Chapter 3. The Solution of Linear Systems $AX = B$

To solve a Linear system, we have 5 direct methods:

1) Gaussian Elimination : $[A \backslash B] \rightsquigarrow [U \backslash C] + \underset{\text{Back substitution}}{B.S.}$

2) Gauss- Jordan elimination : $[A \backslash B] \rightsquigarrow [I \backslash \underset{\text{solution}}{X}]$.

3) Inverse Method : $AX = B \rightsquigarrow X = A^{-1}B$.

4) Cramer's Method : $x_i = \dfrac{|A_i|}{|A|}$.

5) LU- factorization :

$$AX = B \rightsquigarrow \underbrace{(LU)}_{A}X = B$$

Let $UX = Y$, then Solve $LY = B$ for Y.

Using Forward substitution.

Then Solve $UX = Y$ Using Backward substitution

3.3 Upper - Triangular Linear Systems:

Def: An $n \times n$ matrix $A = [a_{ij}]$ is called upper triangular provided that the elements satisfy $a_{ij} = 0$ whenever $i > j$.

The $n \times n$ matrix $A = [a_{ij}]$ is called Lower triangular provided that the elements satisfy $a_{ij} = 0$ whenever $i < j$.

Example:

$$\begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & 0 \\ 4 & -1 & 8 \end{bmatrix}$$ is lower triangular matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & \sqrt{5} & 3 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$ is upper triangular matrix.

Theorem: Assume that A is an $N \times N$ matrix.

The following statements are equivalent.

(1) Given any $n \times 1$ matrix B, the Linear system $AX = B$ has a Unique Solution.

(2) The matrix A is NonSingular (A^{-1} exists).

(3) The system of equations $AX = 0$ has a Unique solution $X = 0$.

(4) $\det(A) = |A| \neq 0$.

Theorem: If the $n \times n$ matrix $A = [a_{ij}]$ is either upper or Lower triangular, then

$$\det(A) = a_{11}\, a_{22} \cdots a_{nn}.$$

If A is an upper-triangular matrix, then $AX = B$ is said to be an upper triangular system of linear equations and has the form:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1\,n-1}x_{n-1} + a_{1n}x_n = b_1$$

$$a_{22}x_2 + a_{23}x_3 + \cdots + a_{2\,n-1}x_{n-1} + a_{2n}x_n = b_2$$

$$a_{33}x_3 + \cdots + a_{3\,n-1}x_{n-1} + a_{3n}x_n = b_3$$

Form (1) ...

$$\vdots$$

$$+ a_{n-1\,n-1}x_{n-1} + a_{n-1\,n}x_n = b_{n-1}$$

$$+ a_{nn}x_n = b_n$$

$$\Longleftrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1\,n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2\,n-1} & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3\,n-1} & a_{3n} \\ 0 & 0 & \vdots & & & \\ 0 & 0 & 0 & \cdots & a_{n-1\,n-1} & a_{n-1\,n} \\ 0 & 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

Theorem (Back Substitution): Suppose that $AX = B$ is an upper triangular system with the form given in (1) Form. If $a_{kk} \neq 0$, $k = 1, 2, \ldots, n$ then there exists a Unique Solution to (1).

Example: Use B.S to Solve

$$
\begin{aligned}
4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\
-2x_2 + 7x_3 - 4x_4 &= -7 \\
6x_3 + 5x_4 &= 4 \\
3x_4 &= 6
\end{aligned}
$$

$\Rightarrow \boxed{x_4 = \frac{6}{3} = 2}$

Using $x_4 = 2$ in the third equation : $\boxed{x_3 = -1}$

Now Using $x_3 = -1$ & $x_4 = 2$ in the 2nd eq. : $\boxed{x_2 = -4}$

Finally, $\boxed{x_1 = 3}$.

Remark: If $a_{kk} = 0$, for some $k = 1, 2, \ldots, n$ then the system either has no solution or infinitly many solutions.

Example: Show that there is no solution for

$$4x_1 - x_2 + 2x_3 + 3x_4 = 20$$
$$0x_2 + 7x_3 - 4x_4 = -7$$
$$6x_3 + 5x_4 = 4$$
$$3x_4 = 6$$

Solving the system: $x_4 = 2$, which is substituted into the second and third equation $\Rightarrow$

$x_3 = \frac{1}{7}$ & $x_3 = -1$, which is a contradiction

So there is No solution for the Linear system.

Example: Show that there are infinitly many solutions for

$$4x_1 - x_2 + 2x_3 + 3x_4 = 20$$
$$0x_2 + 7x_3 + 0x_4 = -7$$
$$6x_3 + 5x_4 = 4$$
$$3x_4 = 6$$

$x_4 = 2$, substitute into the second & third equations.

we get $x_3 = -1$

Equ. (1) $\Rightarrow$ $x_2 = 4x_1 - 16$ which has infinitly many solutions.

Example: Solve

$$\begin{aligned} 2x_1 &= 6 \\ -x + 4x_2 &= 5 \\ 3x_1 - 2x_2 - x_3 &= 4 \\ x_1 - 2x_2 + 6x_3 + 3x_4 &= 2 \end{aligned}$$

$\Rightarrow x_1 = 3, \quad x_2 = 2, \; x_3 = 1, \; x_4 = -1$

Cost (Complexity): is the number of operations $+, -, \times, \div$ needed to complete a certain calculation.

Example: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \text{Cost}(|A|) = 3$

$$|A| = a \times d - b \times c.$$

Example: Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Find the Cost of $|A|$

$$|A| = a \times \underbrace{\begin{vmatrix} e & f \\ h & i \end{vmatrix}}_{\text{cost } 3} - b \underbrace{\begin{vmatrix} d & f \\ g & i \end{vmatrix}}_{\text{cost } 3} + c \underbrace{\begin{vmatrix} d & e \\ g & h \end{vmatrix}}_{\text{cos } 3}$$

$\text{Cost}(|A|) = 14.$

Example : Solve the following system then find the

Cost .
$$3x_1 + 2x_2 + 4x_3 = 9$$
$$4x_2 + 6x_3 = 10$$
$$10x_3 = 10$$

$\Rightarrow \quad x_3 = 1 \quad , x_2 = 1 \quad , x_1 = 1$

Cost :

Step	$+ , -$	$\times , \div$
1	—	1
2	1	2
3	2	3
Σ	3	6

$\therefore$ Total Cost $= 9 = (3)^2$.

