1. Model

1.1 Convert AR(2) to VAR(1)

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \varepsilon_t$$

Forecaster correctly knows this structure and models the following state space model using a vector-form Kalman filter:

- State Equation:

$$x_{t+1} = Ax_t + w_t, \quad x_t = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}, \quad A = \begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix}$$

- Observation Equation:

$$s_t^{(i)} = Hx_t + v_t^{(i)}, \quad H = [1\ 0]$$

1.2 Construct the state space model

linear Gaussian state space model:

- **State transition (VAR(1))**:

$$x_{t+1} = Ax_t + w_t, \quad w_t \sim \mathcal{N}(0, Q)$$

- **Observation equation (signal)**:

$$s_t^{(i)} = Hx_t + v_t^{(i)}, \quad v_t^{(i)} \sim \mathcal{N}(0, R)$$

Where, $H = [1 \quad 0]$, represents we observe y_t, rather than the entire state.

So each forecaster receives a noisy signal:

$$s_t^{(i)} = y_t + \text{noise}$$

1.3 Forecaster uses the vector-form Kalman filter

- **Prediction**:

$$\hat{x}_{t|t-1} = A\hat{x}_{t-1|t-1}$$

$$P_{t|t-1} = AP_{t-1|t-1}A' + Q$$

- **Update**:

$$K_t = P_{t|t-1}H'(HP_{t|t-1}H' + R)^{-1}$$

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t(s_t - H\hat{x}_{t|t-1})$$

$$P_{t|t} = (I - K_tH)P_{t|t-1}$$

> *As for:*
>
> $$\hat{y}_{t+h|t} = E[y_{t+h} \mid \text{data up to time } t]$$
>
> *Since we already have $\hat{x}_{t|t}$, then:*
>
> $$\hat{x}_{t+1|t} = A\hat{x}_{t|t}, \quad \hat{x}_{t+2|t} = A^2\hat{x}_{t|t}, \quad \ldots, \quad \hat{x}_{t+h|t} = A^h\hat{x}_{t|t}$$
>
> *Recall that:*
>
> $$y_{t+h} = \text{the first dimension of } x_{t+h}$$
>
> *which means:*
>
> $$\hat{y}_{t+h|t} = H\hat{x}_{t+h|t} = HA^h\hat{x}_{t|t}, \quad \text{where } H = [1 \quad 0]$$

Therefore, the final forecast is:

$$\hat{y}_{t+h|t} = [1 \quad 0] \cdot A^h\hat{x}_{t|t}$$

1.4 Analyst uses Eq(7) and Eq(8)

Now the analyst gets the forecast from forecaster:

$$\hat{y}_{t+h|t} = \text{from Kalman + VAR(1)} = HA^h\hat{x}_{t|t}$$

$$\hat{y}_{t|t} = H\hat{x}_{t|t}$$

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Then the analyst uses Eq(7) and Eq(8) to test whether the forecast conforms to the structure of an AR(1):

Eq(7):

$$\left(\frac{\tilde{\rho}}{\rho}\right)^h = \frac{\text{Cov}(\hat{y}_{t+h|t}, y_t)}{\text{Cov}(\hat{y}_{t|t}, y_{t+h})}$$

Eq(8):

$$\text{stat}_8 = \frac{\text{Cov}(\hat{y}_{t+h|t}, \hat{y}_{t|t})}{\text{Var}(\hat{y}_{t|t})} \cdot \frac{\text{Var}(y_t)}{\text{Cov}(y_{t+h}, y_t)}$$

2. Different Patterns of ρ_1 and ρ_2

As for AR(2) process, which is a **difference equation**:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \varepsilon_t$$

which has roots:

$$r^2 - \rho_1 r - \rho_2 = 0 \Rightarrow r = \frac{\rho_1 \pm \sqrt{\rho_1^2 + 4\rho_2}}{2}$$

- If roots are real and < 1: monotonic convergence
- If roots are complex: **oscillations**
- If roots > 1: explosive

