MATHEMATICAL FOUNDATIONS OF SIGNAL PROCESSING Fall 2019

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Homework 3

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Exercise 1. MAX SNR BEAFORMER
The array signal vector follows the model

$$x_n = a(\theta)\, s_n + e_n = z_n + e_n\,,$$

where s_n is the desired narrowband source signal, θ is the direction of arrival, $a(\theta)$ is the array response vector at θ, while e_n is the noise vector. Assume that the desired source signal and noise are zero-mean WSS, and that noise covariance matrix R_e is positive definite. Show that the maximum SNR beamformer has the form

$$h = \alpha\, R_e^{-1}\, a(\theta)\,.$$

Note that the SNR in this particular case has the form

$$\mathrm{SNR} = \frac{\mathrm{E}[|h^* z_n|^2]}{\mathrm{E}[|h^* e_n|^2]}\,.$$

Exercise 2. SPLINES
We would like to interpolate the function $f(x) = \frac{1}{x}$ at points $x = 1, 4, 5$ with a cubic spline $s(x)$ that satisfies the clamped boundary conditions:

$$\begin{aligned} s'(a) &= f'(a) \\ s'(b) &= f'(b), \end{aligned} \tag{1}$$

for $a = 1$ and $b = 5$.
Let the cubic spline in the interval from $x = 1$ to $x = 4$ be

$$s_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3\,, \tag{2}$$

and the cubic spline in the interval from $x = 4$ to $x = 5$ be

$$s_2(x) = a_2 + b_2(x-4) + c_2(x-4)^2 + d_2(x-4)^3\,. \tag{3}$$

(i) 3) What are the values of a_1 and a_2?

In order to find the values of b_1, c_1, d_1, b_2, c_2 and d_2 in (2) and (3), we try to reach the following matrix notation:

$$\begin{bmatrix} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{bmatrix} \begin{bmatrix} b_1 \\ c_1 \\ d_1 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{bmatrix}$$

(ii) 7) Fill in the empty boxes in the above equation.

Next, we would like to prove that if $s(x)$ is a *cubic* spline that interpolates a function $f(x) \in C^2[a,b]$ at knots $a = x_1 < x_2 < \cdots < x_n = b$ and satisfies the clamped boundary conditions (1), then

$$\int_a^b [s''(x)]^2 \,\mathrm{d}x \le \int_a^b [f''(x)]^2 \,\mathrm{d}x \,. \tag{4}$$

(iii) 8) Define the difference $d(x) = f(x) - s(x)$ and prove that

$$\int_a^b s''(x) d''(x) \mathrm{d}x = 0 \,. \tag{5}$$

Hint: Divide the interval $[a,b]$ *into subintervals and use integration by parts in each subinterval.*

(iv) 7) Using (5), prove (4).

Exercise 3. Splines

Consider the signal $x(t)$ as shown in Figure 1. Note that $x(t)$ is non-zero only for $[0,5]$.

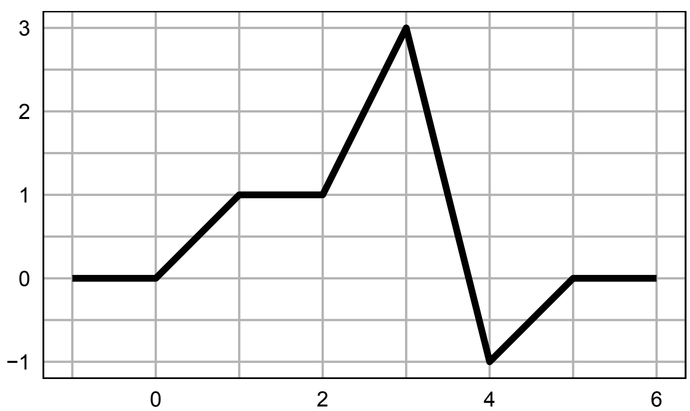

Figure 1: Signal x is non-zero only in interval $[0,5]$.

Recall that $S_{K,\mathbb{Z}}$ is the spline space of degree K with knots at integer values.

1. If we say that $x(t) \in S_{K,\mathbb{Z}}$, what would be the minimum value of K?
2. We can expand $x(t)$ using the causal elementary B-splines of order 1, $\beta_+^{(1)}(t)$:

$$x(t) = \sum_{k\in\mathbb{Z}} a_k \beta_+^{(1)}(t-k) \,.$$

   Find the values of a_k.
3. Define

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau.$$

   The signal $y(t)$ can be expanded as

$$y(t) = \sum_{k\in\mathbb{Z}} b_k \beta_+^{(2)}(t-k) \,.$$

   Find the values of b_k.