Chapter 4.

Interpolation and Polynomial Approximation.

4.2 Introduction to Interpolation.

Given $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, we are Looking for the relation (function) between these points.

\# of points $(n+1)$

Def: Given $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$. "$y_i = f(x_i)$"

Interpolation is an estimation of the Unknown function $f(x)$ by polynomial of degree at most n, $P_n(x)$, that passes through all the given points. (i-e) $f(x_i) = P_n(x_i)$

$i = 0, 1, \ldots, n$

$$\left(P_n(x) \approx f(x) \text{ , while } f(x_i) = P_n(x_i) \right)$$

Remark: when $x_0 < x < x_n$, then $P_n(x)$ is called an Interpolation.

If $x < x_0$ or $x > x_n$, then $P_n(x)$ is extrapolation.

Example: Let $P(x)$ be the polynomial passes through the points: $(1, 1.06)$, $(2, 1.12)$, $(3, 1.34)$ and $(5, 1.78)$. Find the Polynomial $P(x)$.

Sol: Let $P(x) = Ax^3 + Bx^2 + Cx + D$

$$\Rightarrow \quad 1.06 = A + B + C + D$$

$$1.12 = 8A + 4B + 2C + D$$

$$1.34 = 27A + 9B + 3C + D$$

$$1.78 = 125A + 25B + 5C + D$$

$$\Rightarrow \quad A = -0.02, \quad B = 0.2, \quad C = -0.4, \quad D = 1.28$$

$$\Rightarrow \quad P(x) = -0.02x^3 + 0.2x^2 - 0.4x + 1.28$$

Remark: This Method is easy to understand but some times the resolution linear systems is difficult, especially if we have a large number of equations.

4.3 Lagrange Interpolation. (Approximation).

Given $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, we need to find $P_n(x)$ that passes through all the given points. (s.t) $f(x_i) = y_i = P_n(x_i)$ for $i = 0, 1, \dots, n$

Case 1:

Assume $(x_0, y_0), (x_1, y_1)$ are Given, then:

$$y = P_1(x) = y_0 + m(x - x_0)$$

$$\Rightarrow \quad P_1(x) = y_0 + \frac{(y_1 - y_0)}{(x_1 - x_0)}(x - x_0)$$

Some rearrangments, we have

$$\boxed{P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1}$$

$P_1(x)$ is called Lagrange Interpolation of degree 1

$$L_{1,0}(x) := \frac{x - x_1}{x_0 - x_1}$$

$$L_{1,1}(x) := \frac{x - x_0}{x_1 - x_0}$$

Lagrange Coefficients

Notice that :

1) $P_1(x_0) = y_0$ & $P_1(x_1) = y_1$

2) $L_{1,0}(x_0) = 1$ & $L_{1,0}(x_1) = 0$

3) $L_{1,1}(x_0) = 0$ & $L_{1,1}(x_1) = 1$

4) $P_1(x) = \sum_{k=0}^{1} L_{1,k}(x) \cdot y_k$

5) $P_1(x)$ is used to approximate $f(x)$ over $[x_0, x_1]$

5) [The degree of $P_1(x)$] ≤ 1

Example: Consider $f(x) = \cos x$ over $[0, 1.2]$

Use the nodes $x_0 = 0$, $x_1 = 1.2$ to find $P_1(x)$

Sol: $P_1(x) = \frac{(x-1.2)}{(0-1.2)} y_0 + \frac{(x-0)}{(1.2-0)} y_1$

where $y_0 = f(x_0) = \cos 0 = 1$

$y_1 = f(1.2) = \cos 1.2 = 0.362358$

$\Rightarrow P_1(x) = -0.83333(x-1.2) + 0.301965(x)$

Case 2: Assume (x_0, y_0), (x_1, y_1) and (x_2, y_2) are given, then:

$$P_2(x) := \underbrace{\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}}(y_0) + \underbrace{\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}}(y_1) + \underbrace{\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}}(y_2)$$

$$P_2(x) = L_{2,0}(x) \cdot y_0 + L_{2,1}(x) \cdot y_1 + L_{2,2}(x) \cdot (y_2)$$

$$\Rightarrow P_2(x) = \sum_{k=0}^{2} L_{2,k}(x) \cdot y_k .$$

Notice that: ① $P_2(x_i) = y_i \quad , \; i = 0, 1, 2.$

② $L_{2,k}(x_j) = \begin{cases} 1 & , k = j \\ 0 & , k \neq j \end{cases}$

$k, j \in \{0, 1, 2\}$

Example: Let $f(x) = \cos x$ over $[0, 1.2]$

Find Lagrange interpolation of degree 2.

