Adam Scholefield, Mihailo Kolundžija, Reza Parhizkar
Teaching assistants: Michalina Pacholska

Solution to Homework 3

Monday, October 8, 2018

Solution 1. Max SNR beamformer
The SNR at the beamformer's output then has the following form:

$$\mathrm{SNR} = \frac{E[|s_n\, h^* a(\theta)|^2]}{E[|h^* e_n|^2]} = \sigma_s^2 \frac{h^* a(\theta)\, a^*(\theta)\, h}{h^* R_e\, h}\,. \tag{1}$$

Since R_e is a positive definite Hermitian matrix, we can write

$$R_e = B\, B^*\,,$$

which we use to construct

$$u = B^* h$$

and consequently

$$h = (B^*)^{-1} u\,. \tag{2}$$

Substituting (2) into (1) gives

$$\mathrm{SNR} = \sigma_s^2 \frac{u^* B^{-1} a(\theta)\, a^*(\theta) (B^*)^{-1} u}{u^* u} = \sigma_s^2 \frac{|\langle u, B^{-1} a(\theta)\rangle|^2}{\langle u, u\rangle}\,.$$

Using Cauchy-Schwarz inequality, we further have

$$\mathrm{SNR} \leq \sigma_s^2 \frac{\langle B^{-1} a(\theta), B^{-1} a(\theta)\rangle \langle u, u\rangle}{\langle u, u\rangle}\,,$$

and we know that the equality holds when

$$u = \alpha\, B^{-1} a(\theta)\,.$$

This finally gives us

$$h = (B^*)^{-1} u = \alpha\, (B^*)^{-1} B^{-1} u = \alpha\, R_e^{-1} u\,.$$

Solution 2. Splines

1. According to the condition of interpolation on the knots, we need

$$s_1(1) = f(1) \quad \Rightarrow \quad a_1 = 1$$
$$s_2(4) = f(4) \quad \Rightarrow \quad a_2 = \frac{1}{4}$$

2. interpolation condition on the knots gives the following conditions:

$$s_1(4) = f(4) \quad \Rightarrow \quad 1 + 3b_1 + 9c_1 + 27d_1 = \frac{1}{4}$$
$$s_2(5) = f(5) \quad \Rightarrow \quad \frac{1}{4} + b_2 + c_2 + d_2 = \frac{1}{5}$$

continuity of the first and second derivative of the spline in the inner knot gives:

$$\begin{aligned} s_1'(4) = s_2'(4) \quad &\Rightarrow \quad b_1 + 6c_1 + 27d_1 = b_2 \\ s_1''(4) = s_2''(4) \quad &\Rightarrow \quad 2c_1 + 18d_1 = 2c_2 \end{aligned}$$

lastly, the clamped boundary conditions result in:

$$\begin{aligned} s_1'(1) = f'(1) \quad &\Rightarrow \quad b_1 = -1 \\ s_2'(5) = f'(5) \quad &\Rightarrow \quad b_2 + 2c_2 + 3d_2 = -\frac{1}{25} \end{aligned}$$

Thus we have

$$\begin{bmatrix} 3 & 9 & 27 & 0 & 0 & 0 \\ 1 & 6 & 27 & -1 & 0 & 0 \\ 0 & 2 & 18 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ c_1 \\ d_1 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ 0 \\ 0 \\ -1 \\ -\frac{1}{25} \\ -\frac{1}{20} \end{bmatrix}$$

3. Using integration by parts, we have

$$\int_a^b s''(x)d''(x)dx = s''(x)d'(x)|_a^b - \int_a^b s'''(x)d'(x)dx\,.$$

Note that because of the clamped boundary condition $d'(a) = f'(a) - s'(a) = 0 = f'(b) - s'(b) = d'(b)$, thus the first term above is zero.

In order to compute the second term we devide the period into subintervals as follows

$$\begin{aligned} -\int_a^b s'''(x)d'(x)dx = \\ &= -\sum_{k=1}^{n-1} \int_{x_k}^{x_{k+1}} s'''(x)d'(x)dx \\ &= -\sum_{k=1}^{n-1} s'''(x)d(x)|_{x_k}^{x_{k+1}} - \int_{x_k}^{x_{k+1}} s''''(x)d(x)dx \\ &= 0 \end{aligned}$$

But note that the first term above is zero because of the interpolation condition in the knots ($d(x_k) = 0$). The second term is also zero because $s(x)$ is a cubic spline thus $s''''(x) = 0$.

4.

$$\begin{aligned} \int_a^b (f''(x))^2 dx &= \int_a^b (d''(x) - s''(x))^2 dx \\ &\overset{a}{=} \int_a^b (d''(x))^2 dx + \int_a^b (s''(x))^2 dx - 2\int_a^b d''(x)s''(x)dx \\ &= \int_a^b (d''(x))^2 dx + \int_a^b (s''(x))^2 dx \end{aligned}$$

where in (a), we have used the result from the previous part. This proves the statement.

Solution 3. Splines

1. Recall that $S_{K,\mathbb{Z}}$ is the spline space of degree K with knots at integer values. Since x is a piece-wise linear function, then the minimum value for K would be 1.

2. We have $a_0 = 1, a_1 = 1, a_2 = 3, a_4 = -1$. The rest of a_i's are zero.
3. From the course we know that
$$b_i = \sum_{j \leq i} a_j$$
Thus,
$$\cdots, b_{-2} = 0, b_{-1} = 0, b_0 = 1, b_1 = 2, b_2 = 5, b_4 = 4, b_5 = 4, b_6 = 4, \cdots$$