For $n \times n$ System: The Cost is Calculated as:

step	$+, -$	$\times, \div$
1	—	1
2	1	2
$\vdots$	$\vdots$	$\vdots$
k	$k-1$	k
$\vdots$	$\vdots$	$\vdots$
n	$n-1$	n

$$\therefore \quad \text{Cost}\,(+,-) = \sum_{k=1}^{n}(k-1) = \sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$= \frac{n(n+1)}{2} - n = \frac{n^2-n}{2}$$

$$\text{Cost}\,(\times,\div) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

B.S

$$\therefore \text{Total Cost}\,(+,-,\times,\div) = \frac{n^2-n}{2} + \frac{n^2+n}{2} = \frac{2n^2}{2} = \boxed{n^2}$$

Note: The Cost for B.S = Cost for F.S = $\boxed{n^2}$

3.4 Gaussian Elimination and Pivoting

$$AX = B$$

$$[A \mid B] \longrightarrow \underbrace{[\overset{\text{upper triangular}}{U} \mid C]}_{\text{Reducing}} + B.S$$

We can use Row operations:

1) Multiply a row by nonzero scalar

2) Switch any two rows

3) Replace any row by adding to it a nonzero multiple by another row. (i-e)

$$\text{row } r_{new} = \text{row } r_{old} - m_{rp} \text{ row } p$$

where $m_{rp} = \dfrac{a_{rp}}{a_{pp}}$, $r > p$.

Example:

$$x_1 + 2x_2 + x_3 + 4x_4 = 13.$$

$$2x_1 + 0x_2 + 4x_3 + 3x_4 = 28.$$

$$4x_1 + 2x_2 + 2x_3 + x_4 = 20.$$

$$-3x_1 + x_2 + 3x_3 + 2x_4 = 6.$$

$$\begin{array}{r} \text{Pivot} \rightarrow \\[6pt] m_{21} = \frac{2}{1} \\[6pt] m_{31} = \frac{4}{1} \\[6pt] m_{41} = \frac{-3}{1} \end{array}
\left[\begin{array}{cccc|c}
1 & 2 & 1 & 4 & 13 \\[6pt]
2 & 0 & 4 & 3 & 28 \\[6pt]
4 & 2 & 2 & 1 & 20 \\[6pt]
-3 & 1 & 3 & 2 & 6
\end{array}\right]$$

For Finding the 3 multipliers, we have 3 divisions

$$\begin{array}{r} \\[6pt] \text{Pivot} \rightarrow \\[6pt] m_{32} = \frac{-6}{-4} \\[6pt] m_{42} = \frac{7}{-4} \end{array}
\left[\begin{array}{cccc|c}
1 & 2 & 1 & 4 & 13 \\[6pt]
0 & -4 & 2 & -5 & 2 \\[6pt]
0 & -6 & -2 & -15 & -32 \\[6pt]
0 & 7 & 6 & 14 & 45
\end{array}\right]$$

$$\begin{array}{r} \\[6pt] \\[6pt] \text{Pivot} \rightarrow \\[6pt] m_{43} = \frac{9.5}{-5} \end{array}
\left[\begin{array}{cccc|c}
1 & 2 & 1 & 4 & 13 \\[6pt]
0 & -4 & 2 & -5 & 2 \\[6pt]
0 & 0 & -5 & -7.5 & -35 \\[6pt]
0 & 0 & 9.5 & 5.25 & 48.5
\end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 0 & -9 & -18 \end{array}\right]$$

Now B. S $\Rightarrow$ $x_4 = 2$, $x_3 = 4$, $x_2 = -1$, $x_1 = 3$.

The Cost for Reducing

$[A|B] \longrightarrow [U|C]$

is

Step	$+,-$	$\times, \div$
1	4×3	$(4\times 3)+3$
2	3×2	$(3\times 2)+2$
3	2×1	$(2\times 1)+1$
Σ	20	26

Division of Multipliers

$\therefore$ Cost $(+, -, \times, \div)$ for Reducing $= 46$.

$\therefore$ Total Cost for Solving (4x4 system) $= 46 + (4)^2$

$= \boxed{62}$

Reducing + B. S

In General, the Cost for Reducing the Augmented matrix into an upper triangular form is

Step	$+, -$	$\times, \div$
1	$(n-1)\times n$	$(n-1)(n)+(n-1)$
2	$(n-2)\times(n-1)$	$(n-2)(n-1)+(n-2)$
3	$(n-3)\times(n-2)$	$(n-3)(n-2)+(n-3)$
$\vdots$	$\vdots$	$\vdots$
k	$(n-k)(n-k+1)$	$(n-k)(n-k+1)+(n-k)$
$\vdots$	$\vdots$	$\vdots$
$n-1$	1×2	$1\times 2+1$

$$\text{Cost}\,(+,-) = \sum_{k=1}^{n-1}(n-k)(n-k+1)$$

$$= \sum_{k=1}^{n-1}\left(n^2-nk+n-kn+k^2-k\right)$$

$$= \sum_{k=1}^{n-1}(n^2+n) - 2n\sum_{k=1}^{n-1}k + \sum_{k=1}^{n-1}k^2 - \sum_{k=1}^{n-1}k$$

$$= (n^2+n)\sum_{k=1}^{n-1}1 - (2n+1)\sum_{k=1}^{n-1}k + \sum_{k=1}^{n-1}k^2$$

$$= (n^2+n)(n-1) - (2n+1)\left(\frac{(n-1)n}{2}\right) + \frac{(n-1)(n)(2(n-1)+1)}{6}$$

$$= \frac{n^3 - n}{3}$$

Now, Cost for $(\times, \div) = \sum_{k=1}^{n-1} (n-k)(n-k+1) + (n-k)$

$$= \frac{n^3 - n}{3} + \sum_{k=1}^{n-1} n - \sum_{k=1}^{n-1} k$$

$$= \frac{n^3 - n}{3} + n(n-1) - \frac{(n-1)n}{2}$$

$$= \frac{1}{3} n^3 + \frac{n^2}{2} - \frac{5}{6} n$$

$\therefore$ Total Cost for Reducing $(+, -, \times, \div) =$

$$= \frac{n^3 - n}{3} + \left(\frac{1}{3} n^3 + \frac{n^2}{2} - \frac{5}{6} n\right)$$

$$= \frac{2}{3} n^3 + \frac{n^2}{2} - \frac{7}{6} n \quad \approx \frac{2}{3} n^3$$

