Matrix Exponentiation

Matrix Exponentiation

Given a square matrix A, we will calculate A^n.

a_{11}	a_{12}	a_{13}
a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}

Let's revise previous topics

Matrix multiplication

M1

2	4	1	2
8	4	3	6
1	7	9	5

3X4

M2

1	2	3
4	5	6
7	8	9
4	5	6

4X3

Code

```
vector<vector<int>> multiply(vector<vector<int>> a,
vector<vector<int>> b)
{
    int n = a.size();
    vector<vector<int>> ans(n, vector<int>(n, 0));

    for (int i = 0; i < n; i++) {
        for (int j = 0; j < n; j++) {
            for (int k = 0; k < n; k++) {
                ans[i][j] += a[i][k] * b[k][j];
            }
        }
    }
return ans;
```

Let's revise previous topics

Binary Exponentiation

Cases:

First Case: n -> odd

$$A^n = A^{n/2} \times A^{n/2} \times A$$

Second Case: n -> even

$$A^n = A^{n/2} \times A^{n/2}$$

This time A is a matrix instead of integer.

Matrix Exponentiation - Example

Fibonacci Numbers

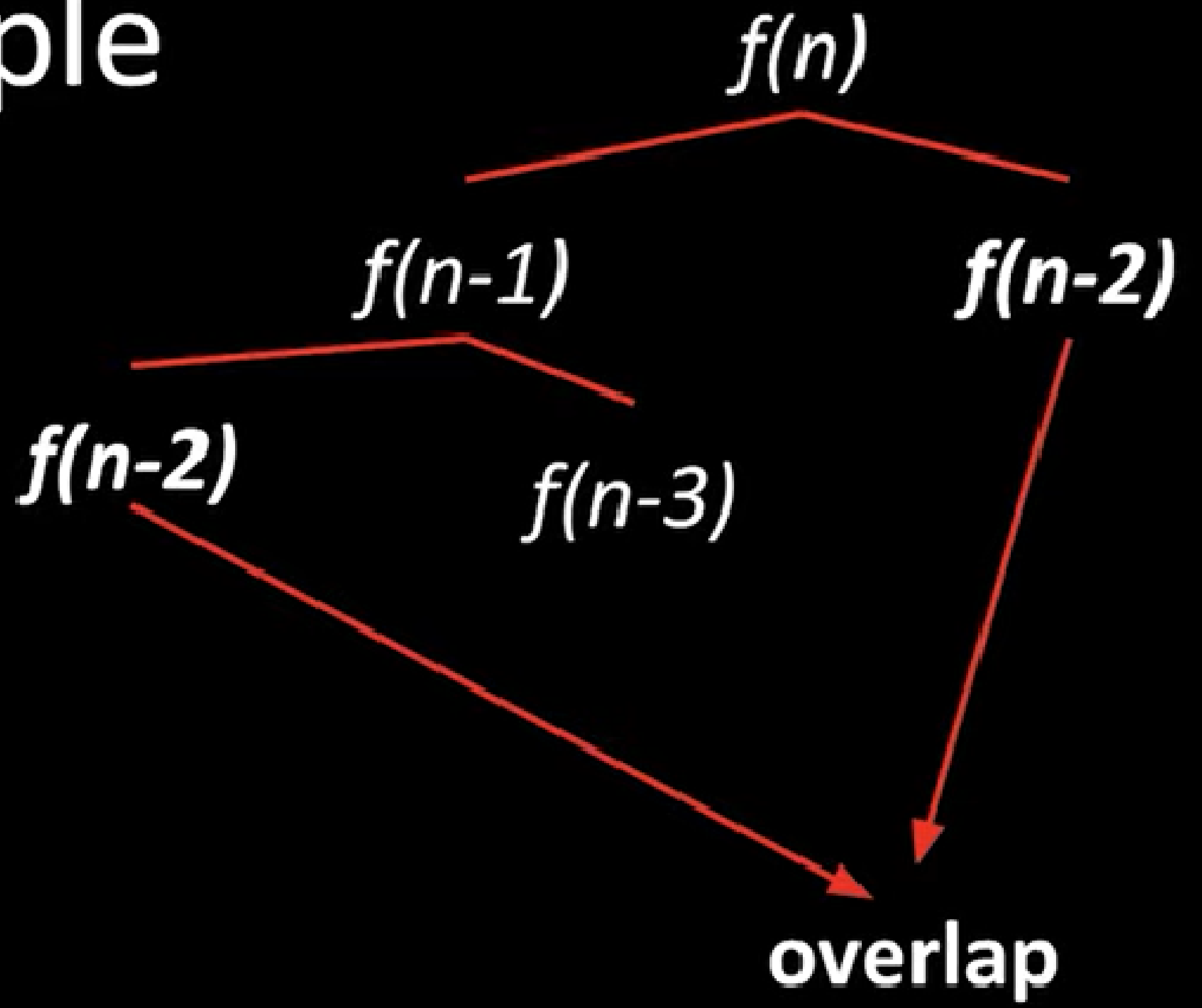
$F(n) = F(n-1) + F(n-2)$

Using Recursion: $O(2^n)$

Using Dynamic Programming: $O(n)$

Using Matrix Exponentiation: $O(\log(n))$

k^3 $\log(n)$



Matrix Exponentiation - Example

Representing fibonacci series in the form of matrix multiplication

$$\underset{2X1}{\begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}} = \underset{2X2}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \times \underset{2X1}{\begin{bmatrix} f(n-1) \\ f(n-2) \end{bmatrix}}$$

Matrix Exponentiation - Example

$$\underset{2X1}{\begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}} = \underset{2X2}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \times \underset{2X1}{\begin{bmatrix} f(n-1) \\ f(n-2) \end{bmatrix}}$$

Matrix Exponentiation - Example

$$\underbrace{\begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}}_{2X1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{2X2} \times \underbrace{\begin{bmatrix} f(n-1) \\ f(n-2) \end{bmatrix}}_{2X1}$$