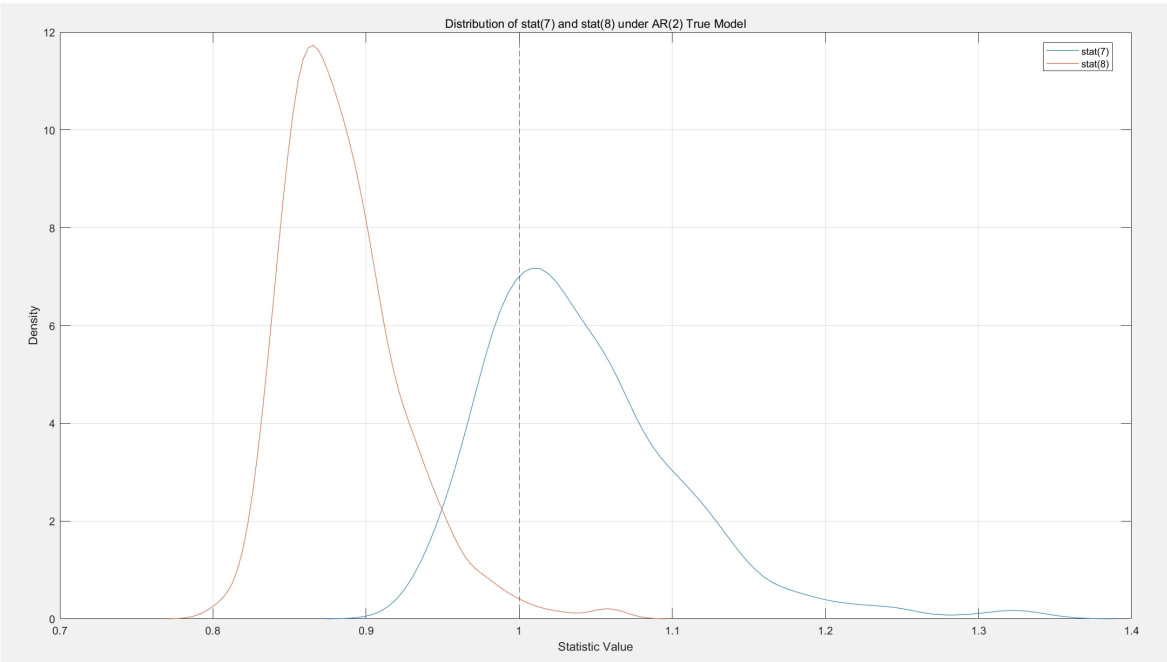
		Dynamic Pattern	Econ Intuition
>1	<0	cyclical convergence	unemployment, output gap
<1	>0	smooth convergence, stable	GDP growth
Small ± (mix)	Small ± (mix)	rapid mean reversion, rapid adjustment	monetary policy instruments

ρ_1	ρ_2	Dynamic Pattern	Econ Intuition
Large +	Large −	high oscillatory/overshooting	consumer expectations

