Multifractal Random Walks With Fractional Brownian Motion via Malliavin Calculus

Alexis Fauth and Ciprian A. Tudor

Abstract—**We introduce a Multifractal Random Walk (MRW) defined as a stochastic integral of an infinitely divisible noise with respect to a dependent fractional Brownian motion. Using the techniques of the Malliavin calculus, we study the existence of this object and its properties. We then propose a continuous time model for financial modeling that captures the main properties observed in the empirical data, including the leverage effect. We illustrate our result by numerical simulations.**

Index Terms—**Fractional Brownian motion, Malliavin calculus, multifractal random walk, scaling, infinitely divisible cascades, leverage effect, high frequency financial data.**

I. Introduction

STARTING with the seminal work of Mandelbrot about cotton price [26], several studies of financial stock prices times series, have allowed to exhibit some particularities of their fluctuations. Without making a comprehensive list, we can mention the appearance in empirical data of the following properties: non Gaussian distributions due to now well known fat tails of financial returns, the so-called volatility clustering that means that the volatility fluctuations are of intermittent and correlated nature, scaling invariance, long run correlation in volatility, leverage effect and so on (see e.g [15], [22], [31] for an extensive review). Thus, constructing theoretical models for financial returns that include all the properties listed before appears as a very interesting challenge. Many scientific works, in economics or mathematics, proposed various models for asset returns. The ARCH model introduced by Engle [19] offers an interesting base of work, and after the seminal work by Engle a vast literature on ARCH and related models has been developed. One of the first extensions of the ARCH model, called the GARCH model, has been introduced by Bollerslev in [9] and it also has been the object of various generalizations. While the GARCH models are capable of capturing volatility clustering, there are some drawbacks of the model. For example, the GARCH models are unable to represent volatility asymmetry. Due to the presence of the squared observed data in the conditional variance equation, the positive and negative values of the lagged innovations have the same effect on the conditional variance. In the finance literature, it has been noticed that volatility often is affected by negative and positive shocks in different ways. Another inconvenient of the GARCH model is the fact that, to ensure positiveness of the conditional variance, non-negative constraints on the coefficients in the variance equation must be imposed. To take into account the asymmetric effects on conditional second moments and to avoid the non-negativity constraints on the coefficients in the variance equation, in [28] the so-called EGARCH model has been proposed by D. Nelson.

On the other hand, many recent empirical studies, based on huge data sets available nowadays, put in light new aspects of the financial returns. For example, they suggested that the fluctuations of the asset process displays multifractal properties (see e.g. [20] or [21]). Taking into account these multifractal character, several authors proposed models based on the on "cascade" random processes and "multifractal random walk (MRW)". We refer, among others, to [4] or [5], [12]. Usually, the noise in these models are defined by

$$Z(t) = Y(X([0,t]), \qquad t \in \mathbb{R} \tag{1}$$

where X is a multifractal random measure and Y is a self-similar process with stationary increments, independent of X. Another construction defines the noise via a stochastic integral as (see e.g. [7], see [1])

$$Z(t) = \int_0^t Q(u)\mathrm{d}Y(u) \tag{2}$$

where Q is a suitable fractal noise and Y is a self-similar process with stationary increments, independent of Q. A natural choice for the process Y, both in (1) and (2), is the fractional Brownian motion (fBm) which includes the Brownian motion as a particular case.

Other works on multifractal random walks models are [17], [18], [24] or [30].

While these multifractal models (1), (2) are able to capture several properties observed on the empirical data (scaling, volatility clustering or long-range dependance), they reproduce poorly the leverage effect. The leverage effect is understood as the correlation between the log-return at a fixed time t and his volatility (that may be defined as the squared or as the absolute log-return) varying around t, for example $t - 100\ \Delta t \le t' \le t + 100\ \Delta t$. Empirical studies (see e.g. [8] or [11]) have shown that this quantity is close to 0 for past volatility and follow

Manuscript received July 6, 2012; revised October 15, 2013; accepted November 17, 2013. Date of publication January 2, 2014; date of current version February 12, 2014. This work was supported in part by the CNCS under Grant PN-II-ID-PCCE-2011-2-0015 and in part by the Université de Panthéon-Sorbonne Paris 1.

A. Fauth is with the SAMM, Université de Panthéon-Sorbonne Paris 1, Paris 75634, France, and in part by Invivoo, Courbevoie 92400, France (e-mail: alexis.fauth@invivoo.com).

C. A. Tudor is with the Laboratoire Paul Painlevé, Université de Lille 1, Villeneuve d'Ascq F-59655, France, and also with the Department of Mathematics, Academy of Economical Studies, Bucharest 010374, Romania (e-mail: tudor@univ-paris1.fr).

Communicated by D. Palomar, Associate Editor for Detection and Estimation.

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TIT.2013.2296785

an exponential law for future volatility. This stylized fact is interpreted as a panic effect, the two quantities are negatively correlated just after t and go back quickly to 0 as t' increases.

II. The New Model

Our purpose is to introduce a generalization of the model (2), able to take into account the leverage effect and which allows a more flexibility for long-range dependence in log-return. That is, in the model (2) we will consider the processes Q to be multifractal process called “infinitely divisible cascading noise” and the process Y to be a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. The same situation has been treated in [1] and [25], but in addition, we construct a model in which the processes Q and B^H are not independent anymore. This introduction of this new model brings several problems, both from the theoretical points of view and the simulation point of view. The theoretical aspects are related to the definition and the properties of the stochastic integral that appears in the expression (2). Notice that this integral is not a Wiener integral anymore (as in the reference [1]) and we need to use the Malliavin calculus in order to define it properly and to study its various properties (scaling, the behavior of the increments etc). From the simulation point of view, dealing with the integral in (2) is also challenging. This stochatic integral, known as the Skorohod or divergence integral in the literature, is known to be hard to be simulated. In our case, taking into account the particular form of the integrand Q, we are able to propose a simulation scheme in Section VII of our paper. Due to the presence of the elements of the Malliavin calculus in our model, our methodology to simulate the model and to forecast the parameters is different from the approaches proposed in the previous works on these topics ([1], [16] or [17], [18]).

An alternative construction has been proposed in [2] but in this reference the integral in (2) is not a true stochastic integral with respect to the fBm. In this work the authors define the multifractal random walk as $\int_0^t Q(u)b^H(u)du$ where Q and b^H have a dependence structure and b^H can be viewed somehow as the formal derivative of the fBm.

We will define the integral (2) using the techniques of the Malliavin calculus and we will study the properties of the fractional Multifractal Random Walk.

Our paper is organized as follows. Section III contains some preliminaries on multifractal process and the basic elements of the Malliavin calculus that we will need in our paper. In Section IV we analyze the stochastic integrals that are used to define the MRW while in Section V we study the existence of the fractional MRW as a limit of a family of stochastic integrals. The properties of the fractional MRW (scaling, moments etc) are discussed in Section VI. The last two sections (Sections VII and VIII) contain a numerical analysis of the data for several financial indices and we compare the simulation of our theoretical model with the real data.

III. Preliminaries

We present here the basic facts related to the infinitely divisible cascading noises and we introduce the basic tools of the Malliavin calculus.

A. Infinitely Divisible Cascading Noise

Let M denote an infinitely divisible, independently scattered random measure on the set $\mathbb{R} \times \mathbb{R}_+$ with generating infinitely divisible distribution G satisfying

$$\int_{\mathbb{R}} e^{qx} G(\mathrm{d}x) = e^{-\rho(q)}$$

for some function ρ and for every $q \in \mathbb{R}$. We assume that M has control measure m on $\mathbb{R} \times \mathbb{R}_+$ meaning that for every Borel set $A \subset \mathbb{R} \times \mathbb{R}_+$ it holds

$$\mathbb{E}e^{qM(A)} = e^{-\rho(q)m(A)} \text{ for every } q \in \mathbb{R}.$$

The fact that M is independently scattered means that the random variables

$$M(A_1), M(A_2), \ldots, M(A_n)$$

are independent whenever the Borel sets $A_1, \ldots, A_n \in \mathbb{R}\times\mathbb{R}_+$ are disjoint. We define the *Infinitely Divisible Cascading noise* (IDC) by

$$Q_r(t) = \frac{e^{M(C_r(t))}}{\mathbb{E}e^{M(C_r(t))}} \tag{3}$$

for every $r > 0$ and $t \in \mathbb{R}$. Here $C_r(t)$ is the cone in $\mathbb{R} \times \mathbb{R}_+$ defined by

$$C_r(t) = \{(t', r'), r \le r' \le 1, t - \frac{r'}{2} \le t' \le t + \frac{r'}{2}\}. \tag{4}$$

The set $C_r(t)$ is empty when $r > 1$.

We will use the following facts throughout our paper. We refer to [1] or [14] for the their proofs. First, let us note the scaling property of the moments of the IDC

$$\mathbb{E}Q_r(t)^q = e^{-\varphi(q)m_r(0)}$$

and the expression of its covariance: for every $r > 0$ and $t, s \in \mathbb{R}$

$$\mathbb{E}Q_r(t)Q_r(s) = e^{-\varphi(2)m_r(|t-s|)} \tag{5}$$

where we denoted by for $u \ge 0, r > 0$.

$$m_r(u) = m\left(C_r(0) \cap C_r(u)\right) \tag{6}$$

and by

$$\varphi(q) = \rho(q) - q\rho(1). \tag{7}$$

The scaling of the moment of Q can be extended to the following scaling property in distribution : for $t \in (0, 1)$

$$\left(Q_{rt}(tu)\right)_{u\in\mathbb{R}} =^{(d)} e^{\Omega_t} \left(Q_r(u)\right)_{u\in\mathbb{R}} \tag{8}$$

where “$=^{(d)}$” means equivalence of finite dimensional distributions. Here Ω_t denotes a random variable independent by Q, which satisfies, if the measure m is given by (16),

$$\mathbb{E}e^{q\Omega_t} = t^{q\varphi(q)}. \tag{9}$$

Remark 1: *As noticed in [1], we have $\varphi(2) < 0$.*

In [1] (see also [25]) the *Multifractal random walk* (MRW) based on fractional Brownian motion is defined as limit when

$r \to 0$ (in some sense) of the family of stochastic integrals $\left(Z_r^H(t)\right)_{r>0}$ defined by

$$Z_r^H(t) = \int_0^t Q_r(u)\mathrm{d}B^H(u), \qquad t \in [0,T] \tag{10}$$

where $(B_t^H)_{t\in[0,T]}$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$. The fractional Brownian motion $(B_t^H)_{t\in[0,T]}$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian process starting from zero with covariance function

$$R^H(t,s) := \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,T]. \tag{11}$$

In [1] it is assumed that M and B^H **are independent.** Therefore the stochastic integral with respect to B^H in (10) behaves mainly as a Wiener integral since, because of the independence, the integrand $Q_r(u)$ can be viewed as deterministic function for the integrator B^H.

