COM-514: Mathematical Foundations of Signal Processing **Fall 2019**

Homework #1 - Due date: 18^{th} October 2019

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Solution 1: Bases and Matrix Representation of Linear Operators

Part (a)

(i) In order to have $\text{span}(\{\psi_0(t)\}) = \text{span}(\{\varphi_0(t)\})$, we need $\psi_0(t)$ to be a constant, say $\psi_0(t) = c$ with $c \in \mathbb{R} \setminus \{0\}$, and we want $\psi_0(t)$ to have unit norm.

$$\|\psi_0(t)\| := \sqrt{\langle \psi_0(t), \psi_0(t) \rangle} = \sqrt{\int_{-1}^{1} c^2 dt} = \sqrt{[c^2 x]_{-1}^{1}} = c\sqrt{2} = 1 \Longleftrightarrow c = \frac{1}{\sqrt{2}} \tag{1}$$

(ii) Let's check that $\varphi_1(t) \perp \psi_0(t)$:

$$\langle \varphi_1(t), \psi_0(t) \rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} t dt = \frac{1}{\sqrt{2}} \left[\frac{t^2}{2} \right]_{-1}^{1} = 0 \tag{2}$$

Now, let $\psi_1(t) = b\varphi_1(t)$ for $b \in \mathbb{R} \setminus \{0\}$ such that $\psi_1(t)$ has unit norm.

$$\|\psi_1(t)\| := \sqrt{\langle \psi_1(t), \psi_1(t) \rangle} = \sqrt{\int_{-1}^{1} b^2 t^2 dt} = \sqrt{\left[b^2 \frac{t^3}{3} \right]_{-1}^{1}} = b\sqrt{\frac{2}{3}} = 1 \Longleftrightarrow b = \sqrt{\frac{3}{2}} \tag{3}$$

(iii) Finally, let $\psi_2(t) = at^2 + bt + c$ for $a, b, c \in \mathbb{R}$ with $a \neq 0$ in order to span all the polynomials up to order 2. We need $\psi_2(t) \perp \psi_0(t)$ and $\psi_2(t) \perp \psi_1(t)$ in order to form a bases:

$$\langle \psi_2(t), \psi_0(t) \rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} at^2 + bt + cdt = \frac{1}{\sqrt{2}} \left[a\frac{t^3}{3} + b\frac{t^2}{2} + ct \right]_{-1}^{1} = \frac{1}{\sqrt{2}}(a\frac{2}{3} + 2c) = 0 \Longleftrightarrow c = \frac{-a}{3} \tag{4}$$

$$\langle \psi_2(t), \psi_1(t) \rangle = \sqrt{\frac{3}{2}} \int_{-1}^{1} at^3 + bt^2 + ctdt = \sqrt{\frac{3}{2}} \left[a\frac{t^4}{4} + b\frac{t^3}{3} + c\frac{t^2}{2} \right]_{-1}^{1} = \sqrt{\frac{3}{2}} b\frac{2}{3} = 0 \Longleftrightarrow b = 0 \tag{5}$$

Finally we want an orthonormal bases so we need $\|\psi_2(t)\| = 1$. Applying the results in (4), (5), we get:

$$\|\psi_2(t)\| := \sqrt{\langle \psi_2(t), \psi_2(t) \rangle} = \sqrt{\int_{-1}^{1} \left(at^2 - \frac{a}{3} \right)^2 dt} = \sqrt{\left[\frac{a^2 t^5}{5} - \frac{2}{3}\frac{a^2 t^3}{3} + \frac{a^2 t}{9} \right]_{-1}^{1}} \tag{6}$$

$$= \sqrt{\frac{a^2 8}{45}} = 1 \Longleftrightarrow a = \frac{3\sqrt{10}}{4} \tag{7}$$

So $\Psi = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} t, \frac{3\sqrt{10}}{4} t^2 - \frac{\sqrt{10}}{4} \right\}$ is an orthonormal bases for H.

