Chapter 2. Solutions of Non Linear equations $f(x) = 0$

2.2 Bisection Method and False position Method:

I Bisection Method: To solve $f(x) = 0$ for continuous function on $[\overset{a_0}{a}, \overset{b_0}{b}]$, where $f(a) \cdot f(b) < 0$

Bolzano's thm.

we take $c_0 = \dfrac{a+b}{2} = \dfrac{a_0 + b_0}{2}$.

1) If $f(c_0) = 0$, then we are done. c_0 is the root.

2) If $f(c_0) f(a_0) < 0$, then the root $r \in [a_0, c_0]$

we call the new Interval $[a_0, c_0] = [a_1, b_1]$.

3) If $f(c_0) f(b_0) < 0$, then the root $r \in [c_0, b_0]$

and we call $[c_0, b_0] = [a_1, b_1]$.

Then according to case (2) or (3) we choose

$c_1 = \dfrac{a_1 + b_1}{2}$, Again we check the 3 posibilites

repeat the process: $c_2 = \dfrac{a_2 + b_2}{2}$

$$\vdots \qquad \vdots$$

$$c_n = \frac{a_n + b_n}{2}$$

Note: 1) Bisection Method is Used if $f(x)$ satisfies Bolzano's theorem.

2) If Bolzano's theorem is satisfied, then Bisection Method is always Converge.

Example: Estimate the root of $e^x - \cos x - 1 = 0$ on $[0,1]$ Using Bisection Method.

$f(0) = -1 < 0$, $f(1) = 1.1779795 > 0$

n	a_n	C_n	b_n	$f(C_n)$
0	0^-	0.5	1^+	-0.22886
1	0.5^-	0.75	1^+	+0.38531
2	0.5^-	0.625	0.75^+	+0.057283
3	0.5^-	0.5625	0.625^+	-0.090870
4	0.5625^-	0.59375	0.625^+	

Example: Let $x \sin x = 1$. Estimate the value of x that lies in $[0, 2]$ that satisfy the equation. Using Bisection Method.

Sol: Let $f(x) = x \sin x - 1 = 0$

so we need to approximate the root of $f(x)$.

Let $a_0 = 0$ and $b_0 = 2$. (Radian Mode)

$f(0) = -1 < 0$ & $f(2) = 2\sin 2 - 1 = 0.818595 > 0$

n	a_n	c_n	b_n	$f(c_n)$
0	0^-	1	2^+	$f(1) = -0.158529$
1	1^-	1.5	2^+	$^+0.496242$
2	1^-	1.25	1.5^+	0.186231
3	1^-	1.125	1.25^+	0.015051
4	1^-	1.0625	1.125^+	-0.071827
5	1.0625^-	1.09375	1.125^+	-0.028362

In this manner we obtain a sequence $\{c_k\}$ that converges to $r \approx 1.114157141$

Bisection Theorem: Assume that $f \in C[a,b]$ and that there exists a number $r \in [a,b]$ such that $f(r)=0$. If $f(a)\cdot f(b) < 0$, and $\{c_n\}_{n=0}^{\infty}$ represents the sequence of midpoints generated by the Bisection process, then:

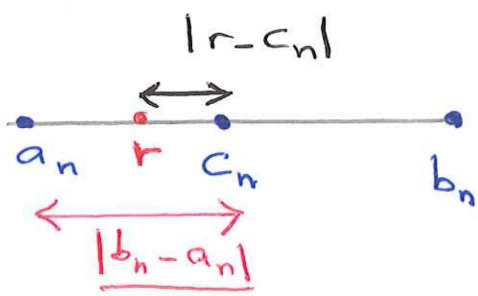
1) $\underbrace{|r - c_n|}_{\text{error}} \leq \dfrac{b-a}{2^{n+1}}$, $n = 0, 1, 2, \ldots$

upper bound of the error "accuracy"

2) $\lim\limits_{n\to\infty} c_n = r$.

proof (1): Since both the zero r & the midpoint c_n lie in $[a_n, b_n]$, then

$$(*)\ldots\ |r - c_n| \leq \frac{b_n - a_n}{2}, \quad \forall n.$$



Observe that the successive interval widths form

$$b_1 - a_1 = \frac{b_0 - a_0}{2}.$$

$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b_0 - a_0}{2^2}.$$

Using Mathematical Induction, we have

$$b_n - a_n = \frac{b_0 - a_0}{2^n}. \quad \text{---}(**)$$

From (*) & (**), we have:

$$|r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}}, \quad \forall n.$$

proof (2): $0 \leq |r - c_n| \leq \dfrac{b_0 - a_0}{2^{n+1}}$.

$$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} |r - c_n| \leq \lim_{n\to\infty} \frac{b_0 - a_0}{2^{n+1}}$$

Using Sandwich theorem: $\lim_{n\to\infty} |r - c_n| = 0$

$$\Leftrightarrow \lim_{n\to\infty} c_n = r.$$

Note: The advantage of Bisection Method is that its always converge.

The disadvantage that is very slow in Convergence.

Example: Find the number of Iterations needed to estimate the root of a function that has a root in $[\overset{a_0}{0.2}, \overset{b_0}{0.8}]$ with error at most $\underbrace{6\times 3^{-8}}_{\delta}$

Sol: $$|r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}} \leq \underset{=}{\delta} \quad \text{(accuracy)}$$

$$\Rightarrow \quad \frac{b_0 - a_0}{\delta} \leq 2^{n+1}$$

$$\Rightarrow \quad \ln\left(\frac{b_0 - a_0}{\delta}\right) \leq \ln 2^{n+1} = (n+1)\ln 2$$

$$\Rightarrow \quad \frac{\ln(b_0 - a_0) - \ln\delta}{\ln 2} \leq n+1$$

$$\Rightarrow \quad \frac{\ln(b_0 - a_0) - \ln\delta}{\ln 2} - 1 \leq n$$

$$\Rightarrow \quad \frac{\ln(0.6) - \ln(6\times 3^{-8})}{\ln 2} - 1 \leq n \qquad \begin{pmatrix} c_0 \\ \downarrow \\ c_9 \end{pmatrix}$$

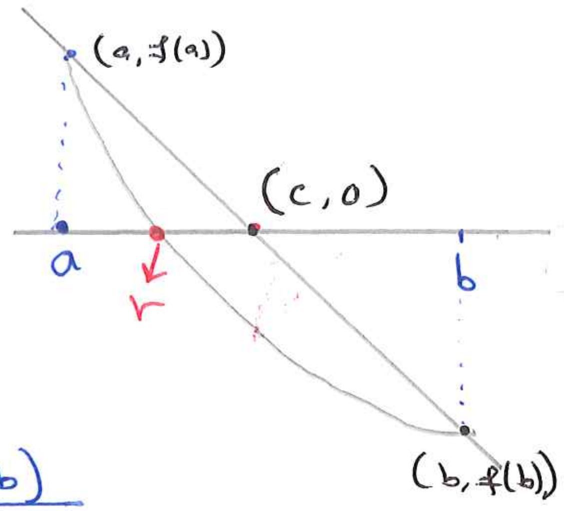
$$\Rightarrow \quad 8.35 \leq n. \quad \Rightarrow \quad n \geq 9$$

Hence, the least number of iteration is $\boxed{10}$

[2] False position Method. "Regula falsi Method"

This Method was developed because the Bisection method convergence is fairly slow.

