

Dynamic Mode Decomposition (DMD)

Workshop on Data -Driven Modelling 2017

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Outline

Introduction

Eigen Values and Vectors

Singular Values and Vectors

Dynamic Mode Decomposition



Dynamic Mode Decomposition (DMD)

- Dynamic systems: Measurements of nonlinear dynamical systems and/or complex systems.
- DMD: Integrating data with dynamical systems theory.
- Has deep connections with many recent innovations in compressed sensing and machine learning.
- Algorithm evolved with Eigen Values and Vectors.



Eigen Values and Vectors

- Eigen Value (λ) = characteristic value
- Eigen Vector (v) = characteristic vector.

$$Av = \lambda v \quad (1)$$

Example

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (2)$$

$$Av = \lambda v \quad (3)$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad (5)$$

$$v_1, v_2 = (1, 1), (1, -1) \quad (6)$$

$$\lambda_1, \lambda_2 = 1, -1 \quad (7)$$

Visualizing an Example

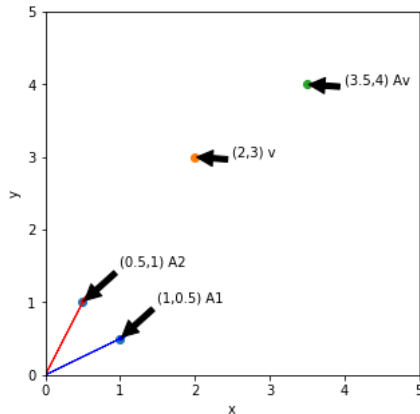


Figure: A is a matrix with A_1 , A_2 column vectors, v is a Eigen vector, Magnitude of the green line is Eigen value

Eigen Spaces

$$A v = \lambda v \quad (8)$$

$$A v = (\lambda I) v \quad (9)$$

$$(A - (\lambda I)) v = 0 \quad (10)$$

$$v \in N(A - \lambda I) \text{ is the nullspace of the matrix } (A - \lambda I) \quad (11)$$

$$N(A - \lambda I) \neq \{0\} \text{ is the eigenspace of } A \text{ with } \lambda \quad (12)$$

Singular Value Decomposition - SVD

- Generalization of the Eigen decomposition

$$U_{m \times m} \Sigma_{m \times n} V_{n \times n}^T = \text{svd}(A_{m \times n}) \quad (13)$$

$$U_{m \times m} = \text{eigen vector} \left(A_{m \times n} A_{n \times m}^T \right) \quad (14)$$

$$V_{n \times n} = \text{eigen vector} \left(A_{n \times m}^T A_{m \times n} \right) \quad (15)$$

$$\Sigma_{m \times n} = \sqrt{\text{eigen value} (A_{m \times n} \text{ or } A_{n \times m}^T)} \quad (16)$$

Singular Value Decomposition - SVD

- Full Matrix

$$U_{m \times m} \Sigma_{m \times n} V_{n \times n}^T = \text{svd}(A_{m \times n}) \quad (17)$$

- Truncated Matrix

$$\tilde{A} \approx U_{m \times r} \Sigma_{r \times r} V_{r \times n}^T \quad (18)$$

Singular Value Decomposition - SVD

$$\begin{pmatrix} \hat{X} \\ \begin{matrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & \\ \vdots & \vdots & \ddots & \\ x_{m1} & & & x_{mn} \end{matrix} \\ m \times n \end{pmatrix} \approx \begin{pmatrix} U \\ \begin{matrix} u_{11} & \dots & u_{1r} \\ \vdots & \ddots & \\ u_{m1} & & u_{mr} \end{matrix} \\ m \times r \end{pmatrix} \begin{pmatrix} S \\ \begin{matrix} s_{11} & 0 & \dots \\ 0 & \ddots & \\ \vdots & & s_{rr} \end{matrix} \\ r \times r \end{pmatrix} \begin{pmatrix} V^T \\ \begin{matrix} v_{11} & \dots & v_{1n} \\ \vdots & \ddots & \\ v_{r1} & & v_{rn} \end{matrix} \\ r \times n \end{pmatrix}$$