Another important fact in the development of this theory is that the IDC Q is a martingale with respect to the argument r. Let us recall the following result (see [14]):

Lemma 1: *For every $u > 0$ the stochastic process $(Q_r(u))_{r>0}$ is a martingale with respect to its own filtration. As a consequence, for every $u, v, r, r' > 0$ with $r < r'$ it holds*

$$\mathbb{E}Q_r(u)Q_{r'}(v) = \mathbb{E}Q_r(u)Q_r(v). \tag{12}$$

The property (12) plays an important role in the construction of the MRW process in [1] or [14].

B. Malliavin Calculus

Let $(W_t)_{t\in T}$ be a classical Wiener process on a standard Wiener space $(\boldsymbol{\Omega}, \mathcal{F}, \mathbf{P})$. By $W(\varphi)$ we denote the Wiener integral of the function $\varphi \in L^2(T)$ with respect to the Brownian motion W. We denote by D the Malliavin derivative operator that acts on smooth functionals of the form $F = g(W(\varphi_1), \ldots, W(\varphi_n))$ (here g is a smooth function with compact support and $\varphi_i \in L^2(T)$ for $i = 1, \ldots, n$)

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(W(\varphi_1), \ldots, W(\varphi_n))\varphi_i.$$

The operator D can be extended to the closure $\mathbb{D}^{p,2}$ of smooth functionals with respect to the norm

$$\|F\|_{p,2}^2 = \mathbb{E}F^2 + \sum_{i=1}^p \mathbb{E}\|D^{(i)}F\|^2_{L^2(T^i)}$$

where the ith Malliavin derivative $D^{(i)}$ is defined iteratively. The adjoint of D is denoted by δ and is called the divergence (or Skorohod) integral. Its domain ($Dom(\delta)$) coincides with the class of stochastic processes $u \in L^2(\Omega \times T)$ such that

$$|\mathbb{E}\langle DF, u\rangle| \le c\|F\|_2$$

for all $F \in \mathbb{D}^{1,2}$ and $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the duality relationship

$$\mathbb{E}(F\delta(u)) = \mathbb{E}\langle DF, u\rangle.$$

For adapted integrands, the divergence integral coincides to the classical Itô integral. A subset of $Dom(\delta)$ is the space $\mathbb{L}^{1,p}$ of the stochastic processes such that u_t is Malliavin differentiable for every t and

$$\|u\|_{1,p}^p := \|u\|^p_{L^p(T\times\Omega)} + \|Du\|^p_{L^p(T^2\times\Omega)} < \infty.$$

We will need Meyer's inequality that allows to estimate the L^p moment of the Skorohod integral

$$\mathbb{E}\,|\delta(u)|^p \le \|u\|_{1,p}^p. \tag{13}$$

For the L^2 moment of the Skorohod integral we have the explicit formula

$$\mathbb{E}\delta(u)^2 = \int_T u_s^2\mathrm{d}s + \int_T\int_T D_r u_s D_s u_r \mathrm{d}r\mathrm{d}s \tag{14}$$

if $u \in \mathbb{L}^{1,2}$. We also recall that the Malliavin derivative satisfies the chain rule

$$Df(F) = f'(F)DF \tag{15}$$

if f is a differentiable function and $F \in \mathbb{D}^{1,2}$.

IV. The Construction of the Fractional Multifractal Random Walk With Dependent Noise

Our purpose is to give a meaning to the stochastic integral (10) in the situation when the IDC Q and the fBm B^H are not independent. We will use techniques related to the Malliavin calculus. In order to apply these type of techniques we will restrict ourselves to the case when the measure M introduced in Section II is Gaussian.

A. The Gaussian Isonormal Noise

We introduce a MRW without independence between the measure M (denote by W in our settings) and the integrator B^H. We will restrict to the case where M is a Gaussian measure. More precisely, we will consider $(W(h), h \in H)$ an isonormal process, that is, a centered Gaussian family with

$$\mathbb{E}W(h)W(g) = \langle h, g\rangle_H$$

for every $g, h \in H$. The Hilbert space H will be

$$H = L^2\left(\mathbb{R}\times\mathbb{R}_+, \mathcal{B}(\mathbb{R}\times\mathbb{R}_+), m\right)$$

where m is the control measure. In this work we will consider

$$m(dt, dr) = dt\frac{cdr}{r^2} \text{ if } 0 < r \le 1 \tag{16}$$

and m vanishes if $r \ge 1$. Here c is a strictly positive constant. This is called in [3] (see also [1]) the exact invariant scaling case.

The following properties of the noise W are immediate: denote $W(A) = W(1_A)$ for $A \in \mathcal{B}(\mathbb{R}\times\mathbb{R}_+)$.

- For every $A \in \mathcal{B}(\mathbb{R}\times\mathbb{R}_+)$ it holds that
$$W(A) \sim N(0, m(A)).$$
- We have
$$\mathbb{E}W(A)W(B) = 0.$$

if the Borel sets $A, B \subset \mathbb{R} \times \mathbb{R}_+$ are disjoint. This implies that the random variables $W(A)$ and $W(B)$ are independent when the sets A and B are disjoint, so the random measure W is independently scattered.

- For every $q \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R} \times \mathbb{R}_+)$ we have
$$\mathbb{E}e^{qW(A)} = e^{-\frac{1}{2}q^2 m(A)}$$
which means that
$$\rho(q) = -\frac{1}{2}q^2 \text{and} \varphi(q) = -\frac{1}{2}q^2 + \frac{q}{2} = -\frac{1}{2}q(q-1).$$
Let us use the following notation:
$$m_1(dt) = dt \quad \text{and} \quad m_2(dr) = c\frac{dr}{r^2}.$$

Above m_1 is the Lebesque measure on $\mathbb{R}$ and m_2 is a measure on $\mathbb{R}_+$. Clearly $m = m_1 \otimes m_2$, the product measure.

For every $t \geq 0$ and $A_2 \in \mathcal{B}(\mathbb{R}_+)$ such that
$$m_2(A_2) = 1$$
(take for example $A_2 = (c, \infty)$) we set
$$W^{(1)}(t) := W(1_{[0,t] \times A_2}). \tag{17}$$

The following result is immediate.

Proposition 1: *If $A_2 \in \mathcal{B}(\mathbb{R}_+)$ is such that $m_2(A_2) = 1$ then the process $(W^{(1)}(t))_{t \geq 0}$ given by (17) is a standard Brownian motion on the same probability space as W.*

Proof: It is clear that $W^{(1)}$ is a Gaussian process. Let us compute its covariance. For every $s, t \geq 0$ it holds that
$$\begin{aligned}\mathbb{E}W^{(1)}(t)W^{(1)}(s) &= \mathbb{E}W(1_{[0,t] \times A_2})W(1_{[0,s] \times A_2}) \\ &= \langle 1_{[0,t] \times A_2}, 1_{[0,s] \times A_2} \rangle_H \\ &= \langle 1_{[0,t]}, 1_{[0,s]} \rangle_{L^2(\mathbb{R})} m_2(A_2) \\ &= t \wedge s\end{aligned}$$
and this implies that $W^{(1)}$ is a Wiener process with respect to its own filtration. ■

B. The Approximating Multifractal Random Walk With Dependent Fractional Brownian Motion

We introduce the Multifractal Random Walk based on the fractional Brownian motion by the formula
$$Z_r^H(t) = \int_0^t Q_r(u) \mathrm{d}B^H(u) \tag{18}$$
where for every $r > 0, u \geq 0$ the integrands $Q_r(u)$ is defined by (3) with W instead of M and the fractional Brownian motion B^H is given by
$$B^H(t) = \int_0^t K^H(t, s) \mathrm{d}W^{(1)}(s) \tag{19}$$
where K^H is the usual kernel of the fractional Brownian motion and $W^{(1)}$ is the Brownian motion defined by (17). By Proposition 1, it is clear that B^H is a fractional Brownian motion. Recall that, when $H > \frac{1}{2}$ the kernel $K^H(t, s)$ has the expression
$$K^H(t, s) = c_H s^{1/2-H} \int_s^t (u - s)^{H-3/2} u^{H-1/2} \mathrm{d}u$$
(see [29]) where $t > s$ and $c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-1/2)}\right)^{1/2}$ and $\beta(\cdot, \cdot)$ is the Beta function. For $t > s$, the kernel's derivative is
$$\partial_1 K^H(t, s) := \frac{\partial K^H}{\partial t}(t, s) = c_H \left(\frac{s}{t}\right)^{1/2-H} (t - s)^{H-3/2}.$$
In the sequel we will simply denote $K^H := K$.

The stochastic integral in (18) is a divergence (Skorohod) integral with respect to B^H as defined in e.g. [29]. Actually, when $H > \frac{1}{2}$ we can write
$$\begin{aligned}Z_r^H(t) &= \int_0^t Q_r(s) \mathrm{d}B^H(s) \\ &= \int_0^t dW^{(1)}(s) \left(\int_s^t \mathrm{d}a \partial_1 K(a, s) Q_r(a)\right).\end{aligned}$$
The last equality is due to the definition of the stochastic integral with respect to B^H (see [29], Chapter 5 for example). We can also express $Z_r(t)$ as a Skorohod integral with respect to the measure W by the formula
$$Z_r^H(t) = \int_0^t \int_{A_2} \mathrm{d}W(s, r') \left(\int_s^t \mathrm{d}a \partial_1 K(a, s) Q_r(a)\right) \tag{20}$$
where A_2 is the Borel set satisfying $m_2(A_2) = 1$ and that appears in the definition of $W^{(1)}$.

Remark 2: *Clearly the fBm B^H and the noise W are dependent. Actually, it can be seen that B^H and W are correlated in general. Indeed, for every Borel set $A \subset \mathbb{R} \times \mathbb{R}_+$ and for every $t \geq 0$*
$$\begin{aligned}\mathbb{E}B^H(t)W(A) &= \mathbb{E}\int_0^t K(t, s) \mathrm{d}W^{(1)}(t) W(A) \\ &= \mathbb{E}\int_0^t \int_0^1 K(t, s) \mathrm{d}W(s, r) W(A) \\ &= \mathbb{E}\int\int_{((0,t) \times (1,\infty)) \cap A} K(t, s) \mathrm{d}s \frac{\mathrm{d}r}{r^2}\end{aligned}$$
and this is in general not zero.

