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Solution to Homework 1

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Solution 1. Bases and matrix representation of linear operators (25 points)

(i) $\Psi = \{\ \sqrt{1/2}, \quad t\sqrt{3/2}, \quad (3t^2-1)\sqrt{5/8}\ \}$

(ii) $\Gamma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(iii)

$$\beta = \Psi^* y = \Psi^* A x = \underbrace{\Psi^* A \Psi}_{\Gamma} \alpha = \underbrace{\begin{bmatrix} \langle A\psi_0, \psi_0\rangle & \langle A\psi_1, \psi_0\rangle & \langle A\psi_2, \psi_0\rangle \\ \langle A\psi_0, \psi_1\rangle & \langle A\psi_1, \psi_1\rangle & \langle A\psi_2, \psi_1\rangle \\ \langle A\psi_0, \psi_2\rangle & \langle A\psi_1, \psi_2\rangle & \langle A\psi_2, \psi_2\rangle \end{bmatrix}}_{\Gamma} \alpha$$

$\langle A\psi_0, \psi_i\rangle = 0$, $i = 0, 1, 2$, since $A\psi_0 = 0$.
$\langle A\psi_i, \psi_2\rangle = 0$, $i = 0, 1, 2$, since we have no quadratic terms after differentiation.
$\langle A\psi_1, \psi_1\rangle = 0$, since an integral of an even*odd = odd function from -1 to 1.
$\langle A\psi_2, \psi_0\rangle = 0$, since an integral of an odd*even = odd function from -1 to 1.
$\langle A\psi_1, \psi_0\rangle = \sqrt{3/4}\int_{-1}^{1} dt = \sqrt{3}$
$\langle A\psi_2, \psi_1\rangle = 6\sqrt{15/16}\int_{-1}^{1} t^2 dt = \sqrt{15}$

$$\Gamma = \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{bmatrix}$$

Solution 2. Norms of Oblique Projections

(i) We can compute

$$(I - \Pi)^T = I^T - \Pi^T = I - \Pi, \tag{1}$$

since Π is orthogonal projection hence self-adjoint. Therefore, $(I - \Pi)$ is self-adjoint as well. Furthermore,

$$(I - \Pi)(I - \Pi) = I - 2\Pi + \Pi\Pi = I - 2\Pi + \Pi = I - \Pi. \tag{2}$$

So $(I - \Pi)$ is also idempotent. Since it is idempotent and self-adjoint, we conclude that it is an orthogonal projection.

As Π is an orthogonal projection, we know that $\|\Pi x\| \le \|x\|$ for any x. Consequently

$$\|\Pi\| = \sup\{\|\Pi x\| \ :\ \|x\| = 1\} \le 1.$$

Take any $q \in \mathcal{R}(\Pi)$ such that $\|q\| = 1$. It holds that $\Pi q = q$. Such a q exists because the range of Π is not $\{0\}$. Clearly $\|\Pi q\| = \|q\| = 1$ so the above bound is achieved and $\|\Pi\| = 1$. Exactly the same line of reasoning holds for $(I - \Pi)$ by noticing that the range of $(I - \Pi)$ is nontrivial (because the range of Π is not $\mathcal{H}$).

(ii) This follows trivially from the definition (clearly $u = x + y$)

$$\|u^2\| = \langle u, u\rangle = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle y, y\rangle + \langle x, y\rangle + \underbrace{\langle y, x\rangle}_{\langle x,y\rangle^*} = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,\langle x, y\rangle.$$

(iii) $(I-P)$ is also a projection, thus $\|I-P\| \geq 1$. If $x = 0$, we have $Pu = 0$, if $y = 0$, we have $Pu = 1$. In both cases $Pu \leq 1 \leq \|I-P\|$.