Assume $f(a) \cdot f(b) < 0$



To determine the Value of C we compute the slope:

$$m = \frac{f(b) - f(a)}{b - a} = \frac{0 - f(b)}{c - b}$$

$$\Rightarrow \quad c = b - \frac{f(b)(b-a)}{f(b) - f(a)}$$

Let $c = c_0$, $a = a_0$, $b = b_0$

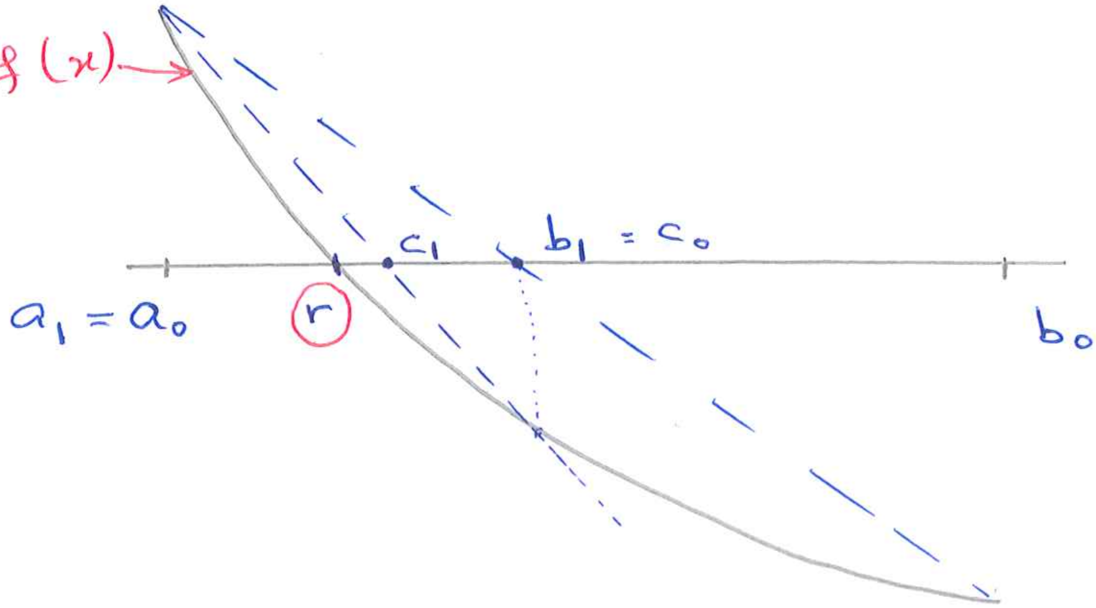
$$\Rightarrow \quad c_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$$

1) If $f(c_0) = 0$, then we are done, c_0 is the root.

2) If $f(c_0) f(a_0) < 0 \Rightarrow r \in [a_0, c_0] = [a_1, b_1]$

3) If $f(c_0) f(b_0) < 0 \Rightarrow r \in [c_0, b_0] = [a_1, b_1]$

Then repeat the process: $C_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$



Note: (1) False position Method is faster than Bisection Method.

(2) We can't know the error without solving.

Example: $e^x - \cos x - 1 = 0$ on $[0, 1]$.

n	a_n	c_n	b_n	$f(c_n)$
0	0^-	0.459	1^+	-0.31372
1	0.459^-	0.5726	1^+	-0. ...

where $c_0 = 1 - \dfrac{f(1)(1-0)}{f(1)-f(0)} = 0.459$

$$c_1 = 1 - \frac{f(1)(1-0.459)}{f(1)-f(0.459)} = 0.5726$$

Less Iteration than Bisection

Example: Estimate the solution of

$$x \sin x = 1 \quad \text{on } [0,2]$$

Sol: Let $f(x) = x\sin x - 1 = 0.$

$f(0) = -1$, $f(2) = 0.81859485.$

$$\Rightarrow c_0 = 2 - \frac{f(2)(2-0)}{0.81859485 - -1} = 1.09975017$$

with $f(c_0) = -0.02001921.$

Therefore, we choose $[a_1, b_1] = [1.09975017, 2]$

$$\Rightarrow c_1 = 2 - \frac{f(2)(2 - 1.09975017)}{0.81859485 - -0.02001921} = 1.12124074$$

with $f(c_1) = 0.00983461.$

Therefore, we choose $[a_2, b_2] = [1.09975017, 1.12124074]$

$\Rightarrow c_2 = 1.11416120$ with $f(c_2) = 0.00000563$

$\Rightarrow c_3 = 1.11415714$ with $f(c_3) = 0.00000000$

Example: Find an approximation of $\sqrt[3]{25}$ correct to within 10^{-2} using:

(a) Bisection Method.

(b) False position Method.

Sol: Let $x = \sqrt[3]{25}$, then $f(x) = x^3 - 25$.

choose $[a_0, b_0] = [2, 3]$ (s.t) Bolzano's is satisfied. Then: $f(2) < 0$, $f(3) > 0$

$$c_0 = \frac{a_0 + b_0}{2} = 2.5 \quad , \quad f(c_0) < 0$$

$\therefore [a_1, b_1] = [2.5, 3]$

$$c_1 = \frac{2.5 + 3}{2} = 2.75 \quad , \text{ with } f(c_1)\ \ 0$$

$\therefore [a_2, b_2] = [2.75, 3]$

$c_2 = 2.875$, $f(c_2) < 0$

$c_3 = 2.9375$, $f(c_3) > 0$

$c_4 = \ldots$ Continue.

(b) Home Work.

2.1 Fixed point Iterations.

Let $f(x) = 0$, split $f(x)$ to two functions such that $f(x) = g(x) - x = 0$.

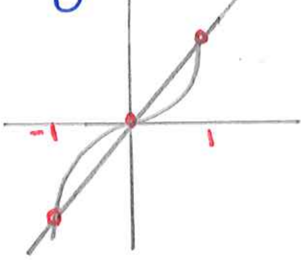
$\Rightarrow$ $f(x) = 0 \iff g(x) = x$. ... (*)

Def: A real number p is called a fixed point of $g(x)$ iff $g(p) = p$.

Note: (*) Implies that finding the roots of $f(x) = 0$ is Equivalent to finding the fixed points of $g(x)$.

Example: Find the fixed points of the following:

1) $g(x) = x^3$ $\Rightarrow$ $x^3 = x$ $\Rightarrow$ $x^3 - x = 0$



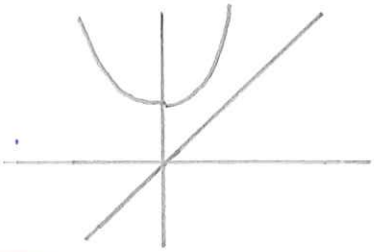
$\Rightarrow$ $x(x^2-1) = x(x-1)(x+1) = 0$

$\Rightarrow$ $x = 0, 1, -1$.

2) $g(x) = x^2+1 \Rightarrow x^2+1 = x$

$\Rightarrow x^2 - x + 1 = 0$, which has no real roots.

$\therefore g(x)$ has No fixed points.