Let us further discuss the dependence between the integrator B^H and the integrand Q in (10). We need to distinguish two situations. It depends on the relations of the set A_2 and the unit interval $(0, 1)$.

The disjoint case: We have $A_2 \cap (0, 1) = \emptyset$ (this happens when $A_2 = (c, \infty)$ with $c \geq 1$ for example).

On the other hand, B_s^H is independent with $W(C_r(t))$ for every s, t, r. Indeed, since $C_r(t) \subset \mathbb{R} \times (0, 1)$ (see (4)) we have
$$\mathbb{E}B_s^H W(C_r(t)) = 0$$
and since $(B^H(s), W(C_r(t)))$ is a Gaussian vector, we obtain the independence.

The non-disjoint case: $A_2 \cap (0, 1) \neq \emptyset$ (this happens when $A_2 = (c, \infty)$ with $c < 1$). In this case $\mathbb{E}B^H(s)W(C_r(t)) \neq 0$ in general and so $B^H(s)$ and $W(C_r(t))$ are dependent.

We will refer throughout this work to the two situations above as the *the disjoint case* and the *non-disjoint case.* Basically, the results in the disjoint case can be obtained by following the arguments in [1] while in the non-disjoint case the context is different because of the appearance of the Malliavin derivatives in the expression of square mean of (10).

We will consider the following assumption, which also appears in the paper [1]. It is needed to prove the existence of the integral (18) . We assume that (recall that φ is given by (7))

$$c\varphi(2) + 2H > 1. \tag{21}$$

Remark 3: *Since $\varphi(2) = -1$, the condition (21) means that $c < 2H - 1$. Since $H > \frac{1}{2}$ we can choose $c > 0$ in order to have (21). By assuming (21), we are in the case (A) in [1].*

Proposition 2: *Suppose $H > \frac{1}{2}$ and assume (21). Then for every $r > 0$ the stochastic Skorohod integral in (18) is well-defined.*

Proof: We will use the representation of (10) of $Z_r^H(t)$ as a Skorohod integral with respect to the measure W. Next we use the bound (13) with $p = 2$. Let us apply it to the process

$$(s, r') \to 1_{[0,t]}(s) 1_{A_2}(r) \int_s^t \mathrm{d}a \partial_1 K(a, s) Q_r(a)$$

(which is a two-parameter process) and to $H = L^2(\mathbb{R} \times \mathbb{R}_+, m)$. The variables t, r and the set A_2 with $m_2(A_2) = 1$ are fixed. Below D is the Malliavin derivative with respect to the isonormal process W (see Section II). We will get, by (13),

$$\begin{aligned}
\mathbb{E}(Z_r^H(t))^2 \leq\ & \mathbb{E}\int_0^t \mathrm{d}s \int_{A_2} \frac{c\mathrm{d}r'}{(r')^2} \left(\int_s^t \mathrm{d}a \partial_1 K(a, s) Q_r(a)\right)^2 \\
& + \mathbb{E}\int_0^t \mathrm{d}s \int_{A_2} \frac{c\mathrm{d}r'}{(r')^2} \int_0^t \mathrm{d}\alpha \int_{A_2} \frac{c\mathrm{d}\beta}{\beta^2} \\
& \times \left(D_{\alpha,\beta} \int_s^t \mathrm{d}a \partial_1 K(a, s) Q_r(a)\right)^2 \\
=\ & \mathbb{E}\int_0^t \mathrm{d}s \int_{A_2} \frac{c\mathrm{d}r'}{(r')^2} \left(\int_s^t \mathrm{d}a \partial_1 K(a, s) Q_r(a)\right)^2 \\
& + \mathbb{E}\int_0^t \mathrm{d}s \int_{A_2} \frac{c\mathrm{d}r'}{(r')^2} \int_0^t \mathrm{d}\alpha \int_{A_2} \frac{c\mathrm{d}\beta}{\beta^2} \\
& \times \left(\int_s^t \mathrm{d}a \partial_1 K(a, s) D_{\alpha,\beta} Q_r(a)\right)^2 \\
=\ & \mathbb{E}\int_0^t \mathrm{d}s \int_{A_2} \frac{c\mathrm{d}r'}{(r')^2} \left(\int_s^t \mathrm{d}a \partial_1 K(a, s) Q_r(a)\right)^2 \\
& + \mathbb{E}\int_0^t \mathrm{d}s \int_{A_2} \frac{c\mathrm{d}r'}{(r')^2} \int_0^t \mathrm{d}\alpha \int_{A_2} \frac{c\mathrm{d}\beta}{\beta^2} \\
& \times \left(\int_s^t \mathrm{d}a \partial_1 K(a, s) Q_r(a) 1_{C_r(a)}(\alpha, \beta)\right)^2 \\
:=\ & T_1 + T_2.
\end{aligned}$$

since, by the chain rule of the Malliavin operator (15)

$$D_{\alpha,\beta} Q_r(a) = Q_r(a) 1_{C_r(a)}(\alpha, \beta). \tag{22}$$

We need again to consider two cases.

The disjoint case: We have $A_2 \cap (0, 1) = \emptyset$ (this happens when $A_2 = (c, \infty)$, $c > 1$ for example). In this case the term denoted by T_2 vanishes because

$$1_{C_r(a)}(\alpha, \beta) 1_{A_2}(\beta) = 0.$$

We need to show that

$$T_1 = m_2(A_2)\mathbb{E}\int_0^t \mathrm{d}s \left(\int_s^t \mathrm{d}a \partial_1 K(a, s) Q_r(a)\right)^2$$

is finite under condition (21) and this is exactly the computation in [1], proof of Proposition 2.1.

The non-disjoint case: Suppose $A_2 \cap (0, 1) \neq \emptyset$ (this happens when $A_2 = (c, \infty)$ with $c < 1$). In this case we need also to show that

$$T_2 < \infty$$

since the term T_1 can be treated as in the disjoint case. By bounding the indicator function $1_{C_r(a)}(\alpha, \beta)$ by 1 (note that $\partial_1 K(a, s) \geq 0$ for every a, s)

$$\begin{aligned}
&\mathbb{E}\left(\int_s^t \mathrm{d}a \partial_1 K(a, s) Q_r(a) 1_{C_r(a)}(\alpha, \beta)\right)^2 \\
&\qquad \leq \mathbb{E}\left(\int_s^t \mathrm{d}a \partial_1 K(a, s) Q_r(a)\right)^2
\end{aligned}$$

and by computing the integrals dr' and $d\beta$

$$\begin{aligned}
T_2 &\leq m_2(A_2)^2 \mathbb{E}\int_0^t \mathrm{d}s \int_0^t d\alpha \left(\int_s^t \mathrm{d}a \partial_1 K(a, s) Q_r(a)\right)^2 \\
&= t m_2(A_2)^2 \mathbb{E}\int_0^t \mathrm{d}s \left(\int_s^t \mathrm{d}a \partial_1 K(a, s) Q_r(a)\right)^2 \\
&= t T_1 m_2(A_2)
\end{aligned}$$

and this is finite under (21). ■

V. The Multifractal Random Walk

The purpose of this section is to study the limit as $r \to 0$ of the family of stochastic processes $\left(Z_r^H(t)\right)$ with fixed $t > 0$. This limit will be called the *the Multifractal Random Walk.*

We will assume throughout this paragraph that $Z_r^H(t)$ is defined by (18) with $W^{(1)}$ given by (17). Moreover we will suppose

$$m_2(A_2) = 1$$

and $A_2 \cap (0, 1) \neq \emptyset$. The disjoint case $A_2 \cap (0, 1) = \emptyset$ follows from [1].

We have the following limit theorem.

Theorem 1: *Assume (21). For every $t > 0$ the sequence of stochastic process $\left(Z_r^H(t)\right)_{r>0}$ defined by (18) converges in $L^2(\Omega)$ to a random variable $Z^H(t)$.*

Proof: Let us fix $r, r' \in (0, 1)$ with $r < r'$. We will first compute the $L^2(\Omega)$ norm of the increment $Z_r^H(t) - Z_{r'}^H(t)$

where $t > 0$ is fixed. We can write, with $W^{(1)}$ given by (17)

$$\begin{aligned}
&\mathbb{E}\left|Z_r^H(t) - Z_{r'}^H(t)\right|^2 \\
&= \mathbb{E}\left[\int_0^t (Q_r(u) - Q_{r'}(u))\,\mathrm{d}B_u^H\right]^2 \\
&= \mathbb{E}\left[\int_0^t \left(\int_u^t (Q_r(a) - Q_{r'}(a))\,\partial_1 K(a,u)\right)\mathrm{d}W_u^{(1)}\right]^2 \\
&= \mathbb{E}\left[\int_0^t \int_{A_2} \left(\int_u^t \mathrm{d}a\,(Q_r(a) - Q_{r'}(a))\,\partial_1 K(a,u)\right)\right. \\
&\qquad \left.\times \mathrm{d}W(u,x)\right]^2 \\
&\leq \mathbb{E}\int_0^t \mathrm{d}u \int_{A_2} \frac{c\mathrm{d}x}{x^2}\left(\int_u^t \mathrm{d}a\,(Q_r(a) - Q_{r'}(a))\,\partial_1 K(a,u)\right)^2 \\
&\quad + \mathbb{E}\int_0^t \mathrm{d}u \int_{A_2} \frac{c\mathrm{d}x}{x^2}\int_0^t \mathrm{d}v \int_{A_2} \frac{c\mathrm{d}y}{y^2}\left[D_{v,y}\int_u^t \mathrm{d}a\,(Q_r(a)\right. \\
&\qquad \left. - Q_{r'}(a))\,\partial_1 K(a,u)\right]^2
\end{aligned}$$

where we used the bound (13) for the L^2 norm of the divergence operator. Using the differentiation rule (22), we get

$$\begin{aligned}
&\mathbb{E}\left|Z_r^H(t) - Z_{r'}^H(t)\right|^2 \\
&\leq m_2(A_2)\int_0^t du \int_u^t \mathrm{d}a \int_u^t \mathrm{d}b\,\partial_1 K(a,u)\partial_1 K(b,u) \\
&\times \mathbb{E}\,(Q_r(a) - Q_{r'}(a))\,(Q_r(b) - Q_{r'}(b)) \\
&+ m_2(A_2)\int_0^t \mathrm{d}u \int_0^t \mathrm{d}v \int_{A_2} \frac{c\mathrm{d}y}{y^2}\left[\int_u^t \mathrm{d}a\,\partial_1 K(a,u)\big(Q_r(a)1_{C_r(a)}\right. \\
&\qquad \left.(v,y) - Q_{r'}(a)1_{C_{r'}(a)}(v,y)\big)\right]^2 \\
&:= A(r,r') + B(r,r').
\end{aligned}$$

