COM-514: Mathematical Foundations of Signal Processing **Fall 2019**

Homework #2 - Due date: 29^{th} November 2019

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Problem 1 - Quick Review of Chapter 2

(i)

- $A(x) := x * h_A, \quad x, h_A \in \ell^2$. It's easy to check that A is a linear operator. Let $y \in \ell^2$ and for the rest of the exercise let $\alpha, \beta \in \mathbb{C}$.

$$(A(\alpha x + \beta y))_n := ((\alpha x + \beta y) * h_A)_n := \sum_{k \in \mathbb{Z}} (\alpha x[k] + \beta y[k]) h_A[n-k] \tag{1}$$

$$= \alpha \sum_{k \in \mathbb{Z}} x[k] h_A[n-k] + \beta \sum_{k \in \mathbb{Z}} y[k] h_A[n-k] =: \alpha (A(x))_n + \beta (A(y))_n \tag{2}$$

Moreover, A is also shift invariant so it's LSI. To see this, let $x'[n] := x[n-n_0]$. Then,

$$(A(x'))_n := x' * h_A := \sum_{k \in \mathbb{Z}} x'[n-k] h_A[k] := \sum_{k \in \mathbb{Z}} x[n-k-n_0] h_A[k] =: (A(x))_{n-n_0} \tag{3}$$

- $B(x)(t) := x(t) + \text{sinc}(t), \quad x \in \mathcal{L}^2(\mathbb{R})$ is clearly not shift invariant. For the rest of the exercise, let $x'(t) := x(t-t_0)$. In this case

$$B(x')(t) := x'(t) + \text{sinc}(t) := x(t-t_0) + \text{sinc}(t) \neq x(t-t_0) + \text{sinc}(t-t_0) =: B(x)(t-t_0) \tag{4}$$

hence B is not LSI.

- $C(x)(t) := x(2t), \quad x \in \mathcal{L}^2(\mathbb{R})$. Note that

$$C(x')(t) := x'(2t) := x(2t - t_0) \neq x(2(t-t_0)) =: C(x)(t-t_0) \tag{5}$$

hence C is not LSI.

- $D(x) := \frac{dx}{dt}, \quad x \in \mathcal{C}^\infty$. It's well-known that the derivative is linear, but for completeness, let $y \in \mathcal{C}^\infty$

$$D(\alpha x + \beta y) := \frac{d}{dt}(\alpha x + \beta y) = \alpha \frac{dx}{dt} + \beta \frac{dy}{dt} =: \alpha D(x) + \beta D(y) \tag{6}$$

so indeed linearity holds for D. By chain rule we have that

$$D(x')(t) := \frac{d}{dt} x'(t) := \frac{d}{dt} x(t-t_0) =: D(x)(t-t_0) \left[\frac{d}{dt}(t-t_0)\right] = D(x)(t-t_0) \tag{7}$$

so D is also LSI.

(iii) By definition of the adjoint operator, we have that

$$\langle C(x)(t), y(t) \rangle_{\mathcal{L}^2(\mathbb{R})} = \int_{-\infty}^{\infty} C(x)(t) y^*(t) dt := \int_{-\infty}^{\infty} x(2t) y^*(t) dt \tag{8}$$

$$= \langle x(t), C^*(y)(t) \rangle_{\mathcal{L}^2(\mathbb{R})} = \int_{-\infty}^{\infty} x(t) (C^*(y)(t))^* dt \tag{9}$$

so we can see that by letting $t' := 2t$,

$$\int_{-\infty}^{\infty} x(2t) y^*(t) dt = \int_{-\infty}^{\infty} x(t') y^* \left(\frac{t'}{2}\right) dt' =: \int_{-\infty}^{\infty} x(t') (C^*(y)(t'))^* dt' \tag{10}$$

and hence $C^*(y)(t) = y\left(\frac{t}{2}\right)$.