Figure: Truncation of the matrix

DMD Formulation

$$\text{Observed Sequences} = \begin{bmatrix} | & | & | & \dots & | & | \\ x_1 & x_2 & x_3 & \dots & x_n & x_{n+1} \\ | & | & | & & | & | \end{bmatrix} \quad (19)$$

$$X_{m \times n} = \begin{bmatrix} | & | & | & \dots & | & | \\ x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \\ | & | & | & & | & | \end{bmatrix} \quad (20)$$

$$Y_{m \times n} = \begin{bmatrix} | & | & | & \dots & | & | \\ x_2 & x_3 & x_4 & \dots & x_n & x_{n+1} \\ | & | & | & & | & | \end{bmatrix} \quad (21)$$

$$Y_{m \times n} = A_{m \times M} X_{M \times n} \quad (22)$$

$$A_{m \times m} = Y_{m \times M} X_{n \times m}^\dagger \quad (23)$$

$$DMD \text{ Modes} = \text{eigen vector} (A_{m \times m}) \quad (24)$$

- Finding pseudo inverse of X

$$\begin{aligned}X^\dagger &= (X^* X)^{-1} X^* \\&= (V \Sigma U^* U \Sigma V^*)^{-1} V \Sigma U^* \\&= (V \Sigma^2 V^*)^{-1} V \Sigma U^* \\&= (V^*)^{-1} \Sigma^{-2} V^{-1} V \Sigma U^* \\&= V \Sigma^{-2} \Sigma U^* \\X^\dagger &= V \Sigma^{-1} U^*\end{aligned}\tag{25}$$

$$X_{n \times m}^\dagger = V_{n \times n} \Sigma_{n \times m}^{-1} U_{m \times m}^* \tag{26}$$

- Projection of the full matrix A onto U

$$Y_{m \times n} = A_{m \times m} X_{m \times n}$$

$$A_{m \times m} = Y_{m \times m} X_{n \times m}^\dagger$$

$$\tilde{A} = U_* A U = U^* Y X^\dagger U \quad (27)$$

$$\tilde{A} = U A U^* = U^* Y V \Sigma^{-1} U^* U \quad (28)$$

$$\tilde{A} = U^* Y V \Sigma^{-1} \quad (29)$$

- Reducing dimension to r (rank)

$$\tilde{A}_{m \times m} = U_{m \times m}^* Y_{m \times n} V_{n \times n} \Sigma_{n \times m}^{-1}$$

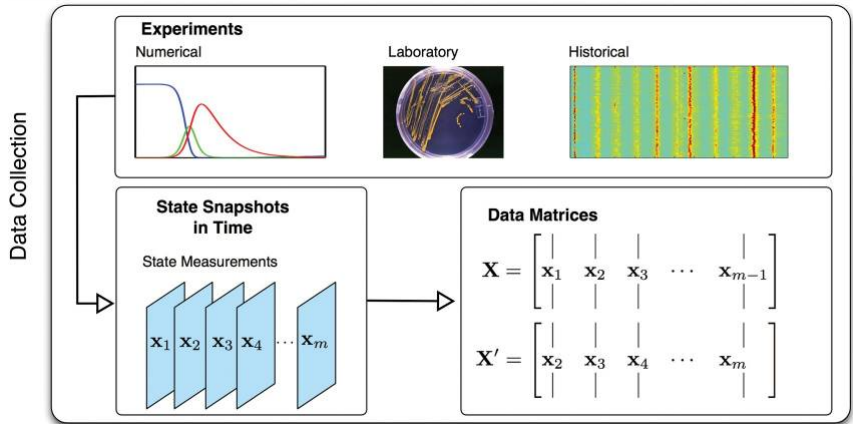
$$\tilde{A}_{r \times r} \approx U_{r \times m}^* Y_{m \times n} V_{n \times r} \Sigma_{r \times r}^{-1} \quad (30)$$

- Eigen Vectors of \tilde{A}

$$W_{r \times r} = \left(\text{eigen vector } \tilde{A}_{r \times r} \right) \quad (31)$$

- DMD Modes

$$\phi_{m \times r} = Y_{m \times n} V_{n \times r} \Sigma_{r \times r} W_{r \times r} \quad (32)$$



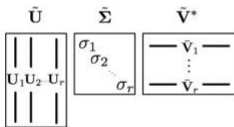
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Dynamic Mode Decomposition (DMD)

Find the dynamic properties of $\tilde{\mathbf{A}}$ by solving: $\mathbf{X}' = \tilde{\mathbf{A}}\mathbf{X}$

