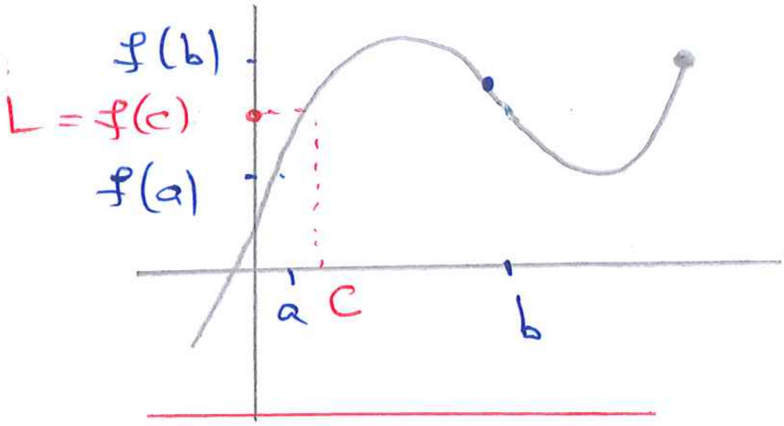
Calculator: Radian Mode

Numerical Methods "Math 330"

1.1 Review of Calculus:

Theorem: Intermediate Value Theorem.

Assume that "$f \in C[a,b]$"⊛ and L is any number between $f(a)$ and $f(b)$. Then there exists a number $c \in (a,b)$, such that $f(c) = L$.



Bolzano's Theorem: If f is continuous on $[a,b]$ and $f(a) \cdot f(b) < 0$, then there exists a number $c \in (a,b)$ such that $f(c) = 0$

"i.e, there exists a root for $f(x)$ in $[a,b]$"

⊛ means: f is Continuous on $[a,b]$.

Example: Let $f(x) = \cos x - x$ on $[0,1]$

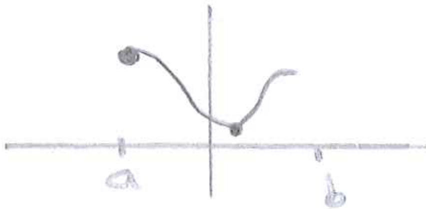
$$\left.\begin{array}{l} f \text{ is Cont. on } [0,1] \\ f(0) = 1 > 0 \\ f(1) = \cos 1 - 1 < 0 \end{array}\right\} \Rightarrow \begin{array}{l} \exists\, c \in (0,1) \\ \text{such that} \\ f(c) = 0. \end{array}$$

Theorem (Extreme Value Theorem For a Continuous Functions).

Assume that $f \in C[a,b]$. Then there exists a lower bound M_1, an upper bound M_2, and two numbers $x_1, x_2 \in [a,b]$ such that

$$M_1 = f(x_1) \leq f(x) \leq f(x_2) = M_2$$

whenever $x \in [a,b]$.



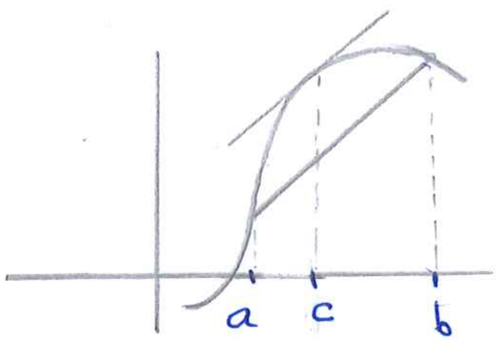
Note: $M_1 = f(x_1) = \underset{[a,b]}{\operatorname{Min}} \{f(x)\}$

and $M_2 = f(x_2) = \underset{[a,b]}{\operatorname{Max}} \{f(x)\}$

Mean Value Theorem: (MVT)

If f is Continuous on $[a,b]$ and differentiable on (a,b) "smooth function", then there exists a number $c \in (a,b)$ such that.