(iv) Let $x, y \in \mathcal{H}$ and let $c \geq 0$ be such that $x(t) = 0$ for $|t| \geq c$ and $d \geq 0$ such that $y(t) = 0$ for $|t| \geq d$. Now by definition of the adjoint operator,

$$\langle D(x)(t), y(t)\rangle_{\mathcal{H}} = \int_{\mathcal{H}} D(x)(t)y^*(t)dt := \int_{-d}^{d} \frac{dx}{dt}(t)y^*(t)dt \tag{11}$$

$$= \langle x(t), D^*(y)(t)\rangle_{\mathcal{H}} = \int_{\mathcal{H}} x(t)(D^*(y)(t))^*dt \tag{12}$$

Using integration by parts, we get that

$$\int_{-d}^{d} \frac{dx}{dt}(t)y^*(t)dt = x(t)y^*(t)\Big|_{-\min(c,d)}^{\min(c,d)} - \int_{\mathcal{H}} x(t)\frac{dy^*}{dt}(t)dt \tag{13}$$

$$= -\int_{\mathcal{H}} x(t)\frac{dy^*}{dt}(t)dt =: \int_{\mathcal{H}} x(t)(D^*(y)(t))^*dt \tag{14}$$

where the penultimate equality follows since $\mathcal{H} \subset \mathcal{L}^2(\mathbb{R})$, so $y^* = y$, and $x(\pm\min(c,d))y(\pm\min(c,d)) = 0$ by the finite support property of $\mathcal{H}$ and the definition of c, d. So $D^*(y)(t) = -\frac{dy}{dt}(t)$.

Problem 2 - LCMV and GSC Derivation

(i) First of all, note that $\arg\max \|\mathbf{x}\| = \arg\max \|\mathbf{x}\|^2 = \arg\max \frac{1}{2}\|\mathbf{x}\|^2$. We can analytically find the local maximum of a function subject to an equality constraint as in this case by using Lagrange multipliers. Let $\mathbf{y}$ be the Lagrange multiplier

$$\mathcal{L}(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\|\mathbf{x}\|^2 + \mathbf{y}^*(\mathbf{b} - A\mathbf{x}) \tag{15}$$

which has its maximum attained at the critical point

$$\nabla_{\mathbf{x}}\mathcal{L}(\mathbf{x}, \mathbf{y}) = \mathbf{x} - A^*\mathbf{y} = 0 \Longleftrightarrow \mathbf{x} = A^*\mathbf{y} \tag{16}$$

By imposing the constraint, $A\mathbf{x} = AA^*\mathbf{y} = \mathbf{b}$.

(ii) When $M \leq N$ and A is of full rank, the matrix AA^* is invertible and hence

$$\mathbf{y} = (AA^*)^{-1}\mathbf{b} \Longrightarrow \mathbf{x} = A^*(AA^*)^{-1}\mathbf{b} \tag{17}$$

(iii) Using again the same trick as before, one can use the equivalent objective function $\frac{1}{2}\mathbf{h}^*R_x\mathbf{h}$, which gives a Lagrangian of

$$\mathcal{L}(\mathbf{h}, \mathbf{y}) = \frac{1}{2}\mathbf{h}^*R_x\mathbf{h} + \mathbf{y}^*(\mathbf{f} - C^*\mathbf{h}) \tag{18}$$

$$\nabla_{\mathbf{h}}\mathcal{L}(\mathbf{h}, \mathbf{y}) = R_x\mathbf{h} - C\mathbf{y} = 0 \Longleftrightarrow R_x\mathbf{h} = C\mathbf{y} \tag{19}$$

Note that R_x is invertible since $R_x \succ 0$, so $\mathbf{h} = R_x^{-1}C\mathbf{y}$. Again we can find the value of the Lagrange multiplier by imposing the constraint

$$C^*\mathbf{h} = C^*R_x^{-1}C\mathbf{y} = \mathbf{f} \tag{20}$$

Assumption 1. *The matrix C is full rank and $P \leq M$, meaning that* $\dim(\mathcal{R}(C)) = \min(M, P) = P$.

Finally, we have that $C^* R_x^{-1} C$ is invertible if Assumption 1 holds. To proof this, I will proceed in a few steps.