midpoint

sol: Let $x_0 = 0$, $x_1 = 0.6$, $x_2 = 1.2$

$$\Rightarrow P_2(x) = \frac{(x-0.6)(x-1.2)}{(0-0.6)(0-1.2)}(\cos 0) + \frac{(x-0)(x-1.2)}{(0.6-0)(0.6-1.2)}(\cos 0.6) + \frac{(x-0)(x-0.6)}{(1.2-0)(1.2-0.6)}(\cos 1.2)$$

$$= 1.388889(x-0.6)(x-1.2) + {}^{-}2.292599(x)(x-1.2)$$
$$+ 0.503275(x)(x-0.6)$$

High order Lagrange Interpolation.

Consider the $n+1$ points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$

Lagrange polynomial of degree n that passes through these points is obtained as follows.

$$P_n(x) = \sum_{k=0}^{n} L_{n,k}(x)\, y_k$$

where $L_{n,k}(x)$ is Lagrange Coefficients defined as:

$$L_{n,k}(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

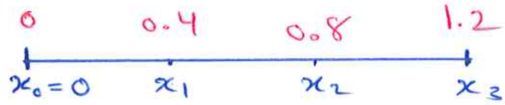
$$= \frac{\prod\limits_{\substack{j=0 \\ j\neq k}}^{n} (x-x_j)}{\prod\limits_{\substack{j=0 \\ j\neq k}}^{n} (x_k-x_j)}.$$

And $L_{n,k}(x_j) = \begin{cases} 1 , & k=j \\ 0 , & k\neq j \end{cases} \quad , k,j=0,1,\dots,n$

$P_n(x_i) = y_i \quad , \; i=0,1,\dots,n.$

Example: Let $f(x) = \cos x$, over $[0, 1.2]$

Estimate $f(0.35)$ Using Lagrange interpolation of degree 3.



sol: To find $P_3(x)$, we need 4 points.

Let length of each subinterval $= \dfrac{\text{Length}[a,b]}{n}$

$\Rightarrow$ h = length of each subinterval $= \dfrac{1.2-0}{3} = 0.4$

$\therefore$ $\boxed{x_0 = 0}$, $\boxed{x_1 = 0.4}$, $\boxed{x_2 = 0.8}$, $\boxed{x_3 = 1.2}$

$y_0 = \cos 0 = 1$, $y_1 = \cos 0.4 \approx 0.921061$

$y_2 = \cos 0.8 \approx 0.696707$, $y_3 = \cos 1.2 \approx 0.362358$

$$\Rightarrow P_3(x) = \frac{(x-0.4)(x-0.8)(x-1.2)}{(0-0.4)(0-0.8)(0-1.2)}\overset{y_0}{(1)} +$$

$$+ \frac{(x-0)(x-0.8)(x-1.2)}{(0.4-0)(0.4-0.8)(0.4-1.2)}\overset{y_1}{(0.921061)} + \frac{(x-0)(x-0.4)(x-1.2)}{(0.8-0)(0.8-0.4)(0.8-1.2)}\overset{y_2}{(0.696707)}$$

$$+ \frac{(x-0)(x-0.4)(x-0.8)}{(1.2-0)(1.2-0.4)(1.2-0.8)}\overset{y_3}{(0.362358)}$$

$$\Rightarrow P_3(0.35) = 0.93960767 \approx \cos(0.35) = 0.939372712$$

Error terms and error bounds

Theorem: (Lagrange polynomial approximation).

Assume that $f \in C^{n+1}[a,b]$ and that

$x_0, x_1, \dots, x_n \in [a,b]$ are $n+1$ nodes.

If $x \in [a,b]$, then:

$$f(x) = P_n(x) + E_n(x)$$

where, $P_n(x)$: Lagrange interpolation for $f(x)$

$$P_n(x) = \sum_{k=0}^{n} L_{n,k}(x)\, y_k$$

and the error term is:

$$E_n(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(c)$$

for some $c \in [a,b]$

Remark: $E_n(x_i) = 0$, $\forall\, i = 0, 1, 2, \dots, n$.

