set of S, denoted by $\mathcal{P}(S)$, is the set of all subsets of S. Eg: Let $S = \{3, 5, 7\}$, then $\mathcal{P}(S) = \{\emptyset, \{3\}, \{5\}, \{7\}, \{3, 5\}, \{3, 7\}, \{5, 7\}, \{3, 5, 7\}\}$

Tuples

- A tuple is a finite ordered list of elements. Eg: $(1, 3), (1, 3, 1), (5, 5)$
- The ordering is important in tuples, and hence the uniqueness property does not hold.
- Based on the length of the tuple it's often referred to as an *n*-tuple. Eg: $(1, 3)$ is a 2-tuple. A 2-tuple is also known as an *ordered pair*.

Operations on Sets

Let $A = \{3, 5, 7, 11, 13\}$, and $B = \{1, 3, 5, 7, 9\}$

- Union: $\cup$ (think joining)
  $A \cup B = \{1, 3, 5, 7, 9, 11, 13\}$
- Intersection: $\cap$ (think commonality)
  $A \cap B = \{3, 5, 7\}$
- Difference: $\setminus$ (think subtraction)
  $A \setminus B = \{11, 13\}$
  $B \setminus A = \{1, 9\}$
- Cartesian Product: $\times$ (think combination)
  $A \times B = \{(x, y) : x \in A \text{ and } x \in B\}$
  $A \times B = \{(3, 1), (3, 3), (3, 5), \ldots, (5, 1), (5, 3), (5, 5), \ldots\}$

Number Systems

- Natural Numbers: $\mathbb{N} = \{1, 2, 3, 4 \ldots\}$ or $\{0, 1, 2, 3, 4 \ldots\}$
- Integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Rational Numbers: The set of all numbers that can be written in the form $\frac{a}{b}$ where $b \neq 0$ and $a, b \in \mathbb{Z}$. Denoted by $\mathbb{Q}$
- Real Numbers: The set of all rational numbers and irrational numbers (such as $\sqrt{2}, \sqrt{5}, \pi$, etc.). Denoted by $\mathbb{R}$

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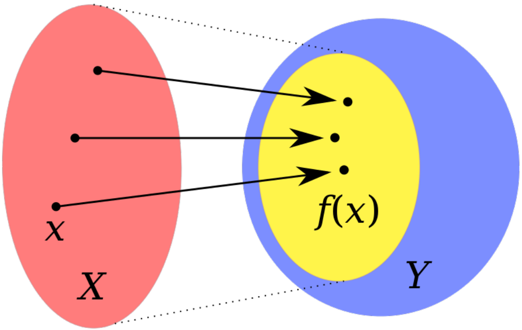
Functions

- A function is a mapping from one set to another where each element of the first set is mapped exactly to one element in the second
- *Notation*: A function, f, which maps elements in set X to set Y is written as:

$$f : X \to Y$$

- *Notation*: The mapping is explicitly specified as $y = f(x)$ read out loud as *"f of x"*. Here y is the *value* or *output* of the function and x is the *argument* or *input* to the function.
- *Example*: Let $X = \{-1, -2, 2, 0, 3\}$ and $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then forall $x \in X$, $f(x) = x^2$ is a function from X to Y

Functions



$$f : X \to Y$$

- Red set is called the *Domain* of the function.
- Blue set is called the *Codomain* of the function.
- Yellow set is a subset of the Codomain called the *range* of the function.

