Kinematic Analysis of Single-Link Inverted Pendulum Model

1 Model Description

The model is represented as a single-link inverted pendulum for human body during gait initiation with the following characteristics:

- Single rotational joint at the ankle
- Body modeled as a rigid link pivoting around the ankle
- Feet assumed to be anchored to the ground
- Motion confined to the sagittal plane
- Body mass (excluding feet) concentrated at center of mass
- Length from ankle to COM is L_{COM}
- Total body length is L_B

2 DH (Denavit-Hartenberg) Parameters

For this single-link system:

Link	θ	d	a	α
1	θ	0	L_{COM}	0

Table 1: DH Parameters for Single-Link Model

Where:

- θ: Joint angle (variable)
- d: Link offset (0 since planar motion)
- a: Link length (L_{COM})
- α: Link twist (0 since planar motion)

3 Forward Kinematics

3.1 Transformation Matrix

Using the DH parameters, the homogeneous transformation matrix from base (ankle) to end-effector (body COM) is:

$$T = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & L_{COM}\cos(\theta) \\ \sin(\theta) & \cos(\theta) & 0 & L_{COM}\sin(\theta) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

3.2 End-Effector Position

The position of the body's center of mass (x_B, y_B) can be calculated as:

$$y_B = L_{COM}\sin(\theta) \tag{2}$$

$$x_B = L_{COM}\cos(\theta) \tag{3}$$

4 Inverse Kinematics

For this single-link system, the inverse kinematics solution is:

$$\theta = \arctan\left(\frac{y_B}{x_B}\right) \tag{4}$$

Modified from paper(For convention used in class):

$$\theta(t) = -(\arctan(L_{ACRx} - L_{LMAx}, L_{ACRy} - L_{LMAy}) - \frac{\pi}{2}) \tag{5}$$

Where L_{ACR} and L_{LMA} are the positions of the acromion and lateral malleolus markers respectively.

5 Jacobian Analysis

The Jacobian J is composed of linear velocity component J_v and angular velocity component J_ω:

$$J = \begin{bmatrix} z_0 \times (o_1 - o_0) \\ z_0 \end{bmatrix} \tag{6}$$

Where z_0 is a unit vector:

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{7}$$

position vectors are:

$$o_1 = \begin{bmatrix} L_{COM}\cos\theta \\ L_{COM}\sin\theta \\ 0 \end{bmatrix}, \quad o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

5.1 Linear Velocity Component

The linear velocity component J_v is calculated as:

$$J_v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} L_{COM}\cos\theta \\ L_{COM}\sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} -L_{COM}\sin\theta \\ L_{COM}\cos\theta \\ 0 \end{bmatrix} \tag{9}$$

5.2 Angular Velocity Component

The angular velocity component J_ω is:

$$J_\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{10}$$

5.3 Complete Jacobian

Therefore, the complete Jacobian matrix is:

$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} = \begin{bmatrix} -L_{COM}\sin\theta \\ L_{COM}\cos\theta \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}_{6\times 1} \tag{11}$$