Proof: We are going to establish the result for $\boxed{n=1}$

Consider the special function

$$g(t) = f(t) - \underset{\text{poly. of degree 1}}{P_1(t)} - E_1(x)\frac{(t-x_0)(t-x_1)}{(x-x_0)(x-x_1)}.$$

Notice that x, x_0 and x_1 are constants w.r.to t.

$$g(x_0) = \underbrace{f(x_0) - P_1(x_0)}_{=} - E_1(x)\,[\,0\,] = 0.$$

$$g(x_1) = \underbrace{f(x_1) - P_1(x_1)}_{=} - E_1(x)\,[\,0\,] = 0.$$

$$g(x) = \underbrace{f(x) - P_1(x)}_{E_1(x)=} - E_1(x) = 0.$$

Let $x \in (x_0, x_1)$, applying Rolle's theorem on $[x_0, x]$ ($0 = g(x_0) = g(x)$)

$\exists\, c_0 \in (x_0, x)$ such that $g'(c_0) = 0$.

Again, applying Rolle's theorem on $[x, x_1]$ ($0 = g(x_1) = g(x)$), then

$\exists\, c_1 \in (x, x_1)$ such that $g'(c_1) = 0$

Now, since $g'(c_0) = g'(c_1) = 0$, we can

again apply Rolle's theorem to $g'(t)$ on $[c_0, c_1]$

then $\exists\, c \in (c_0, c_1)$ such that $g''(c) = 0$.

Calculate: $g'(t) = f'(t) - P_1'(t) - E_1(x) \boxed{\dfrac{(t-x_1)+(t-x_0)}{(x-x_0)(x-x_1)}}$

and $g''(t) = f''(t) - 0 - E_1(x) \dfrac{(2)}{(x-x_0)(x-x_1)}$

$\Rightarrow g''(c) = 0 = f''(c) - E_1(x) \dfrac{(2)}{(x-x_0)(x-x_1)}$

$\Rightarrow E_1(x) = \dfrac{(x-x_0)(x-x_1)}{2!} f''(c)$. ∎

Example: Let $f(x) = \cos x$, $[0, 1.2]$

Find the upper bound of the error when estimating $f(0.35)$ using $P_3(x)$. $x_0 = 0,\ x_1 = 0.4,\ x_2 = 0.8,\ x_3 = 1.2$

$$|E_2(x)| \leqslant \frac{|(x-x_0)(x-x_1)(x-x_2)(x-x_3)|}{4!} \max_{[0,1.2]} |f''''(x)|$$

decreasing

$$|E_2(0.35)| \leqslant \frac{|(0.35-0)(0.35-0.4)(0.35-0.8)(0.35-1.2)|}{4!} [\cos 0]$$

$\Rightarrow |E_2(0.35)| \leqslant 2.7891 \times 10^{-4}$.

Example: Let $f(x) = \dfrac{1}{1-2x}$

(a) Use Lagrange interpolation polynomial with the nodes $x_0 = 2$, $x_1 = 3$ and $x_2 = 3.5$ to find an estimation for $f(2.5)$.

(b) Find an upper bound of the error when estimating $f(2.5)$

(c) Find an upper bound of the error when estimating $f(x)$, $\forall x \in [2, 3.5]$.

Sol: (a) $P_2(x) = \dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \dfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$

$$\Rightarrow f(2.5) \approx P_2(2.5) = \frac{(2.5-3)(2.5-3.5)}{(2-3)(2-3.5)} f(2) +$$

$$+ \frac{(2.5-2)(2.5-3.5)}{(3-2)(3-3.5)} f(3) + \frac{(2.5-2)(2.5-3)}{(3.5-2)(3.5-3)} f(3.5)$$

$$= \frac{0.5}{1.5}\left(-\frac{1}{3}\right) + \frac{-0.5}{-0.5}\left(-\frac{1}{5}\right) + \frac{-0.25}{0.75}\left(-\frac{1}{6}\right)$$

$$= -0.2555556.$$

(b) $$|E_2(x)| \leq \frac{|(x-x_0)(x-x_1)(x-x_2)|}{3!} \underset{[2,3.5]}{\text{Max}} |f^{(3)}(x)|$$

$$f^{(3)}(x) = \frac{48}{(1-2x)^4} \quad , \; x \in [2, 3.5] \text{ which}$$

is decreasing function over $[2, 3.5]$

$$\therefore \underset{[2,3.5]}{\text{Max}} |f^{(3)}(x)| = \left|\frac{48}{(1-4)^4}\right| = 0.59259259$$

$$\therefore |E_2(2.5)| \leq \frac{|(2.5-2)(2.5-3)(2.5-3.5)|}{3!} (0.59259259)$$

$$\Rightarrow |E_2(2.5)| \leq 0.02469$$

(c) $$|E_2(x)| \leq \frac{|(x-x_0)(x-x_1)(x-x_2)|}{3!} \left(\underset{[2,3.5]}{\text{Max}} |f^{(3)}(x)| \right)$$