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



Geometrically:

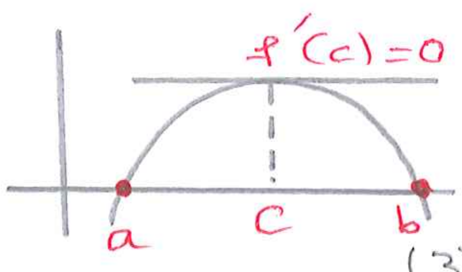
Slope of tangent line at $x = c$

= Slope of the secant line from a to b

(i.e) $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Rolle's Theorem: Assume that $f \in C[a,b]$ and that $f'(x)$ exists for all $x \in (a,b)$. If $f(a)=f(b)=0$, then there exists a number $c \in (a,b)$ such that $f'(c)=0$

"The tangent line is Horizontal"



Def: Taylor series of $f(x)$ at $x=c$ is given by

$$f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2!}+\cdots+\frac{f^{(n)}(c)(x-c)^n}{n!}+\cdots$$

Note: If $c=0$, the series is called Maclaurin series for f.

Taylor polynomial of order n is given by:

$$P_n(x)=f(c)+f'(c)(x-c)+\cdots+\frac{f^{(n)}(c)(x-c)^n}{n!}$$

Examples: Maclaurin series for the following is:

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

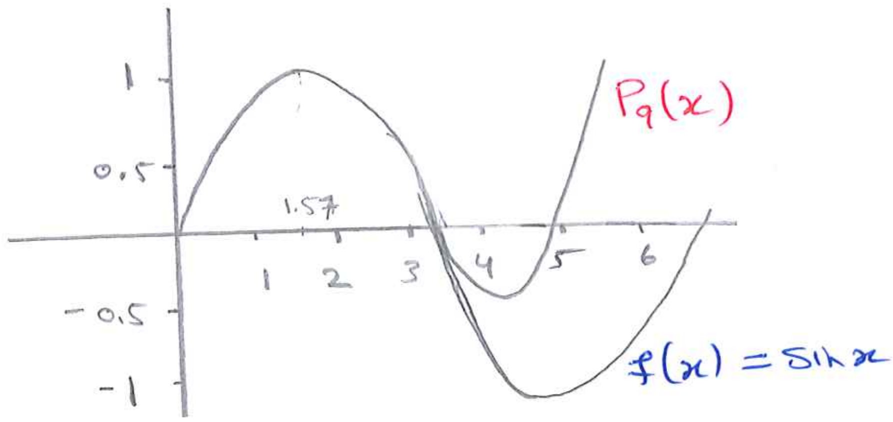
$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots = \sum_{k=0}^{\infty}x^k, \text{ for } |x|<1.$$

Example: If $f(x) = \sin x$, then the Taylor polynomial $P_n(x)$ of degree $n=9$ expanded at $x=0$ is given by:

$$P_9(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$



Taylor's theorem: Assume that $f \in C^{n+1}[a,b]$.

Let $x_0 \in [a,b]$. Then for every $x \in (a,b)$, there exists a number $c = c(x)$ (The value of c depends on x), that lies between x_0 and x such that

$$f(x) = P_n(x) + R_n(x)$$

where $$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

and $$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$ (Error term)

Note : In Taylor Theorem: $c \in (x_0, x)$, so

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

$$\Rightarrow |R_n(x)| \leq \frac{\underset{(x_0, x)}{Max} |f^{(n+1)}(x)| (x - x_0)^{n+1}}{(n+1)!}$$

Upper bound of the error

Example: Find a Linear estimation of $f(x) = e^x$ expanded about $x = 0$, then find the upper bound of the error. when $x = 0.01$.

$$e^x \approx 1 + x$$

$$|R_1(x)| \leq \frac{\underset{(x_0, x)}{Max} |f^{(2)}(x)| (x - 0)^2}{2!} ,$$

$f^{(2)}(x) = e^x$ which is Increasing function

$$\therefore \underset{(0, 0.01)}{Max} |e^x| = e^{0.01} \approx 1.01005$$

$$\therefore |R_1(x)| \leq (1.01005)\frac{(0.01)^2}{2} = 5.05025 \times 10^{-5}$$

1.3 Error Analysis:

Def: Suppose that $\hat{p}$ is an approximation to p.

1) The absolute error is : $E_p = |p - \hat{p}|$

2) The Relative error is : $R_p = \dfrac{|p-\hat{p}|}{|p|}$, $p \neq 0$.

