A Generic Method for the Evaluation of Interval Type-2 Fuzzy Linguistic Summaries

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***Abstract*—Linguistic summarization has turned out to be an important knowledge discovery technique by providing the most relevant natural language-based sentences in a human consistent manner. While many studies on linguistic summarization have handled ordinary fuzzy sets [type-1 fuzzy set (T1FS)] for modeling words, only few of them have dealt with interval type-2 fuzzy sets (IT2FS) even though IT2FS is better capable of handling uncertainties associated with words. Furthermore, the existent studies work with the scalar cardinality based degree of truth which might lead to inconsistency in the evaluation of interval type-2 fuzzy (IT2F) linguistic summaries. In this paper, to overcome this shortcoming, we propose a novel probabilistic degree of truth for evaluating IT2F linguistic summaries in the forms of type-I and type-II quantified sentences. We also extend the properties that should be fulfilled by any degree of truth on linguistic summarization with T1FS to IT2F environment. We not only prove that our probabilistic degree of truth satisfies the given properties, but also illustrate by examples that it provides more consistent results when compared to the existing degree of truth in the literature. Furthermore, we carry out an application on linguistic summarization of time series data of Europe Brent Spot Price, along with a comparison of the results achieved with our approach and that of the existing degree of truth in the literature.**

***Index Terms*—Data mining, interval type-2 fuzzy set, linguistic summarization.**

NOMENCLATURE

Y	The set of objects.
$\mathbb{V}$	The set of attributes.
$\mathbb{X}_k$	The domain for k^{th} attribute.
v_k^m	The value of the k^{th} attribute for the m^{th} objects.
Q	Quantifier labeled with type-1 fuzzy set.
S	Summarizer labeled with type-1 fuzzy set.
S_g	Qualifier (predefined summarizer) labeled with type-1 fuzzy set.
T	The degree of truth (point value).
μ	Membership degree.
$\tilde{A}^I$,$\tilde{B}^I$	Interval type-2 fuzzy sets.
X	Universe of discourse.
x	Primary variable.
u	Secondary variable.
J_x	Primary membership.
$\mu(x,u)$	Secondary membership degree.
$\underline{\mu}$	Lower membership degree.
$\overline{\overline{\mu}}$	Upper membership degree.
$\tilde{Q}^I$	Quantifier labeled with interval type-2 fuzzy set.
$\tilde{S}^I$	Summarizer labeled with interval type-2 fuzzy set.
$\tilde{S}_g^I$	Qualifier (predefined summarizer) labeled with interval type-2 fuzzy set.
α	α-cut.
$\tilde{A}_\alpha^I$	α-cut of interval type-2 fuzzy set A^I.
$\underline{A}_\alpha^I$	The lower crisp set of interval type-2 fuzzy set A^I obtained by α-cut of A^I.
$\overline{A}_\alpha^I$	The upper crisp set of interval type-2 fuzzy set A^I obtained by α-cut of A^I.
Γ	The set of union of α-cuts (levels).
C_{α_i}	The set of possible values for α_i-cut.
$c_\diamond$	The element of the set of possible values.
φ	Probability distribution.
$\tilde{T}$	Interval valued degree of truth.
T^L	The lower bound of interval valued degree of truth.
T^U	The upper bound of interval valued degree of truth.
S^{e_t}	t^{th} embedded type-1 fuzzy set of $\tilde{S}^I$.
$S_g^{e_j}$	j^{th} embedded type-1 fuzzy set of $\tilde{S}_g^I$.
T^{e_t}	The degree of truth (point value) for type-I quantified sentence $Q\ Y's\ are/have\ S^{e_t}$.
$T^{e_{jt}}$	The degree of truth (point value) for type-II quantified sentence $Q\ S_g^{e_j}\ Y's\ are/have\ S^{e_t}$.
$\wp$	Power set.