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3) $g(x) = x \Rightarrow x = x \Rightarrow \exists$ Infinitly many fixed points

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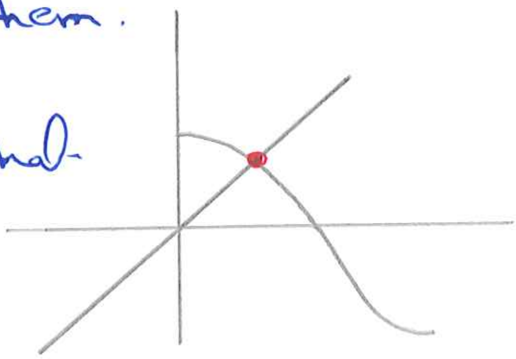
4) $\cos x = x$, we can't find the fixed points although we can sketch them.

From the graph, we conclude that there exists a fixed point.



Fixed point Iteration:

We start with P_0 → Given , then

$P_1 = g(P_0)$, $P_2 = g(P_1)$, $P_3 = g(P_2)$

$\cdots$ $P_n = g(P_{n-1})$, $P_{n+1} = g(P_n)$, $\cdots$

for $n = 0, 1, 2, \cdots$

Fixed point Iteration

Theorem: If g is Continuous and the sequence generated by the fixed point iteration Converges to p, then the number p is the fixed point of g. $[g(p)=p]$

proof: $p = \lim_{n\to\infty} p_{n+1} = \lim_{n\to\infty} g(p_n) \overset{g \text{ is continuous}}{=} g(\lim_{n\to\infty} p_n) = g(p)$

Example: $x^2 - x - 12 = 0$. [The roots are 4 & -3]

① Let $g(x) = x^2 - 12$ $\quad (f(x) = g(x) - x = 0)$

• Let $P_0 = 5$, then:

$P_1 = 5^2 - 12 = 13$, $P_2 = (13)^2 - 12 = 169 - 12 = 157$

$P_3 = (157)^2 - 12 = 24637$, ---

$\Rightarrow$ The sequence diverges. (bad choice of $g(x)$ or P_0).

• Let $P_0 = 3$, then $P_1 = (3)^2 - 12 = -3$

$P_2 = (-3)^2 - 12 = -3 \Rightarrow -3$ is a fixed point.

(2) Let $g(x) = \sqrt{x+12}$.

If $P_0 = 5$, then $P_1 = g(P_0) = \sqrt{5+12} \approx 4.1231$

$P_2 = \sqrt{4.1231+12} \approx 4.0154$

$P_3 \approx 4.00193$

$\vdots$

$P_9 \approx 4.000\,000\,007 \xrightarrow{\text{converge}} 4.$

Since the sequence converges to 4, then 4 is a fixed point for $g(x) = \sqrt{x+12}$

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(3) $x(x-1) = 12 \Rightarrow g(x) = \frac{12}{x-1} = x$

If $P_0 = 5$, then $P_1 = 3$, $P_2 = 6$

$P_3 = 2.4$, $P_4 = 8.57$

$P_5 = 1.585$, $P_6 = 20.51$

$P_7 = 0.6$, $P_8 = -31.1$

$\vdots$

$P_{26} = -3.0317$

$\vdots$

$P_{36} = -3.00178 \xrightarrow{\text{converge}} -3$

Note : 1) Fixed point iteration Can't solve all questions

2) F.P.I is slow (needs a lot of iterations).

Example: Let $P_{k+1} = e^{-P_k}$, $P_0 = 0.5$, then

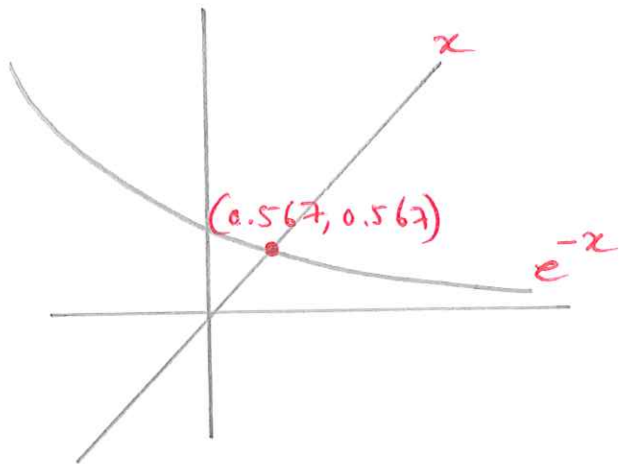
$P_1 = e^{-P_0} = 0.606531$

$P_2 = e^{-P_1} = 0.545239$

⋮

$P_9 = e^{-P_8} = 0.567560$

$P_{10} = e^{-P_9} = 0.566907$.



Further Calculations reveal that $\lim_{n\to\infty} P_n = 0.567143$.

Important Questions:

Is there a fixed point for $g(x)$?

Does this fixed point Unique ?

Does the F.P.I converges to the fixed point ?

The following theorem establishs Conditions for existence & Uniqueness & Convergence of F.P.I.

Theorem: Assume $g \in C[a,b]$.

(1) If $g(x) \in [a,b]$, $\forall x \in [a,b]$, then g has a fixed point in $[a,b]$.

(2) Suppose that $g'(x)$ is defined over (a,b) and the positive constant $K<1$ exists with

$$|g'(x)| \leq K < 1 \quad , \forall x \in (a,b) \text{, then}$$

g has a Unique fixed point P in $[a,b]$.

Proof (1): If $g(a) = a$ or $g(b) = b$, then g has a fixed point.

If Not, then $g(a) \in (a,b]$ $(g(a) > a)$

and $g(b) \in [a,b)$ $(g(b) < b)$

Define $h(x) := g(x) - x$, which is Cont. $[a,b]$

$$h(a) = g(a) - a > 0$$

$$h(b) = g(b) - b < 0$$

$\Rightarrow$ $h(x)$ satisfies Bolzano's theorem. Then

$\exists\, c \in (a,b)$ (s.t) $h(c) = 0$.

$$h(c) = g(c) - c = 0 \iff g(c) = c$$

$\therefore$ $\exists$ a fixed point for g in $[a,b]$.

Proof (2): By Contradiction.

Assume that P_1 and P_2 are two fixed points of g

without loss of generality assume $P_1 < P_2$

where P_1 & $P_2 \in [a,b]$.

g is continuous on $[P_1, P_2] \subset [a,b]$

g is differentiable on $(P_1, P_2) \subset (a,b)$

$\Rightarrow$ g satisfies MVT, then $\exists\, c \in (P_1, P_2)$ (s.t)

$$g'(c) = \frac{g(P_2) - g(P_1)}{P_2 - P_1} = \frac{P_2 - P_1}{P_2 - P_1} = 1. \text{ Contradiction}$$

$|g'(x)| < 1$, $\forall x \in [a,b]$ contradiction.

Example: Show that $g(x) = \cos x$ has a Unique fixed point in $[0,1]$.

Sol: For existence:

(a) g is Continuous on $[0,1]$

(b) $g(x) = \cos x$ which is decreasing function on $[0,1]$

$g(0) = \cos 0 = 1$

$g(1) = \cos 1 \approx 0.99988$

$g(x) \in [0,1] \Leftarrow \forall x \in [0,1]$

Then by part (1) of previous theorem, g has a fixed point in $[0,1]$

For Uniqueness:

Increasing on $[0,1]$

$$g'(x) = -\sin x \Rightarrow |g'(x)| = \sin x \leq \sin 1 = 0.8415 < 1$$

Thus $k = \sin 1 < 1$, so part (2) of the previous theorem is satisfied, then g has a Unique fixed point in $[0,1]$.

