COM-514: Mathematical Foundations of Signal Processing **Fall 2019**

Homework #3 - Due date: 20^{th} December 2019

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Problem 1 - Maximum SNR Beamformer

Let R_z be the covariance matrix of the signal and R_e the covariance matrix of the noise. Given that both are zero-mean WSS,

$$\text{SNR} = \frac{\mathbb{E}[|\mathbf{h}^* z_n|^2]}{\mathbb{E}[|\mathbf{h}^* e_n|^2]} = \frac{\mathbf{h}^* R_z \mathbf{h}}{\mathbf{h}^* R_e \mathbf{h}} \tag{1}$$

To find the maximum, equate the gradient to zero. Given that any covariance matrix is self-adjoint,

$$\nabla_{\mathbf{h}} \text{SNR} = \frac{2R_z \mathbf{h}\mathbf{h}^* R_e \mathbf{h} - 2\mathbf{h}^* R_z \mathbf{h} R_e \mathbf{h}}{(\mathbf{h}^* R_e \mathbf{h})^2} = 0 \tag{2}$$

Given that $R_e \succ 0$, we can invert this matrix and we get the generalized eigenvalue problem.

$$R_z \mathbf{h} = \frac{\mathbf{h}^* R_z \mathbf{h}}{\mathbf{h}^* R_e \mathbf{h}} R_e \mathbf{h} \Longrightarrow R_e^{-1} R_z \mathbf{h} = \frac{\mathbf{h}^* R_z \mathbf{h}}{\mathbf{h}^* R_e \mathbf{h}} \mathbf{h} \tag{3}$$

Now, given that $z_n = \mathbf{a}(\theta) s_n$ and s_n is a narrow-band zero-mean WSS signal, $R_z = \sigma_s^2 \mathbf{a}(\theta)\mathbf{a}^*(\theta)$. Substituting into the latter expression, we get

$$\sigma_s^2 R_e^{-1} \mathbf{a}(\theta)\mathbf{a}^*(\theta)\mathbf{h} = \sigma_s^2 \frac{\mathbf{h}^* \mathbf{a}(\theta)\mathbf{a}^*(\theta)\mathbf{h}}{\mathbf{h}^* R_e \mathbf{h}} \mathbf{h} \tag{4}$$

Rearranging terms, we get the desired result

$$R_e^{-1} \mathbf{a}(\theta) = \frac{\mathbf{h}^* \mathbf{a}(\theta)}{\mathbf{h}^* R_e \mathbf{h}} \mathbf{h} \Longrightarrow \mathbf{h} = \frac{\mathbf{h}^* R_e \mathbf{h}}{\mathbf{h}^* \mathbf{a}(\theta)} R_e^{-1} \mathbf{a}(\theta) \tag{5}$$

Problem 2 - Splines (I)

(i) To find the values a_1, a_2 it's enough to impose the boundary conditions

$$s_1(1) = a_1 = f(1) = \frac{1}{1} \Longrightarrow a_1 = 1 \qquad ; \qquad s_2(4) = a_2 = f(4) = \frac{1}{4} \Longrightarrow a_2 = \frac{1}{4} \tag{6}$$

(ii) The rows of the matrix correspond to the following constraints:

1. $s_1'(1) = f'(1) = \frac{-1}{1^2}$
2. $s_1(4) = f(4) = \frac{1}{4}$
3. $s_1'(4) = s_2'(4)$
4. $s_1''(4) = s_2''(4)$
5. $s_2(5) = f(5) = \frac{1}{5}$
6. $s_2'(5) = f'(5) = \frac{-1}{5^2}$

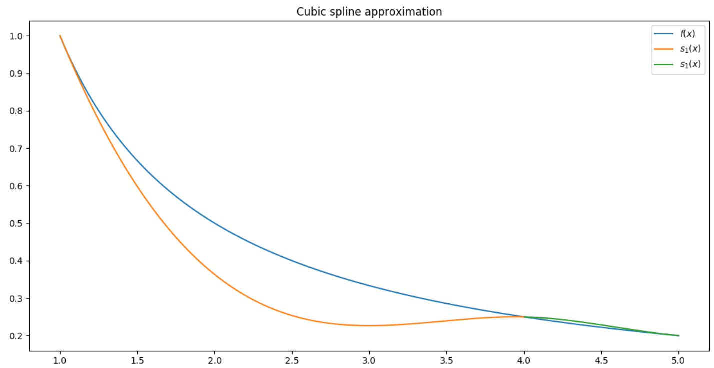

Figure 1: Cubic spline approximation of $f(x) = \frac{1}{x}$ with clamped boundary conditions

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ (4-1) & (4-1)^2 & (4-1)^3 & 0 & 0 & 0 \\ 1 & 2(4-1) & 3(4-1)^2 & -1 & 0 & 0 \\ 0 & 2 & 6(4-1) & 0 & -2 & 0 \\ 0 & 0 & 0 & (5-4) & (5-4)^2 & (5-4)^3 \\ 0 & 0 & 0 & 1 & 2(5-4) & 3(5-4)^2 \end{bmatrix} \begin{bmatrix} b_1 \\ c_1 \\ d_1 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{4} - a_1 \\ 0 \\ 0 \\ \frac{1}{5} - a_2 \\ -\frac{1}{25} \end{bmatrix} \tag{7}$$

Solving the previous linear system, gives the cubic spline approximation depicted in Figure 1.