3. Some Cases

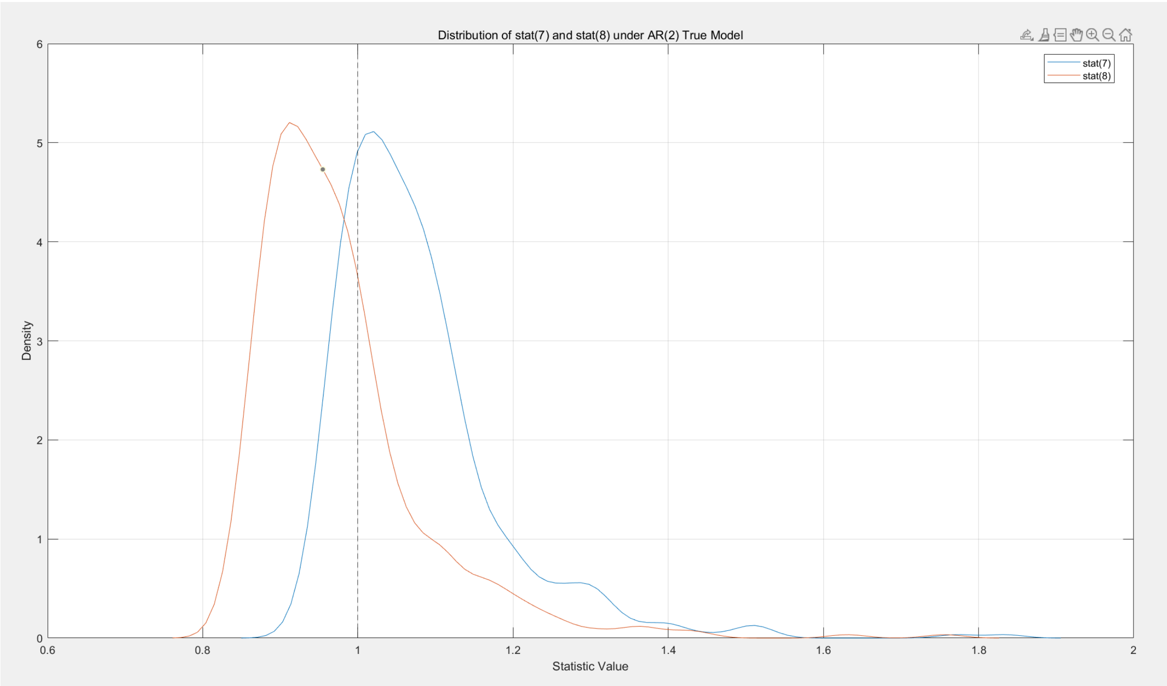
Economic Variable Example	ρ_1	ρ_2	Literature
Unemployment Rate	1.3	-0.4	Stock & Watson (1998)
GDP Growth	0.8	0.1	Hamilton (1994), *Time Series Analysis*
Inflation	1.1	-0.2	Cogley, T., & Sargent, T. J. (2001). *Evolving Post-World War II U.S. Inflation Dynamics.*
Interest Rate / Output Gap	0.6	-0.3	Galí (2015), *Monetary Policy*
Expectational Errors	1.4	-0.6	Angeletos & La'O (2013); Coibion & Gorodnichenko (2015) **(From Chatgpt)**
Price Dynamics under Demand Shocks	0.9	0.3	Bils & Klenow (2004); Nakamura & Steinsson (2008), menu cost literature **(From Chatgpt)**

3.1 Unemployment Rate



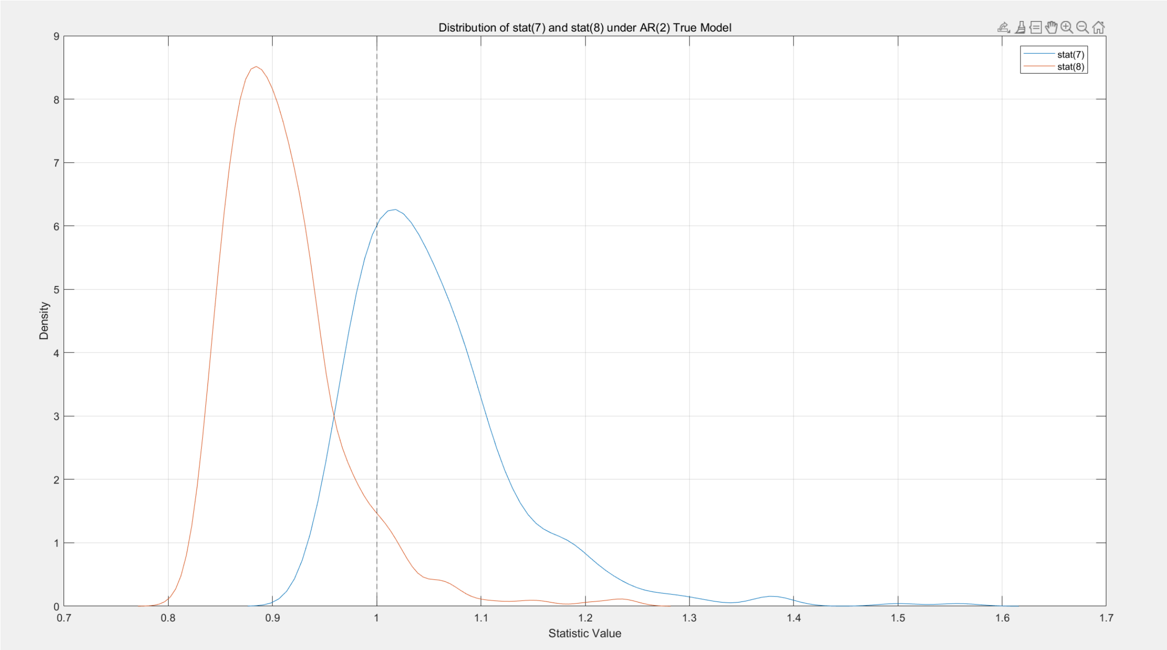
```
=== AR(2) DGP with Analyst Assuming AR(1) ===
Mean of stat(7): 1.0416
Mean of stat(8): 0.8853
Ratio stat(7) / stat(8): 1.1765
```