$\therefore$ Total Cost for Solving $n \times n$ System Using Gaussian Elimination is :

$$\text{Cost} = \frac{2}{3} n^3 + \frac{n^2}{2} - \frac{7}{6} n + \overset{B.S}{n^2} = \boxed{\frac{2}{3} n^3 + \frac{3}{2} n^2 - \frac{7}{6} n}$$

Pivoting to Avoid $a_{pp}^{(p)} = 0$

If $a_{pp}^{(p)} = 0$, we can't use row p to eliminate the elements in Column p below the main diagonal. So, we find row k, where $a_{kp}^{(p)} \neq 0$ and $k > p$, and then interchange row p and row k, so that a non zero pivot element is obtained.

This criteria is called trivial pivoting

Pivoting is Used to Reduce the Error:

Partial pivoting: We choose the pivot element to be the Largest (In magnitude) in the Remaining Column P (on the diagonal and Below), then we locate row k that is:

$$|a_{kp}| = \max\{|a_{pp}|, |a_{p+1\,p}|, \ldots, |a_{np}|\}$$

then we change row p with row k.

Example: Solve the following system Using Gaussian elimination with partial pivoting and four digits rounded.

$$1.133\,x_1 + 5.281\,x_2 = 6.414$$
$$24.14\,x_1 - 1.210\,x_2 = 22.93$$

(Exact $x_1 = 1$, $x_2 = 1$)

Sol:
$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 24.14 & -1.210 & 22.93 \end{array}\right]$$

To use partial pivoting, we change the two rows:
$$\left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 1.133 & 5.281 & 6.414 \end{array}\right]$$

$$\Rightarrow m_{21} = \frac{1.133}{24.14} \approx 0.04693$$

$$\Rightarrow \left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 0 & 5.338 & 5.338 \end{array}\right]$$

$$\Rightarrow x_2 = 1 \quad \& \quad x_1 = 1.$$

Note: If we solve the system (without changing rows) we will get $x_1 = 0.9956$ & $x_2 = 1.001$. (Error bigger).

(2) Gauss - Jordan Elimination: $[A|B] \longrightarrow [I|X]$

Example:

$$3x_1 + 2x_2 + 4x_3 = 9$$
$$x_1 - 2x_2 + 3x_3 = 2$$
$$3x_1 + 4x_2 - x_3 = 6$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 9 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array}\right] \xrightarrow[\div 3]{\text{first row}} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{4}{3} & 3 \\ 1 & -2 & 3 & 2 \\ 3 & 4 & -1 & 6 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2/3 & \frac{4}{3} & 3 \\ 0 & -4 & 5/3 & -1 \\ 0 & 2 & -5 & -3 \end{array}\right] \xrightarrow[\div -4]{\text{second row}} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{4}{3} & 3 \\ 0 & 1 & \frac{5}{-12} & \frac{-1}{4} \\ 0 & 2 & -5 & -3 \end{array}\right]$$

$\rightarrow$ Continue ...

Step	$+, -$	$\times, \div$
1	3×2	$3 \times 2 + 3$
2	2×2	$2 \times 2 + 2$
3	1×2	$1 \times 2 + 1$

Division for Making $a_{11}^{(1)} = 1$

Division for Making $a_{22}^{(2)} = 1$

$\therefore$ Total Cost for Solving 3×3 System Using Gauss-Jordan elimination $= 30$

In General, The Cost for Solving $n \times n$ Using Gauss-Jordan elimination is equal

$$\text{Cost} = n^3 + \frac{n^2}{2} - \frac{n}{2}$$

Step	$+, -$	$\times, \div$
1	$n(n-1)$	$n(n-1)+n$
2	$(n-1)(n-1)$	$(n-1)(n-1)+(n-1)$
$\vdots$	$\vdots$	$\vdots$
k	$(n-k+1)(n-1)$	$(n-k+1)(n-1)+(n-k+1)$
$\vdots$	$\vdots$	$\vdots$
n	$(1)(n-1)$	$(1)(n-1)+1$

$$\text{Cost} = \sum_{k=1}^{n} (n-k+1)(n-1) + \sum_{k=1}^{n} \left[(n-k+1)(n-1)+(n-k+1)\right]$$

$$= \sum_{k=1}^{n} \left[2(n-k+1)(n-1) + (n-k+1)\right]$$

$$= (2n-1)\sum_{k=1}^{n}(n-k+1) = n^3 + \frac{n^2}{2} - \frac{n}{2}.$$

(3) Inverse Method : $AX = B \Rightarrow X = A^{-1}B$

How to find A^{-1} ?

$$[A \mid I] \longrightarrow [I \mid A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ \underbrace{a_{31} \quad a_{32} \quad a_{33}}_{A} & & & \underbrace{0 \quad 0 \quad 1}_{I} & & \end{array}\right] \longrightarrow \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

Step	+, −	×, ÷
1	5×2	5×2 + 5
2	4×2	4×2 + 4
3	3×2	3×2 + 3
Σ	24	36

→ Table for finding the Cost of A^{-1}

$\therefore$ Cost for finding A^{-1} = $\underline{60}$

Now, Cost for $(A^{-1}b)$ = 3 (3 multiplication + 2 addition)
= $\underline{15}$

$\therefore$ Total Cost for Solving 3×3 system Using Inverse Method = $\boxed{75}$