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I. INTRODUCTION

THE representation of knowledge that can be easily understood by the human beings has become very challenging task since the ever increasing developments in information science and technology make easy to collect and to store a huge amount of data from a variety of sources which is beyond the human understanding capabilities. Data mining techniques, capable of extracting previously unknown and potentially useful knowledge from a huge amount of data, have therefore emerged as an important research area. Even though various techniques have been used for different purposes, the main goals of data mining, generally speaking, are categorized into two groups: predictive and descriptive. While in the former the aim is to predict a value of target attribute based on values of other attributes, in the latter, the objective is to explore hidden patterns summarizing the important relationships among attributes [1]. One of the descriptive techniques in data mining is summarization aiming to discover patterns that cover overall aspect of data in a concise manner. Although the simplest form of summarization is based on the statistical methods, understanding the results obtained by them usually provide a limited knowledge to use and sometimes beyond the capacities of human beings. Hence, linguistic summarization that generates natural language statements from data has received a great attention in the literature. Undoubtedly, the most important tool to define linguistic summaries with a flexible manner is the fuzzy set theory coined by Zadeh [2]. The studies on linguistic summarization with fuzzy set have been reported under the different names, such as fuzzy quantification [3]–[10], fuzzy association rules [11]–[14], fuzzy rules [15]–[17], linguistic description of phenomena [18]–[20], and so on.

The crucial point in linguistic summarization with fuzzy set is the evaluation of linguistic summaries in which the degree of truth is used to measure whether obtained linguistic summary is valid and reliable. Most studies have focused on developing a degree of truth since an improper degree of truth leads to extract unreliable knowledge [14]. Zadeh [3], Yager [21]–[25], Bosc and Lietard [26], [27] proposed to use the scalar cardinality for computing the degree of truth. Dubois *et al.* [11], Martin *et al.* [14], and Delgado *et al.* [5] illustrated that the scalar cardinality based degree of truth might often provide inconsistent results in the evaluation. Accordingly, Delgado *et al.* [4], [5] recommended two degree of truths based on a probabilistic fuzzy cardinality and a possibilistic fuzzy cardinality instead of a scalar cardinality. According to Dubois and Prade [28], some types of fuzzy cardinalities are special kind of gradual numbers called as a gradual integer number. Considering this property, Rocacher [29], Rocacher and Bosc [30], and Lietard and Rocacher [31]–[33] described the concept of gradual natural integers to evaluate linguistic summaries. With a similar idea to gradual number, Lietard [34] suggested an approach based on the functional dependency of fuzzy sets for evaluating type-I and type-II quantified sentences. While α-cut representation is a widespread way to represent a fuzzy cardinality, it might fail to properly evaluate a linguistic summary. For example, when the linguistic summary is Some A are $\neg A$ ($\neg$ is negation), the degree of truth with the fuzzy cardinality is greater than zero; but, it should be, in fact, equal to zero. To fix this problem, Sanchez *et al.* [35], [36] and Delgado *et al.* [37] presented the restriction level representation of a fuzzy set instead of α-cuts, which is a general framework for evaluating linguistic summaries. Mass assignment, first introduced by Baldwin [38] and Baldwin *et al.* [39], [40], was also associated with the computation of the degree of truth. Martin and Shen [13] and Martin *et al.* [14] suggested a mass assignment based confidence measure for evaluating association rules, which can also be used for the evaluation of linguistic summary. Mass assignment was additionally exploited in semi-fuzzy quantification to evaluate semi-fuzzy quantified sentences that could be considered as linguistic summary [7]–[10]. Since linguistic summarization is a powerful descriptive knowledge discovery techniques for extracting knowledge from large dataset, it has been applied to a diversity of areas such as time series [41]–[44], qualitative comparative analysis [45], [46], decision support system [47], natural language generation [48], eldercare [49]–[52], energy consumption data [53], social networks [54], network traffic [55], and so on.