3.2 GDP Growth



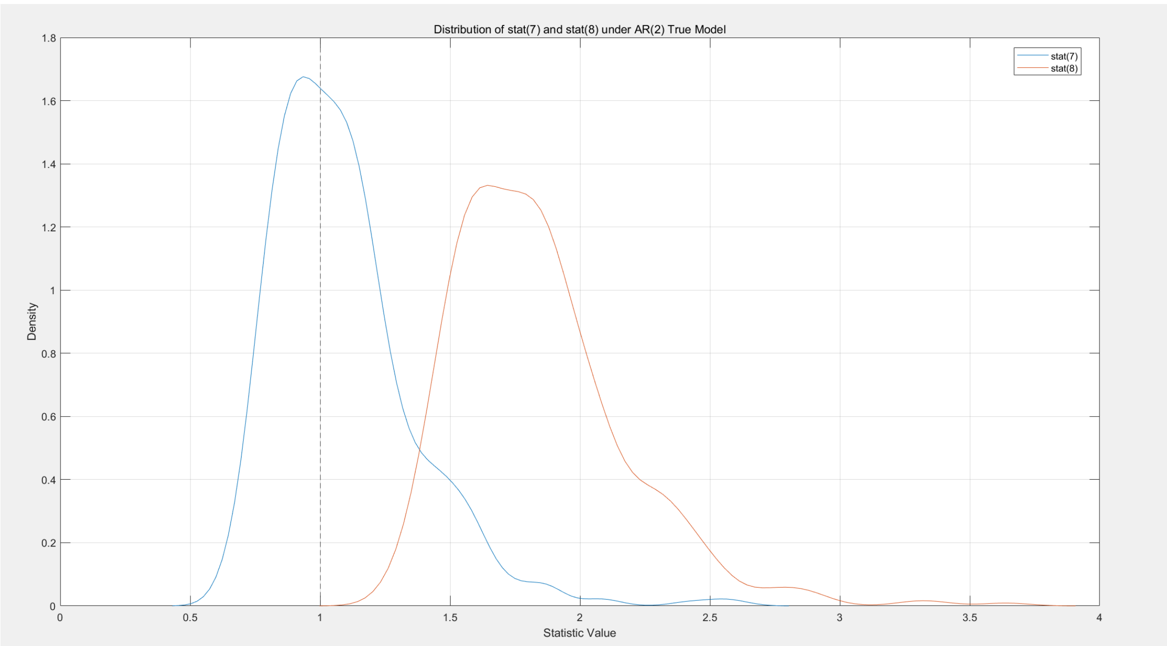

```
=== AR(2) DGP with Analyst Assuming AR(1) ===
Mean of stat(7): 1.0771
Mean of stat(8): 0.9764
Ratio stat(7) / stat(8): 1.1031
```

