Obstructions to maps out of equivariant multiplicative Thom spectra

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Outline

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Motivation

 One popular approach to study of the stable homotopy groups of spheres is through the approach of chromatic homotopy theory.

Theorem (Devinatz-Hopkins [4], 2004)

Certain Lubin–Tate spectra E_n , spectra associated to formal group laws, having group actions by respective Morava stabilizer groups have homotopy fixed point spectra $L_{K(n)}S$ where K(n) is the p-local Morava K-theory.

• Varying n, $\pi_* L_{K(n)} S$ gives a filtration on the p-local stable homotopy groups of spheres.

Motivation (cont.)

- *E_n* are constructed from *MU* via Landweber exact functor theorem and have multiplicative structure understood by obstruction theories of Goerss–Hopkins–Miller [10, 6, 5].
- Genuine equivariant homotopy theory holds rich invariants

Theorem (Hahn–Shi [8], 2017)

There is a map of equivariant A_{∞} -ring spectra $MU_{\mathbb{R}} \to E_n$ where the C_2 action corresponds to inversion of the formal group. The genuine homotopy groups $\underline{\pi}_{\star}(E_n)$ are computed as well as the homotopy fixed point spectral sequence.

Driving question

• More structure desired for progress on similar problems for larger subgroups of the Morava stabilizer group $G \ge C_2$.

Question

Is the map $MU_{\mathbb{R}} \to E_n$ more than a map of A_{∞} ring spectra? They are both C_2E_{∞} .

Non-equivariantly,

Theorem (Chadwick-Mandell, 2015 [3])

Let R be a connective E_2 -ring spectrum with $\pi_*(R)$ concentrated in even degrees. Every ring spectrum map $MU \to R$ is homotopic to an E_2 -ring spectrum map.

• In particular, $MU \rightarrow E_n$ is an E_2 -ring map.

Overview of operads

• From now on *G* is a finite group.

Definition

An operad in G-spaces $\mathcal P$ consists of $G \times \Sigma_n$ -spaces with an identity element $e \in \mathcal P(1)$ and composition operations $\gamma: \mathcal P(k) \times \prod_{i=1}^k \mathcal P(n_i) \to \mathcal O(\sum_{i=1}^n n_i)$ satisfying equivariance, identity, and associativity axioms.

• The little V-disks operad C_V is an operad whose elements in $C_V(n)$ are tuples of n-disjoint disks. Composition: $\gamma(x,(y_i)_{i=1}^k)$ is the result of placing appropriate scaled and translated copies of y_i in the ith disk of x.

Overview of operads (cont.)

Let $\mathcal D$ be a symmetric monoidal category enriched, powered, and copowered in G-spaces satisfying compatibility conditions involving colimits and the monoidal product. Let $\underline{\otimes}$ denote the copowering. For example in G-spaces, $X\underline{\otimes}Y=X\times Y$, while in genuine G-spectra, $X\underline{\otimes}Y=\Sigma_+^{\infty}X\wedge Y$.

Definition

An algebra X over an operad \mathcal{P} consists of action maps $\mathcal{P}(n) \otimes \bigotimes_{i=1}^{n} X \to X$ satisfying equivariance, unit, and associativity axioms.

ullet Given an operad \mathcal{P} , there is an associated monad, consisting of a free algebra functor, equipped with associated unit and multiplication map from the free-forgetful adjunction.

Augmented Objects

Definition

Let U be the unit object in $\mathcal D$ An augmented object U in $\mathcal D$ is an object X equipped with a retract diagram

$$U \xrightarrow{1_U} U$$

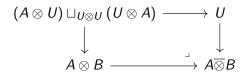
Proposition (May 1972 [9], R.)

If $\mathcal P$ is an operad with $\mathcal P(0)=*$ then there is a free augmented $\mathcal P$ -algebra monad $\mathfrak P$ on the category of augmented objects in $\mathcal D$.

The free algebras on the augmented and non-augmented categories are not the same.

Some notation

- Recall $\underline{\otimes}$ denotes the copowering. (Again, for example in *G*-spaces, $X\underline{\otimes}Y=X\times Y$, while in genuine *G*-spectra, $X\underline{\otimes}Y=\Sigma^\infty_+X\wedge Y$.)
- For two augmented objects A, B in $\mathcal{D}, A \overline{\otimes} B$ is given by the canonical pushout



A key observation

Proposition (May 1972. in CGWH spaces, R.)

For X augmented in \mathcal{C} , there exists a natural map

$$(S^{V}\underline{\otimes} U)\overline{\otimes} \mathfrak{C}_{V}X \to (S^{V}\underline{\otimes} U)\overline{\otimes} X$$

satisfying commutative diagram conditions so that $(S^V \underline{\otimes} U) \overline{\otimes} (-)$ is a right module over \mathfrak{C}_V , the augmented algebra monad associated to C_V . These monads are natural in \mathcal{D} .

• May (1972) [9], and Guillou–May (2017) [7] go on to show that for some C_V -spaces Y (grouplike when V has a copy of the trivial rep.), a two sided simplicial bar construction $Y \simeq \Omega^V B^V Y$ where

$$B^{V}Y = B((S^{V} \underline{\otimes} U) \overline{\otimes} (-), \mathfrak{C}_{V}, Y) = B(\Sigma^{V}, \mathfrak{C}_{V}, Y)$$

and $B^V\Omega^VX\simeq X$ for sufficiently connected spaces X

• The proposition above justifies the construction of B^V .