Type-1 fuzzy sets (T1FS), capable of dealing with only intrapersonal uncertainty, have used in most studies on linguistic summarization; however, it often fails to handle uncertainty associated with words since intrapersonal and interpersonal uncertainties exist for words [56]–[58]. Few studies have emphasized the use of type-2 fuzzy set (T2FS) [59]–[63], and interval type-2 fuzzy set (IT2FS) [58], [64]–[66] on linguistic summarization which in all extended the scalar cardinality-based degree of truth for T1FS to IT2FS or T2FS since they are capable of handling both intrapersonal and interpersonal uncertainty. Inconsistent cases seen with T1FSs, however, also occurred there again. The existing degree of truths have neglected the uncertainty associated with words by only taking into account a few embedded T1FSs of IT2FSs while computing the relative cardinality. Moreover, in some cases, they might not properly model all types of quantifier such as “all”, “about half”, and so on.

In this paper, we introduce a probabilistic degree of truth for evaluating interval type-2 fuzzy (IT2F) linguistic summaries in the forms of type-I and type-II quantified sentences when the summarizers are labeled with IT2FSs and the quantifier is labeled with T1FS. It is first illustrated that an IT2FS can be presented by the union of alpha-cuts. Next, some properties that should be fulfilled by any degree of truth used for evaluating type-I and type-II quantified sentences are extended to IT2FS environment. Finally, we introduce our probabilistic degree of truth along with the series of propositions, which prove that the concerned properties are satisfied. The remainder of this paper is organized as follows. The idea of linguistic summarization with T1FSs is briefly introduced in Section II. Section III presents the existing approaches on linguistic summarization with IT2FSs. Section IV introduces our probabilistic method to evaluate IT2F linguistic summaries in the form of the type-I and type- II quantified sentences. Section V presents an application on the time series data of daily Europe Brent Spot Price from Jan 02, 2003 to Oct 29, 2012 in addition to the discussions on the results obtained with our approach and the existing approach. Finally, the conclusion remarks and future directions are discussed in Section VI.

II. Linguistic Summarization With Type-1 Fuzzy Sets

In this section, we briefly discuss the idea of linguistic summarization in which T1FSs are used to model words. Before representing the idea of linguistic summarization, we briefly give the related definitions on T1FS. A T1FS on X, denoted by A, is defined as $A = \{\langle x, \mu_A(x)\rangle \,|x \in X\}$, where $\mu_A(x)$ is the membership grade of x. The α-cut of A is the crisp set $A_\alpha = \{x \in X \,|\mu_A(x) \geq \alpha\}$.

Let Y be defined as a set of objects $Y = \{y_1, y_2, y_3, \ldots, y_M\}$, $\mathbb{V}$ be defined as a set of attributes $\mathbb{V} = \{v_1, v_2, v_3, \ldots, v_K\}$ and $\mathbb{X}_k$ $(k = 1, 2, ..., K)$ be the domain of v_k. Then, $v_k^m \equiv v_k(y_m) \in \mathbb{X}_k$ is the value of the k^{th} attribute for the m^{th} object. The linguistic summarization studies have usually employed two summary forms based on the fuzzy quantifiers, proposed by Zadeh [3]. The first summary form called as type-I quantified sentence is in the form of “$Q\ Y's\ are/have\ S\ [T]$". Here,

Q is the linguistic quantifier labeled with a T1FS (e.g., about half, most, etc.), Y is the set of objects, S is the summarizer labeled with a T1FS, and T is the degree of truth describing how much data support the summary. The degree of truth for type-I quantified sentences is defined as

$$T = \mu_Q \left(\frac{\sum_{m=1}^{M} \mu_S (v^m)}{R} \right) \tag{1}$$

in which $R = M$ for relative quantifiers such that "most" $R = 1$ for absolute quantifiers such as "about three".