Let $g(x) = (x-2)(x-3)(x-3.5)$, we need to find

$\underset{[2,3.5]}{\text{Max}} |g(x)|$. $\Rightarrow g'(x) = (x-2)(x-3) + (x-2)(x-3.5) + (x-3)(x-3.5)$

$$\Rightarrow g'(x) = 3x^2 - 17x + 23.5$$

$$\Rightarrow g'(x) = 0 \Rightarrow x_{1,2} = \frac{-(-17) \pm \sqrt{(-17)^2 - 4(3)(23.5)}}{2(3)}$$

$\Rightarrow \quad x_1 = 3.2743 \quad$ and $\quad x_2 = 2.3924$

Notice that $\left.\begin{array}{l} g(2) = 0 \\ g(3.5) = 0 \end{array}\right\} \Leftarrow$ اطراف الفترة

$$|g(3.2743)| = 0.07889$$

$$|g(2.3924)| = 0.2641$$

$$\Rightarrow \max_{[2,3.5]} |g(x)| = 0.2641$$

$$\therefore \quad |E_2(x)| \leq \frac{0.2641}{6}(0.59259259) \approx 0.02608$$

Remark : The next theorem address the special case when the nodes for Lagrange polynomial are equally spaced (Uniform partition).

(i.e) $\quad x_k = x_0 + hk \quad , \text{ for } k = 0, 1, \dots, n$

and $\quad h = \dfrac{b-a}{n} \quad$, where $P_n(x)$ is used over the Interval $[x_0, x_n]$

Theorem: (Error bounds for Lagrange Interpolation equally spaced nodes).

Assume that $f(x)$ is defined on $[a,b]$, which contains equally spaced nodes $x_k = x_0 + hk$, $\boxed{h = \frac{b-a}{n}}$

Assume $f^{(n+1)} \in C[a,b]$ and bounded on the special subintervals $[x_0, x_1]$, $[x_0, x_2]$, $[x_0, x_3]$

that is : $|f^{n+1}(x)| \leqslant \operatorname{Max} |f^{(n+1)}(x)| = M_{n+1}$

for $n = 1, 2, 3$. Then the error terms are :

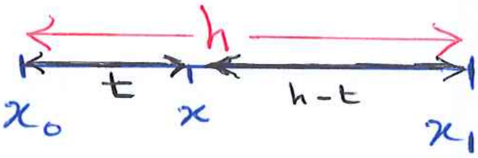
1) $$|E_1(x)| \leqslant \frac{h^2 M_2}{8}, \quad x \in [x_0, x_1]$$

2) $$|E_2(x)| \leqslant \frac{h^3 M_3}{9\sqrt{3}}, \quad x \in [x_0, x_2]$$

3) $$|E_3(x)| \leqslant \frac{h^4 M_4}{24}, \quad x \in [x_0, x_3].$$

Proof (1) :

Consider the change of Variable :



$$t = x - x_0 \Rightarrow h - t = x_1 - x$$

$$\Rightarrow t - h = x - x_1$$

$$\Rightarrow E_1(x) = E_1(x_0 + t) = \frac{t(t-h)}{2!} f''(c) \, , \quad 0 \leq t \leq h$$

Notice that $|f''(c)| \leq M_2 \quad , \quad x_0 \leq c \leq x_1$

Now , we need to find : $\underset{[0,h]}{\mathrm{Max}} |t(t-h)|$

Let $\phi(t) = t^2 - th \Rightarrow \phi'(t) = 2t - h = 0$

$$\Rightarrow \quad t = \frac{h}{2} \quad \text{(critical point)}.$$

$\Rightarrow$ Extreme values : $\left.\begin{array}{l} \phi(0) = 0 \\ \phi(h) = 0 \end{array}\right\}$ أطراف الفترة

$$\phi\left(\frac{h}{2}\right) = \frac{-h^2}{4}$$

$$\Rightarrow \quad |\phi(t)| \leq \frac{h^2}{4}.$$

$$\Rightarrow \quad |E_1(x)| \leq \frac{\frac{h^2}{4}}{2} M_2 = \frac{h^2 M_2}{8}. \quad \blacksquare$$

Example: Consider $y = f(x) = \cos x$, $[0, 1.2]$.