Aside: A question for you

Question

Let Z be a G-space such that the fixed points Z^H is homotopy equivalent to a finite dimensional sphere for every subgroup H of G. Can you construct an operad that fits into a two-sided bar construction B^Z so that B^Z Map_{*} $(Z,X) \simeq X$ and Map_{*} $(Z,B^ZY) \simeq Y$ under analogous conditions?

Slice spheres and the slice tower

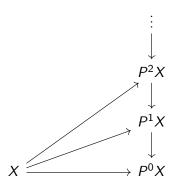
- Suppose H ≤ G. There is an additive induction functor Spectra_H → Spectra_G denoted by G ⋈_H (-)
- A slice sphere is a spectrum of the form $G_+ \ltimes_H S^{n\rho_H}$ for $n \in \mathbb{Z}$.
- Dimension of $G_+ \ltimes_H S^{n\rho_H}$ is n|H| (the dimension of the representation $n\rho_H$.)
- Inclusions $G_+ \ltimes_H S^{n\rho_H} \to G_+ \ltimes_H D^{n\rho_H+1}$ have contractible target. Using slice cells of dimension $\geq n+1$, can construct from X a relative cell complex P^nX such that

$$[G_+ \ltimes_H S^{m\rho_H}, X] \to [G_+ \ltimes_H S^{m\rho_H}, P^n X]$$

is an iso when $m|H| \le n$ and the zero map when m|H| > n.

Slice limit diagram

This is a homotopy limit diagram.



Algebraic cells

For a connective C_V algebra in G-spectra X one can construct C_V algebras with underlying spectra naturally homotopy equivalent to P^nX using C_V -algebra cells $F_{C_V}(G_+ \ltimes_H S^{n\rho_H}) \to F_{C_V}(G_+ \ltimes_H D^{n\rho_H+1})$ where F_{C_V} is the free C_V -algebra functor (m|H| > n).

Slice tower for highly structured equivariant ring spectrum

Theorem (Angeltveit–Blumberg–Gerhardt–Hill–Lawson–Mandell [1], (in preparation))

Let X be a connective C_V spectrum.

$$P^0X = H\underline{\pi}_0(X)$$

If P^nX is constructed, then one can build a $P^{n+1}X$ as a homotopy pullback (after replacements)

$$\begin{array}{ccc}
P^{n+1}X & \longrightarrow & P^{0}X \\
\downarrow & & \downarrow \\
P^{n}X & \longrightarrow & P^{0}X \vee \Sigma P_{n+1}^{n+1}X
\end{array}$$

where $P_{n+1}^{n+1}X$ is the fiber of $P^{n+1}X \to P^nX$ (called the $(n+1)^{th}$ slice of X).

The obstruction question

Question

Let Mf be an E_V Thom spectrum arising from an E_{V+1} -space map $Y \to GL_1S$.

Let R be an E_V -ring spectrum.

Suppose $g: Mf \to R$ is a ring spectrum map such that $Mf \to P^0R$ is a fixed E_V map. Can we use the observation that the target of the k-invariant is a square-zero extension of P^0X to recharacterize the obstruction class?

Answer (Chadwick-Mandell 2007 [3] (non-equiv.), R.)

Yes, such maps are naturally in bijective correspondence with cohomology classes in $(P_{n+1}^{n+1}R)^{V+1}B^{V+1}Y$.

Why does B^V appear?

Theorem (Basterra–Mandell 2011 [2](non-equiv.), R.)

Let k be a commutative ring spectrum. Suppose A is an augmented E_V -ring spectrum in k-modules and suppose $k \vee M$ in the homotopy category is the initial square-zero extension of k under A. Then B^VA is equivalent to $k \vee \Sigma^VA$

Question

Can one use the above result to study ring spectra of the form $R \wedge R$?

Specializing to C_2 , $MU_{\mathbb{R}}$

- ullet Specialize to the case $G=C_2$ and V=
 ho and study $MU_{\mathbb{R}}$
- $P_{2k}^{2k}X \simeq \Sigma^{k\rho}H\underline{\pi}_{k\rho}(X)$
- $P_{2k+1}^{2k+1}X \simeq \Sigma^{k\rho+\sigma}H_{\underline{\pi}_{k\rho+\sigma}}(X)$ where σ is the sign representation.
- $H^*\underline{M}(B^{\rho+1}U_{\mathbb{R}})$ is concentrated in even dimensional representation degrees when M is a constant Mackey functor, i.e. its restriction map is an isomorphism.

Answer to the driving question

Theorem (R.)

Suppose R is a connective E_{ρ} -ring spectrum with each $\underline{\pi}_{k\rho}(R)$ a constant Mackey functor and $\underline{\pi}_{k\rho+\sigma}(R)=0$ for all non-negative k. Any ring map $MU_{\mathbb{R}}\to R$ is homotopic to a map of E_{ρ} -algebras.

Corollary

The Hahn-Shi map $MU_{\mathbb{R}} \to E_n$ is homotopic to an E_{ρ} -algebra map.

Corollary

Every ring map $MU_{\mathbb{R}} \to MU_{\mathbb{R}}$ is homotopic to an E_{ρ} -algebra map. Consequently, the Real Brown–Peterson spectrum $BP_{\mathbb{R}}$ has an E_{ρ} structure.

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