Example: Show that $g(x) = 2^{-x}$ on $[0,1]$ has a Unique fixed point.

Existence: (i) g is continuous on $[0,1]$

Need now to show that $g(x) \in [0,1]$, $\forall x \in [0,1]$.

1) Since $g'(x) = -(\ln 2)\, 2^{-x} < 0$, $\forall x \in [0,1]$

$$\left.\begin{array}{l} \text{then } g \text{ is decreasing on } [0,1] \\ g(0) = 1 \\ g(1) = 2^{-1} = \frac{1}{2} \end{array}\right\} \Rightarrow g(x) \in [0,1] \;\; \forall x \in [0,1]$$

$\therefore$ g has a fixed point in $[0,1]$.

Uniqueness:

$$|g'(x)| = (\ln 2)\, 2^{-x} \overset{\text{decreasing}}{\leq} \ln(2) < \ln e = 1$$

$\forall x \in [0,1]$

$$\Rightarrow K = \ln 2 \approx 0.693147 < 1$$

$\therefore$ g has a Unique fixed point in $[0,1]$

The following theorem can be used to determine whether the fixed point Iteration will produce a convergent or divergent sequence.

Theorem: (Fixed Point Theorem): Assume that.

(i) $g, g' \in C[a,b]$

(ii) $k > 0$

(iii) $P_0 \in (a,b)$

(iv) $g(x) \in [a,b]$, $\forall x \in [a,b]$.

Then:

(1) If $|g'(x)| \leq k < 1$, $\forall x \in [a,b]$,

then the iteration $P_n = g(P_{n-1})$ will converge to the unique fixed point $\underline{P} \in [a,b]$.

In this case, $\underline{P}$ is said to be attractive F.P.

(2) If $|g'(x)| > 1$, $\forall x \in [a,b]$, then the F.P.I will not converge to $\underline{P}$.

In this case, $\underline{P}$ is said to be repelling F.P

Note: If $|g'(x)| = 1$, then F.P.T failed

We can construct an Iteration to know whether it converges or diverges.

Proof: (1) From Previous theorem we conclude that there exists a Unique fixed point P in $[a,b]$.

We will prove it by induction

Since $g(x) \in [a,b]$, $\forall x \in [a,b]$, then

$P_n \in [a,b]$, $\forall n = 0, 1, \ldots$ for $P_0 \in [a,b]$

We will Apply MVT on $[P_{n-1}, P]$:

$\Rightarrow \exists c \in [P_{n-1}, P]$ such that

$$g'(c) = \frac{g(P) - g(P_{n-1})}{P - P_{n-1}} = \frac{P - P_n}{P - P_{n-1}}$$

$$\Rightarrow |P - P_{n-1}| \underbrace{|g'(c)|}_{\text{by hypo. } \leq K} = |P - P_n|$$

$$\Rightarrow |P - P_n| \leq K |P - P_{n-1}|$$

Apply MVT again, we get

$$|P - P_{n-1}| \leq k\,|P - P_{n-2}|$$

By Induction, we get:

$$|P - P_n| \leq k\,|P - P_{n-1}| \leq k^2 |P - P_{n-2}| \leq \cdots \leq k^n |P - P_0|$$

$$\Rightarrow \lim_{n\to\infty} |P - P_n| = \lim_{n\to\infty} k^n |P - P_0| \overset{\text{Since } 0 \leq k < 1}{=} 0$$

$$\Rightarrow \lim_{n\to\infty} |P - P_n| = 0 \iff \lim_{n\to\infty} P_n = P$$

Proof (2): Exercise.

Corollary:

(1) $|P_n - P| \leq k^n |P_0 - P|$, $\forall n \geq 1$

(2) $|P_n - P| \leq \dfrac{k^n |P_1 - P_0|}{1 - k}$, $\forall n \geq 1$

Example: Consider the fixed point iteration

$$P_{n+1} = \sqrt[4]{P_n + 1} = g(P_n)$$

(a) Show that $g(x)$ has a fixed point in $[1, 2]$

(b) Show that if $P_0 \in [1, 2]$, then the fixed point iteration converges.

(c) Estimate the fixed point p starting with $P_0 = 1.5$ with accuracy 10^{-3}.

(d) Find The Least number of iterations need to estimate the fixed point with accuracy 10^{-5}?

sol: (a) Let $g(x) = \sqrt[4]{x+1}$

$g(x)$ is cont. on $[1, 2]$

- $g(x)$ is increasing on $[1, 2]$
- $g(1) = \sqrt[4]{2} \approx 1.1892$
- $g(2) = \sqrt[4]{3} \approx 1.31607$

$\Rightarrow$ $1 \leq g(1) \leq g(x) \leq g(2) \leq 2$

$\therefore g(x) \in [1, 2] \quad \forall x \in [1, 2]$

$\therefore \exists$ a fixed point P in $[1, 2]$.

(b) $g'(x) = \frac{1}{4}(x+1)^{-3/4} = \dfrac{1}{4\cdot\sqrt[4]{(x+1)^3}}$

$|g'(x)| = g'(x)$ is decreasing function.

$\Rightarrow |g'(x)| \leq g'(1) = \dfrac{1}{4\sqrt[3]{2}} \approx 0.14865089$

$\Rightarrow |g'(x)| \leq k = 0.14865089 < 1$

$\therefore$ F.P.I converges for any $P_0 \in [1, 2]$

Note: $k < 1$ also means there exists Unique F.P.

(c) $P_0 = 1.5$

$P_1 = g(P_0) = 1.25743343$

$P_2 = g(P_1) = 1.225755182$

$P_3 = g(P_2) = 1.221432153$

$P_4 = g(P_3) = 1.220838632$

$|P_4 - P_3| < 10^{-3}$

(d) $\dfrac{k^n |P_1 - P_0|}{1-k} \leq 10^{-5} \Leftrightarrow \dfrac{(0.14865089)^n (0.24258857)}{0.85134911} \leq 10^{-5}$

$\Leftrightarrow n \geq 5.3812 \Leftrightarrow n = 6.$

Example: Investigate the nature of the Fixed points for $g(x) = 1 + x - \frac{x^2}{4}$

Sol: F.P: $x = g(x)$

$$x = 1 + x - \frac{x^2}{4}$$

$$\frac{x^2}{4} = 1 \iff x^2 = 4$$

$$\therefore \quad x = 2, -2.$$

$$g'(x) = 1 - \frac{x}{2} \Rightarrow |g'(2)| = 0 < 1$$

$$|g'(-2)| = 2 > 1$$

$\therefore$ $\boxed{x = 2}$ is an attractive fixed point

• If $P_0 = 1.6 \Rightarrow P_1 = 1.96 \Rightarrow P_2 = 1.9996 \Longrightarrow P_3 = 1.99999996$

... converges to 2.