3.3 Inflation



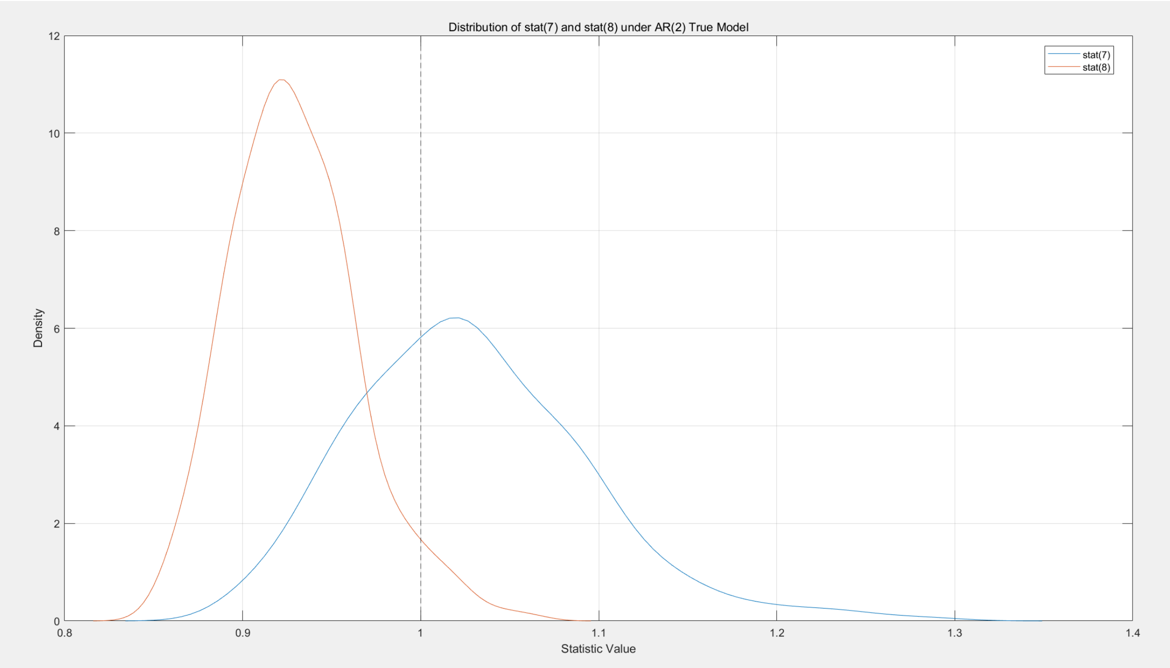

```
=== AR(2) DGP with Analyst Assuming AR(1) ===
Mean of stat(7): 1.0543
Mean of stat(8): 0.9106
Ratio stat(7) / stat(8): 1.1578
```

3.4 Interest Rate / Output Gap



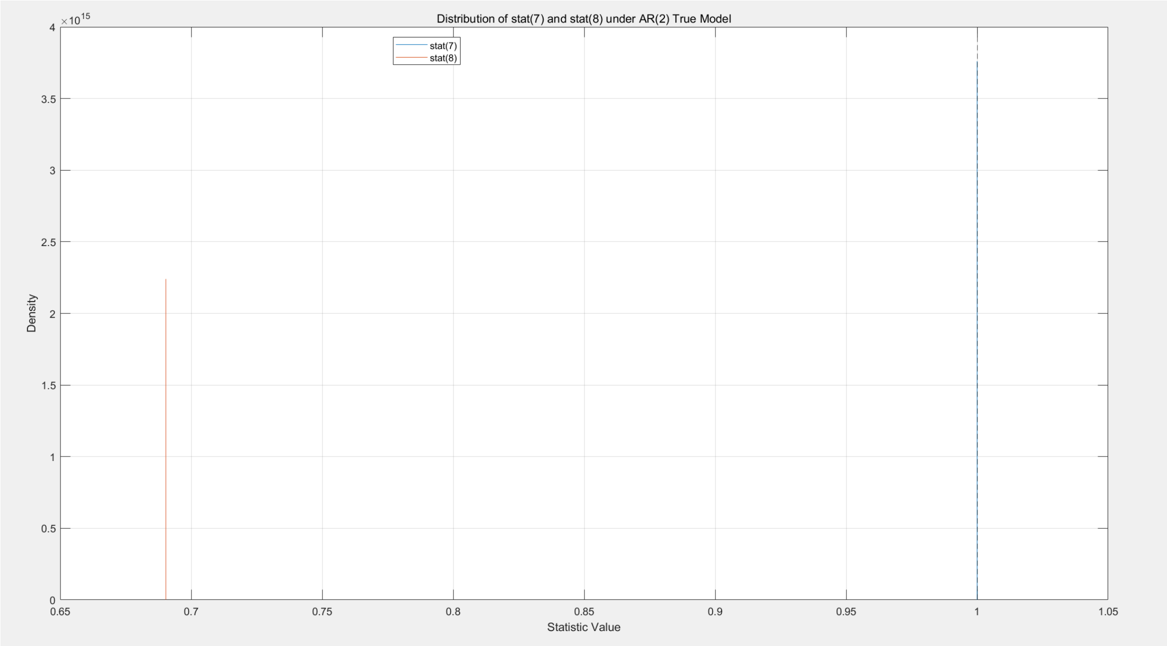
```
=== AR(2) DGP with Analyst Assuming AR(1) ===
Mean of stat(7): 1.0894
Mean of stat(8): 1.8396
Ratio stat(7) / stat(8): 0.5922
```