(iv) We compute directly,

$$\|w\|^2 = \|\tilde{x}\|^2 + \|\tilde{y}\|^2 + 2\,\mathrm{Re}\,\langle \tilde{x}, \tilde{y}\rangle = \frac{\|y\|^2}{\|x\|^2}\|x\|^2 + \frac{\|x\|^2}{\|y\|^2}\|y\|^2 + 2\,\mathrm{Re}\,\langle \frac{\|y\|}{\|x\|}x, \frac{\|x\|}{\|y\|}y\rangle.$$

Thus $\|w\|^2 = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,\langle x, y\rangle = \|u\|^2 = 1$.

(v) Since $(I-P)x = 0$, we have

$$\|(I-P)w\| = \|(I-P)\frac{\|x\|}{\|y\|}y\| = \|x\| = \|Pu\|.$$

Because $\|I-P\| = \max_{x:\|x\|=1} \|(I-P)x\|$, we have that $\|(I-P)w\| \leq \|I-P\|$. Thus $\|Pu\| \leq \|I-P\|$ for any unit norm u, and consequently $\|P\| \leq \|I-P\|$.

(vi) Let $Q = I - P$. Repeating the above reasoning for Q instead of P, we get $\|Q\| \leq \|I-Q\|$, thus $\|I-P\| \leq \|P\|$. Therefore we must have $\|P\| = \|I-P\|$.

Solution 3. DFT MATRIX

(i) Directly from the definition we read that $A = [a_{mn}]$, $B = [b_{mn}]$, with

$$a_{mn} = e^{-j\frac{2\pi}{N}mn}$$

and

$$b_{mn} = \frac{1}{N}e^{j\frac{2\pi}{N}mn}.$$

(ii) We have that $X = Ax = A(BX) = (AB)X$ and $x = BX = B(AX) = (BA)X$. Therefore $AB = BA = I$ and $A = B^{-1}$. From part (i) it follows that $a_{mn} = Nb^*_{mn}$, so the claim follows.

(iii) Denote by b^i the ith column of B. We compute the kth entry of Cb_i,

$$\begin{aligned}
(Cb^i)_k &= \sum_{l=0}^{N-1} (C)_{kl} b^i_l \\
&= \sum_{l=0}^{N-1} \alpha_{(k-l) \bmod N} \left(\frac{1}{N}e^{j\frac{2\pi}{N}il}\right) \\
&= \frac{1}{N}\sum_{l=0}^{N-1} \alpha_l e^{j\frac{2\pi}{N}i((k-l) \bmod N)} \\
&= \frac{1}{N}\sum_{l=0}^{N-1} \alpha_l e^{j\frac{2\pi}{N}i(k-l)} \\
&= \frac{1}{N}e^{j\frac{2\pi}{N}ik} \underbrace{\sum_{l=0}^{N-1} \alpha_l e^{-j\frac{2\pi}{N}il}}_{\text{This does not depend on } k} \\
&= b^i_k \underbrace{\sum_{l=0}^{N-1} \alpha_l e^{-j\frac{2\pi}{N}il}}_{\lambda_i},
\end{aligned}$$

where we used the commutativity of circular convolution to move the modulo operator to the exponent, and then the periodicity of complex exponentials to get rid of the modulo operator.

(iv) Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. Then

$$ACB = AB\Lambda = NB^*B\Lambda = N\frac{1}{N}B^{-1}B\Lambda = \Lambda.$$

As expected, ACB is a diagonal matrix with eigenvalues along the diagonal.

(v) By previous part,

$$\begin{aligned} & Cx = b \\ \Leftrightarrow\ & A^{-1}\Lambda B^{-1}x = b \\ \Leftrightarrow\ & x = B\Lambda^{-1}Ab. \end{aligned}$$

This amounts to two applications of an operator of complexity order $N\log_2(N)$, and a multiplication by Λ^{-1} which is a diagonal matrix so the complexity is of order N. The total complexity is $O(N\log_2 N)$. To solve the linear system using Gaussian elimination, we need $O(N^3)$ operations.