Let us first treat the term denoted by $A(r,r')$. Using property (12),

$$\begin{aligned}
A(r,r') &= m_2(A_2)\int_0^t \mathrm{d}u \int_u^t \mathrm{d}a \int_u^t \mathrm{d}b\,\partial_1 K(a,u)\partial_1 K(b,u) \\
&\quad \times \mathbb{E}\,[Q_r(a)Q_r(b) - 2Q_r(a)Q_r(b) \\
&\qquad + Q_{r'}(a)Q_{r'}(b)] \\
&= m_2(A_2)\int_0^t du \int_u^t \mathrm{d}a \int_u^t \mathrm{d}b\,\partial_1 K(a,u)\partial_1 K(b,u) \\
&\quad \times \mathbb{E}\,[Q_{r'}(a)Q_{r'}(b) - Q_r(a)Q_r(b)] \\
&= \mathbb{E}\int_0^t \mathrm{d}u \left(\int_u^t \mathrm{d}a\,\partial_1 K(a,u)Q_{r'}(a)\right)^2 \mathrm{d}u \\
&\quad - \mathbb{E}\int_0^t \mathrm{d}u \left(\int_u^t \mathrm{d}a\,\partial_1 K(a,u)Q_r(a)\right)^2 \mathrm{d}u \\
&= \mathbb{E}\left(\int_0^t Q_{r'}(u)\mathrm{d}\tilde{B}_u^H\right)^2 - \mathbb{E}\left(\int_0^t Q_r(u)\mathrm{d}\tilde{B}_u^H\right)^2
\end{aligned}$$

where $\tilde{B}^H$ denotes a fractional Brownian motion independent by W. The convergence of this term is similar to the study in [1] we are in the case (A) in their paper (see Remark 3).

The summand $B(r,r')$ will be handled as follows. First, note that

$$\begin{aligned}
&B(r,r') \\
&= m_2(A_2)\int_0^t \mathrm{d}u \int_0^t \mathrm{d}v \int_{A_2} \frac{c\mathrm{d}y}{y^2}\int_u^t \mathrm{d}a \int_u^t \mathrm{d}b\,\partial_1 K(a,u)\partial_1 K(b,u) \\
&\quad \times \mathbb{E}\left(Q_r(a)1_{C_r(a)}(v,y) - Q_{r'}(a)1_{C_{r'}(a)}(v,y)\right) \\
&\quad \times \left(Q_r(b)1_{C_r(b)}(v,y) - Q_{r'}(b)1_{C_{r'}(b)}(v,y)\right) \\
&= m_2(A_2)\int_0^t \mathrm{d}u \int_0^t \mathrm{d}v \int_{A_2} \frac{c\mathrm{d}y}{y^2}\int_u^t \mathrm{d}a \int_u^t \mathrm{d}b\,\partial_1 K(a,u)\partial_1 K(b,u) \\
&\quad \times \mathbb{E}\left[Q_r(a)Q_r(b)1_{C_r(a)}(v,y)1_{C_r(b)}(v,y)\right. \\
&\quad - Q_r(a)Q_r(b)1_{C_r(a)}(v,y)1_{C_{r'}(b)}(v,y) \\
&\quad - Q_r(a)Q_r(b)1_{C_{r'}(a)}(v,y)1_{C_r(b)}(v,y) \\
&\quad \left. + Q_{r'}(a)Q_{r'}(b)1_{C_{r'}(a)}(v,y)1_{C_{r'}(b)}(v,y)\right]
\end{aligned}$$

where we used again (12). By decomposing

$$C_r(a) = C_{r'}(a) \cup (C_r(a) \setminus C_{r'}(a))$$

we will have

$$\begin{aligned}
&B(r,r') \\
&= m_2(A_2)\int_0^t \mathrm{d}u \int_0^t \mathrm{d}v \int_{A_2} \frac{c\mathrm{d}y}{y^2}\int_u^t da \int_u^t \mathrm{d}b\,\partial_1 K(a,u)\partial_1 K(b,u) \\
&\quad \times \mathbb{E}\,[Q_{r'}(a)Q_{r'}(b) - Q_r(a)Q_r(b)] \\
&\quad \times 1_{C_{r'}(a)}(v,y)1_{C_{r'}(b)}(v,y) \\
&\quad + m_2(A_2)\int_0^t \mathrm{d}u \int_0^t \mathrm{d}v \int_{A_2} \frac{c\mathrm{d}y}{y^2}\int_u^t \mathrm{d}a \int_u^t \mathrm{d}b\,\partial_1 K(a,u)\partial_1 K(b,u) \\
&\quad \times \mathbb{E}Q_r(a)Q_r(b)1_{C_r(a)\setminus C_{r'}(a)}(v,y)1_{C_r(b)\setminus C_{r'}(b)}(v,y) \\
&:= B_1(r,r') + B_2(r,r').
\end{aligned}$$

Obviously, by bounding the indicator functions by 1,

$$\begin{aligned}
&B_1(r,r') \\
&\quad \leq \int_0^t \mathrm{d}u \int_0^t \mathrm{d}v \int_{A_2} \frac{c\mathrm{d}y}{y^2}\int_u^t \mathrm{d}a \int_u^t \mathrm{d}b\,\partial_1 K(a,u)\partial_1 K(b,u) \\
&\qquad \times \mathbb{E}\,[Q_{r'}(a)Q_{r'}(b) - Q_r(a)Q_r(b)]
\end{aligned}$$

and therefore it converges to zero as $r, r' \to 0$ by using exactly the same argument as in the case of the term $A(r,r')$.

Concerning $B_2(r,r')$, since for $0 < r < r' < 1$ and $y \in A_2$

$$1_{C_r(a)\setminus C_{r'}(a)}(v,y) \leq 1_{(r,r')\times A_2}(v,y)$$

we can bound it in the following way

$$\begin{aligned}
&B_2(r,r') \\
&\leq m_2(A_2)\int_0^t \mathrm{d}u \int_0^t \mathrm{d}v \int_{A_2} \frac{c\mathrm{d}y}{y^2}\int_u^t \mathrm{d}a \int_u^t \mathrm{d}b\,\partial_1 K(a,u)\partial_1 K(b,u) \\
&\quad \times \mathbb{E}Q_r(a)Q_r(b)1_{(r,r')\times A_2}(v,y) \\
&= m_2(A_2)^2(r'-r)\mathbb{E} \\
&\quad \times \int_0^t \mathrm{d}u \int_u^t \mathrm{d}a \int_u^t \mathrm{d}b\,\partial_1 K(a,u)\partial_1 K(b,u)\mathbb{E}Q_r(a)Q_r(b)
\end{aligned}$$

and by interchanging the order of integration and using

$$\int_0^{a\wedge b} \mathrm{d}u\,\partial_1 K(a,u)\partial_1 K(b,u) = c_H|a-b|^{2H-2}. \tag{23}$$

and since

$$\mathbb{E}Q_r(a)Q_r(b) = e^{-\varphi(2)m_r(|a-b|)} \le e^{c\varphi(2)\log|a-b|} = |a-b|^{c\varphi(2)}$$

we get

$$B_2(r,r') \le m_2(A_2)^2(r'-r)\int_0^t\int_0^t \mathrm{d}a\mathrm{d}b|a-b|^{c\varphi(2)+2H-2}$$

we clearly obtain than $B_2(r,r')$ goes to zero as $r,r' \to 0$ under condition (21). ■

Definition 1: *The process $(Z^H)_{t>0}$ from Theorem 1 will be called as fractional Multifractal Random Walk.*

VI. Properties of the MRW

We will discuss here some immediate properties of the fractional MRW from Definition 1. Basically, as mentioned before, in the disjoint case the fractional MRW Z^H has the same properties as in the situation when Q and B^H are independent: self-similarity, stationarity of increments and long-range dependence (see [1]). But in the non-disjoint case, we will show that even the L^2 moment of the fractional MRW does not scale exactly. We provide an exact calculation in order to show this phenomenon. On the other hand, we can control the L^p norm of the increment of the process and we find some kind of asymptotic scaling even in the non-disjoint case.

A. Scaling of the Second Moment

We first analyze the second moment of the increments of Z^H.

The disjoint case: Following [1], we have

$$\begin{aligned}\mathbb{E}\left(Z^H(t)\right)^2 &= \int_0^t \mathrm{d}s\left(\int_s^t\int_s^t \mathrm{d}a\mathrm{d}b\mathbb{E}(Q_0(a)Q_0(b))\partial_1K(a,u)\partial_1K(b,u)\right)\\ &= \int_0^t \mathrm{d}s\left(\int_s^t\int_s^t \mathrm{d}a\mathrm{d}b e^{-\rho_2 m_0(|a-b|)}\partial_1K(a,u)\partial_1K(b,u)\right)\\ &= c_H\int_0^t\int_0^t \mathrm{d}b|a-b|^{2H-2}\mathbb{E}(Q_0(a)Q_0(b))\end{aligned}$$

where Q_0 is understood, formally, as the limit of Q_r when r goes to zero and the meaning of the quantity $\mathbb{E}Q_0(a)Q_0(b)$ is given by (5) with $r=0$ and where we used the identity (23). Then, for every $h>0$

$$\begin{aligned}\mathbb{E}\left(Z^H(ht)\right)^2 &= c_H\int_0^{ht}\int_0^{ht}\mathrm{d}b|a-b|^{2H-2}\mathbb{E}(Q_0(a)Q_0(b))\\ &= h^{2H}c_H\int_0^t\int_0^t \mathrm{d}b|a-b|^{2H-2}\mathbb{E}(Q_0(ha)Q_0(hb))\\ &= h^{2H}\mathbb{E}e^{2\Omega_h}c_H\int_0^t\int_0^t \mathrm{d}b|a-b|^{2H-2}\mathbb{E}(Q_0(a)Q_0(b))\\ &= h^{2H}\mathbb{E}e^{2\Omega_h}\mathbb{E}\left(Z^H(t)\right)^2 = h^{2H+c\varphi(2)}\mathbb{E}\left(Z^H(t)\right)^2.\end{aligned}$$

where we used the scaling property (8). Actually, it is not difficult to see that for every $p>1$

$$\mathbb{E}\left(Z^H(ht)\right)^p = h^{2Hq}\mathbb{E}e^{p\Omega_t} = h^{2Hp+q\varphi(p)}.$$

Moreover, we have the self-similarity $(Z^H(at))_{t\in[0,1]} =^{(d)} a^{H+\Omega_a}(Z^H(t))_{t\in[0,1]}$, the stationarity of the increments and the long-range dependence in the sense that $\mathbb{E}X_kX_0 \sim \tau^2Hk^{2H}$ where $X_k = Z^H((k+1)\tau) - Z^H(k\tau)$ with $k\ge 0$ integer and $\tau>0$.