3.5 Expectational Errors




```
=== AR(2) DGP with Analyst Assuming AR(1) ===
Mean of stat(7): 1.0281
Mean of stat(8): 0.9290
Ratio stat(7) / stat(8): 1.1067
```

3.6 Price Dynamics under Demand Shocks




```
=== AR(2) DGP with Analyst Assuming AR(1) ===
Mean of stat(7): 1.0000
Mean of stat(8): 0.6903
Ratio stat(7) / stat(8): 1.4486
```

4. Summary & Further Question

4.1 Summary

Variable Type	Analyst AR(1) Misspec. Bias	`stat(8)` Behavior	Interpretation
Unemployment Rate	**Severe**	**Large deviation**	**Analyst is "short-sighted"**
GDP Growth	Mild	Small deviation	Close to AR(1)
Inflation	Mild	Small deviation	Close to AR(1)
Interest Rate / Output Gap	Minimal	near 1	Close to AR(1)

Variable Type	Analyst AR(1) Misspec. Bias	`stat(8)` Behavior	Interpretation
Expectations	**Severe**	**Large deviation**	**Analyst sees too little feedback**
Price/Demand Shock	?	?	Feedback loop via lag ignored (?)

4.2 Special Result: $\rho_1 = 0.9$ and $\rho_2 = 0.3$

Since $\rho_1 + \rho_2 = 1.2 > 1$, according the roots of the characteristic equation:

$$r^2 - \rho_1 r - \rho_2 = 0$$

Suppose:

$$\rho_1 = 0.9, \quad \rho_2 = 0.3$$

$\rho_1 + \rho_2 > 1$ makes the process explosive: variance of y_t grows over time and autocovariances become unreliable

Then the roots are:

$$r = \frac{0.9 \pm \sqrt{0.9^2 + 4 \cdot 0.3}}{2} = \frac{0.9 \pm \sqrt{2.01}}{2}$$

$$r_1 = 1.1589 > 1 \quad r_2 = -0.2589$$

One root is greater than 1 → indicates **non-stationarity**.