Example: Find the absolute and Relative errors for:

(a) $x = 3.141592$ and $\hat{x} = 3.14$.

$$E_x = |3.141592 - 3.14| = 0.001592.$$

$$R_x = \frac{0.001592}{3.141592} = 0.00507.$$

(b) $y = 1\,000\,000$, $\hat{y} = 999\,996$.

$$E_y = |1000000 - 999996| = 4$$

$$R_y = \frac{4}{1000000} = 0.000004.$$

(c) $z = 0.000012$, $\hat{z} = 0.000009$

$$E_z = |0.000012 - 0.000009| = 0.000003$$

$$R_z = \frac{0.000003}{0.000012} = 0.25 .$$

Note: In (a), there is not too much difference between E_x & R_x, and either could be used to determine the accuracy of $\hat{x}$.

In (b) E_y is large, but R_y is small.. In this case $\hat{y}$ is considered as a good approx. to y.

In (c) E_z is small, but R_z is large, thus $\hat{z}$ is a bad approximation of z.

Def: The number $\hat{p}$ is said to approximate p to d significant digits if d is the largest nonnegative integer for which $\dfrac{|p - \hat{p}|}{|p|} < \dfrac{10^{1-d}}{2}$.

Example: Determine the number of significant digits for the approximations in prvious example.

(a) $x = 3.141592$, $\hat{x} = 3.14$, then

$$\frac{|x - \hat{x}|}{|x|} = 0.000507 < \frac{10^{1-3}}{2}. \Rightarrow d = 3$$

If $d = 0$, then $0.000507 < \frac{10}{2} = 5.$ (✓)

If $d = 1$, then $0.000507 < \frac{10^{0}}{2} = \frac{1}{2}.$ (✓)

If $d = 2$, then $0.000507 < \frac{10^{-1}}{2} = 0.05$ (✓)

If $d = 3$, then $0.000507 < \frac{10^{-2}}{2} = 0.005$ (✓)

If $d = 4$, then $0.000507 > \frac{10^{-3}}{2} = 0.0005$.

Then the Longest nonnegative Integer is $= \boxed{3}$

Therefore $\hat{x}$ approximates x to three significant digits.

(b) $y = 1\,000\,000$, $\hat{y} = 999\,996$, then

$$\frac{|y-\hat{y}|}{|y|} = 0.000004 < \frac{10^{1-6}}{2} = 0.000005.$$

$\therefore$ $\boxed{d = 6}$

(c) $\alpha = 1\,000\,000$, $\hat{\alpha} = 999\,994$, then

$$\frac{|\alpha-\hat{\alpha}|}{|\alpha|} = 0.000006 < \frac{10^{1-5}}{2} = 0.00005.$$

$\therefore$ $\boxed{d = 5}$

(d) $z = 0.000012$, $\hat{z} = 0.000009$, then

$$\frac{|z-\hat{z}|}{|z|} = 0.25 < \frac{10^{1-1}}{2} = 0.5.$$

$\therefore$ $\boxed{d = 1}$

Note: As d increases we will have less error and more accuracy.

Truncation Error.

The notion of truncation error usually refers to errors introduced when a more complicated mathematical expression is replaced with a more elementary formula.

Example: Given that $\int_0^{\frac{1}{2}} e^{x^2}\,dx = 0.544987104184 \;\; // \; P$

determine the accuracy of the approximation obtained by replacing $f(x) = e^{x^2}$, with the

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truncated Taylor series $P_8(x) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!}$.

$$\int_0^{\frac{1}{2}} \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!}\right) dx = 0.544986720817 \;\; // \; \hat{P}$$

$\therefore \hat{p}$ approximates p to five significant digits.

since $R_p = 0.70344 2333 \times 10^{-6} < \frac{10^{1-5}}{2} = 5 \times 10^{-5}$.

Round-off Error.

Some other errors formed from using computer or calculator since they have limited number of digits to be used.

For example: $(\sqrt{3})^2 = 3$ (Algebrically).

But $(\sqrt{3})^2 \approx 3$ using computer.

This kind of error is called Round of error.

Def: Consider any real number p that is expressed in normalized decimal form:

$$p = \pm\, 0.d_1 d_2 d_3 \cdots d_k d_{k+1} \cdots \times 10^n,$$

where $1 \leq d_1 \leq 9$ and for $j>1$, $0 \leq d_j \leq 9$.