& $\boxed{x = -2}$ is Repulsive fixed point

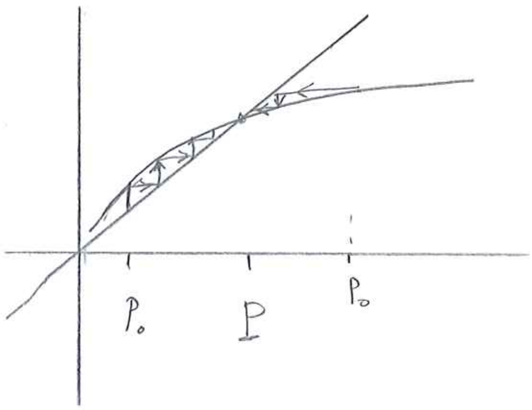
If $P_0 = -2.05 \Longrightarrow P_1 = -2.100625$

$P_2 = -2.20378 \Rightarrow P_3 = -2.4179444$

... $P_6 = -6.8581296$

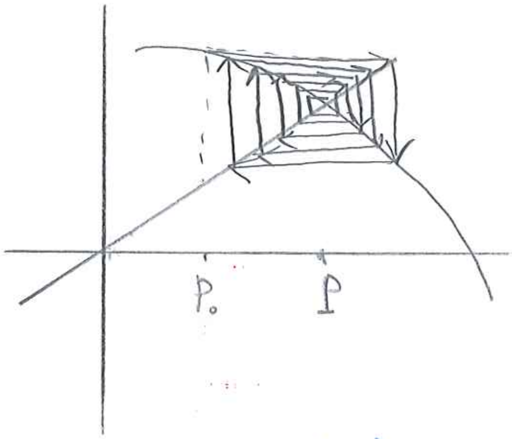
$\therefore$ The Iteration does not Converge to -2.

Graphical Interpretation of F. P. I.



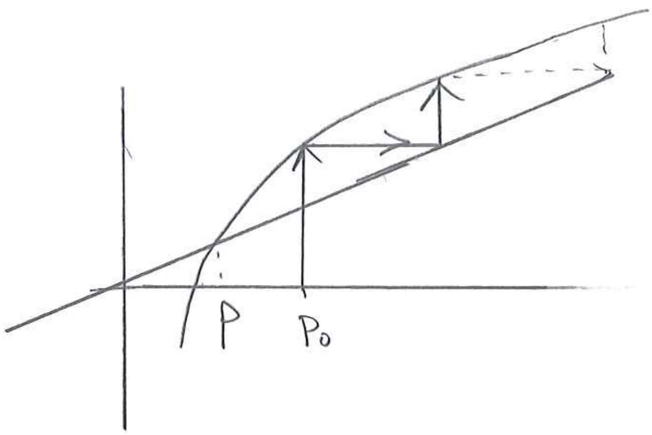
Montone Convergence

$0 < g'(p) < 1$



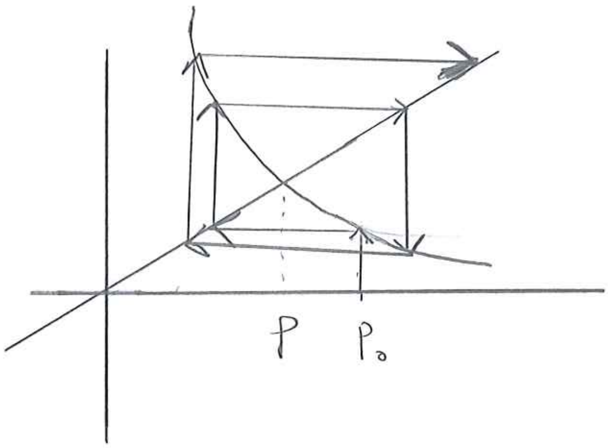
oscillating Convergence

$-1 < g'(p) < 0$



Monotone divergence

$1 < g'(p)$



oscillating Divergence

$g'(p) < -1$

2.3 Stopping Criteria:

1) $|f(c_n)| < \epsilon$

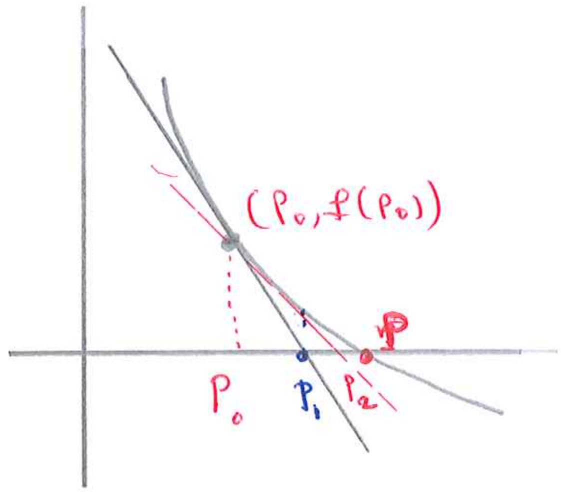
2) $|c_n - c_{n-1}| < \delta$

3) $\dfrac{|P_{n+1} - P_n|}{|P_n|} < \xi$.

2.4 Newton-Raphson and Secant Methods.

Newton Method: Given $f(x)=0$ and p_0, then

$$f'(p_0) = \frac{0 - f(p_0)}{p_1 - p_0}$$



solving for p_1 :

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}.$$

repeat this process to obtain a sequence $\{p_k\}$ that converges to p

$$\therefore \quad \boxed{p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}}.$$

Example: $e^x - \cos x - 1 = 0$ with $p_0 = 1$.

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{(e^1 - \cos 1 - 1)}{e^1 + \sin 1}$$

$p_1 = 0.669083898$, $p_2 = 0.603760843$

$p_3 = 0.601349991$, $p_4 = 0.601346767$.

Newton Raphson theorem:

Assume that $f \in C^2[a,b]$ and there exists a number $p \in [a,b]$, where $f(p)=0$.

If $f'(p) \neq 0$, then there exists a $\delta > 0$ such that the sequence $\{p_k\}_{k=0}^{\infty}$ defined by the iteration

$$\boxed{p_k \underset{\substack{\downarrow \\ F.P}}{=} g(p_{k-1}) = p_{k-1} - \frac{f(p_{k-1})}{f'(p_{k-1})}}$$

for $k = 1, 2, \ldots$ will converge to p for any $p_0 \in [p-\delta, p+\delta]$.

Remark: We can define $g(x) = x - \frac{f(x)}{f'(x)}$ such that $g(x)$ is called Newton Raphson iteration function. Since $f(p) = 0$, we can see $g(p) = p$. Thus we conclude that Newton Raphson Method iteration for finding the root of the equation $f(x)=0$ is accomplished by finding fixed point of $g(x)$.

Proof: Consider $g(x) = x - \dfrac{f(x)}{f'(x)}$, $f(p)=0$.

$$g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}$$

$$g'(p) = 1 - \frac{(f'(p))^2}{(f'(p))^2} = 1-1 = 0 < 1$$

By hypothesis, since $f(p)=0$ and $g'(p)=0<1$ and $g(x)$ is continuous on $[a,b]$, then Using fixed point theorem, there exists a sufficient $P_0 \in (p-\delta, p+\delta)$ for $\delta > 0$ such that the sequence $\{P_k\}_{k=0}^{\infty}$ converges to the fixed point P, but P is a root of $f(x)$, therefore Newton Raphson iteration converges to P.

The Most Important Question is :

How fast does the Iteration Converge ?

Def : Assume P is a root of $f(x)$. The Multiplicity of P is M iff $f(p) = f'(p) = \dots = f^{(M-1)}(p) = 0$ and $f^{(M)}(p) \neq 0$.

