Research Statement

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1 Introduction

My work in algebraic topology, rooted in the study of bordism, using tools of equivariant homotopy theory and operad theory, seeks to answer questions about the algebraic properties of vector bundle orientations. Bordism is the study of smooth closed manifolds of fixed dimension with additional structure, up to an equivalence, where equivalence means the two manifolds are joined by forming the boundary of a third manifold. Bordism classes under disjoint union generate abelian bordism groups. Usually, cartesian product endows bordism groups with the structure of a graded ring. The first example of framed bordism, for manifolds with trivialization of their stable tangent bundle studied by Pontryagin [22], was used for the first surgery theory of manifolds. Brouder gave a reformulation of the "Kervaire Invariant One" problem, a problem whose resolution would complete the classification of smooth structures on spheres of dimension not equal to 4 due to the work of Milnor and Kervaire [16], in terms of framed bordism classes. The Kervaire Invariant One problem was eventually resolved by Hill, Hopkins, and Ravenel (HHR) [14] in all dimensions besides 126. Central to their proof is a deep study of Real bordism between manifolds with an involution, complex structures on their stable tangent bundles, with involution inducing a conjugate-linear map on fibers.

Bordism is linked to the study of extraordinary homology and cohomology theories, a source of powerful methods of algebraic topology. These theories act as functions taking a space or analogous object as input, and output a collection of abelian groups, often with an additional multiplicative operation relating them. These invariants are often computable, are important to the structural understanding of spaces, and often have compelling geometric interpretations. Groundbreaking work of Thom clarified that bordism groups of various kinds also assemble into the homotopy groups of certain spaces, and that study of these spaces is a powerful computational method for bordism [28], eventually recast in the language of homology theories [2]. For sophisticated study of cohomology theories, the theory of spectra was developed. The term "spectrum" refers, roughly, to indexed sequences of spaces with compatibility mappings that realize spaces further down the sequence as spaces of continuous maps of spheres into previous ones. All cohomology theories of spaces [6] and most studied homology theories arise from the homotopy groups of spectra and multiplicative structures on spectra are common among important examples. Framed bordism corresponds to the sphere spectrum $\mathcal{S}$ with genuine G-spectrum analogues $\mathcal{S}_G$, and spectra are always modules over it, while Real bordism corresponds the genuine C_2-spectrum $MU_{\mathbb{R}}$ obtained from a map $BU_{\mathbb{R}} \to BGL_1(\mathcal{S}_{C_2})$ arising from the action of the unitary groups on the one-point compactifications of complex vector spaces.

For comparing multiplicative operations on spectra and spaces, May's notion of an operad [18] has been most useful. Instead, a short list of operations and axioms as usually found in algebra, the operads of algebraic topology use spaces to parameterize multiplicative operations of fixed input, closed under composition. For a non-zero finite-dimensional G-representation V there is a corresponding E_V operad where operations correspond to configurations of distinct points in V. These E_V operads can be put together to form GE_∞ operads, the nearest well-behaved operad approximating the operad of an ordinary commutative and associative operation. The sphere spectra S_G are algebras of GE_∞ operads and $MU_{\mathbb{R}}$ is a C_2E_∞-algebra where C_2 is the group of order 2. A map of spectra being a ring map is a weaker condition than being an $E_{\mathbb{R}}$ map.

The Adams-Novikov spectral sequence is a fundamental computational tool relating classes in the algebraic object $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*(X))$ where BP constructed is a summand of the p-localization of MU to certain subquotients in the homotopy groups of the p-localization of the spectrum X [21] and the Real analogue of the 2-local case was developed by Hu and Kriz in [15] with the role BP taken by $BP_{\mathbb{R}}$. Notably, BP is known to be $E_{\mathbb{R}^4}$ [26] and not E_∞ [27]. However, before my work, $BP_{\mathbb{R}}$ was not known to be E_V for any non-zero V.

Another related spectral sequence of Morava [20] gave rise to some of the major insights of chromatic homotopy theory. The framed bordism groups, or stable homotopy groups of spheres, admit what is known as the chromatic filtration [29] and with associated graded homotopy groups of spectra. $L_{K(n)}\mathcal{S}$ [24]. Work of Hopkins, and Devinatz realized $L_{K(n)}\mathcal{S}$ as the homotopy fixed point spectrum of certain Lubin–Tate spectra E_n under the action of their respective Morava stabilizer group G_n [25, 8, 9, 23], which acts by E_∞ maps [11, 10].

Using an important computational tool from the HHR work [14], the slice tower and the associated slice spectral sequence, Hahn and Shi computed the homotopy groups of homotopy fixed points of Lubin-Tate spectra under a canonical element of order 2, $E(k,\Gamma)^{hC_2}$, by studying the extra structure on the spectral sequence obtained by a ring map $MU_{\mathbb{R}} \to E(k,\Gamma)$ [13]. This project is continued by Beaudry, Hill, Shi, and Zeng [5], studying the equivariant homotopy for larger cyclic 2-subgroups of the Morava stabilizer group and using rings maps from the spectra from the Hill-Hopkins-Ravenel norm $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$.