In General, the Cost for finding A^{-1} for $n \times n$ matrix $= 3n^3 - \frac{5}{2}n^2 + \frac{n}{2}$.

step	$+, -$	$\times, \div$
1	$(2n-1)(n-1)$	$(2n-1)(n-1)+(2n-1)$
2	$(2n-2)(n-1)$	$(2n-2)(n-1)+(2n-2)$
$\vdots$	$\vdots$	$\vdots$
k	$(2n-k)(n-1)$	$(2n-k)(n-1)+(2n-k)$
$\vdots$	$\vdots$	$\vdots$
n	$(2n-\overset{n}{n})(n-1)$	$n(n-1)+n$

$\longrightarrow$ Table for finding the cost of A^{-1}

Now $\text{Cost}(A^{-1}) = \sum_{k=1}^{n} \left[2(2n-k)(n-1) + (2n-k) \right]$

$= 3n^3 - \frac{5}{2}n^2 + \frac{n}{2}$

$\text{Cost}(A^{-1}b) = n\left((n)\text{multip.} + (n-1)\text{ addition}\right)$

$= n(2n-1)$

$\therefore$ Total Cost for Solving $n \times n$ System Using Inverse Method $= 3n^3 - \frac{n^2}{2} - \frac{n}{2}$

4) Cramer's Method: $AX = B$

$$x_i = \frac{|A_i|}{|A|}, \quad |A| \neq 0, \quad i = 1, 2, \ldots, n$$

Where A_i is the matrix obtained by removing the Column i and Replace it by Column B

Case 1: If we assume we have 2×2 system.

then $\text{Cost}(|A|) = \text{Cost}(a\times d - c\times d) = 3$ $\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

So the Cost for Solving 2×2 system Using Cramer's Method $= 3D_2 + 2$ division

$$= 3(3) + 2 = 11$$

$D_2 = \text{Cost of } |A_{2\times2}|$

Case 2: If we have 3×3 system, then

$$\text{Cost}(|A|) = 3D_2 + 3 \text{ multip.} + 2 \text{ addition}$$
$$= 3(3) + 3 + 2 = 14$$

$\therefore$ The Cost for Solving 3×3 system Using Cramer's Method $= 4D_3 + 3$ division

$$= 4(14) + 3 = 59$$

where $D_3 = \text{Cost of } |A_{3\times3}|$

In General : The Cost of the one determinant

of $A_{n \times n}$ is

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$= [n!\,e - 2]$ ← (Greatest Integer function)

مثال: $[11.7] = 11$

$$= \underbrace{n! \sum_{k=1}^{n-1} \frac{1}{k!}}_{\text{cost}(\times, \div)} + \underbrace{(n! - 1)}_{\text{cost}(+, -)}$$

Therefore, the total Cost of Solving $n \times n$ System

Using Cramer's Method =

$= (n+1) D_n + n$ → # divisions

where $D_n = \text{cost}(|A_{n \times n}|)$.

3.5 Triangular Factorization (L U factorization)

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}}$$

Example:

$$x_1 + 2x_2 + 4x_3 + x_4 = 21$$
$$2x_1 + 8x_2 + 6x_3 + 4x_4 = 52$$
$$3x_1 + 10x_2 + 8x_3 + 8x_4 = 79$$
$$4x_1 + 12x_2 + 10x_3 + 6x_4 = 82$$

(*)...

$$\underset{A}{\begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 8 & 6 & 4 \\ 3 & 10 & 8 & 8 \\ 4 & 12 & 10 & 6 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 4 & 1 & 2 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 4 & -2 & 2 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix}}$$

Let $LY = B$ $\Rightarrow$

F.S

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 4 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 52 \\ 79 \\ 82 \end{bmatrix}$$

$$\Rightarrow \boxed{y_1 = 21}, \boxed{y_2 = 10}, \boxed{y_3 = 6}, \boxed{y_4 = -24}$$

Now, Solve $UX \overset{\overset{B.S}{\uparrow}}{=} Y \Longrightarrow$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 4 & -2 & 2 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 10 \\ 6 \\ -24 \end{bmatrix}$$

$$\Rightarrow x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad x_4 = 4$$

Note: In $(*)$, $m_{21} = 2$, $m_{31} = 3$, $m_{41} = 4$

The Cost in the previous Example is obtained by

$$\text{Cost}(L) + \text{Cost}(U) + \text{Cost}(F.S) + \text{Cost}(B.S)$$

$$= 0 + \text{Cost}(U) + (n^2 - \underset{\substack{\uparrow \\ \text{Diagonal}=1}}{n}) + n^2$$

$$= \text{Cost}(U) + 2n^2 - n$$

$$= \text{Cost}(U) + 2(4)^2 - 4$$

$$= \text{Cost}(U) + 28$$

Need the Cost (U):

The Cost for factoring $A_{4\times4}$ into LU factorization

Step	+, −	×, ÷
1	3×3	3×3 + 3
2	2×2	2×2 + 2
3	1×1	1×1 + 1
Σ	14	20

$\therefore$ Cost (U) = 34

$\therefore$ Total Cost for Solving $A_{4\times4}$ Using LU factorization

$$= 34 + 28 = \boxed{62}$$

In General, The Cost for Solving $A_{n\times n}$

Using LU factorization =

$$= \text{Cost}(L) + \text{Cost}(U) + \text{Cost}(B.S) + \text{Cost}(F.S)$$

$$= 0 + ?? + n^2 + (n^2 - n)$$

Cost (U) :

Step	$+, -$	$\times, \div$
1	$(n-1)\times(n-1)$	$(n-1)(n-1)+(n-1)$
2	$(n-2)(n-2)$	$(n-2)(n-2)+(n-2)$
$\vdots$	$\vdots$	$\vdots$
k	$(n-k)(n-k)$	$(n-k)(n-k)+(n-k)$
$\vdots$	$\vdots$	$\vdots$
$n-1$	1×1	$1\times 1+1$

$$\therefore \text{ Cost for } (U) = \sum_{k=1}^{n-1}(n-k)(n-k) + \sum_{k=1}^{n-1}\left[(n-k)(n-k)+(n-k)\right]$$

$$= \frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n.$$

$\therefore$ Total Cost for Solving $A_{n\times n}$ Using LU factorization

$$= 0 + \left(\frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n\right) + (n^2) + (n^2-n)$$

$$= \frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$$

Summary of the Costs for Solving Linear System:

