Dynamic Mode Decomposition (DMD)

Workshop on Data -Driven Modelling 2017

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Outline

Introduction

Eigen Values and Vectors

Singular Values and Vectors

Dynamic Mode Decomposition

Dynamic Mode Decomposition (DMD)

- Dynamic systems: Measurements of nonlinear dynamical systems and/or complex systems.
- DMD: Integrating data with dynamical systems theory.
- Has deep connections with many recent innovations in compressed sensing and machine learning.
- Algorithm evolved with Eigen Values and Vectors.

Eigen Values and Vectors

- Eigen Value (λ) = characteristic value
- Eigen Vector (v) = characteristic vector.

$$Av = \lambda v \tag{1}$$

Example

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

$$Av = \lambda v \tag{3}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{5}$$

$$v1, v2 = (1,1), (1,-1) \tag{6}$$

$$\lambda 1, \lambda 2 = 1, -1 \tag{7}$$

Visualizing an Example

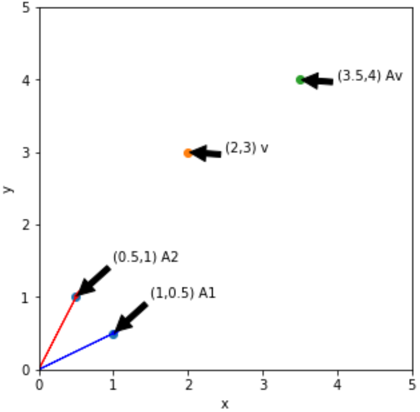


Figure: A is a matrix with A1, A2 column vectors, v is a Eigen vector, Magnitude of the green line is Eigen value

Eigen Spaces

$$A = \lambda v \tag{8}$$

$$Av = (\lambda I)v \tag{9}$$

$$(A - (\lambda I))v = 0 \tag{10}$$

$$v \epsilon N(A - \lambda I) \ is \ the \ nullspace \ of \ the \ matrix \ (A - \lambda I) \tag{11}$$

$$N(A - \lambda I) \neq \{0\} \ \ is \ the \ eigenspace \ of \ A \ with \ \lambda \tag{12}$$

Singular Value Decomposition - SVD

- Generalization of the Eigen decomposition

$$U_{m\times m}\Sigma_{m\times n}V^T_{n\times n} = svd\left(A_{m\times n}\right) \tag{13}$$

$$U_{m\times m} = eigen\ \ vector\left(A_{m\times n}A^T_{n\times m}\right) \tag{14}$$

$$V_{n\times n} = eigen\ \ vector\left(A^T_{n\times m}A_{m\times n}\right) \tag{15}$$

$$\Sigma_{m\times n} = \sqrt{eigen\ \ value\left(A_{m\times n}\ \ or\ \ A^T_{n\times m}\right)} \tag{16}$$

Singular Value Decomposition - SVD

- Full Matrix

$$U_{m\times m}\Sigma_{m\times n}V_{n\times n}^{T} = svd\left(A_{m\times n}\right) \tag{17}$$

- Truncated Matrix

$$\tilde{A} \approx U_{m\times r}\Sigma_{r\times r}V_{r\times n}^{T} \tag{18}$$

Singular Value Decomposition - SVD

$$\underset{m \times n}{\overset{\hat{X}}{\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & \\ \vdots & \vdots & \ddots & \\ x_{m1} & & & x_{mn} \end{pmatrix}}} \approx \underset{m \times r}{\overset{U}{\begin{pmatrix} u_{11} & \dots & u_{1r} \\ \vdots & \ddots & \\ u_{m1} & & u_{mr} \end{pmatrix}}} \underset{r \times r}{\overset{S}{\begin{pmatrix} s_{11} & 0 & \dots \\ 0 & \ddots & \\ \vdots & & s_{rr} \end{pmatrix}}} \underset{r \times n}{\overset{V^{\mathsf{T}}}{\begin{pmatrix} v_{11} & \dots & v_{1n} \\ \vdots & \ddots & \\ v_{r1} & & v_{rn} \end{pmatrix}}}$$