Note : A root P of Multiplicity $M = 1$ is called a simple root, and if $M > 1$, it is called a multiple root.

Lemma : If the equation $f(x) = 0$ has a root of multiplicity M at $x = p$, then there exists a continuous function $h(x)$ such that

$$f(x) = (x - P)^M h(x) \quad , \quad h(p) \neq 0.$$

Example : $f(x) = (x-1)\ln x$, $P = 1$ is a root.

$$f'(x) = \frac{(x-1)}{x} + \ln x \Rightarrow f'(1) = 0$$

$$f''(x) = \frac{x-(x-1)}{x^2} + \frac{1}{x} \Rightarrow f''(1) = 2 \Rightarrow \boxed{M = 2}$$

Example: $f(x) = x^3 - 3x + 2$

$P = 1$ is a root

$$\begin{array}{r|l} & x^2 + x - 2 \\ \hline x - 1 & x^3 - 3x + 2 \\ & -\left(x^3 \mp x^2\right) \\ \hline & x^2 - 3x + 2 \\ & -\left(x^2 \mp x\right) \\ \hline & -2x + 2 \\ & \pm\left(-2x \mp 2\right) \\ \hline & 0 \end{array}$$

$$\Rightarrow f(x) = x^3 - 3x + 2$$
$$= (x^2 + x - 2)(x - 1)$$
$$= (x + 2)(x - 1)(x - 1)$$
$$= (x + 2)(x - 1)^2$$

$\Rightarrow$ for $P = 1 \Rightarrow M = 2$ (double root (Multiple))

$P = -2 \Rightarrow M = 1$ (simple root)

Remark: Newton Method for Simple root $(M = 1)$ is faster than for Multiple root. $(M > 1)$.

Def: (Order of Convergence):

Assume that $\{P_n\}_{n=0}^{\infty}$ converges to P. Set $E_n = P - P_n$, $\forall\, n \geq 0$. If two positive constants $A \neq 0$ and $R > 0$ exist. and

$$\lim_{n \to \infty} \frac{|P - P_{n+1}|}{|P - P_n|^R} = \lim_{n \to \infty} \frac{|E_{n+1}|}{|E_n|^R} = A,$$

then the sequence is said to converge to p with order of convergence R.

The number A is called the asymptotic error constant.

Note: The cases $R = 1, 2$ are given special consideration.

If $R = 1$, the convergence of $\{P_n\}_{n=0}^{\infty}$ is Linear

If $R = 2$, the convergence of $\{P_n\}_{n=0}^{\infty}$ is quadratic

Note: If R is large, the sequence $\{P_n\}$ converges rapidly to P; that is for large values of n we have $|E_{n+1}| \approx A |E_n|^R$

For example: If $R = 2$ and $|E_n| \approx 10^{-2}$, then $|E_{n+1}| \approx A \times 10^{-4}$.

Example: show that $P_n = \frac{1}{n^3}$ Converges to Zero Linearly. $P = 0$, $R = 1$

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^1} = \lim_{n\to\infty} \frac{\left|0 - \frac{1}{(n+1)^3}\right|}{\left|0 - \frac{1}{n^3}\right|}$$

$$= \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = 1 = A.$$

$\Rightarrow$ $\frac{1}{n^3}$ Converges Linearly to 0.

Example: Find the order of Convergence for $P_n = 10^{-n}$. We know $10^{-n} \xrightarrow[n\to\infty]{} 0$

$$\lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^{R}} = \lim_{n\to\infty} \frac{|0 - 10^{-(n+1)}|}{|0 - 10^{-n}|^R}$$

$$= \lim_{n\to\infty} \frac{(10^{-n})(10^{-1})}{10^{-nR}} = \lim_{n\to\infty} \frac{1}{10} \cdot 10^{n(R-1)} = \frac{1}{10}, \boxed{R=1}$$

$$= \begin{cases} \infty, & R > 1 \\ \frac{1}{10}, & R = 1 \\ 0 & , 0 < R < 1 \end{cases}$$

$A \neq 0$, $A \neq \infty$

Example: Find the order of Convergence of the sequence:

$$\overset{P_0}{1.5},\ \overset{P_1}{1.3733333},\ \overset{P_2}{1.365262015}$$

$$\overset{P_3}{1.365230014},\ \overset{P_4}{1.365230013}$$

Since the Convergence is very fast, then $\boxed{R = 2}$, to show that:

Let $|E_1| = |P_1 - P_0| = 0.12666667$

$|E_2| = |P_2 - P_1| = 0.008071315$

$|E_3| = |P_3 - P_2| = 0.000032001$

$|E_4| = |P_4 - P_3| = 0.000000001 \approx 0$

Now:

$$\frac{|E_2|}{|E_1|^2} = 0.50306$$

$$\frac{|E_3|}{|E_2|^2} = 0.49122$$

$\Rightarrow$ $R = 2$ and $A = 0.5$

$$\frac{|E_4|}{|E_3|^2} = A \not\approx 0$$

Note: We can set $E_i = |P_4 - P_{i-1}|$. $i = 1, 2, 3, 4$

Theorem: (Convergence Rate for Newton Iteration)

Assume that Newton-Raphson iteration produces a sequence $\{P_n\}_{n=0}^{\infty}$ that converges to the root p of the function $f(x)$.

1) If p is a simple root $(M=1)$, then

$$R = 2 \quad \text{and} \quad A = \frac{|f''(p)|}{|2f'(p)|}$$

2) If p is a multiple root $(M>1)$, then

$$R = 1 \quad \text{and} \quad A = \frac{M-1}{M}$$

Proof (1): Recall $P_{n+1} = g(P_n) = P_n - \dfrac{f(P_n)}{f'(P_n)}$

Let $g(x) = x - \dfrac{f(x)}{f'(x)}$

Note that $g(p) = p$ (p is a fixed point for g)

Now, Take Taylor second expansion of g about p:

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(c_n)(x-p)^2}{2!}$$

substitute p_n in x

$$\Rightarrow p_{n+1} = g(p_n) = \underbrace{g(p)}_{= p} + \underbrace{g'(p)}_{= 0}(p_n - p) + \frac{g''(c_n)}{2}(p_n - p)^2$$

(check) !!

$$\Rightarrow |p_{n+1} - p| = \frac{g''(c_n)}{2}(p_n - p)^2$$

$$\Rightarrow \frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{|g''(c_n)|}{2}$$

Find $g''(x)$ then $g''(p)$

$$\Rightarrow \lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \frac{|g''(p)|}{2} = \frac{|f''(p)|}{|2f'(p)|}.$$

$$\Rightarrow R = 2 \text{ and } A = \frac{|f''(p)|}{|2f'(p)|}.$$

Proof (2) : Since f has P as a multiple root with $(M>1)$, then there exists a Continuous function $h(x)$ such that

$$f(x) = (x-P)^M h(x) \quad , \quad h(p) \neq 0$$

Recall : $P_{n+1} = g(p_n) = P_n - \dfrac{f(P_n)}{f'(P_n)}$

Let $g(x) = x - \dfrac{f(x)}{f'(x)} = x - \dfrac{(x-P)^M h(x)}{(x-P)^M h'(x) + M(x-P)^{M-1} h(x)}$

Notice that $g(p) = p$ and $g'(p) = \dfrac{M-1}{M}$

Take Taylor first expansion of g about P:

$\Rightarrow \; g(x) = g(p) + g'(c_n)(x-p)$

Substitute P_n in x

$\Rightarrow \; P_{n+1} = g(P_n) = p + g'(c_n)(p_n - p)$

$\Rightarrow \; |P_{n+1} - p| = |g'(c_n)| \, |P_n - p|$

$\Rightarrow \; \lim\limits_{n\to\infty} \dfrac{|P_{n+1} - p|}{|P_n - p|} = |g'(p)| = \dfrac{M-1}{M} \Rightarrow \begin{array}{l} R = 1 \\ A = \dfrac{M-1}{M} \end{array}$

Example: (Quadratic Convergence at a Simple Root)

Let $f(x) = x^3 - 3x + 2 = (x+2)(x-1)^2$

Start with $P_0 = -2.4$ and use Newton Method to find the root $p = -2$.