Determine the upper bounds of the error for Lagrange polynomials $P_1(x)$, $P_2(x)$ and $P_3(x)$.

Sol: For $P_1(x)$: $x_0 = 0$, $x_1 = 1.2$, $h = \frac{1.2-0}{1}$

$$\Rightarrow |E_1(x)| \leq \frac{h^2 M_2}{8} = \frac{(1.2)^2}{8} \max_{[0,1.2]} |f''(x)|$$

$$\Rightarrow |E_1(x)| \leq \frac{(1.2)^2}{8}(1) = 0.18.$$

For $P_2(x)$: $x_0 = 0$, $x_1 = 0.6$, $x_2 = 1.2$, $h = \frac{1.2-0}{2} = 0.6$

$$\& \quad M_3 = \max_{[0,1.2]} |f'''(x)| = |\sin(1.2)| = 0.932039$$

$$\Rightarrow |E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}} = \frac{(0.6)^3 (0.932039)}{9\sqrt{3}} = 0.012915$$

For $P_3(x)$: $x_0 = 0$, $x_1 = 0.4$, $x_2 = 0.8$, $x_3 = 1.2$

$$h = \frac{1.2-0}{3} = 0.4 \quad \text{and} \quad M_4 = \max_{[0,1.2]} |f^{(4)}(x)| = |\cos 0| = 1$$

$$\Rightarrow |E_3(x)| \leq \frac{h^4 M_4}{24} = \frac{(0.4)^4 (1)}{24} = 0.001067.$$

4.4. Newton Polynomials.

It is sometimes useful to find several approximating polynomials $P_1(x)$, $P_2(x)$, ..., $P_n(x)$ and then choose the one that suits our need.

If Lagrange polynomials are used, there is no constructive relationship between $P_{n-1}(x)$ and $P_n(x)$. Each polynomial has to be constructed individually. So, we will construct Newton polynomials that have the recursive pattern: Given $(x_0, y_0), \ldots, (x_n, y_n)$

$$P_1(x) = a_0 + a_1(x - x_0).$$

$$\begin{aligned} P_2(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &= P_1(x) + a_2(x - x_0)(x - x_1). \end{aligned}$$

$$\begin{aligned} P_3(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \\ &= P_2(x) + a_3(x - x_0)(x - x_1)(x - x_2). \end{aligned}$$

Hence :

$$P_n(x) = P_{n-1}(x) + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}).$$

$P_n(x)$ is called Newton polynomial with centers $x_0, x_1, \ldots, x_{n-1}$, and degree $\leq n$. (Unique poly.)

Now, we need to determine the a_i's , $i=0,\ldots,n$.

Def: $a_k = f[x_0, x_1, \ldots, x_k]$, $k = 0, 1, \ldots, n$.

$$\boxed{a_0} = f[x_0] = f(x_0) = y_0 .$$ (Zero divided difference)

$$\boxed{a_1} = f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

(First divided difference).

$$\boxed{a_2} = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - \overbrace{f[x_0, x_1]}^{a_1}}{x_2 - x_0}$$

$$= \frac{\dfrac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \dfrac{f(x_1) - f(x_0)}{(x_1 - x_0)}}{x_2 - x_0}$$

(Second divided difference).

In General : $\boxed{a_k} = f[x_0, x_1, \ldots, x_{k-1}, x_k]$

$$= \frac{f[x_1, \ldots, x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}$$

(kth divided difference).