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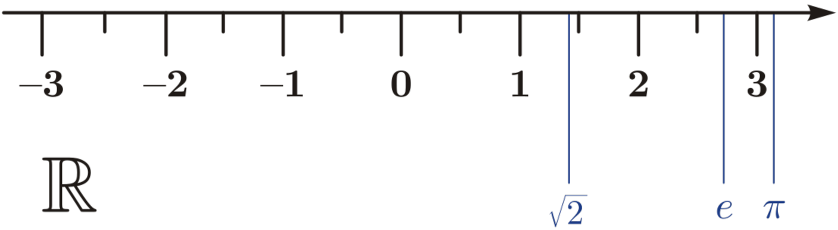
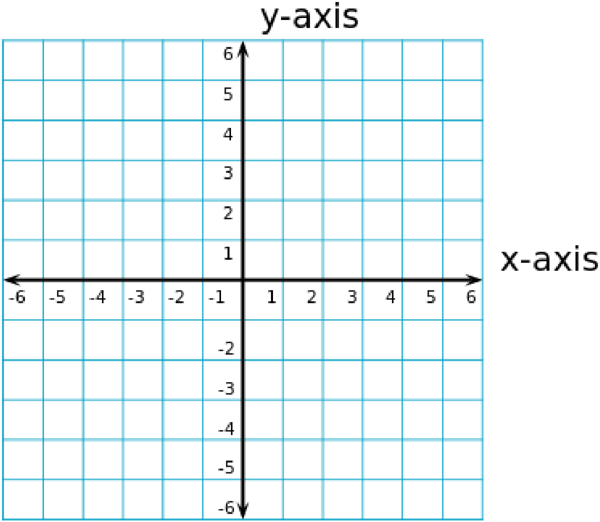
1-D Space



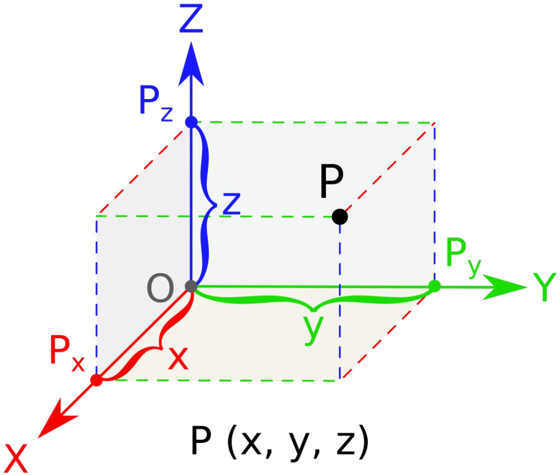
2-D Space

$\mathbb{R}^2$



3-D Space

$\mathbb{R}^3$



N-D Space

- A point in $\mathbb{R}^n$ space would have n coordinates.
- A line in $\mathbb{R}^2$ would be a plane in $\mathbb{R}^3$, and a hyper-plane in $\mathbb{R}^n$
- To visualize an N-D space, think of a 3-D space and call it N-D :)

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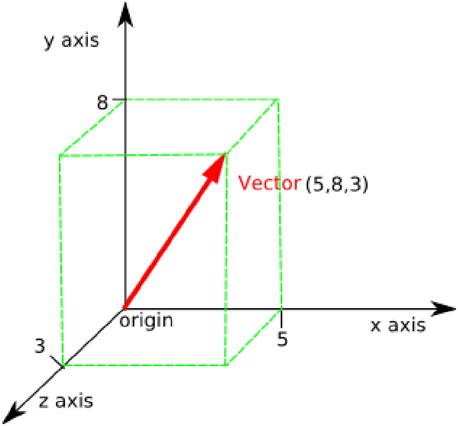
Scalars and Vectors

- *Scalar*: A single number. Usually written with lower case non-bold variable names. Eg: $3, 4.5, \frac{1}{3}, a$ for all $a \in \mathbb{N}$
- *Vector*: An array of ordered numbers. Can be written as a column of numbers, or as an n-tuple. Usually written with lower case bold variable names.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ OR } \mathbf{x} = (x_1, x_2, \ldots, x_n)$$

Cartesian Coordinate View



A vector can be visualized as an object identifying points in space, with each component of the vector giving the coordinate along a different axis.

Vector space

- A real valued vector space is a set V on which two operations $+$ and $\cdot$ are defined, called vector addition and scalar multiplication, and satisfy the following three properties for all scalars $c \in \mathbb{R}$.
- Closure under addition: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in V then $\mathbf{u} + \mathbf{v}$ should also be in V.
- Closure under multiplication: If c is any scalar in $\mathbb{R}$ and $\mathbf{u}$ is a vector in V then $c \cdot \mathbf{u}$ is also in V.
- The zero vector $\mathbf{0}$ is in V.

Vector space