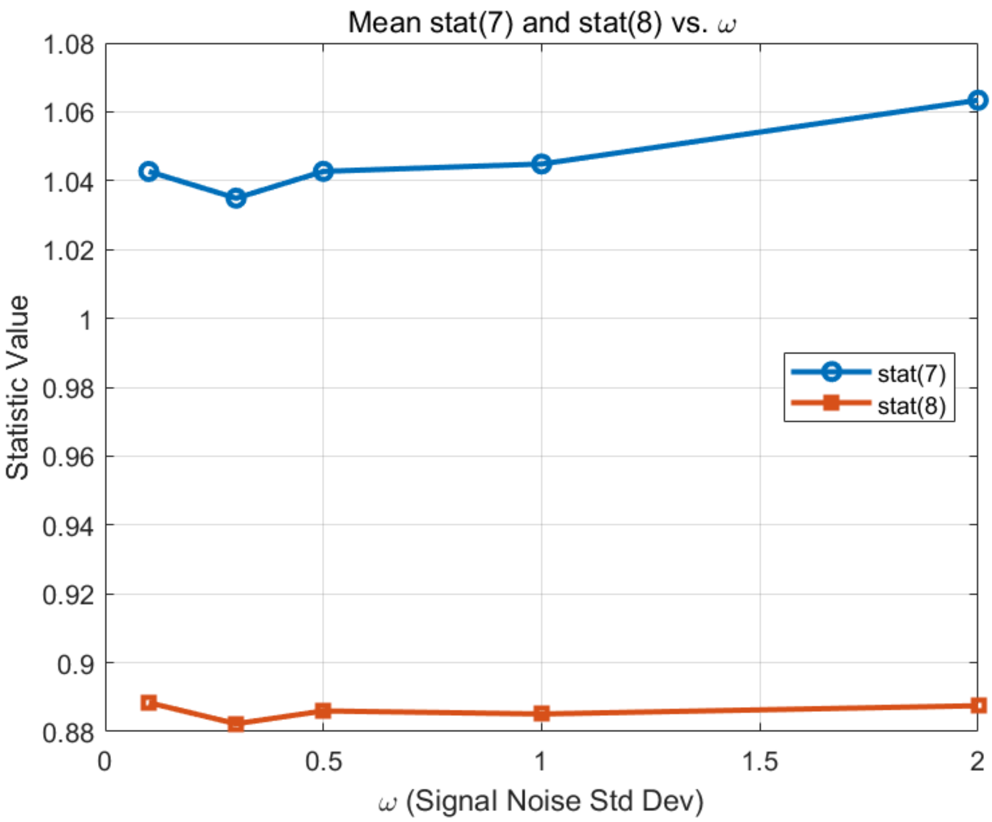
If both roots lie within the unit circle (i.e., $|r_1| < 1, |r_2| < 1$), then the process is stationary, even if $\rho_1 + \rho_2 > 1$.

4.3 Different ω

4.3.1 Mean stat(7) and stat(8) vs. ω

=== Summary Table of stat(7) and stat(8) ===

Omega	Mean_Stat7	Mean_Stat8
0.1	1.0427	0.88838
0.3	1.0349	0.88226
0.5	1.0427	0.88599
1	1.0448	0.8851
2	1.0634	0.8875