The iteration formula $P_{n+1} = P_n - \dfrac{P_n^3 - 3P_n + 2}{3P_n^2 - 3}$

Theoritically: Since $P = -2$ is a simple root then $\boxed{R = 2}$ and $A = \dfrac{|f''(-2)|}{|2f'(-2)|} = \dfrac{|(-2)(6)|}{|2(3(-2)^2 - 3)|} =$

$\boxed{A \approx 0.6666667}$

Numerically:

we use this Column since we know $P = -2$

n	P_n	$\lvert P_{n+1} - P_n \rvert$	$\lvert E_n \rvert = \lvert P - P_n \rvert$	$\dfrac{\lvert E_{n+1} \rvert}{\lvert E_n \rvert^2}$
0	-2.4	$P_1 - P_0$ 0.323809524	0.4	0.476190475 ← $\dfrac{\lvert E_1 \rvert}{\lvert E_0 \rvert^2}$
1	-2.076190476	$P_2 - P_1$ 0.072594465	0.076190476	0.619469086
2	-2.003596011	$P_3 - P_2$ 0.003587422	0.003596011	0.664202613
3	-2.000008589	$P_4 - P_3$ 0.000008589	0.000008589	
4	-2.000000000	0.00000000	0.0000000	

Example: (Linear Convergence at a Double root):

Let $f(x) = x^3 - 3x + 2 = (x+2)(x-1)^2$

start with $p_0 = 1.2$ and Use Newton Method to find the root $P = 1$.

The iteration formula: $P_{n+1} = P_n - \dfrac{P_n^3 - 3P_n + 2}{3P_n^2 - 3}$

Theoritically: Since $P = 1$ is a multiple root with $M = 2 > 1$, then $R = 1$ & $A = \frac{2-1}{2} = \frac{1}{2}$

Numerically:

We know $p = 1$

k	P_k	$\lvert P_{k+1} - P_k \rvert$	$\lvert P - P_k \rvert$	$\dfrac{\lvert E_{k+1} \rvert}{\lvert E_k \rvert}$
0	1.2	0.096969697	0.2	0.515151515
1	1.103030303	0.050673883	0.103030303	0.508165253
2	1.052356420	0.025955609	0.052356420	0.496751115
3	1.026400811	0.013143081	0.026400811	0.509753688
4	1.013257730	0.006614311	0.013257730	0.501097775
5	1.006643419	0.003318055	0.006643419	0.500550093
⋮	⋮	⋮	⋮	⋮

Example: Consider the equation $x = \cos x$.

(a) Use Newton's Method with $P_0 = 0.2$ to estimate the solution of this equation with error less than 10^{-5}

(b) Find the order of Convergence and the asymptotic error Constant both theoritically and Numerically.

Sol: $f(x) = x - \cos x$, $f'(x) = 1 + \sin x$, $f''(x) = \cos x$

Newton's Formula: $P_{n+1} = P_n - \dfrac{P_n - \cos P_n}{1 + \sin P_n}$

(a) $P_0 = 0.2 \Rightarrow P_1 = P_0 - \dfrac{(P_0 - \cos P_0)}{1 + \sin P_0} = 0.850777122.$

$P_2 = 0.741530193$, $P_3 = 0.739086449$

$P_4 = 0.739085133$, Notice $|P_4 - P_3| = 0.0000013 < 10^{-5}$

(b) Theoritically: Take $P \approx P_4 = 0.739085133$.

$f'(P) = f'(0.739085133) = 1.673612029 \neq 0$

$\therefore$ P is a simple root, Therefore

$R = 2$ and $A = \dfrac{|f''(P)|}{|2f'(P)|} = 0.220805395$

Numerically:

n	P_n	$\lvert P_{n+1} - P_n \rvert$	$\frac{\lvert E_{n+1} \rvert}{\lvert E_n \rvert^2}$
0	0.2	$P_1 - P_0$ 0.650777122	0.257955435 $\leftarrow \frac{\lvert P_2 - P_1 \rvert}{\lvert P_1 - P_0 \rvert^2}$
1	0.850777122	$P_2 - P_1$ 0.109246929	0.204756281
2	0.741530193	$P_3 - P_2$ 0.002443744	0.220365941
3	0.739086449	$P_4 - P_3$ 0.000001316	
4	0.739085133	⋮	

Example: Show that the Bisection Method Converges Linearly on $[a, b]$.

$$\lim_{n \to \infty} \frac{\lvert P - P_{n+1} \rvert}{\lvert P - P_n \rvert} \leq \frac{\frac{b-a}{2^{n+2}}}{\frac{b-a}{2^{n+1}}} = \frac{2^{n+1}}{2^{n+2}} = \frac{1}{2}$$

$\therefore$ $R = 1$ and $A = 0.5$ for Bisection Method.

Corollary: For Bisection Method $R = 2$ & $A = 0.5$.

Theorem: (Acceleration of Newton-Raphson Iteration for Multiple root ($M>1$).

Suppose that the Newton-Raphson algorithm produces a sequence that converges linearly to the root $x = p$ of multiplicity $M > 1$. Then the iteration

$$p_{k+1} = p_k - \frac{M f(p_k)}{f'(p_k)}$$

will produce a sequence that converges quadratically to p. [(i-e) for p with $M>1 \Rightarrow R = 2$].

Proof: Since p is of multiplicity $M>1$, then there exists a continuous function $h(x)$ (s.t)

$$f(x) = (x-p)^M h(x) \quad , \quad h(p) \neq 0.$$

Let $g(x) = x - \dfrac{M f(x)}{f'(x)}$.

Notice that $g(p) = p$ and $g'(p) = 0$

(Check)!!

