

Homework #2 - Due date: 29th November 2019

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PROBLEM 1 - QUICK REVIEW OF CHAPTER 2

- (i) • $A(x) = x * h_A$, $x, h_A \in \ell^2$. It's easy to check that A is a linear operator. Let $y \in \ell^2$ and for the rest of the exercise let $\alpha, \beta \in \mathbb{C}$.

$$(A(\alpha x + \beta y))_n = ((\alpha x + \beta y) * h_A)_n = \sum_{k \in \mathbb{Z}} (\alpha x[k] + \beta y[k]) h_A[n - k] \quad (1)$$

$$= \alpha \sum_{k \in \mathbb{Z}} x[k] h_A[n - k] + \beta \sum_{k \in \mathbb{Z}} y[k] h_A[n - k] = \alpha (A(x))_n + \beta (A(y))_n \quad (2)$$

Moreover, A is also shift invariant so it's LSI. To see this, let $x'[n] := x[n - n_0]$. Then,

$$(A(x'))_n = x' * h_A := \sum_{k \in \mathbb{Z}} x'[n - k] h_A[k] := \sum_{k \in \mathbb{Z}} x[n - k - n_0] h_A[k] = (A(x))_{n - n_0} \quad (3)$$

- $B(x)(t) = x(t) + \text{sinc}(t)$, $x \in \mathcal{L}^2(\mathbb{R})$ is clearly not shift invariant. For the rest of the exercise, let $x'(t) := x(t - t_0)$. In this case

$$B(x')(t) = x'(t) + \text{sinc}(t) := x(t - t_0) + \text{sinc}(t) \neq B(x)(t - t_0) \quad (4)$$

hence B is not LSI.

- $C(x)(t) = x(2t)$, $x \in \mathcal{L}^2(\mathbb{R})$. Note that

$$C(x')(t) = x'(2t) := x(2t - t_0) \neq x(2(t - t_0)) = C(x)(t - t_0) \quad (5)$$

hence C is not LSI.

- $D(x) = \frac{dx}{dt}$, $x \in \mathcal{C}^\infty$. It's well-known that the derivative is linear, but for completeness, let $y \in \mathcal{C}^\infty$

$$D(\alpha x + \beta y) = \frac{d}{dt}(\alpha x + \beta y) = \alpha \frac{dx}{dt} + \beta \frac{dy}{dt} = \alpha D(x) + \beta D(y) \quad (6)$$

so indeed linearity holds for D . By chain rule we have that

$$D(x')(t) = \frac{d}{dt} x'(t) := \frac{d}{dt} x(t - t_0) = D(x)(t - t_0) \left[\frac{d}{dt} (t - t_0) \right] = D(x)(t - t_0) \quad (7)$$

so D is also LSI.

- (iii) By definition of the adjoint operator, we have that

$$\langle C(x)(t), y(t) \rangle_{\mathcal{L}^2(\mathbb{R})} = \int_{-\infty}^{\infty} C(x)(t) y^*(t) dt = \int_{-\infty}^{\infty} x(2t) y^*(t) dt \quad (8)$$

$$= \langle x(t), C^*(y)(t) \rangle_{\mathcal{L}^2(\mathbb{R})} = \int_{-\infty}^{\infty} x(t) (C^*(y)(t))^* dt \quad (9)$$

so we can see that by letting $t' := 2t$,

$$\int_{-\infty}^{\infty} x(2t) y^*(t) dt = \int_{-\infty}^{\infty} x(t') y^* \left(\frac{t'}{2} \right) dt' = \int_{-\infty}^{\infty} x(t') (C^*(y)(t'))^* dt' \quad (10)$$

and hence $C^*(y)(t) = y \left(\frac{t}{2} \right)$.

- (iv) Let $x, y \in \mathcal{H}$ and let $c \geq 0$ be such that $x(t) = 0$ for $|t| \geq c$ and $d \geq 0$ such that $y(t) = 0$ for $|t| \geq d$. Now by definition of the adjoint operator,

$$\langle D(x)(t), y(t) \rangle_{\mathcal{H}} = \int_{\mathcal{H}} D(x)(t) y^*(t) dt = \int_{-d}^d \frac{dx}{dt}(t) y^*(t) dt \quad (11)$$

$$= \langle x(t), D^*(y)(t) \rangle_{\mathcal{H}} = \int_{\mathcal{H}} x(t) (D^*(y)(t))^* dt \quad (12)$$

Using integration by parts, we get that

$$\int_{-d}^d \frac{dx}{dt}(t) y^*(t) dt = x(t) y^*(t) \Big|_{-\min(c,d)}^{\min(c,d)} - \int_{\mathcal{H}} x(t) \frac{dy^*}{dt}(t) dt \quad (13)$$

$$= - \int_{\mathcal{H}} x(t) \frac{dy^*}{dt}(t) dt \quad (14)$$

where the last equality follows since $\mathcal{H} \subset \mathcal{L}^2(\mathbb{R})$, so $y^* = y$, and $x(\pm \min(c, d)) y(\pm \min(c, d)) = 0$ by the finite support property of \mathcal{H} and the definition of c, d . So $D^*(y)(t) = -\frac{dy}{dt}(t)$.