The non-disjoint case: The situation is different in the non-disjoint case and we will see that even the second moment of the fractional MRW does not scale. We can compute exactly the L^2 norm of $Z^H(t)$.

$$\begin{aligned}\mathbb{E}\left|Z^H(t)\right|^2 &= \int_0^t \mathrm{d}s\left(\int_s^t \mathrm{d}a\,Q_0(a)\partial_1K(a,s)da\right)^2\\ &\quad+\int_0^t \mathrm{d}s\int_{A_2}\frac{c\mathrm{d}r}{r^2}\int_0^t\mathrm{d}\alpha\int_{A_2}\frac{c\mathrm{d}\beta}{\beta^2}\\ &\quad\times\left(D_{\alpha,\beta}\int_s^t \mathrm{d}b\,Q_0(b)\partial_1K(b,s)\mathrm{d}b\right)\\ &\quad\times\left(D_{s,r}\int_\alpha^t \mathrm{d}a\,Q_0(a)\partial_1K(a,\alpha)\mathrm{d}a\right)\\ &= \int_0^t \mathrm{d}s\left(\int_s^t \mathrm{d}a\,Q_0(a)\partial_1K(a,s)\mathrm{d}a\right)^2\\ &\quad+\int_0^t \mathrm{d}s\int_{A_2}\frac{c\mathrm{d}r}{r^2}\int_0^t\mathrm{d}\alpha\int_{A_2}\frac{c\mathrm{d}\beta}{\beta^2}\\ &\quad\times\left(\int_\alpha^t \mathrm{d}a\,Q_0(a)\partial_1K(a,\alpha)1_{C_0(a)}(s,r)\right)\\ &\quad\times\left(\int_s^t \mathrm{d}b\,Q_0(b)1_{C_0(b)}(\alpha,\beta)\partial_1K(b,s)\mathrm{d}b\right)\\ &= I(t)+J(t)\end{aligned}$$

Consider $h>0$. Then, applying the above formula to $t=ht$ and making several changes of variables, we will get

$$\begin{aligned}\mathbb{E}\left|Z^H(ht)\right|^2 &= h^{2H+c\varphi(2)}I(t)\\ &\quad+h^{2H+1+c\varphi(2)}\int_0^t\mathrm{d}s\int_{A_2}\frac{c\mathrm{d}r}{r^2}\int_0^t d\alpha\int_{A_2}\frac{c\mathrm{d}\beta}{\beta^2}\\ &\quad\times\left(\int_\alpha^t \mathrm{d}a\,Q_0(a)\partial_1K(a,\alpha)1_{C_0(ha)}(hs,r)\right)\\ &\quad\times\left(\int_s^t \mathrm{d}b\,Q_0(b)1_{C_0(hb)}(h\alpha,\beta)\partial_1K(b,s)\mathrm{d}b\right).\end{aligned}$$

The integral with respect to $d\beta$ and dr can be computed explicitly. For example, when $A_2=(c,\infty)$, $c<1$, we find

$$\int_{A_2}\frac{c\mathrm{d}\beta}{\beta^2}1_{C_0(hb)}(h\alpha,\beta) = c\left[\frac{1}{c\vee 2h|b-\alpha|}-1\right]$$

and it is clear that the term $J(t)$ does not scale exactly as $I(t)$.

B. The Control of the Increments

Let us estimate the L^p norm of the increment $Z^H(t)-Z^H(s)$ of the fractional Multifractal Random

Walk introduced in Definition 1. Fix $s, t \in [0,1]$ with $t > s$. We will not insist on the disjoint case because the calculations in [1] still hold, so we will have

$$\mathbb{E}\left|Z^H(t) - Z^H(s)\right|^p \sim C_p|t-s|^{2Hp+c\varphi(p)}$$

with φ given by (7). The symbol $\sim$ means that the two sides have the same behavior when $t-s$ is small.

Let us consider the non-disjoint case. In this case $Z^H(t)$, which can formally be written as $\int_0^t Q_0(y)dB^H(u)$ is an anticipating (Skorohod integral). We need to use Meyer's inequalities (13) in order to estimate its L^p norm.

Actually

$$Z^H(t) - Z^H(s) = \int_0^1 \mathrm{d}W^{(1)}(u)F_{s,t}(u)$$

where we denoted by

$$\begin{aligned} F_{s,t}(u) = &\, 1_{(0,t)}(u)\int_u^t \partial_1 K(a,u)Q_0(a)\mathrm{d}a \\ &-1_{(0,s)}(u)\int_u^s \partial_1 K(a,u)Q_0(a)\mathrm{d}a. \end{aligned}$$

By Meyer's inequality (13)

$$\begin{aligned} &\mathbb{E}\left|Z^H(t) - Z^H(s)\right|^p \\ &\quad\leq \mathbb{E}\int_0^1 \mathrm{d}u\int_{A_2}\frac{c\mathrm{d}r}{r^2}\left|F_{s,t}(u)\right|^p \\ &\quad\quad+\mathbb{E}\int_0^1 \mathrm{d}u\int_{A_2}\frac{c\mathrm{d}r}{r^2}\int_0^1 \mathrm{d}\alpha\int_{A_2}\frac{c\mathrm{d}\beta}{\beta^2}\left|D_{\alpha,\beta}F_{s,t}(u)\right|^2 \\ &\quad:= A(t,s) + B(t,s). \end{aligned}$$

The terms A is exactly the L^p norm in the disjoint case, so

$$A(t,s) \leq c_p|t-s|^{2Hp+c\varphi(p)}.$$

Concerning the summand denoted by $B(t,s)$, we proceed as in the proof of Proposition 2: we use first the Malliavin differentiation $D_{\alpha,\beta}Q_r(a) = Q_r(a)1_{C_0}(\alpha,\beta)$, then we bound the indicator function $1_{C_0}(\alpha,\beta)$ by 1, then we integrate dr and $d\beta$ and we obtain

$$B(t,s) \leq c_p \int_0^1 \mathrm{d}u\left|F_{s,t}(u)\right|^p \leq c_p|t-s|^{2Hp+c\varphi(p)}$$

because the right hand side is equal, modulo a constant, to $A(t,s)$. Taking into account the above estimates, we conclude that

$$\mathbb{E}\left|Z^H(t) - Z^H(s)\right|^p \leq c_p|t-s|^{2Hp+c\varphi(p)} \tag{24}$$

for every t, s.

VII. Simulation Scheme

As mentioned in the Introduction, the multifractal random walks appears nowadays as a serious candidate to model the financial time series. In order to compare its behavior with real data, one needs to simulate it. The main difficulty consists in the fact that, in our construction, the variables Q_r and B^H (that appear in the stochastic integral (18)) are dependent. From the theoretical point of view, the Malskorohodliavin calculus offers convenient techniques but usually the mathematical objects (Malliavin derivatives, Skorohod integrals) related to it are difficult to be simulated. The purpose of this section is to explain how we simulated the stochastic integral that defines the fractional MRW.

Recall that the fractional MRW is defined by

$$Z^H(t) = \int_0^t Q_0(s)\mathrm{d}B^H(s) \tag{25}$$

this expression being formal (recall that Q_0 needs to be understood, formally, as the limit of Q_r when r goes to zero).

Let us assume that the underlying infinitely divisible law G is a Gaussian $\mathcal{N}(\mu,\sigma^2)$, which is an usual choice in the literature. Then, by (7) and and using the fact that $\rho(q)$ is the moment generating function of a random variable drawn from G, we get

$$\varphi(q) = \frac{\sigma^2}{2}q(1-q).$$

Also recall that we have chosen to work with the control measure $\mathrm{d}m(t,r) = \frac{c}{r^2}\mathrm{d}r\mathrm{d}t$ for $0 < r \leq 1$ (this situation is called the exact scale invariance case). In this case, we have (see e.g. [1])

$$\begin{aligned} &m_r(|t-s|) \\ &\quad= \begin{cases} -c\log r + c|t-s|(1-\frac{1}{r}) & \text{if } 0 < |t-s| \leq r \\ -c\log|t-s| + c(|t-s|-1) & \text{if } r < |t-s| \leq 1 \\ 0 & \text{if } |t-s| > 1. \end{cases} \end{aligned}$$

where m_r is defined by (6). By using (5) and the above formula for m_r, we can easily obtain

$$\begin{aligned} &Cov(\log Q_r(t), \log Q_r(t+\tau)) \\ &\quad= \begin{cases} c\sigma^2\left(\log\left(\frac{1}{r}\right)+\tau-\frac{\tau}{r}\right) & \text{if } 0 < \tau \leq r \\ c\sigma^2\left(\log\left(\frac{1}{\tau}\right)+\tau-1\right) & \text{if } r < \tau \leq 1 \\ 0 & \text{if } \tau > 1. \end{cases} \end{aligned} \tag{26}$$

In order to simulate a log-normal IDC noise $(Q_r(t), t \geq 0)$, we just have to consider (26). Actually, we have to simulate correlated Gaussian random variables with covariance structure (26). For a comprehensive list of simulation methods, we refer e.g. to [6]. Here, we use the circulant matrix embedding method (see [32]). See also [14] for another approach on the synthesis of a log-normal IDC noise.