Suppose that k is the maximum number of decimal digits carried in the floating-point computations of a computer, then p is represented in two ways:

1) Chopped floating-point representation "$fl_{chop}(p)$"

$$fl_{chop}(p) = \pm 0.d_1 d_2 d_3 \cdots d_k \times 10^n \quad , \ d_1 \neq 0.$$

2) Round floating-point representation "$fl_{round}(p)$".

$$fl_{round}(p) = \pm 0.d_1 d_2 d_3 \cdots r_k \times 10^n, \quad d_1 \neq 0.$$

Where r_k is obtained by rounding the number

$d_k d_{k+1} d_{k+2} \cdots$ to the nearst integer.

Example: $p = \frac{22}{7} = 3.142857142857\overline{1}$.

The Six digits representations is

$$fl_{chop}(p) = 0.314285 \times 10^1$$

$$fl_{round}(p) = 0.314286 \times 10^1$$

Example: $x = 0.5234456$. The four digits repre.

$fl_{chop}(x) = 0.5234$, $fl_{round}(x) = 0.5235$

Example: Use 3 significant digits rounded to approximate:

$$\frac{\frac{2}{7}+\frac{8}{3}+\frac{9}{11}}{\underbrace{467 \times 9.4}_{\substack{\| \\ 4389.8 = 4390}}} = \frac{0.286+2.67+0.818}{0.439 \times 10^{4}}$$

$$= \frac{2.96+0.818}{0.439 \times 10^{4}} = \frac{3.78}{0.439 \times 10^{4}} = 8.61 \times 10^{-4} = 0.861 \times 10^{-3}.$$

Loss of Significant

Let $p = 3.1415926536$

$q = 3.1415957341$

then $p - q = -0.0000030805 = -3.0805 \times 10^{-6}$

$\Longrightarrow \quad p - q \approx 0.$

So, using some calculators the difference is equal 0, while it is not true.

This phenomenon is called loss of significant.

Example: Let $f(x) = x\left(\sqrt{x+1} - \sqrt{x}\right)$

$$g(x) = \frac{x}{\sqrt{x+1} + \sqrt{x}}$$

Use 6 significant digits rounded to estimate $f(500), g(500)$.

sol: $f(500) = 500\left(\sqrt{501} - \sqrt{500}\right)$

$$= 500\,(22.3830 - 22.3607)$$

$$= 500\,(0.0223) = 11.1500.$$

$$g(500) = \frac{500}{\sqrt{501} + \sqrt{500}} = \frac{500}{22.3830 + 22.3607}$$

$$= \frac{500}{44.7437} = 11.1748.$$

$\Rightarrow \quad f(500) \approx g(500)$.

Although $f(x)$ and $g(x)$ are Algebrically equivalent.

$$f(x) = x\left(\sqrt{x+1} - \sqrt{x}\right)\cdot\left(\frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}\right)$$

$$= \frac{x\,(\cancel{x}+1-\cancel{x})}{\sqrt{x+1} + \sqrt{x}} = \frac{x}{\sqrt{x+1} + \sqrt{x}} = g(x).$$

Note: Sometimes the loss of significant error can be avoided by rearranging terms in the function using a known identity from trigonometry or algebra.

Example: Find an equivalent formula for the following functions that avoids a loss of significance.

(a) $\ln(x+1) - \ln x \quad , x > 0$

$$\ln(x+1) - \ln x = \ln\left(\frac{x+1}{x}\right)$$

(b) $$\left(\sqrt{x^2+1} - x\right)\left(\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x}\right) = \frac{1}{\sqrt{x^2+1} + x}$$

(c) $$\cos^2 x - \sin^2 x = \cos(2x) \quad , \quad x \approx \frac{\pi}{4}$$

(d) $$\sqrt{\frac{1+\cos x}{2}} = \cos\left(\frac{x}{2}\right) \quad , \quad x \approx \pi .$$

Order of Approximation $O(h^n)$

Def: Assume that $f(h)$ is approximated by the function $p(h)$ and that there exists a real constant $M > 0$ and a positive integer n, so that

$$\frac{|f(h) - p(h)|}{|h^n|} \leq M \quad , \quad \text{for sufficiently small } h.$$

We say that $p(h)$ approximates $f(h)$ with order of approximation $O(h^n)$ and we write it as:

$$f(h) = p(h) + O(h^n).$$

Note: It is instructive to consider $p(x)$ to be the nth taylor polynomial approximation of $f(x)$. Then the reminder term is denoted by $O(h^{n+1})$.