The second summary form called as type-II quantified sentence is in the form of "$Q\ S_g\ Y's\ are/have\ S\ [T]$". S_g is a qualifier (predefined summarizer) labeled with T1FS. "Most tall people are blonde" can be given as an example for type-II quantified sentences. Here, "most" is the linguistic quantifier (Q), "people" is the set of objects (Y), "tall" is the qualifier (S_g), and "blonde" is the summarizer (S). T is the degree of truth defined as follows:

$$T = \mu_Q \left(\frac{\sum_{m=1}^{M} \min\left(\mu_S (v^m), \mu_{S_g}(v_g^m)\right)}{\sum_{m=1}^{M} \mu_{S_g}\left(v_g^m\right)} \right) \tag{2}$$

In this summary form, only relative quantifier can be used. Another type of linguistic summary form is if/then fuzzy rules. It is stated that an "if/then" fuzzy rule is a special type of quantified sentence when the quantifier is chosen "all" [67]. The linguistic summary form using if/then rule is defined as If "v_g are/have S_g then v are/have S [T]".

III. Linguistic Summarization With Interval Type-2 Fuzzy Sets

In this section, two available approaches, Niewiadomski's Approach [64], [65] and Wu and Mendel's Approach [58], on linguistic summarization with IT2FS are briefly represented. In order to easily understand them, we first give the basic definitions related to IT2FS.

A. Interval Type-2 Fuzzy Set

An IT2FS, denoted by $\tilde{A}^I$ in X, is a kind of T2FS where all secondary membership grades are equal to one ($\mu_{\tilde{A}^I}(x, u) = 1$) [68], [69]. IT2FS $\tilde{A}^I$ is expressed as

$$\tilde{A}^I = \int_{x \in X} \int_{u \in J_x} 1/(x, u) \quad J_x \subseteq [0, 1] \tag{3}$$

where $\int \int$ denotes over all admissible x and u. x is the primary variable, u is the secondary variable with domain J_x that is the primary membership function of x.

Definition 1 [68], [69]: An IT2FS can be fully described by its footprint of uncertainty (FOU) since the secondary membership grades convey no new information. FOU is bounded by two T1FSs, named lower membership function, denoted by $\underline{\mu}_{\tilde{A}^I}(x)$, and upper membership function, denoted by $\overline{\mu}_{\tilde{A}^I}(x)$

$$\underline{\mu}_{\tilde{A}^I}(x) = \underline{FOU(\tilde{A}^I)} \quad \forall x \in X \tag{4}$$

$$\overline{\mu}_{\tilde{A}^I}(x) = \overline{FOU(\tilde{A}^I)} \quad \forall x \in X \tag{5}$$

Note that $J_x = [\underline{\mu}_{\tilde{A}^I}(x), \overline{\mu}_{\tilde{A}^I}(x)]$ for an IT2FS. An IT2FS $\tilde{A}^I$ can be expressed by its FOU

$$\tilde{A}^I = 1/FOU(\tilde{A}^I) = 1/\bigcup_{\forall x \in X} [\underline{\mu}_{\tilde{A}^I}(x), \overline{\mu}_{\tilde{A}^I}(x)] \tag{6}$$

Definition 2 [68]: Let $\tilde{A}^I$ and $\tilde{B}^I$ be two IT2FSs in X. The union and the intersection of $\tilde{A}^I$ and $\tilde{B}^I$ are defined as follows, respectively:

$$\tilde{A}^I \cup \tilde{B}^I = 1/\left[\underline{\mu}_{\tilde{A}^I}(x) \vee \underline{\mu}_{\tilde{B}^I}(x), \bar{\mu}_{\tilde{A}^I}(x) \vee \bar{\mu}_{\tilde{B}^I}(x)\right]$$

$$\tilde{A}^I \cap \tilde{B}^I = 1/\left[\underline{\mu}_{\tilde{A}^I}(x) \wedge \underline{\mu}_{\tilde{B}^I}(x), \bar{\mu}_{\tilde{A}^I}(x) \wedge \bar{\mu}_{\tilde{B}^I}(x)\right]$$

Generally, maximum is used for the union, and minimum is used for the intersection.