1) Gaussian Elimination : $\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$

2) Gauss-Jordan Elimination : $n^3 + \frac{n^2}{2} - \frac{n}{2}$

3) Inverse Method : $3n^3 - \frac{n^2}{2} - \frac{n}{2}$

4) Cramer's Method : $(n+1)[n!\,e - 2] + n$

5) LU factorization : $\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$

So, we conclude that the cost for solving $A_{n \times n}$ system using Gaussian elimination = LU factorization

$\therefore$ Gaussian = LU fact. Less Cost than Gauss-Jordan Less Cost than Inverse Method Less Cost than Cramer's Method.

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↑

$\Rightarrow$ Gaussian = LU fact. < Gauss-Jordan < Inverse < Cramer's.

Example: Let A be 4×4 matrix and b_1, b_2 are 4×1 vectors. Find the Cost of solving the linear systems $Ax = b_1$ and $Ax = b_2$ Using LU factorization

Sol: $\text{Cost}(LU) = \frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n \Big|_{n=4} = 34$

Now, $\text{Cost}(Ax = b_1) = \text{Cost}(LU) + \text{Cost}(B.S) + \text{Cost}(F.S)$

$= 34 + (4)^2 + ((4)^2 - 4)$

$= 62$

Now, $\text{Cost}(Ax = b_1) = \text{Cost}(B.S) + \text{Cost}(F.S)$

$= (4)^2 + (4^2 - 4)$

$= 28$ ---(*)

$\therefore$ Total Cost $(Ax = b_1)$ & $(Ax = b_2)$ =

$= 62 + 28 = 90$

Note: In (*), we didn't Compute the Cost of (LU) again, since we compute it in the Cost $(Ax = b_1)$.

Summary for some special sums:

(1) $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(2) $$\sum_{k=1}^{n-1} k = \frac{(n-1)(n)}{2}$$

(3) $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

(4) $$\sum_{k=1}^{n-1} k^2 = \frac{(n-1)(n)(2n-1)}{6}$$

Remark: Let A, B be $n\times n$ matrices, b an $n\times 1$ vec and $\alpha \in \mathbb{R}$, P is a positive integer.

1) $\text{Cost}(A \pm B) = n^2$.

2) $\text{Cost}(\alpha A) = n^2$.

3) $\text{Cost}(AB) = 2n^3 - n^2$.

4) $\text{Cost}(Ab) = 2n^2 - n$.

5) $\text{Cost}(A^P) = (p-1)(2n^3 - n^2)$.

3.6 Iterative Methods for Linear Systems.

(1) Jacobi Iteration.

$f_1(x,y) = g_1(x,y) - x = 0$

$f_2(x,y) = g_2(x,y) - y = 0$

Consider the Linear system:

$$\left.\begin{aligned} f_1(x,y) &= 0 \\ f_2(x,y) &= 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} g_1(x,y) &= x \\ g_2(x,y) &= y \end{aligned}$$

Def: A point (p,q) is a fixed point of the system

$$\left.\begin{aligned} x &= g_1(x,y) \\ y &= g_2(x,y) \end{aligned}\right\} \text{ if } \left\{\begin{aligned} p &= g_1(p,q) \\ q &= g_2(p,q) \end{aligned}\right.$$

Similarly, A point (p,q,r) is a fixed point of the system:

$$\left.\begin{aligned} x &= g_1(x,y,z) \\ y &= g_2(x,y,z) \\ z &= g_3(x,y,z) \end{aligned}\right\} \text{ if } \left\{\begin{aligned} p &= g_1(p,q,r). \\ q &= g_2(p,q,r). \\ r &= g_3(p,q,r). \end{aligned}\right.$$

Such that $\left\{\begin{aligned} f_1(x,y,z) &= g_1(x,y,z) - x = 0 \\ f_2(x,y,z) &= g_2(x,y,z) - y = 0 \\ f_3(x,y,z) &= g_3(x,y,z) - z = 0 \end{aligned}\right.$

Example: Consider the Linear system of equations

$$4x - y + z = 7$$
$$4x - 8y + z = -21$$
$$-2x + y + 5z = 15.$$

(Exact Sol. (2, 4, 3))

These equations can be written in the form

$$x = \frac{7 + y - z}{4} = g_1(x, y, z)$$

$$y = \frac{21 + 4x + z}{8} = g_2(x, y, z)$$

$$z = \frac{15 + 2x - y}{5} = g_3(x, y, z)$$

Jacobi iteration is defined as:

$$x_{k+1} = \frac{7 + y_k - z_k}{4}, \quad \text{Let } (x_k, y_k, z_k) = (p_k, q_k, r_k)$$

$$y_{k+1} = \frac{21 + 4x_k + z_k}{8}$$

$$z_{k+1} = \frac{15 + 2x_k - y_k}{5}.$$

starting with $(p_0, q_0, r_0) = (1, 2, 2) \Rightarrow (p_1, q_1, r_1) = (1.75, 3.375, 3)$

After 19 Iterations, the Iteration Converges to $(2, 4, 3)$

k	p_k	q_k	r_k
0	1	2	2
1	1.75	3.375	3
⋮	⋮	⋮	⋮
15	1.99999993	3.99999985	2.99999993
⋮	⋮	⋮	⋮
19	2.000000000	4.000000000	3.000000000

Remark: Sometimes the Jacobi Method does Not work.