Take Taylor second expansion of $g(x)$ about p:

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(c_n)}{2!}(x-p)^2$$

Substitute P_n in x

$$\Rightarrow P_{n+1} = g(P_n) = p + \frac{g''(c_n)}{2}(P_n - P)^2$$

$$\Rightarrow |P_{n+1} - P| = \frac{|g''(c_n)|}{2} |P_n - P|^2$$

$$\Rightarrow \lim_{n\to\infty} \frac{|P_{n+1} - P|}{|P_n - P|^2} = \frac{|g''(p)|}{2} \Rightarrow R = 2$$

A or Numerically

Example: Let $f(x) = x^3 - 3x + 2 = (x+2)(x-1)^2$

Start with $P_0 = 1.2$ to find $P = 1$ Using Accelerated Newton Method. Formula which is

$$P_{k+1} = P_k - \frac{M(P_k^3 - 3P_k + 2)}{3P_k^2 - 3}, \quad M = 2$$

n	P_k	$\lvert P_{k+1} - P_k\rvert$	$\lvert P - P_k\rvert$	$\frac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert^2} = R$
0	1.2	0.19393939	0.2	0.1515151515
1	1.0060606	0.006054519	0.0060606	0.165718578 = A
2	1.000006087	0.000006087	0.000006087	

Example: Show that if $g(p) = p$ and $g'(p) = g''(p) = 0$, then the sequence generated by $p_{n+1} = g(p_n)$ Converges at least Cubically.

proof: Take Taylor Third expasion of g about p:

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(p)}{2!}(x-p)^2 + \frac{g'''(c_n)}{3!}(x-p)^3$$

Substitute p_n in x

$$\Rightarrow p_{n+1} = g(p_n) = \underbrace{g(p)}_{=p} + \frac{g'''(c_n)}{3!}(p_n - p)^3$$

$$\Rightarrow |p_{n+1} - p| = \frac{|g'''(c_n)|}{3!}|p_n - p|^3$$

Then, $\displaystyle\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^3} = \frac{|g'''(p)|}{6}$

If $g'''(p) \neq 0$, then the convergence is $\underbrace{\text{Cubically}}_{R=3}$. If $g'''(p) = 0$, then $R > 3$.

Theorem: Order of Convergence for Fixed point Iteration. Let p be a fixed point of $g(x)$

If $g'(p) = g''(p) = \cdots = g^{(k-1)}(p) = 0$

and $g^{(k)}(p) \neq 0$, then the fixed point Iteration of $g(x)$ will Converge to p with $R = k$, $A = \left|\frac{g^{(k)}(p)}{k!}\right|$

proof: Take Taylor expansion of $g(x)$ about p.

$$g(x) = g(p) + g'(p)(x-p) + \cdots + \frac{g^{(k-1)}(p)}{(k-1)!}(x-p)^{k-1} + \frac{g^{(k)}(c_k)}{k!}(x-p)^{k}$$

Substitute P_n in x:

$$\Rightarrow P_{n+1} = g(P_n) = p + \frac{g^{(k)}(c_k)}{k!}(p_n - p)^{k}$$

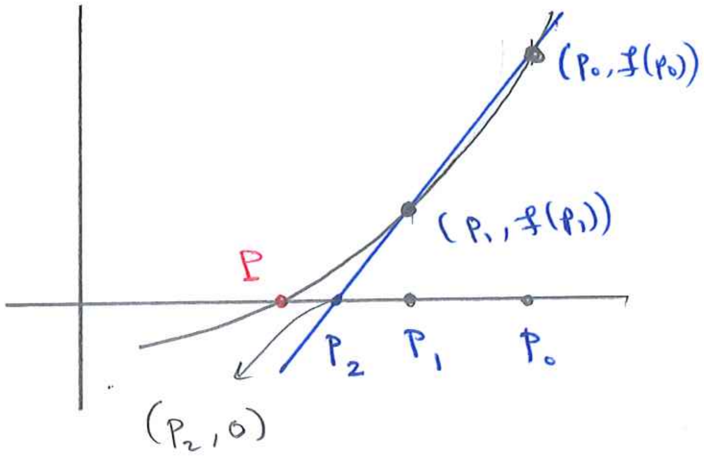
$$\Rightarrow |P_{n+1} - p| = \left|\frac{g^{(k)}(c_n)}{k!}\right| \, |P_n - p|^{k}$$

$$\Rightarrow \lim_{n\to\infty} \frac{|P_{n+1} - p|}{|P_n - p|^{k}} = \frac{|g^{(k)}(p)|}{k!}.$$

Secant Method:

Let $f(x) = 0$, P_0 and P_1 are Given

Find the slope: P_0, P_1, P_2



$$\frac{f(P_1) - 0}{P_1 - P_2} = \frac{f(P_1) - f(P_0)}{P_1 - P_0}$$

Solving for P_2, we get:

$$P_2 = P_1 - \frac{f(P_1)(P_1 - P_0)}{f(P_1) - f(P_0)}$$

$$\vdots$$

$$P_{n+1} = P_n - \frac{f(P_n)(P_n - P_{n-1})}{f(P_n) - f(P_{n-1})}$$

Remark: Secant Method is Similar to False position Method, but its faster since P_0 and P_1 are given.

Theorem : Order of Convergence for Secant Method.

Let P be a simple root $(M=1)$ for $f(x)$.

Then the sequence $\{P_n\}_{n=2}^{\infty}$ generated by Secant Method will Converge to P with

$$R = 1.618 \quad \text{and} \quad A = \left| \frac{f''(p)}{2 f'(p)} \right|^{0.618}$$

Example: Consider the equation $x = \cos x$

(a) If $P_0 = 0.5$, $P_1 = \frac{\pi}{4}$, use the secant method to approximate the root of the equation with accuracy of 10^{-4}.

(b) Find the order of Convergence and the asymptotic error Constant both theoritically and Numerically.

Sol: Let $f(x) = x - \cos x$, $f'(x) = 1 + \sin x$, $f''(x) = \cos x$

The secant formula is

$$P_{n+2} = P_{n+1} - \frac{f(P_{n+1})(P_{n+1} - P_n)}{f(P_{n+1}) - f(P_n)}.$$

(b) Numerically.

n	P_n	$\lvert P_{n+1} - P_n\rvert$	$\dfrac{\lvert E_{n+1}\rvert}{\lvert E_n\rvert^{1.618}}$
0	0.5	$P_1 - P_0$ 0.285398163	0.372735166
1	0.785398163	$P_2 - P_1$ 0.049014025	0.351741034
2	0.736384138	$P_3 - P_2$ 0.002674	0.393005788 ← $\dfrac{\lvert P_4 - P_3\rvert}{\lvert P_3 - P_2\rvert^{1.618}}$
3	0.739058138	$P_4 - P_3$ 0.00002701134	
4	0.739085149		

Notice that $|P_4 - P_3| < 10^{-4}$

so $P \approx P_4 = 0.739085149$

(b) Theoritically: $f'(P) \cong f'(0.739085149)$

$$= 1.673612041 \neq 0$$

$\therefore$ P is a simple root, so $R = 1.618$

and $$A = \left|\frac{f''(P)}{2f'(P)}\right|^{0.618} = 0.393185938.$$

Numerically: The Table above Column (3).

Remark : ① The number $1.618 \approx \frac{\sqrt{5}+1}{2}$ which is called the golden number.

② For multiple root using secant Method, $\boxed{R=1}$ and A can be found numerically.

Summary : Comparison for the Order of Convergence.

Method	Special Consideration	R & A resp.
Bisection	—	1 & 0.5
Regula falsi	—	1 & Numerically
Secant Method	Multiple root	1 & Numerically
Secant Method	Simple root	1.618 & $\left\lvert \frac{f''(p)}{2f'(p)} \right\rvert^{0.618}$
Newton Method	Multiple root	1 & $\frac{M-1}{M}$
Newton Method	Simple root	2 & $\left\lvert \frac{f''(p)}{2f'(p)} \right\rvert$
Accelerated Newton	Multiple root	2 & Numerically.

"End of Chapter Two"