Taylor's Theorem: Assume $f \in C^{n+1}[a,b]$.

If both x_0 and $x_0 + h \in [a,b]$, then:

$$f(x_0 + h) = \sum_{k=0}^{n} \frac{f^k(x_0)}{k!} h^k + O(h^{n+1}).$$

Example: $e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + O(h^4)$.

$$\cos(h) = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + O(h^6)$$

$$\sin(h) = h + O(h^3)$$

where $O(h^i)$ means "constant $\times h^i$", $i = 3, 4, 6$.

Example: Show that $p(h) = 1+h$ approximate $f(h) = e^h$ with order of approximation $O(h^2)$?

Sol: $\dfrac{|e^h - (1+h)|}{|h^2|} \overset{???}{\leq} M$ (Need to find M).

$$\frac{|\cancel{1} + \cancel{h} + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots - \cancel{1} - \cancel{h}|}{|h^2|} = \frac{|\frac{h^2}{2!} + \frac{h^3}{3!} + \cdots|}{|h^2|}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

$$= \frac{1}{2!} + \frac{h}{3!} + \frac{h^2}{4!} + \cdots < \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$< \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{2}\left(\frac{1}{1 - \frac{1}{2}}\right) = \boxed{1}$$

Theorem: Assume that $f(h) = p(h) + O(h^n)$,

$g(h) = q(h) + O(h^m)$, and $r = \min\{m, n\}$.

Then

$$f(h) \pm g(h) = p(h) \pm q(h) + O(h^r)$$

$$f(h) \cdot g(h) = p(h) \cdot q(h) + O(h^r)$$

$$\frac{f(h)}{g(h)} = \frac{p(h)}{q(h)} + O(h^r), \quad \begin{array}{l} g(h) \neq 0 \\ q(h) \neq 0 \end{array}$$

Example: Let $e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + O(h^4)$

and $\cos(h) = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + O(h^6)$. Then

$$e^h + \cos(h) = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + O(h^4)$$

$$+ 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + O(h^6)$$

$$= 2 + h + \frac{h^3}{3!} + \underbrace{\frac{h^4}{4!} + O(h^4) + O(h^6)}$$

$$\Rightarrow e^h + \cos(h) = 2 + h + \frac{h^3}{3!} + O(h^4).$$

Now: $e^h \cdot \cos(h) =$

$$= \left(1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+O(h^4)\right)\left(1-\frac{h^2}{2!}+\frac{h^4}{4!}+O(h^6)\right)$$

$$= \left(1+h+\frac{h^2}{2!}+\frac{h^3}{3!}\right)\left(1-\frac{h^2}{2!}+\frac{h^4}{4!}\right)+O(h^4)$$

$$= 1+h-\frac{h^3}{3}-\frac{5h^4}{24}-\frac{h^5}{24}+\frac{h^6}{48}+\frac{h^7}{144}+O(h^4)$$

$$= 1+h-\frac{h^3}{3}+O(h^4).$$

Propagation of Error:

If $p=\hat{p}+\epsilon_p$ & $q=\hat{q}+\epsilon_q$, then

$$p+q=(\hat{p}+\epsilon_p)+(\hat{q}+\epsilon_q)=(\hat{p}+\hat{q})+(\epsilon_p+\epsilon_q).$$

$$p\cdot q=(\hat{p}+\epsilon_p)\cdot(\hat{q}+\epsilon_q)=\hat{p}\hat{q}+\hat{p}\,\epsilon_q+\hat{q}\,\epsilon_p+\epsilon_p\epsilon_q.$$

Therefore if $|\hat{p}|>1$ & $|\hat{q}|>1$, then the terms $\hat{p}\,\epsilon_q$ & $\hat{q}\,\epsilon_p$ show that there is a possibility of magnification of the original errors ϵ_p & ϵ_q.

"End of Chapter 1"