Part (b)

(iv) Let $\Gamma_\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the matrix that performs differentiation on the expansion coefficients of Φ. Then, as we are using the usual base for polynomials, we know that $\frac{d}{dt}\Phi = \{0, 1, 2t\}$, and thus

$$\Gamma_\Phi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

(v) (a) Let $A : H \to H$ be the differentiation operator,

$$\Gamma_\Psi = \Psi^* A \Psi = \begin{bmatrix} \langle A\psi_0, \psi_0\rangle & \langle A\psi_1, \psi_0\rangle & \langle A\psi_2, \psi_0\rangle \\ \langle A\psi_0, \psi_1\rangle & \langle A\psi_1, \psi_1\rangle & \langle A\psi_2, \psi_1\rangle \\ \langle A\psi_0, \psi_2\rangle & \langle A\psi_1, \psi_2\rangle & \langle A\psi_2, \psi_2\rangle \end{bmatrix} = \begin{bmatrix} \langle \frac{d}{dt}\psi_0, \psi_0\rangle & \langle \frac{d}{dt}\psi_1, \psi_0\rangle & \langle \frac{d}{dt}\psi_2, \psi_0\rangle \\ \langle \frac{d}{dt}\psi_0, \psi_1\rangle & \langle \frac{d}{dt}\psi_1, \psi_1\rangle & \langle \frac{d}{dt}\psi_2, \psi_1\rangle \\ \langle \frac{d}{dt}\psi_0, \psi_2\rangle & \langle \frac{d}{dt}\psi_1, \psi_2\rangle & \langle \frac{d}{dt}\psi_2, \psi_2\rangle \end{bmatrix} \tag{9}$$

since Ψ defines an orthonormal bases.

(b) Numerically, we have that

$$\Gamma_\Psi = \begin{bmatrix} \langle 0, \frac{1}{\sqrt{2}}\rangle & \langle \sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}}\rangle & \langle \frac{3\sqrt{10}}{2}t, \frac{1}{\sqrt{2}}\rangle \\ \langle 0, \sqrt{\frac{3}{2}}t\rangle & \langle \sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}t\rangle & \langle \frac{3\sqrt{10}}{2}t, \sqrt{\frac{3}{2}}t\rangle \\ \langle 0, \frac{3\sqrt{10}}{4}t^2 - \frac{\sqrt{10}}{4}\rangle & \langle \sqrt{\frac{3}{2}}, \frac{3\sqrt{10}}{4}t^2 - \frac{\sqrt{10}}{4}\rangle & \langle \frac{3\sqrt{10}}{2}t, \frac{3\sqrt{10}}{4}t^2 - \frac{\sqrt{10}}{4}\rangle \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

Solution 2: Norms of Oblique Projections

(i)

Fact 1. *If $\Pi : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection in a Hilbert space $\mathcal{H}$,*

$$\|\Pi x\| \leq \|x\|, \quad \forall x \in \mathcal{H} \tag{11}$$

Using Fact 1, we have that,

$$\|\Pi\| := \max_{x:\|x\|=1} \|\Pi x\| \leq \max_{x:\|x\|=1} \|x\| = 1 \tag{12}$$

Π is an orthogonal projection iff $\Pi\Pi = \Pi$ and $\Pi^T = \Pi$. Mixing both properties, we have that $\Pi^T\Pi = \Pi\Pi = \Pi$. Let's prove that $I - \Pi$ is also an orthogonal projection

$$(I - \Pi)^T = I^T - \Pi^T = I - \Pi \tag{13}$$

$$(I - \Pi)(I - \Pi) = II - \Pi I - I\Pi + \Pi\Pi = I - 2\Pi + \Pi = I - \Pi \tag{14}$$

By idempotence (so this also holds for oblique projections) and definition of the spectral norm,

$$\|\Pi x\| = \|\Pi(\Pi x)\| \leq \|\Pi\| \, \|\Pi x\| \iff \|\Pi\| \geq 1 \tag{15}$$

So putting together (12), (15) we get that $\|\Pi\| = 1$. Moreover, since $(I - \Pi)$ is also an orthogonal projection as proved before,

$$\|\Pi\| = \|I - \Pi\| = 1 \tag{16}$$

(ii)

$$
\begin{aligned}
\|x\|^2 + \|y\|^2 &:= \langle Pu, Pu\rangle + \langle (I-P)u, (I-P)u\rangle && (17)\\
&= \langle Pu, Pu\rangle + \langle u, (I-P)u\rangle - \langle Pu, (I-P)u\rangle && (18)\\
&= \langle Pu, Pu\rangle + \langle (I-P)u, u\rangle^* - \langle (I-P)u, Pu\rangle^* && (19)\\
&= \langle Pu, Pu\rangle^* + \langle u, u\rangle^* - \langle Pu, u\rangle^* - \langle u, Pu\rangle^* + \langle Pu, Pu\rangle^* && (20)
\end{aligned}
$$

where I only used fundamental properties of the inner product and in the last equality, I use the fact that a norm is real (so $\|x\|^2 = \langle x, x\rangle = \langle x, x\rangle^*$).