(iii) Let $d(x) := f(x) - s(x)$. Then, using integration by parts in each subinterval,

$$\begin{aligned} \int_a^b s''(x)d''(x)dx &:= \int_a^b s''(x)(f''(x) - s''(x))dx && (8) \\ &= \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} s''(x)(f''(x) - s''(x))dx && (9) \\ &= \sum_{i=1}^{n-1} s''(x)(f'(x) - s'(x))\Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} s'''(x)(f'(x) - s'(x))dx && (10) \end{aligned}$$

The spline is chosen so that $f(x_i) = s(x_i)\ \forall i \in [1, 2, \ldots, n]$. Moreover, given that s is a cubic Spline, the function s''' is a constant, namely $c \in \mathbb{C}$. Hence,

$$\begin{aligned} \int_{x_i}^{x_{i+1}} s'''(x)(f'(x) - s'(x))dx &= c \int_{x_i}^{x_{i+1}} (f'(x) - s'(x))dx && (11) \\ &= c[f(x_i) - f(x_{i+1}) - (s(x_i) - s(x_{i+1}))] = 0 && (12) \end{aligned}$$

Mixing (10) and (12) and telescoping,

$$\begin{aligned} \int_a^b s''(x)d''(x)dx &:= \sum_{i=1}^{n-1} s''(x)(f'(x) - s'(x))\Big|_{x_i}^{x_{i+1}} && (13) \\ &= \sum_{i=1}^{n-1} s''(x_{i+1})(f'(x_{i+1}) - s'(x_{i+1})) - s''(x_i)(f'(x_i) - s'(x_i)) && (14) \\ &= s''(b)(f'(b) - s'(b)) - s''(a)(f'(a) - s'(a)) = 0 && (15) \end{aligned}$$

where the last equality follows by the clamped boundary conditions.

(iv) Using (15), we have that

$$\int_a^b s''(x)(s''(x) - f''(x))dx = 0 \iff \int_a^b [s''(x)]^2 dx = \int_a^b s''(x)f''(x)dx \tag{16}$$

Given the latter, we can write

$$\begin{aligned}
\int_a^b [f''(x) - s''(x)]^2 dx &= \int_a^b [f''(x)]^2 dx - 2\int_a^b f''(x)s''(x)dx + \int_a^b [s''(x)]^2 dx && (17)\\
&\overset{(16)}{=} \int_a^b [f''(x)]^2 dx - 2\int_a^b [s''(x)]^2 dx + \int_a^b [s''(x)]^2 dx && (18)\\
&= \int_a^b [f''(x)]^2 dx - \int_a^b [s''(x)]^2 dx \geq 0 && (19)
\end{aligned}$$

where the last inequality follows since we are integrating a non-negative function. The proof concludes by rearranging these terms:

$$\int_a^b [f''(x)]^2 dx \geq \int_a^b [s''(x)]^2 dx \tag{20}$$

Problem 3 - Splines (II)

1. The minimum value of K would be 1, since the signal $x(t)$ is piece-wise linear.

2. Let $x(t) = \sum_{k\in\mathbb{Z}} \alpha_k \beta_+^{(1)}(t-k)$. Given that the B-spline of degree 1 is defined as a triangle of height 1 from 0 to 1, and the signal $x(t)$ is piece-wise linear, it's easy to find the values of α_k graphically.

$$\alpha = \begin{bmatrix} \cdots & 0 & \boxed{1} & 1 & 3 & -1 & 0 & \cdots \end{bmatrix}^T \tag{21}$$

3. Using Fubini's theorem,

$$\int_{-\infty}^t x(\tau)d\tau = \int_{-\infty}^t \sum_{k\in\mathbb{Z}} \alpha_k \beta_+^{(1)}(\tau-k)d\tau = \sum_{k\in\mathbb{Z}} \alpha_k \int_{-\infty}^t \beta_+^{(1)}(\tau-k)d\tau = \sum_{k\in\mathbb{Z}} \alpha_k \int_{-\infty}^{t-k} \beta_+^{(1)}(s)ds \tag{22}$$

Given that the causal elementary B-spline of degree K is defined as $\beta_+^{(K)} := \beta_+^{(K-1)} * \beta_+^{(0)}$,

$$\begin{aligned}
\int_{-\infty}^t x(\tau)d\tau &= \sum_{k\in\mathbb{Z}} \alpha_k \int_{-\infty}^{t-k} \beta_+^{(1)}(s)ds = \sum_{k\in\mathbb{Z}} \alpha_k \sum_{m=0}^{\infty} \int_{t-k-m-1}^{t-k-m} \beta_+^{(1)}(s)ds && (23)\\
&=: \sum_{k\in\mathbb{Z}} \alpha_k \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} \beta_+^{(1)}(s)\beta_+^{(0)}(t-k-m-s)ds && (24)\\
&= \sum_{k\in\mathbb{Z}} \alpha_k \sum_{m=0}^{\infty} (\beta_+^{(1)} * \beta_+^{(0)})((t-k-m)) && (25)\\
&=: \sum_{k\in\mathbb{Z}} \alpha_k \sum_{m=0}^{\infty} \beta_+^{(2)}(t-k-m) && (26)\\
&= \sum_{k\in\mathbb{Z}} \alpha_k \sum_{n=k}^{\infty} \beta_+^{(2)}(t-n) = \sum_{n\in\mathbb{Z}} \left[\sum_{k=-\infty}^{n} \alpha_k\right] \beta_+^{(2)}(t-n) && (27)\\
&=: \sum_{n\in\mathbb{Z}} b_n \beta_+^{(2)}(t-n) && (28)
\end{aligned}$$

So, using the latter and (21), we have that

$$\alpha = \begin{bmatrix} \cdots & 0 & \boxed{1} & 2 & 5 & 4 & 4 & \cdots \end{bmatrix}^T \tag{29}$$