Definition 3 [70]: The scalar cardinality ($card$) of an IT2FS $\tilde{A}^I$ is expressed as

$$card\left(\tilde{A}^I\right) = \left[card\left(\underline{A}\right), card\left(\overline{A}\right)\right] \tag{7}$$

in which $card(\underline{A}) = \sum_{x \in X} \underline{\mu}_{\tilde{A}^I}(x)$ and $card(\bar{A}) = \sum_{x \in X} \bar{\mu}_{\tilde{A}^I}(x)$. The average cardinality (AC) of an IT2FS $\tilde{A}^I$ is the average of its minimum and maximum cardinalities

$$AC\left(\tilde{A}^I\right) = \frac{card\left(\underline{A}\right) + card\left(\overline{A}\right)}{2} \tag{8}$$

Definition 4 [71]: An IT2FS $\tilde{A}^I$ is normal if and only if at least one $x \in X$ such that $\overline{\mu}_{\tilde{A}^I}(x) = 1$.

B. Linguistic Summarization with Interval Type-2 Fuzzy Sets: Niewiadomski's Approach

Niewiadomski [64], [65] stated that at least one of Q and S should be labeled with a T1FS in linguistic summarization, and the degree of truth is equal to an interval value. When the quantifier is labeled with an IT2FS and the summarizers are labeled with T1FSs, type-I quantified sentence becomes "$\tilde{Q}^I\ Y's\ are/have\ S\ [T]$" and type-II quantified sentence becomes "$\tilde{Q}^I\ S_g\ Y's\ are/have\ S\ [T]$". If the linguistic summarizers are labeled with IT2FSs, the quantifier and the qualifier are labeled with T1FSs, then type-I quantified sentence becomes "$Q\ Y's\ are/have\ \tilde{S}^I\ [T]$" and type-II quantified sentence becomes "$Q\ S_g\ Y's\ are/have\ \tilde{S}^I\ [T]$". Niewiadomski [64], [65] extended the scalar cardinality based degree of truth proposed by Zadeh [3] to IT2F environment for evaluating linguistic summaries for the both cases.

C. Linguistic Summarization with Interval Type-2 Fuzzy Sets: Wu and Mendel's Approach

Wu [67] and Wu and Mendel [58] were also interested in linguistic summarization with IT2FS. Wu [67] stated that an if/then fuzzy rule is a special kind of type-II quantified sentences "$Q\ \tilde{S}_g^I\ Y's\ are/have\ \tilde{S}^I$" when the quantifier "all" is chosen, and it is modeled as $Q(x) = x$. if/then fuzzy rule is described as

$$\text{If } v_g \text{ are/have } \tilde{S}_g^I \text{ then } v \text{ are/have } \tilde{S}^I\ [T] \tag{9}$$

where v_g and v are the names of attributes, $\tilde{S}_g^I$ and $\tilde{S}^I$ are summarizers labeled with IT2FSs, and T is the degree of truth. The degree of truth is obtained based on the cardinality of IT2FS as follows:

$$T = \mu_Q\left(\frac{AC\left(\tilde{S}_g^I, \tilde{S}^I\right)}{AC\left(\tilde{S}_g^I\right)}\right) = \frac{AC\left(\tilde{S}_g^I, \tilde{S}^I\right)}{AC\left(\tilde{S}_g^I\right)} \tag{10}$$

IV. Evaluation of IT2F Linguistic Summaries in the Forms of Type-I and Type-II Quantified Sentences Using a Probabilistic Degree of Truth

In this section, we put forward a probabilistic degree of truth for evaluating linguistic summaries in the forms of type-I and type-II quantified sentences in which both the qualifier and the summarizers are labeled with IT2FS while the quantifier is labeled with T1FS. Since the probabilistic degree of truth is mainly based on the α-cut representation, we first show that an IT2FS can be presented by the union of α-cuts. Then, we discuss the properties proposed by Delgado *et al.* [5] should be extended to IT2F environment to perform the evaluation of linguistic summaries. Finally, we illustrate the application of our probabilistic degree of truth by giving a series of examples. We also show the validity of our probabilistic degree of truth with a comparison only to that of proposed by Wu and Mendel [58] since Niewiadomski approach [64], [65] is not capable of handling linguistic summaries when the summarizers and the qualifier are labeled with IT2FSs.

