Obstructions to maps out of equivariant multiplicative Thom spectra

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Outline

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3. Equivariant ring spectrum obstructions
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Motivation

- One popular approach to study of the stable homotopy groups of spheres is through the approach of chromatic homotopy theory.

Theorem (Devinatz–Hopkins [4], 2004)

Certain Lubin–Tate spectra E_n, spectra associated to formal group laws, having group actions by respective Morava stabilizer groups have homotopy fixed point spectra $L_{K(n)}\mathcal{S}$ where $K(n)$ is the p-local Morava K-theory.

- Varying n, $\pi_* L_{K(n)}\mathcal{S}$ gives a filtration on the p-local stable homotopy groups of spheres.

Motivation (cont.)

- E_n are constructed from MU via Landweber exact functor theorem and have multiplicative structure understood by obstruction theories of Goerss–Hopkins–Miller [10, 6, 5].
- Genuine equivariant homotopy theory holds rich invariants

Theorem (Hahn–Shi [8], 2017)

There is a map of equivariant A_∞-ring spectra $MU_{\mathbb{R}} \to E_n$ where the C_2 action corresponds to inversion of the formal group. The genuine homotopy groups $\underline{\pi}_\star(E_n)$ are computed as well as the homotopy fixed point spectral sequence.

Driving question

- More structure desired for progress on similar problems for larger subgroups of the Morava stabilizer group $G \geq C_2$.

Question

Is the map $MU_{\mathbb{R}} \to E_n$ more than a map of A_∞ ring spectra? They are both $C_2 E_\infty$.

- Non-equivariantly,

Theorem (Chadwick–Mandell, 2015 [3])

Let R be a connective E_2-ring spectrum with $\pi_*(R)$ concentrated in even degrees. Every ring spectrum map $MU \to R$ is homotopic to an E_2-ring spectrum map.

- In particular, $MU \to E_n$ is an E_2-ring map.

Overview of operads

- From now on G is a finite group.

Definition

An operad in G-spaces $\mathcal{P}$ consists of $G \times \Sigma_n$-spaces with an identity element $e \in \mathcal{P}(1)$ and composition operations $\gamma : \mathcal{P}(k) \times \prod_{i=1}^{k} \mathcal{P}(n_i) \to \mathcal{O}(\sum_{i=1}^{n} n_i)$ satisfying equivariance, identity, and associativity axioms.

- The little V-disks operad C_V is an operad whose elements in $C_V(n)$ are tuples of n-disjoint disks. Composition: $\gamma(x, (y_i)_{i=1}^{k})$ is the result of placing appropriate scaled and translated copies of y_i in the i^{th} disk of x.

Overview of operads (cont.)

Let $\mathcal{D}$ be a symmetric monoidal category enriched, powered, and copowered in G-spaces satisfying compatibility conditions involving colimits and the monoidal product. Let $\underline{\otimes}$ denote the copowering. For example in G-spaces, $X \underline{\otimes} Y = X \times Y$, while in genuine G-spectra, $X \underline{\otimes} Y = \Sigma^\infty_+ X \wedge Y$.

Definition

An algebra X over an operad $\mathcal{P}$ consists of action maps $\mathcal{P}(n) \otimes \bigotimes_{i=1}^n X \to X$ satisfying equivariance, unit, and associativity axioms.

- Given an operad $\mathcal{P}$, there is an associated monad, consisting of a free algebra functor, equipped with associated unit and multiplication map from the free-forgetful adjunction.

Augmented Objects

Definition

Let U be the unit object in $\mathcal{D}$ An augmented object U in $\mathcal{D}$ is an object X equipped with a retract diagram

$$\begin{tikzcd} U \arrow[r] \arrow[rr, bend left, "1_U"] & X \arrow[r] & U \end{tikzcd}$$

Proposition (May 1972 [9], R.)

If $\mathcal{P}$ is an operad with $\mathcal{P}(0) = *$ then there is a free augmented $\mathcal{P}$-algebra monad $\mathfrak{P}$ on the category of augmented objects in $\mathcal{D}$.

The free algebras on the augmented and non-augmented categories are not the same.

Some notation

- Recall $\underline{\otimes}$ denotes the copowering. (Again, for example in G-spaces, $X \underline{\otimes} Y = X \times Y$, while in genuine G-spectra, $X \underline{\otimes} Y = \Sigma^{\infty}_{+} X \wedge Y$.)
- For two augmented objects A, B in $\mathcal{D}$, $A \overline{\otimes} B$ is given by the canonical pushout

$$\begin{tikzcd}
(A \otimes U) \sqcup_{U \otimes U} (U \otimes A) \arrow[r] \arrow[d] & U \arrow[d] \\
A \otimes B \arrow[r] & A \overline{\otimes} B \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

A key observation

Proposition (May 1972. in CGWH spaces, R.)

For X augmented in $\mathcal{C}$, there exists a natural map

$$(S^V \underline{\otimes} U)\overline{\otimes}\mathfrak{C}_V X \to (S^V \underline{\otimes} U)\overline{\otimes} X$$

satisfying commutative diagram conditions so that $(S^V \underline{\otimes} U)\overline{\otimes}(-)$ is a right module over $\mathfrak{C}_V$, the augmented algebra monad associated to C_V. These monads are natural in $\mathcal{D}$.

- May (1972) [9], and Guillou–May (2017) [7] go on to show that for some C_V-spaces Y (grouplike when V has a copy of the trivial rep.), a two sided simplicial bar construction $Y \simeq \Omega^V B^V Y$ where

$$B^V Y = B((S^V \underline{\otimes} U)\overline{\otimes}(-), \mathfrak{C}_V, Y) = B(\Sigma^V, \mathfrak{C}_V, Y)$$

  and $B^V \Omega^V X \simeq X$ for sufficiently connected spaces X
- The proposition above justifies the construction of B^V.