Example:

$$-2x + y + 5z = 15$$
$$4x - 8y + z = -21$$
$$4x - y + z = 7$$

(rearrange the previous equations)

$$\Rightarrow \quad x = \frac{-15 + y + 5z}{2}, \quad y = \frac{21 + 4x + z}{8}, \quad z = 7 - 4x + y. \quad \Rightarrow$$

Starting with $(p_0, q_0, r_0) = (1, 2, 2)$ then, the iteration will diverge away from $(2, 4, 3)$.

(2) Gauss-Seidel Iteration.

Assume :

$$\left.\begin{aligned} f_1(x,y,z) &= 0 \\ f_2(x,y,z) &= 0 \\ f_3(x,y,z) &= 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} x &= g_1(x,y,z) \\ y &= g_2(x,y,z) \\ z &= g_3(x,y,z) \end{aligned}$$

Starting with (p_0, q_0, r_0) , then:

$$p_1 = g_1(p_0, q_0, r_0)$$

$$q_1 = g_2(p_1, q_0, r_0)$$

$$r_1 = g_3(p_1, q_1, r_0)$$

In General:

$$p_{n+1} = g_1(p_n, q_n, r_n)$$

$$q_{n+1} = g_2(\underbrace{p_{n+1}}, q_n, r_n)$$

$$r_{n+1} = g_3(\underbrace{p_{n+1}}, \underbrace{q_{n+1}}, r_n).$$

Remark: Gauss-Seidel iteration is faster than Jacobi Iteration.

Example:

$$\left.\begin{aligned} x &= \frac{7+y-z}{4} \\ y &= \frac{21+4x+z}{8} \\ z &= \frac{15+2x-y}{5} \end{aligned}\right\} \Rightarrow \begin{aligned} p_{k+1} &= \frac{7+q_k-r_k}{4} \\ q_{k+1} &= \frac{21+4p_{k+1}+r_k}{8} \\ r_{k+1} &= \frac{15+2p_{k+1}-q_{k+1}}{5} \end{aligned}$$

Starting with $(p_0, q_0, r_0) = (1, 2, 2)$, then:

k	p_k	q_k	r_k
0	1	2	2
1	1.75	3.75	2.95
2	1.95	3.96875	2.98625
⋮	⋮	⋮	⋮
8	1.99999983	3.99999988	2.99999996
9	1.99999998	3.99999999	3.00000000
10	2.00000000	4.00000000	3.00000000

Def: A matrix $A_{n\times n}$ is said to be strictly diagonally dominant provided that

$$|a_{kk}| > \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}| \qquad \text{for } k = 1, 2, \ldots, n.$$

Example: $\begin{bmatrix} 4 & -1 & 1 \\ 4 & -8 & 1 \\ -2 & 1 & 5 \end{bmatrix}$ is strictly diagonally dominant, since

$|a_{11}| > |a_{12}| + |a_{13}|$

$|a_{22}| > |a_{21}| + |a_{23}|$

$|a_{33}| > |a_{31}| + |a_{33}|$

$|4| > |-1| + |1| = 2$

$|-8| > |4| + |1| = 5$

$|5| > |-2| + |1| = 3$

Theorem: (Jacobi Iteration and Gauss_Seidel method).

Suppose that A is strictly diagonally dominant.

Then $AX = B$ has a Unique solution $X = P$.

That is the sequence generated by Jacobi iteration or Gauss_Seidel iteration will converge to P for any choice of the Initial vector P_0.

3.7 Iteration for Nonlinear Systems :

(1) Fixed Point Iteration.

Recall : A point (p, q) is a fixed point of the system:

$$x = g_1(x, y) \qquad \text{if} \qquad p = g_1(p, q)$$
$$y = g_2(x, y) \qquad\qquad q = g_2(p, q)$$

Recall : $f_1(x, y) = g_1(x, y) - x = 0$

$f_2(x, y) = g_2(x, y) - y = 0$

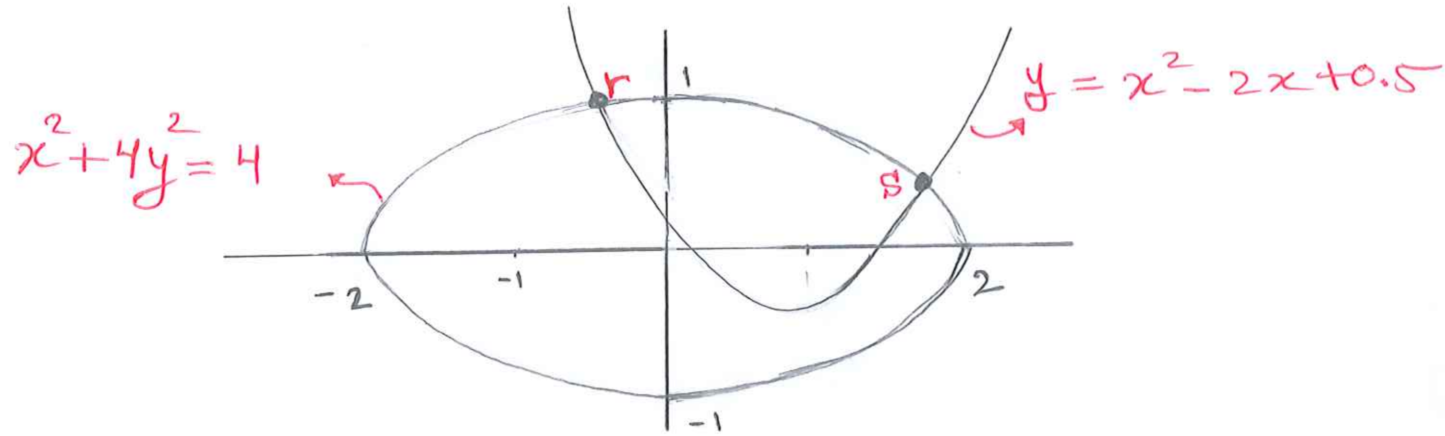
Fixed point Iteration : If we start with (p_0, q_0)

then
$$p_1 = g_1(p_0, q_0)$$
$$q_1 = g_2(p_0, q_0)$$
$$\vdots$$
$$p_{n+1} = g_1(p_n, q_n)$$
$$q_{n+1} = g_2(p_n, q_n)$$

Note: Fixed point Iteration for Linear System is called Jacobi iteration.