SVD and Truncation

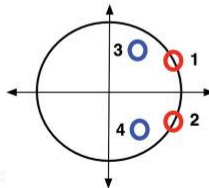
$$\mathbf{X} \approx \tilde{\mathbf{U}}\tilde{\Sigma}\tilde{\mathbf{V}}^*$$



Eigenvalue Spectrum

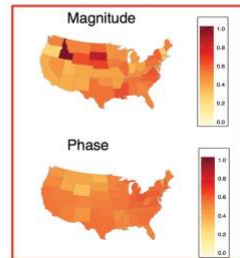
$$\tilde{\mathbf{A}} = \tilde{\mathbf{U}}\mathbf{X}'\tilde{\mathbf{V}}\tilde{\Sigma}^{-1}$$

$$\tilde{\mathbf{A}}\mathbf{W} = \mathbf{W}\Lambda$$



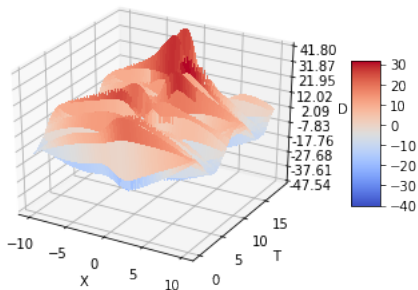
Dynamic Modes

$$\Phi = \mathbf{X}'\mathbf{V}\Sigma^{-1}\mathbf{W}$$

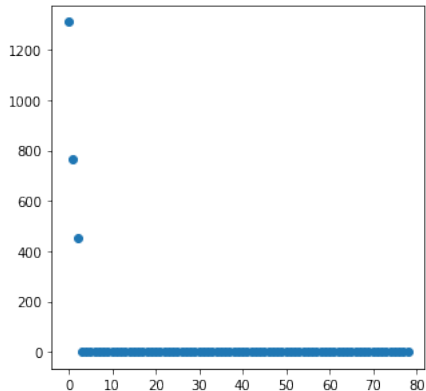


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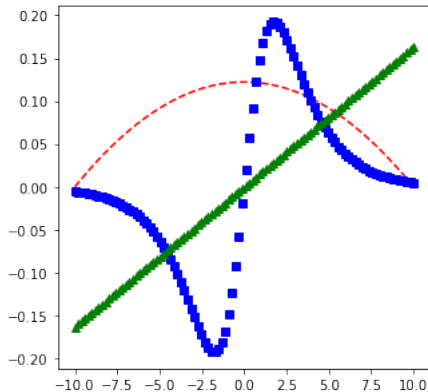
Dynamic System



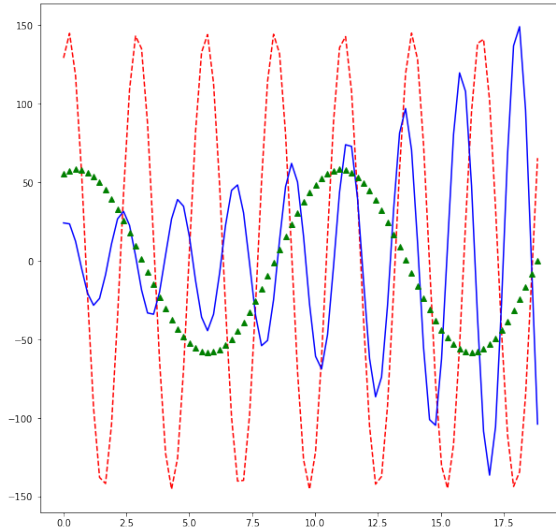
Singular Values



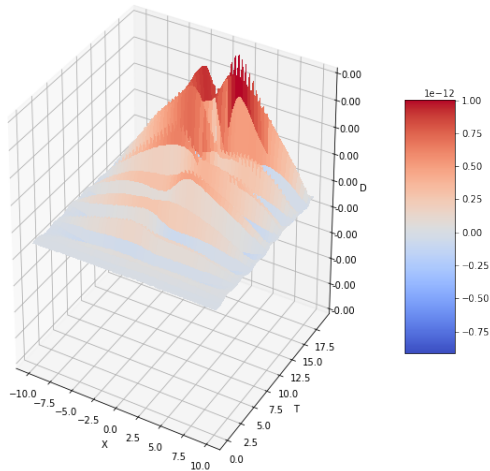
DMD Modes



Time Evolution



Reconstructed Error



Thank You.

you can follow me through:

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