A. The Representation of Interval Type-2 Fuzzy Sets

The α-cuts representation is capable of decomposing T1FSs into a collection of crisp sets in fuzzy set theory. With a similar idea, some authors have extended α-cut representation to interval-valued fuzzy set (IVFS),[1] IT2FS, and T2FS. Zeng and Shi [75] and Zeng *et al.* [76], [77] have investigated a variety of interval α-cut decomposition for interval-valued fuzzy sets (FSs). Yager [78] proposed interval α-cuts for a representation of an IVFS in terms of crisp level sets. On the other hand, Hamrawi *et al.* [79] introduced another representation for IVFSs and IT2FSs with α-cuts and the associated lower and the upper crisp sets. In this paper, we utilize α-cut representation of IT2FS. Therefore, we first introduce the following definition.

Definition 5 [79] : Let $\tilde{A}^I$ be an IT2FS in X. The α-cut of $\tilde{A}^I$ is defined as follows:

$$\tilde{A}_\alpha^I = \left(\underline{A}_\alpha^I, \bar{A}_\alpha^I\right) \tag{11}$$

in which $\underline{A}_\alpha^I$ and $\bar{A}_\alpha^I$ are the lower and the upper crisp sets of $\tilde{A}^I$, and $\underline{A}_\alpha^I \subseteq \bar{A}_\alpha^I$ always holds.

Theorem 1 [79]: Let $\tilde{A}^I$ be an IT2FS in X. An IT2FS $\tilde{A}^I$ can be represented by the union of all its α-cuts and the associated crisp sets

$$\tilde{A}^I = \bigcup_{\alpha\in[0,1]} \alpha\tilde{A}_\alpha^I = \bigcup_{\alpha\in[0,1]} \alpha\left(\underline{A}_\alpha^I, \bar{A}_\alpha^I\right) \tag{12}$$

[1]It is advocated that IVFSs are equivalent to IT2FSs [72]–[74]

Proof: Let the lower and the upper degree of membership of an IT2FS $\tilde{A}^I$ be denoted by $\underline{\mu}_{\tilde{A}^I}(x)$ and $\overline{\mu}_{\tilde{A}^I}(x)$, respectively. $\tilde{A}^I$ can be represented as

$$\tilde{A}^I = \left(\bigcup_{\alpha\in[0,1]} \alpha\underline{A}_\alpha^I, \bigcup_{\alpha\in[0,1]} \alpha\bar{A}_\alpha^I\right) = \left(\sup_{\alpha\in[0,1]}\left\{\alpha\underline{A}_\alpha^I\right\}, \sup_{\alpha\in[0,1]}\left\{\alpha\bar{A}_\alpha^I\right\}\right)$$

α can be equal to the following values: $\alpha \in [0, \underline{\mu}_{\tilde{A}^I}(x)]$, $\alpha \in (\underline{\mu}_{\tilde{A}^I}(x), \overline{\mu}_{\tilde{A}^I}(x)]$, and $\alpha \in (\overline{\mu}_{\tilde{A}^I}(x), 1]$. It is easy to see that