Let us now explain the idea to simulate the integral (25). This integral is a divergence integral and in principle its simulation is difficult. But in our case, taking into account the particular form of the integrand, we can use the following approach. Let $\{0 = t_0 < t_1 < \cdots < t_n = t\}$, $\tau = t_{i+1} - t_i$, a partition of the interval $[0,t]$. In the independent case (meaning when W and B^H are independent), (25) can be naturally approximated by the Riemann sum (we use the

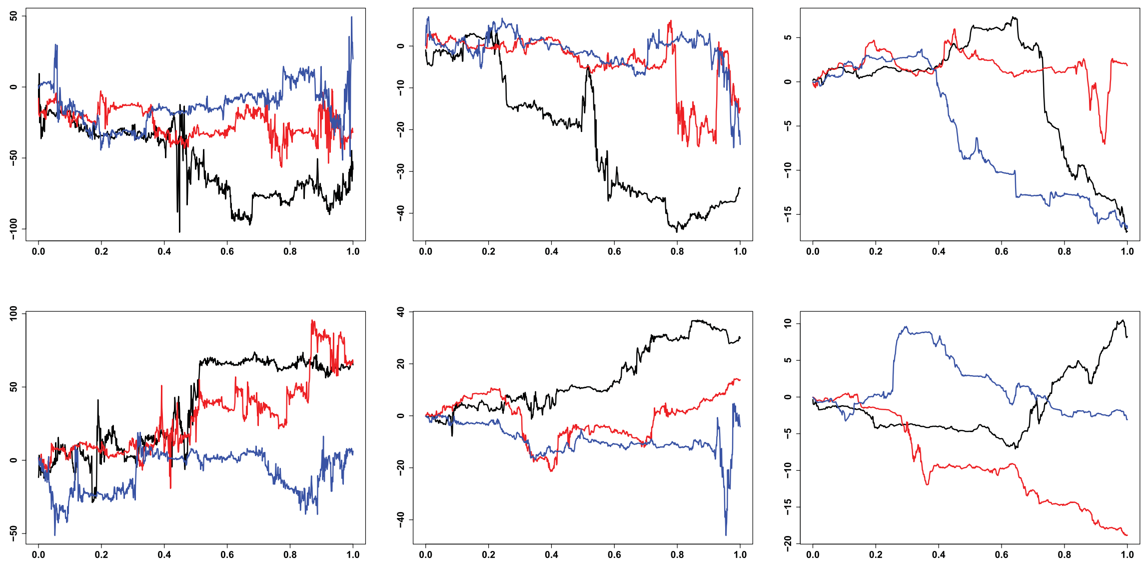

Fig. 1. Paths of MRW, with $\tau = 10^{-4}$, $\sigma^2 = 0.02$, $r = 10^{-8}$. From left to right we considered the following values for H: $H = 0.25, 0.5, 0.75$. From top to the bottom we took: $a = 5.10^{-4}$ and $a = 5.10^{-2}$.

notation $Q_r(t) = \frac{e^{w_r(t)}}{\mathbb{E}e^{w_r(t)}})$

$$\sum_{i=0}^{n-1} Q_r(t_i)(B^H(t_{i+1}) - B^H(t_i)) = \sum_{i=0}^{n-1} \frac{e^{w_r(t_i)}}{\mathbb{E}e^{w_r(t_i)}}(B^H(t_{i+1}) - B^H(t_i)).$$

Since the simulation of B^H is well-known (see e.g. [6] for an explicit algorithm), we can generate the above sum. In the dependent case, which is the situation treated in our paper, using the integration by parts formula $\delta(Fu) = F\delta(u) - \langle DF, u\rangle$, δ being the Skorohod integral with respect to B^H (see [29]), the sum that approximates the integral (18) can be expressed as

$$\begin{aligned}&\sum_{i=0}^{n-1}\left(\frac{e^{w_r(t_i)}}{\mathbb{E}e^{w_r(t_i)}}(B^H(t_{i+1}) - B^H(t_i)) - \langle D\frac{e^{w_r(t_i)}}{\mathbb{E}e^{w_r(t_i)}}, 1_{(t_i,t_{i+1})}\rangle\right)\\ &=\sum_{i=0}^{n-1}\left(\frac{e^{w_r(t_i)}}{\mathbb{E}e^{w_r(t_i)}}(B^H(t_{i+1}) - B^H(t_i))\frac{e^{w_r(t_i)}}{\mathbb{E}e^{w_r(t_i)}}\langle Dw_r(t_i), 1_{(t_i,t_{i+1})}\rangle\right)\\ &=\sum_{i=0}^{n-1}\left(\frac{e^{w_r(t_i)}}{\mathbb{E}e^{w_r(t_i)}}(B^H(t_{i+1}) - B^H(t_i)) - a_{r,i}\frac{e^{w_r(t_i)}}{\mathbb{E}e^{w_r(t_i)}}\right) \qquad (27)\end{aligned}$$

where we denoted by

$$a_{r,i} = a_i := \langle Dw_r(t_i), 1_{(t_i,t_{i+1})}\rangle. \qquad (28)$$

The scalar product is in the Hilbert space associated to the fBm and, since $w(t_i)$ is Gaussian, $Dw(t_i)$ is deterministic and it dependent on the set A_2 (more exactly on the intersection of A_2 and the interval (0,1) which is in principle very small). Thus, a possibility to simulate (25) is to approximate it by $\sum_{i=0}^{n-1}\left(e^{w_r(t_i)}(B^H(t_{i+1}) - B^H(t_i)) - a_i e^{w_r(t_i)}\right)$ with suitable coefficients a_i and with suitably chosen r and $\tau = t_{i+1} - t_i$.

The coefficients a_i defined above in (28) represents a measure of the dependence of the IDC Q and of the integrator (the fBm B^H) in (18). These coefficients are equal to zero in the case when Q and B^H are independent (the situation treated in [1] for example). The main challenge of our simulation methology is how to estimate properly these coefficients a_i. The details on the choice of these coefficients can be found in the next section (Section VIII) of our paper.

Let us summarize our simulation approach:

- we first suitably choose the lag τ and r very small.
- we simulate a Gaussian vector with covariance structure (26).
- we simulate the fractional MRW Z^H approximating it by the sequence (27) with a suitable choice for the coefficients a_i given by (28) and for σ and H.

We present several paths of the fractional MRW in Fig. 1.

VIII. Financial Statistics

Let $S(t)$ be the price at time t of an financial asset, $X(t) = \ln S(t)$ the log price and then, log-returns at lag τ are given by

$$\delta_\tau X(t) = X(t) - X(t-\tau) = \ln\left(\frac{S(t)}{S(t-\tau)}\right),$$

for sake of simplicity and no loss of generality, we assume that log-return are centered, $\mathbb{E}[\delta_\tau X(t)] = 0$, thus, we can interpret the volatility as the squared log-returns. The log-price will be modeled by a fractional MRW.

To judge the viability of our model, we will analyze the fluctuations of the financial data, the so-called stylized facts in finance.

The random walk that we introduce in our work allows the conditional variance and the random noise, which is a fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$, to be dependent. This dependence should involve the so-called leverage effect, that is, the correlation between the log-return at time t and the squared log-return (the volatility) in the future. The presence of

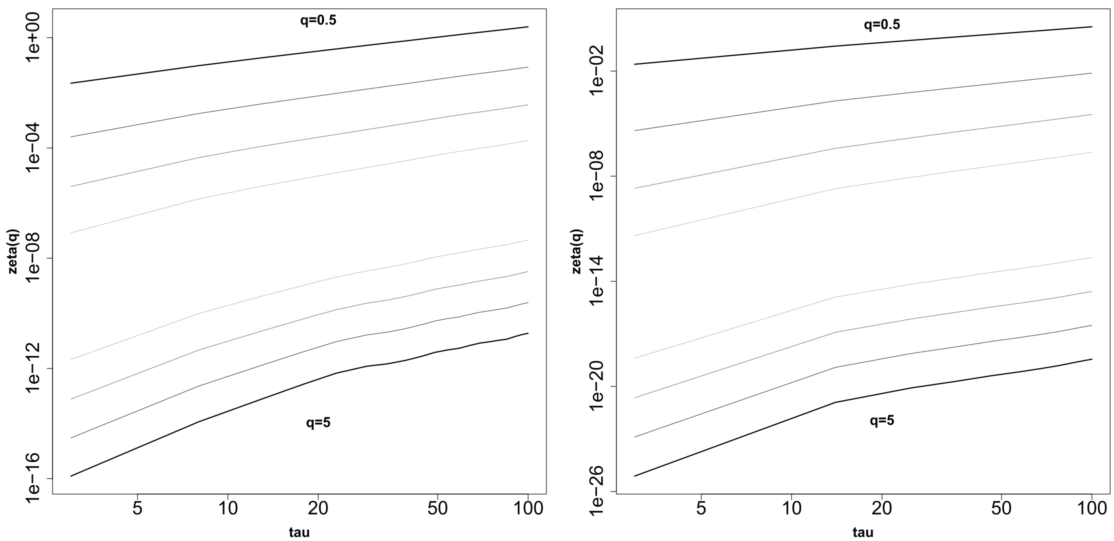

Fig. 2. Left: Multifractal spectrum of the simulated MRW with $\tau = 10^{-4}$, $H = 0.55$, $a = 5.10^{-4}$, $r = 10^{-8}$ and $\sigma^2 = 0.02$. Right: Multifractal spectrum of the S&P 500 index at 15 seconds. In both cases, q varies from 0.5 to 5 and τ from 1 to 100. We used a log-log scale.

the coefficients a_i in the formula (27) will induce the leverage effect and also a negative skewness.

The case $H > 1/2$ implies that the noise B^H exhibits long-range dependence, whereas $H < 1/2$ induce a mean-reverting effect (notice that the fractional MRW Z^H from Definition 1 is not defined theoretically for $H < \frac{1}{2}$; but we found useful to discuss it empirically). This should involve a long-range dependence or mean-reverting effect of the fluctuations of the financial time series. These facts (leverage effect, long-range dependence and mean-reverting effect) will be checked on the data. Actually the leverage effect is mainly empirically observed on financial index such as the S&P 500 (US market), Nikkei 225 (Japan market), FTSE 100 (UK market), CAC 40 (French market), etc. But it does not appear on the Forex market and on the commodities, for example. We notice the absence of auto-correlation in log-returns if we consider low frequency financial data; but this auto-correlation clearly appears in the case of square or absolute log-returns. Nevertheless, at high frequency, we observe a dependence relation which is going to be formed (see also e.g. [23] or [31]).

We will now regard the empirical behavior of the fractional MRW constructed by using a dependent IDC. Taking into account the definition of the MRW, that involves a stochastic integral of the divergence type, it is hard to check theoretically the multifractal character of the MRW. We only proved an upper bound in equation (24). Nevetheless, we can check empirically the multifractal properties of the fractional MRW. We give in Fig. 2 the multifractal spectrum $\zeta(q)$ of this process, defined by the equation (see Section VI-B)

$$\mathbb{E}[|\delta_\tau X(t)|]^q \sim c_q \tau^{\zeta(q)}$$

obtained empirically via the variations of the process X

$$\zeta(q) = \frac{\log \frac{1}{n} \sum_{i=1}^{n} |X(t+\tau) - X(t)|^q}{\log \tau}.$$

In Fig. 2, the multifractal character is empirically verified. On the other hand, it is known that, if a process X is self-similar, then it satisfies the so-called Castaing equation [13]. That means that at a low frequency τ the density of the process is close to the Gaussian density while at high frequency the density presents a more pronounced kurtosis. In Fig. 3 we give the emprical density of the fractional MRW for various values of τ and we compare it with the empirical density of the S&P 500 index. The particularity due to the Castaing equations seems satisfied and more the lag τ is small, more the distribution is leptokurtic and when the lag τ is large, the distribution becomes closer to the normal distribution.