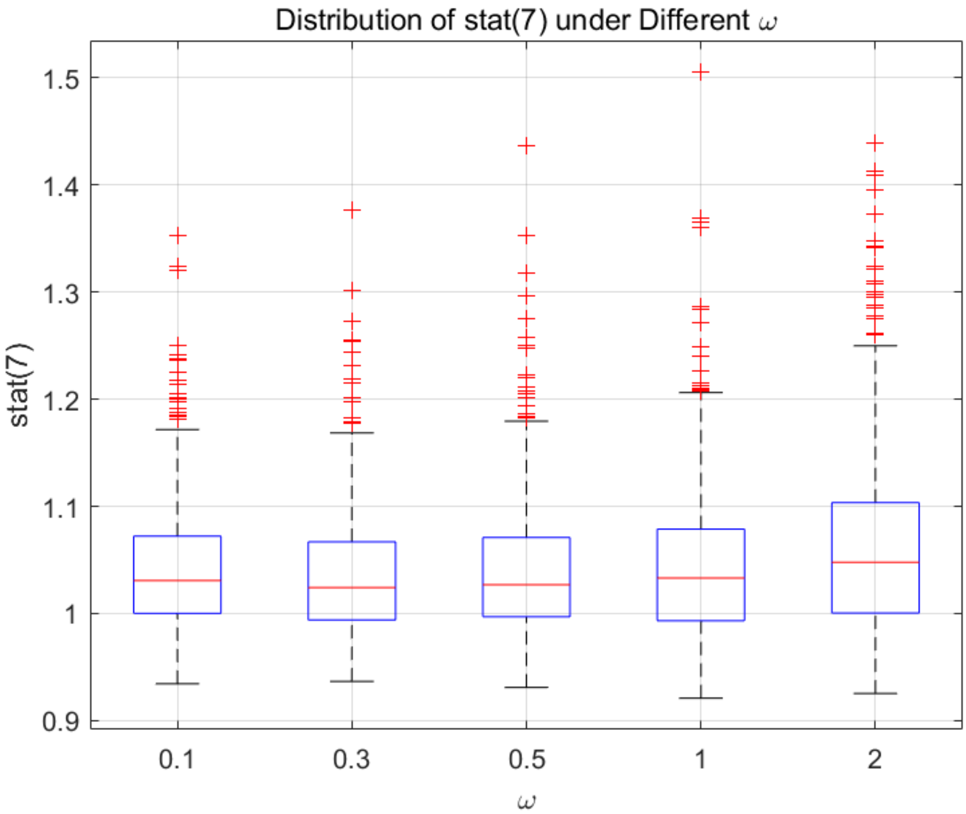


`stat(7)` remains close to **1.03–1.06** across all values of ω, and even slightly increases as noise grows.

`stat(8)` is consistently lower, around **0.88–0.89**, and essentially flat across the range.

Implication: `stat(7)` is robust and stat(8) is structurally biased.

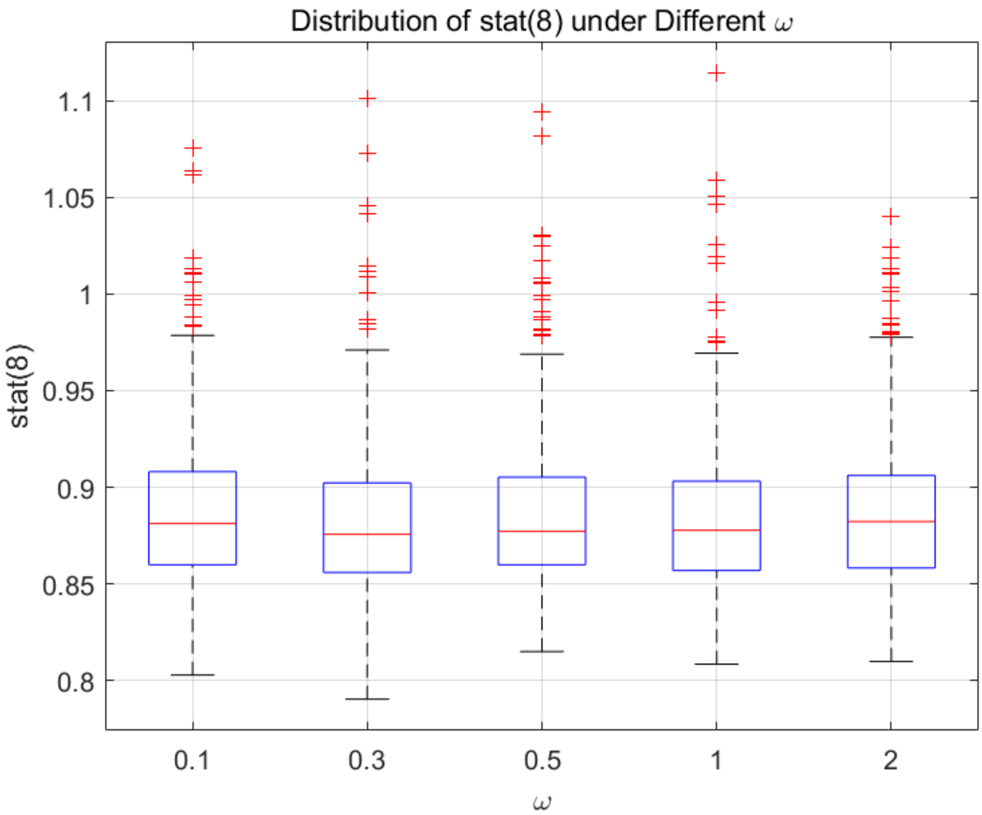
4.3.2 Boxplot of `stat(7)` and `stat(8)` across ω



Median values remain close to **1.0** at every ω level.

Variance and outliers **slightly increase** as ω increases, for example, extreme outliers (>1.3) become more common at $\omega = 2.0$.

As signal noise increases, the Kalman filter becomes less precise?



All distributions are centered around 0.88. Dispersion slightly increases as ω grows, but not dramatically.

The bias remains a regardless of signal precision, showing that this is a structural attribution error, not a noise problem.