Divided - Difference Table for $y = f(x)$

x_k	$f[x_k]$	$f[-,-]$	$f[-,-,-]$	$f[-,-,-,-]$	$f[-,-,-,-,-]$
x_0	$f[x_0]$ $\leftarrow a_0$				
x_1	$f[x_1]$	$f[x_0, x_1]$ $\leftarrow a_1$			
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$ $\leftarrow a_2$		
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$ $\leftarrow a_3$	
x_4	$f[x_4]$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$	$f[x_0, x_1, x_2, x_3, x_4]$ $\leftarrow a_4$

Example: $f(x) = \cos x$.

$$f[1,2,3] = \frac{f[2,3] - f[1,2]}{3 - 1}$$

$$= \frac{\frac{f(3) - f(2)}{(3-2)} - \frac{f(2) - f(1)}{(2-1)}}{3 - 1} = \frac{\cos 3 - 2\cos 2 + \cos 1}{2} = \cdots$$

Theorem: (Newton Polynomial). Suppose that $x_0, x_1, \dots, x_n$ are $n+1$ distinct numbers in $[a,b]$. There exists a Unique polynomial $P_n(x)$ of degree at most n with the property that

$$\left(\begin{array}{l}\text{no error}\\ \text{at the nodes}\end{array}\right) \leftarrow f(x_i) = P_n(x_i), \quad i = 0, 1, \dots, n.$$

The Newton form of this polynomial is:

$$P_n(x) = a_0 + a_1(x-x_0) + \dots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}),$$

where $a_k = f[x_0, \dots, x_k]$, $k = 0, 1, \dots, n$.

Corollary: Assume that $P_n(x)$ is Newton polynomial is used to approximate $f(x)$, that is:

$$f(x) = P_n(x) + E_n(x).$$

If $f \in C^{n+1}[a,b]$, then $\forall x \in (a,b)$, $\exists\, c = c(x)$ in (a,b) such that the error term has the form:

$$E_n(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} f^{(n+1)}(c).$$

Remark: The error term $E_n(x)$ is the same as the one for Lagrange Interpolation.

Example: Let $f(x) = x^3 - 4x$. Construct the divided-difference table based on the nodes:

$x_0 = 1,\ x_1 = 2,\ x_2 = 3,\ x_3 = 4,\ x_4 = 5,\ x_5 = 6.$

then find $P_3(x)$.

x_k	$f[x_k]$	1st D.D	2nd D.D	3rd D.D	4th D.D	5th D.D
1	$-3 = a_0$	—	—	—	—	—
2	0	$3 = a_1$	—	—	—	—
3	15	15	$6 = a_2$	—	—	—
4	48	33	9	$1 = a_3$	—	—
5	105	57	12	1	$0 = a_4$	—
6	192	87	15	1	0	$0 = a_5$

$$\Rightarrow P_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$= -3 + 3(x-1) + 6(x-1)(x-2) + 1(x-1)(x-2)(x-3).$$

$$\Rightarrow \boxed{P_3(x) = x^3 - 4x}.$$

Remark: If $f(x)$ = polynomial of degree n

then:

1) $P_n(x) = f(x)$

2) $a_{n+i} = 0$, $\forall i = 1, 2, \dots$

3) a_n can be found directly from $f(x)$,

such that a_n = Coefficient of x^n.

4) a_i , $\forall i < n$ Can be found from table.

Example: If $f(x) = x^5 - 3x^2 + 4x - 8$,

then: ① $P_5(x) = f(x)$, Using the nodes

$x_0, x_1, \dots, x_8$.

② $a_5 = 1$

③ $a_i = 0$, $\forall i > 5$

④ $a_1 \neq 4$, $a_0 \neq -8$, $a_2 \neq -3$

$\Rightarrow$ a_j , $j < 5$ Can be found from table.

Example: Construct a divided-difference table for $f(x) = \cos x$, based on the points $(k, \cos k)$, for $k = 0, 1, 2, 3, 4$. Then find $P_1(x)$, $P_2(x)$, $P_3(x)$ and $P_4(x)$. [Using 5 significant digits rounded].

Sol:

x_k	$f[x_k]$	$f[-,-]$	$f[-,-,-]$	$f[-,-,-,-]$	$f[-,-,-,-,-]$
0	1	—	—	—	—
1	0.54030	-0.4597	—	—	—
2	-0.41615	-0.95645	-0.24838	—	—
3	-0.98999	-0.57384	0.19131	0.14656	—
4	-0.65364	0.33635	0.4551	0.08793	-0.014658

$$\Rightarrow P_1(x) = 1 - 0.4597(x-0)$$

$$P_2(x) = P_1(x) - 0.24838(x-0)(x-1)$$

$$P_3(x) = P_2(x) + 0.14656(x-0)(x-1)(x-2)$$

$$P_4(x) = P_3(x) - 0.014658(x-0)(x-1)(x-2)(x-3).$$

"End of Chapter 4"