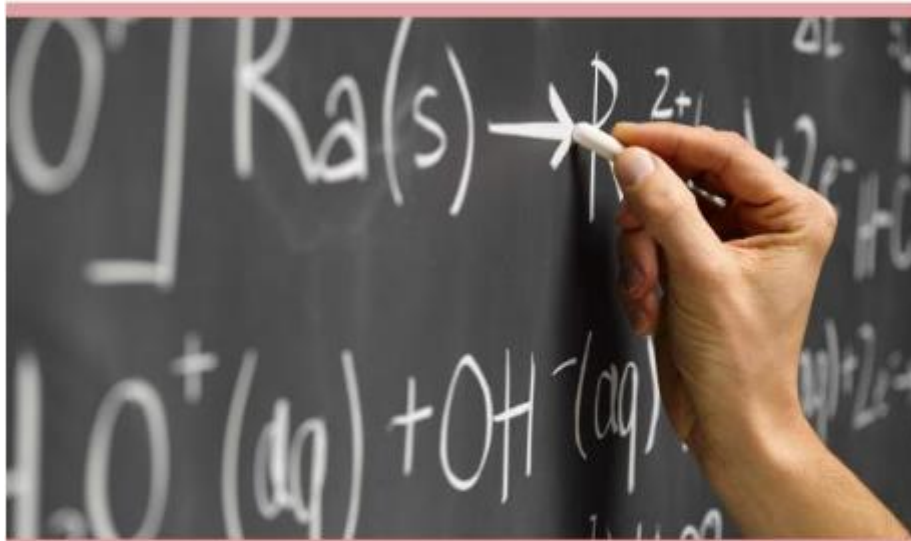


LECTURES ON
PROBABILITY THEORY
AND MATHEMATICAL
STATISTICS



3 R D E D I T I O N

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