Stat5504, Fall 2015: Homework 1

(Due on September 15th, 2015)

1. Consider the estimation of parameters $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$in the Ridge regression:

$$\hat{\boldsymbol{\beta}} = \text{argmin}_{\boldsymbol{\beta}} \left\{ \sum_{i=1}^{n} (y_i - \sum_{j=1}^{p} x_{ij}\beta_j)^2 + \sum_{j=1}^{p} \lambda_j \beta_j^2 \right\},$$

where $\lambda_j > 0$. Suppose the design matrix $\boldsymbol{X} = (x_{ij})_{n\times p}$ is orthogonal, i.e., $\boldsymbol{X}^T\boldsymbol{X} = diag(s_1, \ldots, s_p)$ with $s_i > 0, i = 1, \ldots, p$. Please derive the explicit solution of $\hat{\boldsymbol{\beta}}$.

2. Let $X_1 \sim N(0, 1)$ and

$$X_2 = \begin{cases} -X_1, & \text{if } -c < X_1 < c \text{ ;} \\ X_1, & \text{otherwise,} \end{cases}$$

for some c.

   (a) Show that X_2 also has $N(0, 1)$ distribution.

   (b) Do you think (X_1, X_2) has a bivariate normal distribution? Prove your conclusion (numerically or theoretically).

   (c) Show that there exists $c > 0$ such that $Cov(X_1, X_2) = 0$, but X_1 and X_2 are not independent.

3. Recall the logistic regression with the data $(\boldsymbol{x}_i, y_i), i = 1, \ldots, n$ where $\boldsymbol{x} \in \mathcal{R}^p$ and $y \in \{0, 1\}$. Note that the model is

$$y = \begin{cases} 1, & w.p.\ p(\boldsymbol{x}) \\ 0, & w.p.\ 1 - p(\boldsymbol{x}) \end{cases} \quad \text{with } p(\boldsymbol{x}) = \frac{\exp(\boldsymbol{x}'\boldsymbol{\beta})}{1 + \exp(\boldsymbol{x}'\boldsymbol{\beta})}.$$

Denote the log-likelihood function to be $\log L(\boldsymbol{\beta})$.

   (a) Derive the Fisher information matrix for $\log L(\boldsymbol{\beta})$, i.e., $-\frac{\partial^2 \log L(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}^T}$.

   (b) Code the iterative reweighted least squares (IWLS) algorithm for estimating the parameter $\boldsymbol{\beta}$. Examine your code using the iris flower data set. Please report the fitted model and classification error.

   `http://en.wikipedia.org/wiki/Iris_flower_data_set`

4. Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix}$ be a partition matrix. Here $\boldsymbol{A}, \boldsymbol{A}_{11}, \boldsymbol{A}_{22}$ are $n \times n$, $n_1 \times n_1$, $n_2 \times n_2$ matrices, respectively.

(a) Show that matrix $\boldsymbol{A}$ is symmetric if and only if $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are symmetric and $\boldsymbol{A}_{12}' = A_{21}$.

(b) Assume that matrix $\boldsymbol{A}_{22}$ is nonsingular. Verify the following

$$\begin{pmatrix} \boldsymbol{I} & -\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21} & \boldsymbol{I} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_{22} \end{pmatrix},$$

where $\boldsymbol{I}$ denotes the identity matrix of an appropriate order. Show that

$$|\boldsymbol{A}| = |\boldsymbol{A}_{22}||\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}|$$

(c) Show that

$$\boldsymbol{A}^{-1} = \begin{pmatrix} (\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21})^{-1} & -(\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21})^{-1}\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1} \\ -\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}(\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21})^{-1} & \boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21}(\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{A}_{21})^{-1}\boldsymbol{A}_{12}\boldsymbol{A}_{22}^{-1} + \boldsymbol{A}_{22}^{-1} \end{pmatrix}.$$