Example: $f_1(x,y) = x^2 - 2x - y + 0.5 = 0.$

$f_2(x,y) = x^2 + 4y^2 - 4. = 0.$



The solution is the points of intersection that are around $(-0.2, 1)$ and $(1.9, 0.3)$.

Let

(***) ...
$$\begin{cases} g_1(x,y) = \dfrac{x^2 - y + 0.5}{2} \\[2ex] g_2(x,y) = \dfrac{-x^2 - 4y^2 + 8y + 4}{8} \end{cases}$$

(adding $-8y$ for the two sides)

Start with $(p_0, q_0) = (0, 1)$, then:

$$p_1 = g_1(0,1) = \frac{0^2 - 1 + 0.5}{2} = -0.25$$

$$q_1 = g_2(0,1) = \frac{-0^2 - 4(1)^2 + 8(1) + 4}{8} = 1$$

See table (1)

k	p_k	q_k
0	0	1
1	-0.25	1
2	-0.21875	0.9921875
3	-0.2221680	0.9939880
4	-0.2223147	0.9938121
5	-0.2221941	0.9938029
6	-0.2222163	0.9938095
7	-0.2222147	0.9938083
8	-0.2222145	0.9938084
9	-0.2222146	0.9938084

Table (1)

k	p_k	q_k
0	2	0
1	2.25	0
2	2.78125	-0.1328125
3	4.184082	-0.6085510
4	9.307547	-2.4820360
5	44.80623	-15.891091
6	1011.995	-392.60426
7	512263.2	-205477.8

Table (2)

In table (1), We see that the sequence generated by F.P.I converges to $(-0.2, 1)$

While in table (2), if we start with $(p_0, q_0) = (2, 0)$ The sequence generated by F.P.I diverges.

Example: $f_1(x,y) = x^2 - 2x - y + 0.5 = 0$ $\quad [-2x]$ adding

$f_2(x,y) = x^2 + 4y^2 - 4 = 0$ $\quad [-11y]$

$$g_1(x,y) = \frac{-x^2 + 4x + y - 0.5}{2} = x$$

$$g_2(x,y) = \frac{-x^2 - 4y^2 + 11y + 4}{11} = y$$

Again start with $(p_0, q_0) = (2, 0)$:

k	p_k	q_k
0	2	0
1	1.75	0
⋮	⋮	⋮
12	1.900924	0.3112267
⋮	⋮	⋮
24	1.900677	0.3112186

$\Rightarrow$ The sequence generated by F. P. I converges to $(1.9, 0.3)$.

Theorem: (Fixed - Point Iteration).

Assume that the functions $g_1(x,y), g_2(x,y)$

$(\text{similarly}, \ g_1(x,y,z), g_2(x,y,z), g_3(x,y,z))$

and their partial derivatives are Continuous on a region that Contains the fixed point (p,q) (p,q,r), resp. . If the starting point is chosen sufficiently close to the fixed point, then:

1) If (p_0, q_0) is sufficiently close to (p,q) and

if $\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| < 1$ at $(x,y) = (p,q)$

and $\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| < 1$ at $(x,y) = (p,q)$

then the Iteration converges to the fixed point (p,q).

2) If (p_0, q_0, r_0) is sufficiently close to (p,q,r)

and if

$$\left.\begin{aligned} \left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| + \left|\frac{\partial g_1}{\partial z}\right| &< 1 \\ \left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| + \left|\frac{\partial g_2}{\partial z}\right| &< 1 \\ \left|\frac{\partial g_3}{\partial x}\right| + \left|\frac{\partial g_3}{\partial y}\right| + \left|\frac{\partial g_3}{\partial z}\right| &< 1 \end{aligned}\right\} \text{ at } (p,q,r)$$

Then the Iteration Converges to (p,q,r).

Remark: The previous theorem can be Used to show why the iteration (***) Converged to the fixed point near $(-0.2, 1)$.

$$g_1(x,y) = \frac{x^2 - y + 0.5}{2} \quad , \quad g_2(x,y) = \frac{-x^2 - 4y^2 + 8y + 4}{8}$$

$$\frac{\partial g_1}{\partial x} = x \quad \text{and} \quad \frac{\partial g_1}{\partial y} = -\frac{1}{2}$$

$$\frac{\partial g_2}{\partial x} = -\frac{x}{4} \quad \text{and} \quad \frac{\partial g_2}{\partial y} = -y+1$$

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| < 1 \Rightarrow |x| + \left|-\frac{1}{2}\right| < 1$$

$$\Rightarrow -0.5 < x < 0.5$$

Now, $\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| < 1 \Rightarrow \left|-\frac{x}{4}\right| + |-y+1| < 1$

substitute $|x| < 0.5 \Rightarrow$

$$\left|-\frac{x}{4}\right| + |-y+1| < \frac{1}{8} + |-y+1| < 1$$

$$\Rightarrow \quad |-y+1| < \frac{7}{8} \Rightarrow 0.125 < y < 1.875$$

$\Rightarrow$ The Iteration will converge to $(-0.2, 1)$

But the Convergence is Not guaranteed for $(1.9, 0.3)$

(2) Seidel Iteration

$$p_{k+1} = g_1(p_k, q_k, r_k)$$

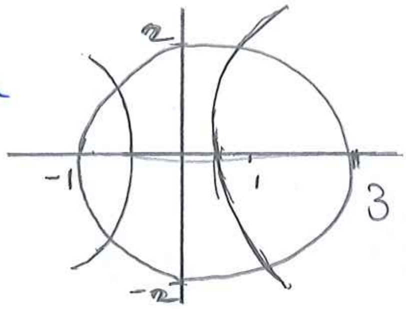
$$q_{k+1} = g_2(\underline{p_{k+1}}, q_k, r_k)$$

$$r_{k+1} = g_3(\underline{p_{k+1}}, \underline{q_{k+1}}, r_k)$$

Example: $x = g_1(x, y) = \dfrac{8x - 4x^2 + y^2 + 1}{8}$. (hyperbola)