We do not present the empirical density obtained for the coefficient a_i (28) varying from 10^{-4} to 10^{-2} (they are available on request). At the end of this section we describe how a_i are estimated from the data. There is not significant difference between the densities. Nevertheless, let us mention that the skewness becomes more and more negative when the coefficients a_i take a bigger value. If we consider a_i to be close to 1, then the empirical density will present a strong negative skewness. Taking into account the expression (27) of the Riemann sum that approximates the fractional MRW, it is clear that the choice of a large a_i will conduct to strictly decreasing trajectories and this will not correspond to the evolution of the financial assets.

Let us now discussed other stylized facts of the financial data. In particular, we will discuss the effect of the Hurst parameter H and of the coefficients a_i on the auto-correlation function (ACF in the sequel)

$$R_1(k) = \mathrm{Cor}(\delta_\tau X(t), \delta_\tau X(t+k)), \quad k \geq 0,$$

where Cor denotes the correlation. We will make the same analysis for the ACF of the squared log-returns (denoted R_2) and for the leverage effect, that is, that correlation between the log-returns and the future volatility given by

$$\mathcal{L}(t, t') = \mathrm{Cor}\left(\delta_\tau X(t), (\delta_\tau X(t'))^2\right), \quad t - k \leq t' \leq t + k. \tag{29}$$

These three quantities are presented in Fig. 4. The first interesting observation is that the a_i' do not affect the ACF, they

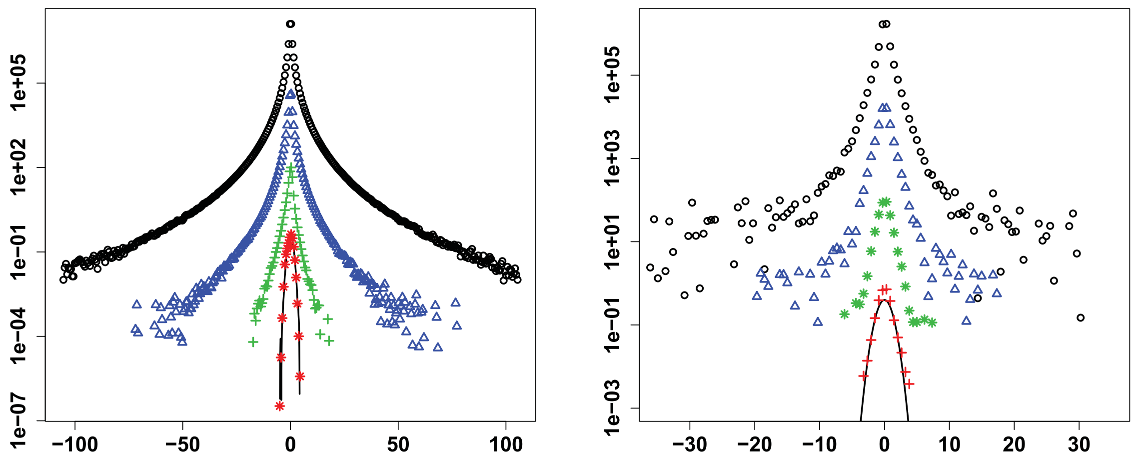

Fig. 3. Left: Empirical density of the MRW with the following values of τ from top to bottom: $\tau = 10^{-4}, 10^{-2}, 1, 10^{2}$ (we fixed $H = 0.5$, $a = 5.10^{-4}$, $r = 10^{-8}$ and $\sigma^2 = 0.02$). Right: Empirical density of S&P 500, with (from top to bottom) $\tau = 15$ seconds, 30 minutes, 4 hours and 1 day. In both cases, the black line corresponds to the Gaussian prediction. We used the logarithmic scale.

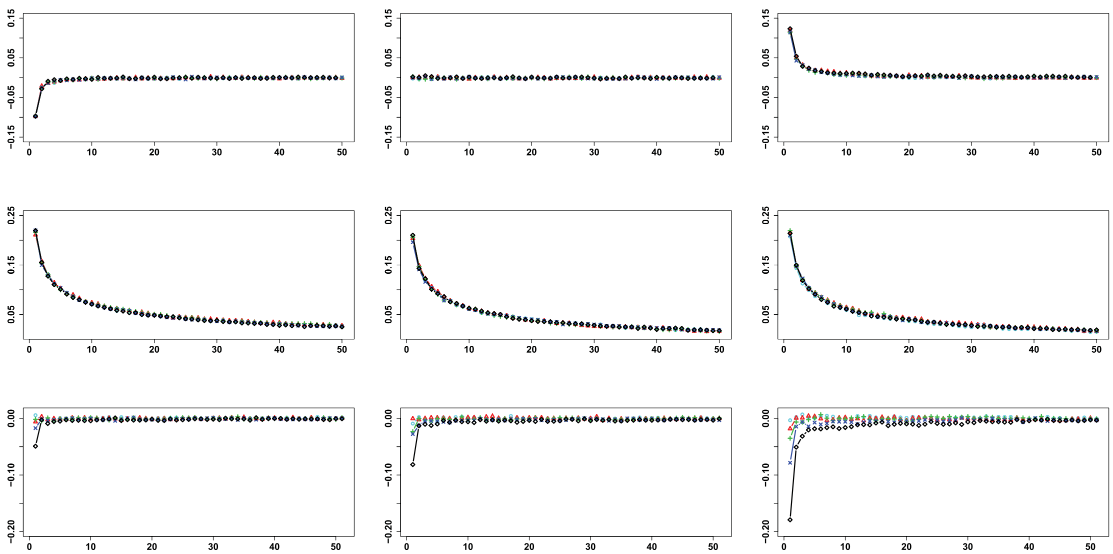

Fig. 4. The top line: the ACF of the log-returns; The middle line: the ACF of the squared log-returns; The bottom line: the leverage effect. On these three lines we considered (from the left to the right): H=0.4, 0.5 and 0.6. On every figure we traced the results obtained for different values of a: a = 0, a = 0.0005, a = 0.001, a = 0.005 and a = 0.01 with the corresponding colours light blue, red, green, blue and black respectively.

have influence only on the leverage effect. Indeed, the values of the empirical ACF of the log-returns and of the squared log-returns do not change with a_i. The second interesting observation is that the leverage effect does not depend only on a_i but also on the self-similarity index H of the fractional Brownian motion. When H is small, close to zero, the leverage effect is weak and when H is close to 1 this leverage effect becomes more important. Moreover, the parameter σ^2 does not affect the leverage effect. As expected, the leverage effect increases with a_i since (as discussed before), the coefficients a_i are the measure of the dependence of the noise B^H and of the integrand Q in the definition of the fractional MRW. Finally, one observes a mean-reverting character when $H < \frac{1}{2}$ (this observation is only empirical, recall that our MRW is not defined when $H < \frac{1}{2}$), the presence of long-memory for $H > \frac{1}{2}$ and one notice the absence of any dependence for $H = \frac{1}{2}$.

Let us now explain how we choose the parameters of our model. The idea is to find the combinaison of the parameters (σ^2, H, a) that minimize the quadratic error

$$(\hat{\sigma}^2, \hat{H}, \hat{a}) = \underset{\sigma^2, H, a}{\operatorname{argmin}} \Big\{ ||R_1 - \tilde{R}_1||_2^2 + ||R_2 - \tilde{R}_2||_2^2 + ||\mathcal{L} - \tilde{\mathcal{L}}||_2^2 + ||\zeta - \tilde{\zeta}||_2^2 + |\gamma - \alpha\tilde{\gamma}| \Big\}. \tag{30}$$

where γ represents the skewness, which is usually negative in the financial fluctuations. The parameter α allows to keep the logic of the minimization. If the empirical skewness is negative, this parameter takes the value +1, if the skewness of the simulations is also negative, then α is -1 and similarly if the skewness is positive. The spectrum ζ is calculated for 5 values of q going from 0.5 to 5. a is a vector, nonetheless, like H or σ, we assume that a_i is constant for all i. Otherwise, it will force us to re-calibrate the model at each new observation. Furthermore, as on icing on the cake, empirical

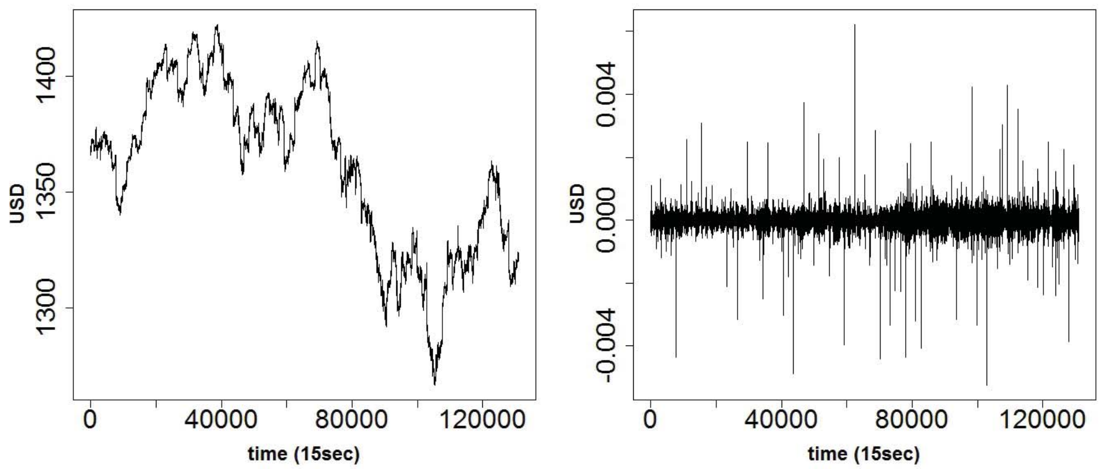

Fig. 5. Price and log-return of the S&P 500 index from 2012-02-28 to 2012-06-26, 131011 points.