First of all note that $R_x^{-1} \succ 0$. This is easier to see when analysing the eigenvalue decomposition of R_x. Given that the covariance matrix is hermitian, it can be written as $R_x = U \Lambda U^*$, where U is a unitary matrix and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix. Given that R_x is positive definite, $R_x^{-1} = U \Lambda^{-1} U^*$ is also positive definite and hence invertible since $\Lambda^{-1} = (\lambda_1^{-1}, \ldots, \lambda_n^{-1})$ and $\lambda_i > 0 \Longrightarrow \lambda_i^{-1} > 0$. Note that invertibility is trivial to see since $(R_x^{-1})^{-1} = R_x$.

By definition of positive definiteness, we say that $R_x^{-1} \succ 0$ if $\mathbf{x} R_x^{-1} \mathbf{x} > 0 \quad \forall \mathbf{x} \in \mathbb{R}^M \setminus \{\mathbf{0}\}$. The projection $\tilde{\mathbf{x}} := C\mathbf{y}$ is non-zero $\forall \mathbf{y} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ since the nullspace of C is trivial by the rank-nullity theorem. In this case, this translates to $\dim(\mathcal{N}(C)) + \dim(\mathcal{R}(C)) = P$, meaning that $\dim(\mathcal{N}(C)) = 0$. Hence,

$$\tilde{\mathbf{x}} := C\mathbf{y} = \mathbf{0} \Longleftrightarrow \mathbf{y} = \mathbf{0} \tag{21}$$

Now we can see that $\tilde{\mathbf{x}}^* R_x^{-1} \tilde{\mathbf{x}} > 0 \quad \forall \tilde{\mathbf{x}} \neq \mathbf{0}$ following from the fact that $R_x^{-1} \succ 0$. So using the latter and the equivalence of (21),

$$\mathbf{y}^* C^* R_x^{-1} C \mathbf{y} := \tilde{\mathbf{x}}^* R_x^{-1} \tilde{\mathbf{x}} > 0 \quad \forall \mathbf{y} \in \mathbb{R}^N \setminus \{0\} \tag{22}$$

and thus $C^* R_x^{-1} C \succ 0$ and hence invertible.

So we can write

$$\mathbf{y} = (C^* R_x^{-1} C)^{-1} \mathbf{f} \Longrightarrow \mathbf{h} = R_x^{-1} C (C^* R_x^{-1} C)^{-1} \mathbf{f} \tag{23}$$

Just as a sanity check, note that by setting $R_x = I$, $A = C^*$ and $\mathbf{f} = \mathbf{b}$, we recover the solution found in section (ii):

$$R_x^{-1} C (C^* R_x^{-1} C)^{-1} \mathbf{f} = A^* (AA^*)^{-1} \mathbf{b} = \mathbf{x} \tag{24}$$

which can be computed if Assumption 1 holds.

(iv) In this last case we have an unconstrained problem so we can use simple derivation of the objective function to find where its minimum is attained.

$$f(\mathbf{h}_n) := (\mathbf{h}_0 - C_n \mathbf{h}_n)^* R_x (\mathbf{h}_0 - C_n \mathbf{h}_n) \tag{25}$$

Note that we also need Assumption 1 to hold, since computing $\mathbf{h}_0$ needs the matrix $C^* C$ to be invertible.

Given that a covariance matrix is hermitian,

$$\nabla_{\mathbf{h}_n} f(\mathbf{h}_n) = 2 C_n^* R_x (C_n \mathbf{h}_n - \mathbf{h}_0) = 0 \Longleftrightarrow C_n^* R_x C_n \mathbf{h}_n = C_n^* R_x \mathbf{h}_0 \tag{26}$$

Moreover, by previous observation and under Assumption 1, $C_n^* R_x C_n$ is invertible, so

$$\mathbf{h}_n = (C_n^* R_x C_n)^{-1} C_n^* R_x \mathbf{h}_0 \tag{27}$$

as claimed.