Figure: Truncation of the matrix

DMD Formulation

$$Observed\ Sequences = \begin{bmatrix} | & | & | & & | & | \\ x_1 & x_2 & x_3 & \ldots & x_n & x_{n+1} \\ | & | & | & & | & | \end{bmatrix} \tag{19}$$

$$X_{m\times n} = \begin{bmatrix} | & | & | & & | & | \\ x_1 & x_2 & x_3 & \ldots & x_{n-1} & x_n \\ | & | & | & & | & | \end{bmatrix} \tag{20}$$

$$Y_{m\times n} = \begin{bmatrix} | & | & | & & | & | \\ x_2 & x_3 & x_4 & \ldots & x_n & x_{n+1} \\ | & | & | & & | & | \end{bmatrix} \tag{21}$$

$$Y_{m\times n} = A_{m\times m}X_{m\times n} \tag{22}$$

$$A_{m\times m} = Y_{m\times n}X^{\dagger}_{n\times m} \tag{23}$$

$$DMD\ Modes = eigen\ vector\,(A_{m\times m}) \tag{24}$$

- Finding pseudo inverse of X

$$X^{\dagger} = (X^{*}X)^{-1} X^{*} \tag{25}$$

$$= (V\Sigma U^{*} U\Sigma V^{*})^{-1} V\Sigma U^{*}$$

$$= \left(V\Sigma^{2} V^{*}\right)^{-1} V\Sigma U^{*}$$

$$= (V^{*})^{-1} \Sigma^{-2} V^{-1} V\Sigma U^{*}$$

$$= V\Sigma^{-2}\Sigma U^{*}$$

$$X^{\dagger} = V\Sigma^{-1} U^{*}$$

$$X^{\dagger}_{n\times m} = V_{n\times n}\Sigma^{-1}_{n\times m} U^{*}_{m\times m} \tag{26}$$

- Projection of the full matrix A onto U

$$Y_{m\times n} = A_{m\times m}X_{m\times n}$$

$$A_{m\times m} = Y_{m\times m}X^{\dagger}_{n\times m}$$

$$\tilde{A} = U^{*}AU = U^{*}YX^{\dagger}U \tag{27}$$

$$\tilde{A} = UAU^{*} = U^{*}YV\Sigma^{-1}U^{*}U \tag{28}$$

$$\tilde{A} = U^{*}YV\Sigma^{-1} \tag{29}$$

- Reducing dimension to r (rank)

$$\tilde{A}_{m\times m} = U^*_{m\times m} Y_{m\times n} V_{n\times n} \Sigma^{-1}_{n\times m}$$

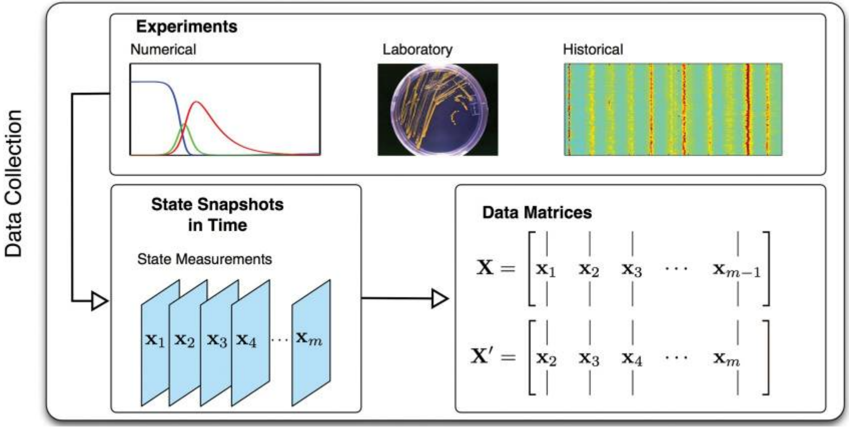
$$\tilde{A}_{r\times r} \approx U^*_{r\times m} Y_{m\times n} V_{n\times r} \Sigma^{-1}_{r\times r} \tag{30}$$

- Eigen Vectors of $\tilde{A}$

$$W_{r \times r} = eigen\ vector\left(\tilde{A}_{r \times r}\right) \tag{31}$$

- DMD Modes

$$\phi_{m \times r} = Y_{m \times n} V_{n \times r} \Sigma_{r \times r} W_{r \times r} \tag{32}$$