PROBLEM 2 - LCMV AND GSC DERIVATION

- (i) First of all, note that $\arg \max \|\mathbf{x}\| = \arg \max \|\mathbf{x}\|^2 = \arg \max \frac{1}{2} \|\mathbf{x}\|^2$. We can analytically find the local maximum of a function subject to an equality constraint as in this case by using Lagrange multipliers. Let \mathbf{y} be the Lagrange multiplier

$$\mathcal{L}(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \|\mathbf{x}\|^2 + \mathbf{y}^*(\mathbf{b} - A\mathbf{x}) \quad (15)$$

which has its maximum attained at the critical point

$$\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \mathbf{y}) = \mathbf{x} - A^* \mathbf{y} = 0 \iff \mathbf{x} = A^* \mathbf{y} \quad (16)$$

By imposing the constraint, $A\mathbf{x} = AA^* \mathbf{y} = \mathbf{b}$.

- (ii) When $M \leq N$ and A is of full rank, the matrix AA^* is invertible and hence

$$\mathbf{y} = (AA^*)^{-1} \mathbf{b} \implies \mathbf{x} = A^* (AA^*)^{-1} \mathbf{b} \quad (17)$$

- (iii) Using again the same trick as before, one can use the equivalent objective function $\frac{1}{2} \mathbf{h}^* R_x \mathbf{h}$, which gives a Lagrangian of

$$\mathcal{L}(\mathbf{h}, \mathbf{y}) = \frac{1}{2} \mathbf{h}^* R_x \mathbf{h} + \mathbf{y}^*(\mathbf{f} - C^* \mathbf{h}) \quad (18)$$

$$\nabla_{\mathbf{h}} \mathcal{L}(\mathbf{h}, \mathbf{y}) = R_x \mathbf{h} - C \mathbf{y} = 0 \iff R_x \mathbf{h} = C \mathbf{y} \quad (19)$$

Note that R_x is invertible since $R_x \succ 0$ and hence all its eigenvalues are strictly positive, so $\mathbf{h} = R_x^{-1} C \mathbf{y}$. Again we can find the value of the Lagrange multiplier by imposing the constraint

$$C^* \mathbf{h} = C^* R_x^{-1} C \mathbf{y} = \mathbf{f} \quad (20)$$

Assumption 1. The matrix C is full rank and $P \leq M$, meaning that $\dim(\mathcal{R}(C)) = \min(M, P) = P$.

Finally, note that $C^*R_x^{-1}C$ is invertible if Assumption 1 holds. By definition of positive definiteness, we say that $R_x^{-1} \succ 0$ if $\mathbf{x}R_x^{-1}\mathbf{x} > 0 \quad \forall \mathbf{x} \in \mathbb{R}^M \setminus \{\mathbf{0}\}$. The projection $\tilde{\mathbf{x}} := C\mathbf{y}$ is non-zero $\forall \mathbf{y} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ since the nullspace of C is trivial by the rank-nullity theorem, i.e. $\dim(\mathcal{N}(C)) + \dim(\mathcal{R}(C)) = P$ so $\dim(\mathcal{N}(C)) = 0$ in this case, and hence

$$\tilde{\mathbf{x}} := C\mathbf{y} = \mathbf{0} \iff \mathbf{y} = \mathbf{0} \quad (21)$$

Now we can see that $\tilde{\mathbf{x}}^*R_x^{-1}\tilde{\mathbf{x}} > 0 \quad \forall \tilde{\mathbf{x}} \neq \mathbf{0}$ following from the fact that $R_x^{-1} \succ 0$. So using the latter and (21),

$$\mathbf{y}^*C^*R_x^{-1}C\mathbf{y} := \tilde{\mathbf{x}}^*R_x^{-1}\tilde{\mathbf{x}} > 0 \quad \forall \mathbf{y} \in \mathbb{R}^N \setminus \{\mathbf{0}\} \quad (22)$$

and thus $C^*R_x^{-1}C \succ 0$ and hence invertible.

So we can write

$$\mathbf{y} = (C^*R_x^{-1}C)^{-1}\mathbf{f} \implies \mathbf{h} = R_x^{-1}C(C^*R_x^{-1}C)^{-1}\mathbf{f} \quad (23)$$

Just as a sanity check, note that by setting $R_x = I$, $A = C^*$ and $\mathbf{f} = \mathbf{b}$, we recover the solution found in section (ii):

$$R_x^{-1}C(C^*R_x^{-1}C)^{-1}\mathbf{f} = A^*(AA^*)^{-1}\mathbf{b} = \mathbf{x} \quad (24)$$

which can be computed if Assumption 1 holds.

- (iv) In this last case we have an unconstrained problem so we can use simple derivation of the objective function to find where its minimum is attained.

$$f(\mathbf{h}_n) = (\mathbf{h}_0 - C_n\mathbf{h}_n)^*R_x(\mathbf{h}_0 - C_n\mathbf{h}_n) \quad (25)$$

Note that Assumption 1 also holds since computing \mathbf{h}_0 needs the matrix C^*C to be invertible.

Given that a covariance matrix is hermitian,

$$\nabla_{\mathbf{h}_n} f(\mathbf{h}_n) = 2C_n^*R_x(C_n\mathbf{h}_n - \mathbf{h}_0) = 0 \iff C_n^*R_xC_n\mathbf{h}_n = C_n^*R_x\mathbf{h}_0 \quad (26)$$

Moreover, by previous observation and under Assumption 1 $C_n^*R_xC_n$ is invertible, so

$$\mathbf{h}_n = (C_n^*R_xC_n)^{-1}C_n^*R_x\mathbf{h}_0 \quad (27)$$

as claimed.