2 Relevant Work to Date

With respect to these recent trends of homotopy theory, my main contributions are the following.

Theorem 1. *Let ρ denote the regular representation of C_2. Suppose Y is an E_ρ spectrum with trivial odd slices and satisfying another common hypothesis in Real orientation theory. Then, every map of ring spectra $MU_{\mathbb{R}} \to Y$ is homotopic to a map of E_ρ spectra.*

In particular Lubin-Tate spectra satisfy these conditions, so the Hahn–Shi map $MU_{\mathbb{R}} \to E(k,\Gamma)$ has an E_ρ structure. As one may expect, the representative example $MU_{\mathbb{R}}$ does as well, leading to the following.

Corollary 2. *The Real Brown–Peterson spectrum $BP_{\mathbb{R}}$ has an E_ρ multiplicative structure and the Quillen map $MU_{\mathbb{R}} \to BP_{\mathbb{R}}$ is an E_ρ morphism.*

These results required finding equivariant analogues of the result of Chadwick and Mandell [7] and another technical result of Basterra and Mandell [3].

The strategy of the proof is by obstruction theory. Suppose $\mathcal{O}$ is a unital operad. The maps of $\mathcal{O}$-spectra $X \to Y$ can be understood through the slice tower of Y by work of Basterra, Blumberg, Hill, Lawson, and Mandell [4]. A map to $X \to P^nY$ the n^{th} slice approximation of Y lifts to a map $X \to P^{n+1}Y$ if the canonical composite $X \to P^nY \to P^0Y \vee \Sigma P^{n+1}_{n+1}Y$ is equivalent to the composite $X \to P^0Y \to P^0Y \vee \Sigma P^{n+1}_{n+1}Y$. Here $P^{n+1}_{n+1}(Y)$ is the $(n+1)^{st}$ slice of Y and the spectrum $P^0Y \vee \Sigma P^{n+1}_{n+1}Y$ is an augmented P^0Y-module. Beginning with a map $X \to P^0Y$ and successively lifting it to maps $X \to P^nY$ allows one to construct a map $X \to Y$ in the homotopy category. In analogy with Chadwick–Mandell [7], I proved that these obstructions appear in nearly-ordinary equivariant cohomology groups:

Theorem 3. *Suppose X is an E_V space satisfying a connectivity condition which is used to build a Thom spectrum Mf, and Y is a connective E_V-algebra. Then, a map to the n^{th} slice section of Y, $Mf \to P^nY$, lifts to a map $Mf \to P^{n+1}Y$ if and only if a certain induced element of a cohomology group*

$$(P^{n+1}_{n+1}Y)^{V\oplus\mathbb{R}}(B^VX)$$

is 0.

The above result required showing a relationship between E_V-Topological Andre-Quillen homology with coefficients in a slice and ordinary cohomology in a slice.

While the previous result used inductive constructions, I adapted the construction of May's work [18] with equivariant generalization in [12] to the setting of augmented spectra. The V-fold loop space functor Ω^V and the V-fold classifying space functor B^V form an equivalence between S^V-connected spaces and group-like E_V-algebras.

Theorem 4. *For an augmented E_V-algebra R in the category of modules of the commutative ring spectrum A, there is an equivalence $A \vee \Sigma^V \mathrm{TAQ}^A(R) \simeq B^VR$ where $\mathrm{TAQ}^A(R)$ is the derived A-algebra indecomposibles of R, also known as the topological Andre-Quillen homology of R.*

The proof of the above theorem circumvents a technical feature of the corresponding non-equivariant proof in [3] that uses constructions defined inductively by dimension. The approach May V-fold classifying space functor made it possible to show the result for representations that are not sums of dimension 1 representations.

The technical aspects of the above result required the development of comparisons between B^VR and an analogous construction $\tilde{B}^VZR$ where ZR is the augmentation ideal of R and $\tilde{B}^V(-)$ is an analogue of B^V in the category of non-unital E_V-algebras.

3 Future Work

Several paths forward have emerged from this investigation. The most straightforward is to investigate more implications of Theorem 3 and find conditions for guaranteeing maps of ring spectra are structured for other bordism theories such as $M\,\mathrm{String}$, $M\,\mathrm{Spin}$, and their equivariant relatives, as well as the equivariant complex bordism theory MU_{C_2} and not-yet-constructed periodic Real bordism theory $MU_{\mathbb{R}}P$.