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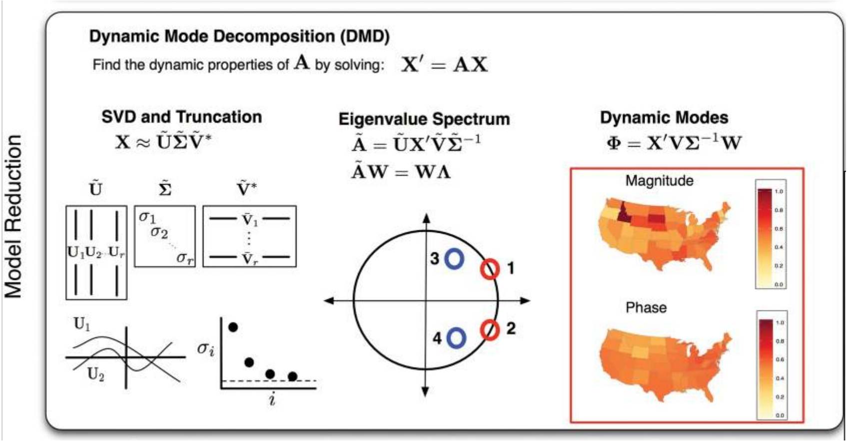

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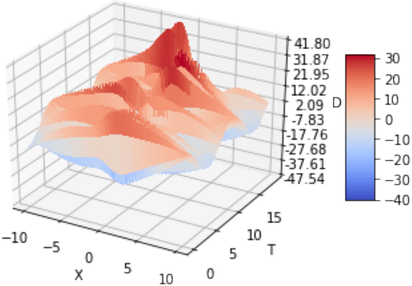
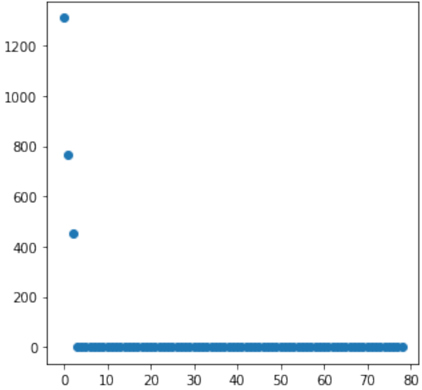
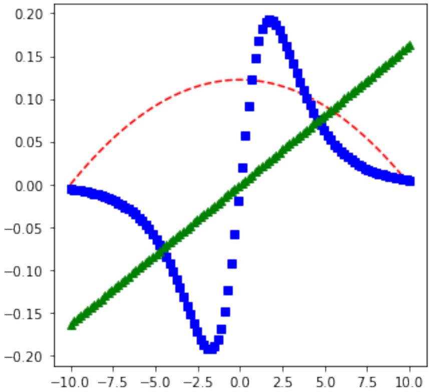
Dynamic System



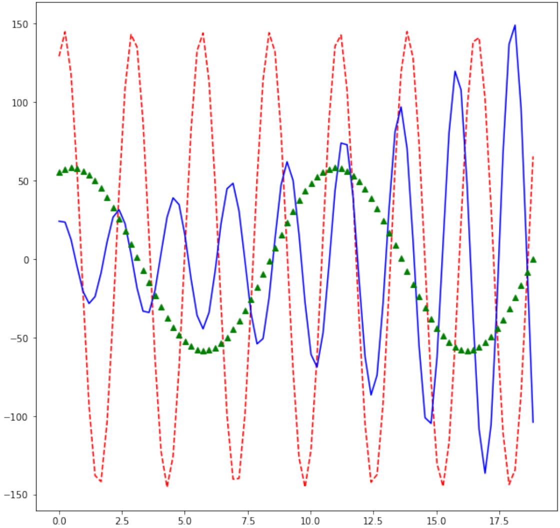
Singular Values



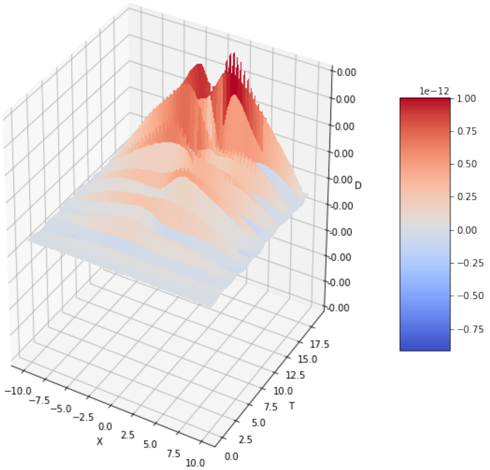
DMD Modes



Time Evolution



Reconstructed Error



Thank You.

you can follow me through:

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