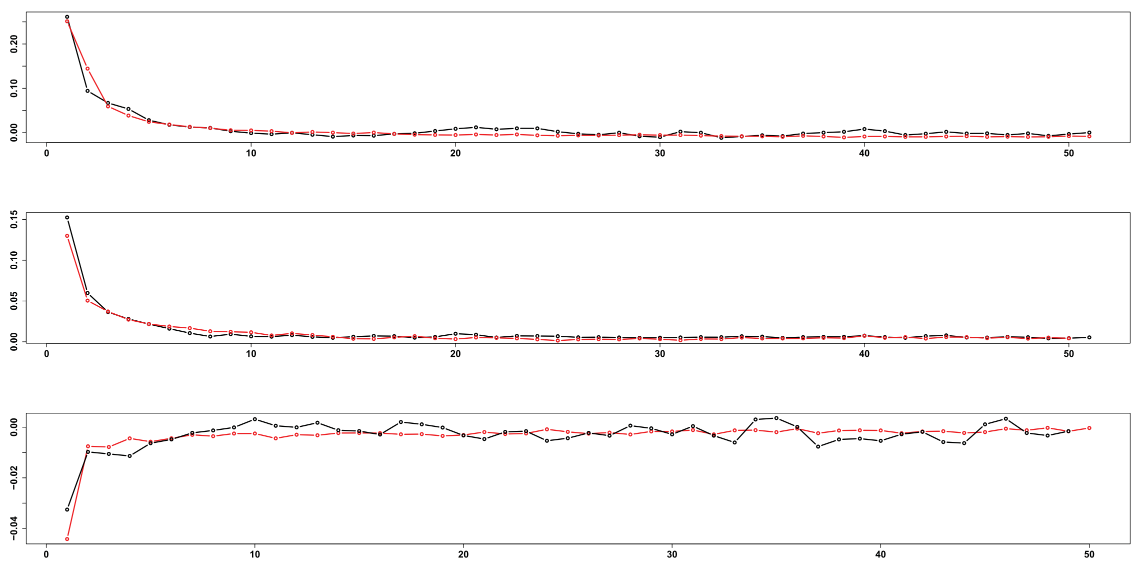

Fig. 6. From top to bottom: ACF of the log-returns, of the squared log-returns and the leverage effect; In black: the S&P 500 index, in red: the simulated fractional MRW.

investigation show us a negligible variation of a_i for the same asset. The quantities denoted by tilde are estimated by using 500 simulations of the MRW with the same parameters and of size 10^7. It is of course possible to minimize (30) by testing one by one the set of the combinaisons of parameters. But in order to save time, we used a genetic algorithm.

The data that we have chosen in order to check the capacity of the dependent fractional MRW to reproduce the stylized facts of the financial data are taken from the S&P 500 index at 15 seconds frequency from 2012-02-28 to 2012-06-26, 131011 points (see Fig. 5). Also, we fixed $\tau = 10^{-4}$, $r = 10^{-8}$. To use our genetic algorithm, we have chosen $\sigma^2 \in [0.0001, 0.1]$, $H \in (1/2, 1)$ and $a \in [10^{-4}, 10^{-2}]$. As it can be seen, during the concerned period we observe a positive auto-correlation and this allows to consider the Hurst parameter bigger than $\frac{1}{2}$, which considerably relax the algorithm. The coefficient a is chosen bigger than 10^{-4} (this will lead to the presence of the leverage effect) and less than 10^{-2} in order to not have the log-returns mostly negative. As before, we make 500 simulations of size 10^7.

After optimization, we found $\sigma^2 = 0.023$, $H = 0.59$ and $a = 0.002$. The comparaison between the stylized facts and the data is included in Fig. 6.

References

[1] P. Abry, P. Chainais, L. Coutin, and V. Pipiras, "Multifractal random walks as fractional Wiener integrals," *IEEE Trans. Inf. Theory*, vol. 55, no. 8, pp. 3825–3840, Aug. 2009.

[2] E. Bacry, L. Duvernet, and J. F. Muzy, "Continuous-time skewed multifractal processes as a model for financial returns," *J. Appl. Probab.*, vol. 49, no. 2, pp. 482–502, 2012.

[3] E. Bacry and J.-F. Muzy, "Multifractal stationary random measures and multifractal walks with log-infinitely divisible scaling laws," *Phys. Rev. E*, vol. 66, no. 5, pp. 056121-1–056121-16, Nov. 2002.

[4] E. Bacry, A. Kozhemyak, and J.-F. Muzy, "Continuous cascade models for asset returns," *J. Econ. Dyn. Control*, vol. 32, no. 1, pp. 156–199, Jan. 2008.

[5] E. Bacry, J. F. Muzy, and J. Delour, "Multifractal random walks," *Phys. Rev. E*, vol. 64, no. 2, pp. 026103–026106, Jul. 2001.

[6] J. M. Bardet, G. Lang, G. Oppenheim, A. Philippe, and M. Taqqu, "Generators of long-range dependent processes: A survey," in *Long-Range Dependence: Theory and Applications*. Berlin, Germany: Birkhäuser, 2003.

[7] J. Barral and B. B. Mandelbrot, "Multifractal products of cylindrical pulses," *Probab. Theory Rel. Fields*, vol. 124, no. 3, pp. 409–430, Nov. 2002.

[8] F. Black, "Studies of stock price volatility changes," in *Proc. Meetings Amer. Statist. Assoc., Bus. Econ. Statist. Sect.*, Jan. 1976, pp. 177–181.

[9] T. Bollerslev, "Generalized autoregressive conditional heteroskedasticity," *J. Econ.*, vol. 31, no. 3, pp. 307–327, Apr. 1986.

[10] T. Bollerslev, J. Litvinova, and G. Tauchen, "Leverage and volatility feedback effects in high-frequency data," *J. Financial Econ.*, vol. 4, no. 3, pp. 353–384, May 2006.

[11] J.-P. Bouchaud, A. Matacz, and M. Potters, "The leverage effect in financial markets: Retarded volatility and market panic," *Phys. Rev. Lett.*, vol. 87, no. 22, pp. 22870-1–22870-4, Nov. 2001.

[12] L. Calvet and A. Fisher, *Multifractal Volatility: Theory, Forecasting and Princing*. Cambridge, U.K.: Cambridge Univ. Press, 2008.

[13] B. Castaing, "The temperature of turbulent flows," *J. Phys. II France*, vol. 6, no. 1, pp. 105–114, Jan. 1996.

[14] P. Chainais, R. Riedi, and P. Abry, "On non-scale invariant infinitely divisible cascades," *IEEE Trans. Inf. Theory*, vol. 51, no. 3, pp. 1063–1083, Mar. 2005.

[15] A. Chakraborti, I. M. Toke, M. Patriarca, and F. Abergel, "Econophysics review: I. Empirical facts," *Quant. Finance*, vol. 11, no. 7, pp. 991–1012, Jun. 2011.

[16] J. Duchon, R. Robert, and V. Vargas, "Forecasting volatility with the multifractal random walk model," *Math. Finance*, vol. 22, no. 1, pp. 83–108, Jan. 2012.

[17] L. Duvernet, "Convergence of the structure function of a multifractal random walk in a mixed asymptotic setting," *Stoch. Anal. Appl.*, vol. 28, no. 5, pp. 763–792, Aug. 2010.

[18] L. Duvernet, C. Robert, and M. Rosenbaum, "Testing the type of a semi-martingale: Itō against multifractal," *Electron. J. Statist.*, vol. 4, pp. 1300–1323, Sep. 2010.

[19] R. F. Engle, "Autoregressive conditional heteroskedasticity with estimates of the variance of the U.K. inflation," *Econometrica*, vol. 50, no. 4, pp. 987–1108, Jul. 1982.

[20] A. Fisher, L. Calvet, and B. Mandelbrot, "Multifractality of deutschemark/US dollar exchange rates," Yale Univ., Cowles Found. Ser. Res. Econ., New Haven, CT, USA, Tech. Rep. 1166, Sep. 1997.

[21] S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner, and Y. Dodge, "Turbulent cascades in foreign exchange markets," *Nature*, vol. 381, pp. 767–770, Jun. 1996.

[22] P. Gopikrishnan, V. Plerou, L. A. Amaral, M. Meyer, and H. E. Stanley, "Scaling of the distribution of fluctuations of financial market indices," *Phys. Rev. E*, vol. 60, no. 5, pp. 5305–5316, 1999.

[23] F. Lillo and J. D. Farmer, "The long memory of the efficient market," *Stud. Nonlinear Dyn. Econ.*, vol. 8, no. 3, pp. 1558–3708, 2004.

[24] C. Ludena, "Confidence intervals for the scaling function of multifractal random walks," *Statist. Probab. Lett.*, vol. 79, no. 9, pp. 1186–1193, May 2009.

[25] C. Ludena, "L^p variations for multifractal fractional random walks," *Ann. Appl. Probab.*, vol. 18, no. 3, pp. 1138–1163, Jun. 2008.

[26] B. Mandelbrot, "The variation of certain speculative prices," *J. Bus.*, vol. 36, no. 4, pp. 394–419, Oct. 1963.

[27] J. F. Muzy, J. Delour, and E. Bacry, "Modeling fluctuations of financial time series: From cascade process to stochastic volatility model," *Euro. Phys. J. B, Condensed Matter Complex Syst.*, vol. 17, no. 3, pp. 537–548, Oct. 2000.

[28] D. Nelson, "Conditional heteroskedasticity in asset returns: A new approach," *Econometrica*, vol. 59, no. 2, pp. 347–370, Mar. 1991.

[29] D. Nualart, *The Malliavin Calculus and Related Topics*, 2nd ed. New York, NY, USA: Springer-Verlag, 2006.

[30] B. Pochart and J.-P. Bouchaud, "The skewed multifractal random walk with applications to option smiles," *Quant. Finance*, vol. 2, no. 4, pp. 303–314, Aug. 2002.

[31] J. Voit, *The Statistical Mechanics of Financial Markets*, 3rd ed. New York, NY, USA: Springer-Verlag, 2005.

[32] A. Wood and G. Chan, "Simulation of stationary Gaussian processes in $[0,1]^d$," *J. Comput. Graph. Statist.*, vol. 3, no. 4, pp. 409–432, Dec. 1994.

Alexis Fauth is PH. D. candidate at the University of Paris 1, Panthéon Sorbonne, and head of quantitive research at the company Invivoo in Paris, France since 2011. He holds a MSc degree in probability and finance from University Paris 1 and a MSc degree in statistics from University Paris 6, Pierre et Marie Curie, in 2010. His research interest includes self-similar processes, point processes and mathematical finance.

Ciprian A. Tudor received the PH. D. in Mathematics from the University of La Rochelle, France, in 2002. From 2002 to 2004, he has been Assistant Professor at the University Pierre et Marie Curie, Paris, France and from 2004 to 2009 he worked as Associate Professor at the University of Paris 1, France. He is currently Full Professor at the University of Lille 1 and research collaborator at the Academy for Economical Sciences, Bucharest, Romania. His research interest includes stochastic analysis, self-similar processes and applications.