$$\underbrace{\begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}}_{2X1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{2X2} \times \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{2X2} \times \underbrace{\begin{bmatrix} f(n-2) \\ f(n-3) \end{bmatrix}}_{2X1}$$

Matrix Exponentiation - Example

$$\underset{2X1}{\begin{bmatrix} f(n) \\ f(n-1) \end{bmatrix}} = \underset{2X2}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \times \underset{2X2}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \times \ \ldots\ldots\ldots \ \times \underset{2X1}{\begin{bmatrix} f(2) \\ f(1) \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} f_{n-1} \\ f_{n-2} \end{bmatrix} = \begin{bmatrix} f_{n-1} + f_{n-2} \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} f_{n-1} \\ f_{n-2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \times \begin{bmatrix} f_{n-2} \\ f_{n-3} \end{bmatrix} \\ &= \vdots \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

$$g_n = g_{n-1} + 5 \times g_{n-2} + 3 \times g_{n-3}$$

ম্যাট্রিক্স দিয়ে প্রকাশ করলে সেটা হবে এরকম

$$\begin{bmatrix} 1 & 5 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} g_{n-1} \\ g_{n-2} \\ g_{n-3} \end{bmatrix} = \begin{bmatrix} g_n \\ g_{n-1} \\ g_{n-2} \end{bmatrix}$$

যদি লিনিয়ার রিকারেন্সটাকে এরকম হয়

$$g_n = a_1 \times g_{n-1} + a_2 \times g_{n-2} + a_3 \times g_{n-3} + \cdots + a_k \times g_{n-k}$$

তাহলে সেটাকে ম্যাট্রিক্সে দিয়ে প্রকাশ করলে সেটা হবে -

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_k \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \times \begin{bmatrix} g_{n-1} \\ g_{n-2} \\ g_{n-3} \\ g_{n-4} \\ \vdots \\ g_{n-k} \end{bmatrix} = \begin{bmatrix} g_n \\ g_{n-1} \\ g_{n-2} \\ g_{n-3} \\ \vdots \\ g_{n-k+1} \end{bmatrix}$$

আর প্রথম k টা এলিমেন্ট যদি b(1), b(2), ..., b(k) হয় তাহলে জেনারালাইজড রিকারেন্সটা হবে এরকম -

$$\begin{bmatrix} g_n \\ g_{n-1} \\ g_{n-2} \\ g_{n-3} \\ \vdots \\ g_{n-k+1} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_k \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^{n-k} \times \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix}$$