$$\begin{aligned}
\sup_{\alpha\in[0,1]}\left\{\alpha\underline{A}_\alpha^I\right\} &= \max\Bigg[\sup_{\alpha\in[0,\underline{\mu}_{\tilde{A}^I}(x)]}\left\{\alpha\underline{A}_\alpha^I\right\}, \\
&\qquad \sup_{\alpha\in(\underline{\mu}_{\tilde{A}^I}(x),\overline{\mu}_{\tilde{A}^I}(x)]}\left\{\alpha\underline{A}_\alpha^I\right\}, \sup_{\alpha\in(\overline{\mu}_{\tilde{A}^I}(x),1]}\left\{\alpha\underline{A}_\alpha^I\right\}\Bigg] \\
&= \max\Bigg[\sup_{\alpha\in[0,\underline{\mu}_{\tilde{A}^I}(x)]}\{\alpha\wedge 1\}, \\
&\qquad \sup_{\alpha\in(\underline{\mu}_{\tilde{A}^I}(x),\overline{\mu}_{\tilde{A}^I}(x)]}\{\alpha\wedge 0\}, \sup_{\alpha\in(\overline{\mu}_{\tilde{A}^I}(x),1]}\{\alpha\wedge 0\}\Bigg] \\
&= \max\left[\underline{\mu}_{\tilde{A}^I}(x), 0, 0\right] = \underline{\mu}_{\tilde{A}^I}(x)
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\alpha\in[0,1]}\left\{\alpha\bar{A}_\alpha^I\right\} &= \max\Bigg[\sup_{\alpha\in[0,\underline{\mu}_{\tilde{A}^I}(x)]}\left\{\alpha\bar{A}_\alpha^I\right\}, \\
&\qquad \sup_{\alpha\in(\underline{\mu}_{\tilde{A}^I}(x),\overline{\mu}_{\tilde{A}^I}(x)]}\left\{\alpha\bar{A}_\alpha^I\right\}, \sup_{\alpha\in(\overline{\mu}_{\tilde{A}^I}(x),1]}\left\{\alpha\bar{A}_\alpha^I\right\}\Bigg] \\
&= \max\Bigg[\sup_{\alpha\in[0,\underline{\mu}_{\tilde{A}^I}(x)]}\{\alpha\wedge 1\}, \\
&\qquad \sup_{\alpha\in(\underline{\mu}_{\tilde{A}^I}(x),\overline{\mu}_{\tilde{A}^I}(x)]}\{\alpha\wedge 1\}, \sup_{\alpha\in(\overline{\mu}_{\tilde{A}^I}(x),1]}\{\alpha\wedge 0\}\Bigg] \\
&= \max\left[\underline{\mu}_{\tilde{A}^I}(x), \overline{\mu}_{\tilde{A}^I}(x), 0\right] = \overline{\mu}_{\tilde{A}^I}(x)
\end{aligned}$$

This proves that $\tilde{A}^I$ can be represented by the union of all its α-cuts and the lower and the upper crisp sets. ■

B. The Properties for Evaluation of Type-I Quantified Sentences

In this subsection, we extend some properties proposed by Delgado *et al.* [5] for the evaluation of type-I quantified sentences for T1FS to IT2F environment.

Let $\tilde{S}^I$ be a normal IT2FS such that

$$\tilde{S}^I = \left\{\frac{\left[\underline{\mu}_{\tilde{S}^I}\left(v^1\right), \bar{\mu}_{\tilde{S}^I}\left(v^1\right)\right]}{y_1}, \ldots, \frac{\left[\underline{\mu}_{\tilde{S}^I}\left(v^M\right), \bar{\mu}_{\tilde{S}^I}\left(v^M\right)\right]}{y_M}\right\}$$

Property 1.1: If $\tilde{S}^I$ is a crisp set, then the degree of truth should be

$$Q\left(\frac{\left|\tilde{S}^I\right|}{|Y|}\right) \tag{13}$$

when Q is a relative, and

$$Q\left(\left|\tilde{S}^I\right|\right) \tag{14}$$

when Q is an absolute quantifier.

Property 1.2: A degree of truth should be coherent with IT2F logic when the quantifiers are selected as "all" and "exist". Type-I quantified sentences "$Q\ Y's\ are/have\ \tilde{S}^I$" with $Q = \exists$ should be represented and evaluated by IT2F logic as follows:

$$\left[\bigvee_{m=1}^{M} \underline{\mu}_{\tilde{S}^I}(v^m), \bigvee_{m=1}^{M} \bar{\mu}_{\tilde{S}^I}(v^m)\right] \tag{15}$$

$\vee$ is the fuzzy union. In the case of $Q = \forall$, a degree of truth should be represented and evaluated by IT2F logic as follows:

$$\left[\bigwedge_{m=1}^{M} \underline{\mu}_{\tilde{S}^I}(v^m), \bigwedge_{m=1}^{M} \bar{\mu}_{\tilde{S}^I}(v^m)\right] \tag{16}$$

in which $\wedge$ is the fuzzy intersection.