$y = g_2(x, y) = \dfrac{2x - x^2 + 4y - y^2 + 3}{4}$. (circle)

Starting with $(p_0, q_0) = (1.1, 2)$, then



1) Fixed point iteration:

$(p_1, q_1) = (g_1(1.1, 2), g_2(1.1, 2)) = (1.12, 1.9975)$

$(p_2, q_2) = (1.1166, 1.9964)$, $(p_3, q_3) = (1.1164, 1.9966)$

2) Seidel Iteration: $(p_1, q_1) = (g_1(p_0, q_0), g_2(p_1, q_0))$ p_1

$\Rightarrow (p_1, q_1) = (1.12, 1.9964)$

$(p_2, q_2) = (g_1(p_1, q_1), g_2(p_2, q_1)) = (1.1160, 1.9966)$ p_2

$(p_3, q_3) = (g_1(p_2, q_2), g_2(p_3, q_2)) = (1.1160, 1.9966)$ p_3

Newton's Method for Nonlinear Systems.

Def: Assume that $f_1(x,y)$ and $f_2(x,y)$ are functions of the independent variables x and y then their Jacobian matrix $J(x,y)$ is

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

Similarly, if $f_1(x,y,z)$ and $f_2(x,y,z)$ and $f_3(x,y,z)$ are functions of the independent variables x, y and z then their Jacobian matrix $J(x,y,z)$ is

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}$$

Example: Find the Jacobian matrix $J(x, y, z)$ of order 3×3 at the point $(1,3,2)$ for:

$$f_1(x,y,z) = x^3 - y^2 + y - z^4 + z^2$$

$$f_2(x,y,z) = xy + yz + xz$$

$$f_3(x,y,z) = \frac{y}{xz}.$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} = \begin{bmatrix} 3x^2 & -2y+1 & -4z^3+2z \\ y+z & x+z & y+x \\ \frac{-y}{x^2 z} & \frac{1}{xz} & \frac{-y}{xz^2} \end{bmatrix}$$

$$\Rightarrow \quad J(1,3,2) = \begin{bmatrix} 3 & -5 & -28 \\ +5 & 3 & 4 \\ -\frac{3}{2} & \frac{1}{2} & -\frac{3}{4} \end{bmatrix}$$

Assume $\left.\begin{aligned} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{aligned}\right\} \Rightarrow$ Starting with (p_0, q_0) which is close to (p, q) We need to construct Iteration.

We can find (p_1, q_1) Using Direct Newton Method :

$$-\begin{bmatrix} f_1(p_0, q_0) \\ f_2(p_0, q_0) \end{bmatrix}_{2\times 1} = J_{2\times 2}\Big|_{(p_0,q_0)} \cdot \begin{bmatrix} (p_1 - p_0) \\ (q_1 - q_0) \end{bmatrix}_{2\times 1}$$

, Let $\Delta p = p_1 - p_0$

$\Delta q = q_1 - q_0$

In General : To find (p_{n+1}, q_{n+1}) Using Direct Newton Method:

$$-\begin{bmatrix} f_1(p_n, q_n) \\ f_2(p_n, q_n) \end{bmatrix}_{2\times 1} = J_{2\times 2}\Big|_{(p_n,q_n)} \cdot \begin{bmatrix} (p_{n+1} - p_n) \\ (q_{n+1} - q_n) \end{bmatrix}_{2\times 1}$$

Which is equivelant to the Inverse Newton Method.

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix}_{2\times 1} = \begin{bmatrix} p_n \\ q_n \end{bmatrix}_{2\times 1} - J^{-1}_{2\times 2}\Big|_{(p_n,q_n)} \cdot \begin{bmatrix} f_1(p_n, q_n) \\ f_2(p_n, q_n) \end{bmatrix}_{2\times 1}$$

Example: Solve the following system Using Newton's Method

$$f_1(x,y) = x^2 - 2x - y + 0.5$$

$$f_2(x,y) = x^2 + 4y^2 - 4$$

Starting with $(p_0, q_0) = (2, 0.25)$.

$$f_1(2, 0.25) = (2)^2 - 2(2) - (0.25) + 0.5 = 0.25$$

$$f_2(2, 0.25) = (2)^2 + 4(0.25)^2 - 4 = 0.25$$

$$J = \begin{bmatrix} 2x-2 & -1 \\ 2x & 8y \end{bmatrix} \Rightarrow J\Big|_{(2,0.25)} = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$$

Direct:

$$-\begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta q \end{bmatrix}$$

$\Delta p = p_1 - p_0$

$\Delta q = q_1 - q_0$

Using Cramer's:

$$\Delta p = \frac{\begin{vmatrix} -0.25 & -1 \\ -0.25 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}} = \frac{-0.75}{8} = -0.09375$$

$$\Rightarrow -0.09375 = p_1 - 2 \Rightarrow p_1 = 1.90625$$

$$\Delta q = \frac{\begin{vmatrix} 2 & -0.25 \\ 4 & -0.25 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}} = \frac{0.5}{8} = 0.0625$$

$\Rightarrow \quad 0.0625 = q_1 - 0.25 \quad \Rightarrow \quad q_1 = 0.3125$

$\Rightarrow \quad (p_1, q_1) = (1.90625, 0.3125).$

We can find (p_2, q_2) by substituting (p_1, q_1) in place of (p_0, q_0) $\Rightarrow$ $(p_2, q_2) = (1.900691, 0.311213)$.

or Inverse: $\quad J^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.125 \\ -0.5 & 0.25 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0.25 \end{bmatrix} - \begin{bmatrix} 0.25 & 0.125 \\ -0.5 & 0.25 \end{bmatrix}\begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0.25 \end{bmatrix} - \begin{bmatrix} 0.09375 \\ -0.0625 \end{bmatrix} = \begin{bmatrix} 1.90625 \\ 0.3125 \end{bmatrix}$$

$\Rightarrow \quad (p_1, q_1) = (1.90625, 0.3125)$

"End of Chapter 3"