By rearranging terms in (20), we get that

$$
\begin{aligned}
\|x\|^2 + \|y\|^2 &= \|u\| - (\langle Pu, u\rangle^* - \langle Pu, Pu\rangle^*) - (\langle u, Pu\rangle^* - \langle Pu, Pu\rangle^*) && (21)\\
&= \|u\| - (\langle u, Pu\rangle - \langle Pu, Pu\rangle) - \langle (I-P)u, Pu\rangle^* && (22)\\
&= \|u\| - \langle (I-P)u, Pu\rangle - \langle Pu, (I-P)u\rangle && (23)\\
&= \|u\| - (\langle Pu, (I-P)u\rangle^* + \langle Pu, (I-P)u\rangle) && (24)\\
&:= \|u\| - 2\mathbb{R}e\langle x, y\rangle && (25)
\end{aligned}
$$

(iii) In the case that $x = 0$,

$$
\|Pu\| := \|x\| = 0 \le \|I - P\| \qquad (26)
$$

which follows from non-negativity of norms.

Let $P' := I - P$. We have that

$$
P'^2 = (I-P)(I-P) = I - P - P + P^2 = I - P := P' \qquad (27)
$$

so P' is an oblique projection.

If $y := (I-P)u = 0$, we have that $u = Pu$, and hence

$$
\|Pu\| = \|u\| = 1 \le \|I - P\| \qquad (28)
$$

where the inequality holds since the norm of any projection is greater or equal than one as proved in (15).

(iv)

$$
\begin{aligned}
\|w\|^2 &:= \|\tilde{x} + \tilde{y}\|^2 := \langle \tilde{x} + \tilde{y}, \tilde{x} + \tilde{y}\rangle = \langle \tilde{x}, \tilde{x} + \tilde{y}\rangle + \langle \tilde{y}, \tilde{x} + \tilde{y}\rangle && (29)\\
&= \langle \tilde{x}, \tilde{x}\rangle^* + \langle \tilde{y}, \tilde{x}\rangle^* + \langle \tilde{x}, \tilde{y}\rangle^* + \langle \tilde{y}, \tilde{y}\rangle^* = \|y\|^2 + \langle y, x\rangle^* + \langle x, y\rangle^* + \|x\|^2 && (30)\\
&= \|y\|^2 + \langle x, y\rangle + \langle x, y\rangle^* + \|x\|^2 = \|x\|^2 + \|y\|^2 + 2\mathbb{R}e\langle x, y\rangle && (31)\\
&= \|u\|^2 && (32)
\end{aligned}
$$

where the last step follows from (25). Moreover, since $\|u\| = 1$ by construction, $\|w\| = 1$.

(v)

$$
\begin{aligned}
(I-P)w &:= (I-P)\frac{\|y\|}{\|x\|}x + (I-P)\frac{\|x\|}{\|y\|}y && (33)\\
&= \frac{\|y\|}{\|x\|}x - \frac{\|y\|}{\|x\|}Px + \frac{\|x\|}{\|y\|}y - \frac{\|x\|}{\|y\|}Py && (P^2 = P \Rightarrow Px := PPu = Pu := x) \quad (34)\\
&= \frac{\|x\|}{\|y\|}y - \frac{\|x\|}{\|y\|}Py := \frac{\|x\|}{\|y\|}(I-P)u - \frac{\|x\|}{\|y\|}P(I-P)u && (35)\\
&= \frac{\|x\|}{\|y\|}u - \frac{\|x\|}{\|y\|}Pu - \frac{\|x\|}{\|y\|}Pu + \frac{\|x\|}{\|y\|}Pu && (P^2 = P) \quad (36)\\
&= \frac{\|x\|}{\|y\|}(I-P)u := \frac{\|x\|}{\|y\|}y && (37)
\end{aligned}
$$

So using (37), we have that

$$\|(I-P)w\| = \frac{\|x\|}{\|y\|}\|y\| = \|x\| := \|Pu\| \tag{38}$$

So taking equations (32), (38) and by definition of the spectral norm,

$$\|Pu\| = \|(I-P)w\| \leq \|I-P\|\,\|w\| = \|I-P\| \tag{39}$$

Applying again the definition of spectral norm and given that $\|u\| = 1$,

$$\|I-P\| \geq \|Pu\| \geq \|P\|\,\|u\| = \|P\| \tag{40}$$

which concludes the proof.