Problem 1. *Prove analogues of Theorem 1 for other bordism theories.*

In all cases, this short project would require performing additional equivariant cohomology computations. For constructing $MU_{\mathbb{R}}P$, one would need to extend the constructions of Thom spectra from maps $X \to \mathrm{Pic}(R)$ of [1] to the equivariant setting. The operads $\mathcal{P}_{\mathcal{V}}$ of Theorem 1 together with the parameterized homotopy theory of May and Siggurdson [19] are probably invaluable recent tools toward this goal.

More ambitiously, I hope to contribute directly to the study of homotopy fixed points of Lubin–Tate spectra $E(k,\Gamma)^{hC_{2^n}}$. The strategy would be to first identify a C_{2^n}-representation V, such that ring maps $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}} \to E(k,\Gamma)$ is guaranteed to be E_V. Then, by the Hahn–Shi results, we would obtain a canonical E_V map. Next, I plan to study homotopy operations on E_V-algebras, and to test their power on the understanding of the homotopy groups of $E(k,\Gamma)$, and homotopy fixed point spectral sequence. The first two stages of this program are independent of each other.

The relationship between the HHR norm and operadic multiplicative structures is not yet understood but can be approached from the point of view of obstruction theory and geometry.

Problem 2. *If X is an E_V-algebra in H-spectra and H is a subgroup of G, is $N_H^G X$ an $E_{F(V)}$-algebra in G spectra for some functorial G-representation V? If not, is it the case in a restricted class of spectra such as $N_{C_2}^G MU_{\mathbb{R}}$ modules?*

The second problem is an investigation of homotopy operations on E_V-algebras. Non-equivariantly, there is n operation called the Browder bracket on the homotopy groups of an E_n-algebra A. Namely, there are maps

$$\pi_p(A) \otimes \pi_q(A) \to \pi_{p+q-(n-1)}(A)$$

arising from a map induced by a homotopy equivalence between S^{n-1} and the space of binary operations of the E_n operad

$$\Sigma^\infty S^{n-1}_+ \wedge A \wedge A \to A.$$

The Browder bracket for $n \geq 2$ gives $\pi_*(A)$ the structure of a Poisson n-algebra and specifies enough homotopy operations so that $\pi_*(\Sigma^\infty_+ \Omega_n \Sigma^n X)$ is a free Poisson n-algebra [17]. Equivariant analogues of the Browder bracket and Poisson n-algebras have not been yet explored.

Problem 3. *Suppose X is an E_V-algebra in spectra. What multiplicative operations exist in their homotopy groups? Identify the orbit types of $G \times \Sigma_k$ occurring in k^{th} space of the E_V-operad and describe the homotopy commutative diagrams of spectra between the corresponding tensor powers of X. Then, identify analogues of the Browder bracket operation and Poisson n-algebra axioms satisfied by the resulting system in the homotopy groups of X.*

When sufficient progress on the previous two questions is made, I hope to seek collaborators in pursuing the following goal, new computational tools in hand.

Problem 4. *Investigate the C_{2^n} equivariant homotopy groups of $E(k,\Gamma)$. What can we say about the homotopy fixed point spectral sequence?*

For more powerful applications of Theorem 3, one likely needs information about the equivariant Dyer-Lashof operations, cohomology operations that conceptually dual to the product and Browder bracket operations discussed above. Basterra and Mandell applied the E_4 Dyer-Lashof algebra to a similar study of BP and found that they are E_4 [26]. It is plausible that the Real analogue $BP_{\mathbb{R}}$ is $E_{2\rho}$, however equivariant computations lag behind their non-equivariant counterparts.

Problem 5. *For the group C_2, compute the $\underline{\mathbb{F}_2}$, E_V-Dyer–Lashof operations, following the ideas of Wilson [30]. Is $BP_{\mathbb{R}}$ an $E_{2\rho}$ ring spectrum where 2ρ is the direct sum of two copies of the regular representation?*

Lawson in his expository writing notes that the theory of non-equivariant E_n Dyer-Lashof operations is overdue for modern reformulations [17]. In preparation for work on Problem 2 and to connect the theory of E_V-spectra to modern tools, I am working on developing multiplicative properties of their module categories.

Problem 6. *Let A be an E_V algebra. For a well-behaved E_V operad $\mathcal{O}$, define families of definitions of multilinear maps of A-modules indexed by the spaces of the operad. For a subrepresentation W of V, construct the category of E_W-algebras in the category of A modules. Use this to generalize Theorems 3 and 4 to cases where $W^\perp$ includes a copy of the trivial representation $\mathbb{R}$.*

Progress on Problem 6 would, for example, provide additional tools to understand $E_{\mathbb{R}}$ algebras under spectra BP and $BP_{\mathbb{R}}$. Other potential applications of Problem 6 is as a stepping stone to aspects of Problem 1 that require parameterized homotopy theory and the constructions required for Problem 2. While Problem 6 remains unsolved, I have made partial progress towards a homotopical solution without a strict unit axiom.

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