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Graded Homework 1

Due: Friday 15h00, October 18, 2019

Exercise 1. Bases and matrix representation of linear operators (20 points)
Let H be the space of real polynomials of maximum degree 2 with the standard inner product and norm on $L^2([-1,1])$. Additionally, let $\Phi = \{\varphi_i(t)\}_{i=0,1,2}$, where $\varphi_i(t) = t^i$. Similarly, let $\Psi = \{\psi_i(t)\}_{i=0,1,2}$.

Part (a)

In the first half of this question, we'll construct functions $\psi_i(t)$, $i = 0,1,2$ to make Ψ an orthonormal bases for H.

(i) Find $\psi_0(t)$ such that $\|\psi_0(t)\| = 1$ and $\operatorname{span}(\{\psi_0(t)\}) = \operatorname{span}(\{\varphi_0(t)\})$.

(ii) Find $\psi_1(t)$ such that $\|\psi_1(t)\| = 1$, $\langle\psi_0(t), \psi_1(t)\rangle = 0$ and $\operatorname{span}(\{\psi_0(t), \psi_1(t)\}) = \operatorname{span}(\{\varphi_0(t), \varphi_1(t)\})$.
Hint: Start from $\varphi_1(t)$ and make it orthogonal to $\psi_0(t)$ (if it is not already). Then, make the resulting vector have unit length.

(iii) Find $\psi_2(t)$ such that $\|\psi_2(t)\| = 1$, $\langle\psi_0(t), \psi_2(t)\rangle = 0$, $\langle\psi_1(t), \psi_2(t)\rangle = 0$ and $\operatorname{span}(\Psi) = \operatorname{span}(\Phi) = H$.

Part (b)

In the second half of this question, we'll find matrices to perform differentiation. To do this, let $A : H \to H$ be the differentiation operator so that $\frac{dx(t)}{dt} = Ax(t)$ for all $x(t) \in H$.

(iv) Write down the matrix $\Gamma : \mathbb{R}^3 \to \mathbb{R}^3$ that performs differentiation on the expansion coefficients of Φ; i.e., if $x(t) = \Phi\alpha$ and $\frac{dx(t)}{dt} = \Phi\beta$ write down Γ such that $\beta = \Gamma\alpha$.
Hint: Since you know how to differentiate polynomials in this basis, you can just write out the matrix with no calculations.

(v) Let $\Gamma : \mathbb{R}^3 \to \mathbb{R}^3$ be the matrix that performs differentiation on the expansion coefficients of Ψ; i.e., if $x(t) = \Psi\alpha$ and $\frac{dx(t)}{dt} = \Psi\beta$, Γ is such that $\beta = \Gamma\alpha$.

   (a) Write Γ in terms of inner products.

   (b) Find the matrix Γ by calculating the inner products.

Exercise 2. Norms of Oblique Projections
Orthogonal projections are contractions—they either make a vector shorter, or they don't change its length. If $\Pi : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection in a Hilbert space $\mathcal{H}$, we can write this as

$$\|\Pi x\| \leq \|x\|, \ \forall x \in \mathcal{H}.$$

Equivalently, we can say that the spectral norm of Π is upper bounded by 1, and in fact it is equal to one.

(i) Show that $I - \Pi$ is also an orthogonal projection, and that we have

$$\|\Pi\| = \|I - \Pi\| = 1.$$

Assume that the range of Π is non-trivial, and that it is not the whole space $\mathcal{H}$.

For oblique projections, it no longer holds that they are contractions—they can make a vector longer as well. Thus for a general projection $P : \mathcal{H} \to \mathcal{H}$, we will have

$$\|P\| \geq 1.$$

However, it still holds that $\|P\| = \|I - P\|$, and it is the goal of the rest of this exercise to prove this.
Let P be a projection (not necessarily orthogonal), such that neither $\mathcal{R}(P)$ nor $\mathcal{N}(P)$ equal $\mathcal{H}$. Let $u \in \mathcal{H}$ be such that $\|u\| = 1$. Let $x = Pu$ and $y = (I - P)u$.

(ii) Show that $\|u\|^2 = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,\langle x, y\rangle$.

(iii) We want to show that $\|Pu\| \leq \|I - P\|$. Show first that this holds in special cases when $x = 0$ or $y = 0$ (the general claim when $x \neq 0$ and $y \neq 0$ is the subject of the next two subquestions).
Hint: Recall that the norm of any projection is greater than or equal to 1.

(iv) Assume that $x \neq 0$ and $y \neq 0$. Define $w \in \mathcal{H}$ as $w = \tilde{x} + \tilde{y}$, with

$$\tilde{x} = \frac{\|y\|}{\|x\|}x, \quad \tilde{y} = \frac{\|x\|}{\|y\|}y.$$

Prove that $\|w\| = 1$.

(v) Show that $\|Pu\| = \|(I - P)w\|$, and then use the definition of the spectral norm to prove that $\|Pu\| \leq \|I - P\|$. Use it again to prove $\|P\| \leq \|I - P\|$.

(vi) Use symmetry to show that $\|I - P\| \leq \|P\|$. Conclude finally that $\|I - P\| = \|P\|$.

Exercise 3. DFT MATRIX (20 POINTS)
The DFT of a length-N sequence x_n is given by

$$X_k = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}kn}, \qquad k \in \{0, 1, \ldots, N-1\},$$

and x_n is computed from X_k by the inverse DFT as

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{N}kn}, \qquad n \in \{0, 1, \ldots, N-1\}.$$

1. Find matrices A and B, such that

$$\begin{aligned} X &= Ax \\ x &= BX\,. \end{aligned}$$

2. Verify that

$$A \overset{a}{=} B^{-1} \overset{b}{=} NB^*,$$

where B^* is the conjugate-transpose of B.

3. Let C be an $N \times N$ circulant matrix. That is, $c_{ij} = \alpha_{(i-j) \bmod N}$ for some length-N sequence α_n.

Show that the columns of B are the eigenvectors of C and compute the corresponding eigenvalues.

4. Compute the product
$$ACB\,.$$

5. Assume that the computational complexity of applying operators A and B is of order $N \log_2 N$. What is the computational complexity of solving the system
$$Cx \;=\; b\,?$$
Compare this to general (non-circulant) C.

   Compare these costs for $N = 1024$.