(vi) Take $P' := I - P$ as in (27), where it was proved that P' is an oblique projection so (40) holds and we have that

$$\|I-P\| := \|P'\| \leq \|I-P'\| := \|I-(I-P)\| = \|P\| \tag{41}$$

By mixing (40) and (41), we have that $\|I-P\| = \|P\|$.

Solution 3: DFT Matrix

1. This matrices have coefficients

$$A_{k,n} = e^{-j\frac{2\pi kn}{N}} \quad ; \quad B_{n,k} = \frac{1}{N}e^{j\frac{2\pi kn}{N}} \tag{42}$$

$$\begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \quad ; \quad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} \tag{43}$$

with $n, k \in \{1, 2, \ldots, N\}$.

2. Note that equality (b) is trivial to prove, since the coefficients of the matrices A and B as expressed in (42) satisfy $A_{k,n} = NB^*_{n,k}$, so $A = NB^*$.

If equality (a) is satisfied, then $AB = BA = I_{N\times N}$. Let's start by proving that A is a left-inverse of B.

$$(AB)_{i,j} = \frac{1}{N}\sum_{l=0}^{N-1} e^{-j\frac{2\pi il}{N}} e^{j\frac{2\pi lj}{N}} \tag{44}$$

So here we can distinguish two cases. On the one hand, if $i = j$, sum simplifies to

$$(AB)_{i,j} = \frac{1}{N}\sum_{l=0}^{N-1} 1 = 1 \tag{45}$$

On the other hand, if $i \neq j$,

$$(AB)_{i,j} = \frac{1-(e^{j\frac{2\pi N}{N}})^{j-i}}{1-e^{j\frac{2\pi(j-i)}{N}}} = \frac{1-1^{j-i}}{1-e^{j\frac{2\pi(j-i)}{N}}} = 0 \tag{46}$$

so A is a left-inverse of B. Now, let's see if it's actually a proper inverse.

$$(BA)_{i,j} = \frac{1}{N}\sum_{l=0}^{N-1} e^{-j\frac{2\pi lj}{N}} e^{j\frac{2\pi il}{N}} = (AB)_{j,i} = \mathbb{1}_{\{i=j\}} \tag{47}$$

and hence we can conclude that $A = B^{-1}$.

3. Let $b_k(n) := \frac{1}{N} e^{j\frac{2\pi k n}{N}}$. If $b_k := [b_k(0), \dots, b_k(n)]^T$ is an eigenvector of C, then $Cb_k = \lambda_k b_k$.

   Let $\mathcal{A}_k$ be the sequence resulting of applying the Inverse Discrete Fourier Transform to the sequence α_n for $k, n \in \{0, 1, \dots, N-1\}$.

$$(Cb_k)_0 = \sum_{l=0}^{N-1} b_k(l)\alpha_l = \alpha_0 b_k(0) + \alpha_1 b_k(1) + \cdots \tag{48}$$

$$:= \mathcal{A}_k = N\mathcal{A}_k b_k(0) \tag{49}$$

   so it seems $\lambda_k = N\mathcal{A}_k$.

