Homework 2 (Graded)

Due: Friday, November 29, 2019

Exercise 1. Quick Review of Chapter 2

This exercise uses only concepts from Chapter 2 and should be relatively straight forward for you by now. It should be mostly and exercise in writing clear and complete proofs. Please keep that in mind when writing your solutions.

Important: Answer only **two** of Parts (ii)-(iv). If you hand answers to all three, we will decide which two of them we grade and *we might not decide in your favour.*

In this exercise, C^∞ is a space of functions for which all derivatives exist and are continuous.

(i) Are the following operators Linear Shift Invariant? Prove that they are or provide a counter example.

- $A(x) = x * h_A$, $x, A_h \in \ell^2$
- $B(x)(t) = x(t) + \operatorname{sinc}(t)$, $x \in \mathcal{L}^2(\mathbb{R})$
- $C(x)(t) = x(2t)$, $x \in \mathcal{L}^2(\mathbb{R})$
- $D(x) = \frac{dx}{dt}$, $x \in C^\infty$

(ii) Let $\mathcal{T} : \mathcal{L}^2[a,b] \to \mathcal{L}^2[a,b]$ be the bounded operator defined by the continuous kernel function $T(.,.)$:

$$\mathcal{T}f(x) = \int_a^b T(x,y)f(y)dy.$$

Compute the kernel representation $T^*(.,.)$ of the adjoint operator.

(iii) Calculate the adjoint operator to the operator $C(x)(t) = x(2t)$, $C : \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$.

(iv) Let $\mathcal{H} \subset \mathcal{L}^2(\mathcal{R})$ be a space of smooth functions with finite support, that is $\mathcal{H} \subset C^\infty$ and for each x in $\mathcal{H}$ there exist $c \geq 0$ such that $x(t) = 0$ for $|t| \geq c$. Find the adjoint operator to $D(x) = \frac{dx}{dt}$, $D : \mathcal{H} \to \mathcal{H}$.

Exercise 2. LCMV and GSC derivation

(i) Let A be an $M \times N$ matrix. Prove that if

$$x = \underset{x}{\operatorname{argmin}} \|x\|_2 \quad \text{subject to} \quad Ax = b\,,$$

then there exists y such that

$$AA^*y = b \quad \text{and} \quad x = A^*y\,.$$

(ii) What is x equal to when $M \leq N$ and A is of full rank?

(iii) Assume that R_x is a positive definite covariance matrix of size $M \times M$, and C is an $M \times P$ matrix, also of full rank. Show that if

$$h = \underset{h}{\operatorname{argmin}}\; h^*\, R_x\, h \quad \text{subject to} \quad C^*\, h = f\,,$$

then

$$h = R_x^{-1}\, C \left(C^* R_x^{-1} C\right)^{-1} f\,.$$

Hint: Eigenvalue decomposition of R_x may facilitate your proof.

(iv) Show that the solution of the generalized sidelobe canceler unconstrained optimization

$$\arg\min_{h_n} \, (h_0 - C_n\, h_n)^* \, R_x \, (h_0 - C_n\, h_n) \quad \text{with} \quad h_0 = C(C^* C)^{-1} f\,,$$

where R_x and C_n are of full rank, is given by

$$h_n = (C_n^*\, R_x\, C_n)^{-1} C_n^*\, R_x\, h_0\,.$$

Exercise 3. SEE JUPYTER NOTEBOOK ON MOODLE