Aside: A question for you

Question

Let Z be a G-space such that the fixed points Z^H is homotopy equivalent to a finite dimensional sphere for every subgroup H of G. Can you construct an operad that fits into a two-sided bar construction B^Z so that $B^Z \operatorname{Map}_*(Z, X) \simeq X$ and $\operatorname{Map}_*(Z, B^Z Y) \simeq Y$ under analogous conditions?

Slice spheres and the slice tower

- Suppose $H \leq G$. There is an additive induction functor $\mathrm{Spectra}_H \to \mathrm{Spectra}_G$ denoted by $G \ltimes_H (-)$
- A slice sphere is a spectrum of the form $G_+ \ltimes_H S^{n\rho_H}$ for $n \in \mathbb{Z}$.
- Dimension of $G_+ \ltimes_H S^{n\rho_H}$ is $n|H|$ (the dimension of the representation $n\rho_H$.)
- Inclusions $G_+ \ltimes_H S^{n\rho_H} \to G_+ \ltimes_H D^{n\rho_H+1}$ have contractible target. Using slice cells of dimension $\geq n+1$, can construct from X a relative cell complex $P^n X$ such that

$$[G_+ \ltimes_H S^{m\rho_H}, X] \to [G_+ \ltimes_H S^{m\rho_H}, P^n X]$$

is an iso when $m|H| \leq n$ and the zero map when $m|H| > n$.

Slice limit diagram

This is a homotopy limit diagram.

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P^2X \arrow[d] \\
 & P^1X \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & P^0X
\end{tikzcd}$$

Algebraic cells

For a connective C_V algebra in G-spectra X one can construct C_V algebras with underlying spectra naturally homotopy equivalent to $P^n X$ using C_V-algebra cells $F_{C_V}(G_+ \ltimes_H S^{n\rho_H}) \to F_{C_V}(G_+ \ltimes_H D^{n\rho_H+1})$ where F_{C_V} is the free C_V-algebra functor $(m|H| > n)$.

Slice tower for highly structured equivariant ring spectrum

Theorem (Angeltveit–Blumberg–Gerhardt–Hill–Lawson–Mandell [1], (in preparation))

Let X be a connective C_V spectrum.

$$P^0 X = H\underline{\pi}_0(X)$$

If $P^n X$ is constructed, then one can build a $P^{n+1}X$ as a homotopy pullback (after replacements)

$$\begin{tikzcd} P^{n+1}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & P^0 X \arrow[d] \\ P^n X \arrow[r] & P^0 X \vee \Sigma P^{n+1}_{n+1} X \end{tikzcd}$$

where $P^{n+1}_{n+1}X$ is the fiber of $P^{n+1}X \to P^n X$ (called the $(n+1)^{th}$ slice of X).

The obstruction question

Question

Let Mf be an E_V Thom spectrum arising from an E_{V+1}-space map $Y \to GL_1\mathcal{S}$.
Let R be an E_V-ring spectrum.
Suppose $g : Mf \to R$ is a ring spectrum map such that $Mf \to P^0R$ is a fixed E_V map. Can we use the observation that the target of the k-invariant is a square-zero extension of P^0X to recharacterize the obstruction class?

Answer (Chadwick-Mandell 2007 [3] (non-equiv.), R.)

Yes, such maps are naturally in bijective correspondence with cohomology classes in $(P_{n+1}^{n+1}R)^{V+1}B^{V+1}Y$.

Why does B^V appear?

Theorem (Basterra–Mandell 2011 [2](non-equiv.), R.)

Let k be a commutative ring spectrum. Suppose A is an augmented E_V-ring spectrum in k-modules and suppose $k \vee M$ in the homotopy category is the initial square-zero extension of k under A. Then $B^V A$ is equivalent to $k \vee \Sigma^V A$

Question

Can one use the above result to study ring spectra of the form $R \wedge R$?

Specializing to C_2, $MU_{\mathbb{R}}$

- Specialize to the case $G = C_2$ and $V = \rho$ and study $MU_{\mathbb{R}}$
- $P^{2k}_{2k} X \simeq \Sigma^{k\rho} H\underline{\pi}_{k\rho}(X)$
- $P^{2k+1}_{2k+1} X \simeq \Sigma^{k\rho+\sigma} H\underline{\pi}_{k\rho+\sigma}(X)$ where σ is the sign representation.
- $H^{\star}\underline{M}(B^{\rho+1}U_{\mathbb{R}})$ is concentrated in even dimensional representation degrees when M is a constant Mackey functor, i.e. its restriction map is an isomorphism.

Answer to the driving question

Theorem (R.)

Suppose R is a connective E_ρ-ring spectrum with each $\underline{\pi}_{k\rho}(R)$ a constant Mackey functor and $\underline{\pi}_{k\rho+\sigma}(R) = 0$ for all non-negative k. Any ring map $MU_{\mathbb{R}} \to R$ is homotopic to a map of E_ρ-algebras.

Corollary

The Hahn-Shi map $MU_{\mathbb{R}} \to E_n$ is homotopic to an E_ρ-algebra map.

Corollary

Every ring map $MU_{\mathbb{R}} \to MU_{\mathbb{R}}$ is homotopic to an E_ρ-algebra map. Consequently, the Real Brown–Peterson spectrum $BP_{\mathbb{R}}$ has an E_ρ structure.

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