   Note that $b_k(n+1) = b_k(n)e^{j\frac{2\pi k}{N}}$ and that $b_k(N) := \frac{1}{N}e^{j2\pi k} = \frac{1}{N} = b_k(0)$.

$$\begin{aligned}
(Cb_k)_1 &= \sum_{l=0}^{N-1} b_k(l)\alpha_{(1-l) \mod N} = \alpha_{N-1}b_k(0) + \alpha_0 b_k(1) + \alpha_1 b_k(2) + \cdots && (50)\\
&= \alpha_{N-1}b_k(0) + e^{j\frac{2\pi k}{N}}(\alpha_0 b_k(0) + \alpha_1 b_k(1) + \cdots) && (51)\\
&= \alpha_{N-1}b_k(0) + e^{j\frac{2\pi k}{N}} N\mathcal{A}_k b(0) - e^{j\frac{2\pi k}{N}}\alpha_{N-1}b(N-1) = \alpha_{N-1}b_k(0) + N\mathcal{A}_k b(1) - \alpha_{N-1}b(N) && (52)\\
&= N\mathcal{A}_k b(1) && (53)
\end{aligned}$$

   The general form of $(Cb_k)_i$ is

$$(Cb_k)_i = \sum_{l=0}^{N-1} b_k(l)\alpha_{(i-l) \mod N} \tag{54}$$

   Let's finish the proof by induction on i. We can do so since by the circular property of the matrix, we can express $(Cb_k)_{i+1}$ in terms of $(Cb_k)_i$.

   I already proved the base cases $i = 0, 1$ before. Now suppose by induction hypothesis that $(Cb_k)_i = N\mathcal{A}_k b_k(i)$ holds for the $i-$th coefficient and let $\beta = (i+1) \mod N = 0$. Note that the sequence $\alpha_{(i+1-l) \mod N}$ is $\alpha_\beta, \alpha_{\beta+1}, \dots, \alpha_{\beta+N-1}$ and the sequence $\alpha_{(i-l) \mod N}$ is $\alpha_{\beta+1}, \alpha_{\beta+2}, \dots, \alpha_{\beta+N-1}, \alpha_\beta$.

$$\begin{aligned}
(Cb_k)_{i+1} &= \sum_{l=0}^{N-1} b_k(l)\alpha_{(i+1-l) \mod N} && (55)\\
&= \alpha_\beta b_k(0) + \alpha_{\beta+1}b_k(1) + \alpha_{\beta+2}b_k(2) + \cdots && (56)\\
&= \alpha_\beta b_k(0) + e^{j\frac{2\pi k}{N}}(\alpha_{\beta+1}b_k(0) + \alpha_{\beta+2}b_k(1) + \cdots) && (57)\\
&= \alpha_\beta b_k(0) + e^{j\frac{2\pi k}{N}} N\mathcal{A}_k b_k(i) - e^{j\frac{2\pi k}{N}}\alpha_\beta b(N-1) && \text{By induction hypothesis} \quad (58)\\
&= \alpha_\beta b_k(0) + e^{j\frac{2\pi k}{N}} N\mathcal{A}_k b_k(i) - \alpha_\beta b(N) && (59)\\
&= N\mathcal{A}_k b_k(i)e^{j\frac{2\pi k}{N}} = N\mathcal{A}_k b_k(i+1) && (60)
\end{aligned}$$

4. By the findings of the previous section, we know that

$$CB = C\begin{bmatrix} | & & | \\ b_0 & \dots & b_{N-1} \\ | & & | \end{bmatrix} = N\begin{bmatrix} | & & | \\ \mathcal{A}_0 b_0 & \dots & \mathcal{A}_{N-1}b_{N-1} \\ | & & | \end{bmatrix} \tag{61}$$

   Since $A = B^{-1}$, $AB = I$ so

$$ACB = N\mathrm{diag}(\mathcal{A}) = N\begin{bmatrix} \mathcal{A}_0 & & 0 \\ & \ddots & \\ 0 & & \mathcal{A}_{N-1} \end{bmatrix} \tag{62}$$

5. Given that we can run `FFT` in $\mathcal{O}(n \log_2 n)$, using the analytical result of the operation ACB obtained in (62) is computationally much cheaper than computing the direct matrix product in case of not having a circular matrix. In this case, we would require $\mathcal{O}(n^3)$ operations (or in the best case $\mathcal{O}(n^{2.373})$, which is the asymptotically fastest known algorithm to perform $n \times n$ matrix multiplication).

   Numerically, for $N = 1024$, which is a common number of DFT coefficients (power of 2 so `fft` performs better) the comparison is shown in Table 1.

Algorithm	Asymptotic number of operations
`FFT` (C circulant)	$10,240$
Matrix multiplication	$1,073,741,824$
Efficient matrix multiplication	$\sim 13,913,673$

Table 1: Comparison with non-circulant C