Property 1.3: A degree of truth should be consistent in terms of quantifiers inclusion. If $Q \subseteq Q'$ then it should be $\tilde{T}(Q\ Y's\ are/have\ \tilde{S}^I) \leq \tilde{T}(Q'\ Y's\ are/have\ \tilde{S}^I)$.

Property 1.4 : Any quantifier should be used in the evaluation of type-I quantified sentence, i.e., any possibility distribution over nonnegative integers or interval [0, 1].

C. The Properties for Evaluation of Type-II Quantified Sentences

In this subsection, we extend some properties proposed by Delgado *et al.* [5] for the evaluation of type-II quantified sentences for T1FS to IT2F environment.

Let $\tilde{S}_g^I$ be a normal IT2FS and $\tilde{S}^I$ be an IT2FS such that

$$\tilde{S}_g^I = \left\{ \frac{\left[\underline{\mu}_{\tilde{S}_g^I}(v_g^1), \bar{\mu}_{\tilde{S}_g^I}(v_g^1)\right]}{y_1}, \ldots, \frac{\left[\underline{\mu}_{\tilde{S}_g^I}(v_g^M), \bar{\mu}_{\tilde{S}_g^I}(v_g^M)\right]}{y_M} \right\}$$

$$\tilde{S}^I = \left\{ \frac{\left[\underline{\mu}_{\tilde{S}^I}(v^1), \bar{\mu}_{\tilde{S}^I}(v^1)\right]}{y_1}, \ldots, \frac{\left[\underline{\mu}_{\tilde{S}^I}(v^M), \bar{\mu}_{\tilde{S}^I}(v^M)\right]}{y_M} \right\}$$

Property 2.1: If $\tilde{S}^I$ and $\tilde{S}_g^I$ are crisp sets and Q is a relative quantifier, then the degree of truth should be

$$Q\left(\frac{\left|\tilde{S}_g^I \cap \tilde{S}^I\right|}{\left|\tilde{S}_g^I\right|}\right) \tag{17}$$

Property 2.2: Type-I quantified sentence is a special case of type-II quantified sentence when $\tilde{S}_g^I = Y$ and relative quantifiers are used. Thus, a degree of truth for type-II quantified sentence should be valid for the evaluation of type-I quantified sentence.

Property 2.3: A degree of truth should be equal to $Q(0)$, when $\tilde{S}_g^I \cap \tilde{S}^I = \emptyset$.

Property 2.4: A degree of truth should be coherent with IT2F logic for the quantifier "exist" such as

$$\left[\bigvee_{m=1}^{M}\left(\underline{\mu}_{\tilde{S}_g^I}(v_g^m) \wedge \underline{\mu}_{\tilde{S}^I}(v^m)\right), \bigvee_{m=1}^{M}\left(\overline{\mu}_{\tilde{S}_g^I}(v_g^m) \wedge \overline{\mu}_{\tilde{S}^I}(v^m)\right)\right] \tag{18}$$

TABLE I

LOWER AND UPPER CRISP SETS

α	$(\underline{S}^I)_{\alpha_i}$	$(\bar{S}^I)_{\alpha_i}$
α_1	$(\underline{S}^I)_{\alpha_1}$	$(\bar{S}^I)_{\alpha_1}$
α_2	$(\underline{S}^I)_{\alpha_2}$	$(\bar{S}^I)_{\alpha_2}$
...	...	...
α_{m-1}	$(\underline{S}^I)_{\alpha_{m-1}}$	$(\bar{S}^I)_{\alpha_{m-1}}$
α_m	$(\underline{S}^I)_{\alpha_m}$	$(\bar{S}^I)_{\alpha_m}$

Property 2.5: Any quantifier should be used in the evaluation of type-II quantified sentence, i.e., any possibility distribution over nonnegative integers or interval [0, 1].

D. A Probabilistic Degree of Truth for Evaluation of Type-I Quantified Sentences

In this subsection, we introduce a probabilistic degree of truth for evaluating type-I quantified sentences in which the summarizers are labeled with IT2FSs and the quantifier is labeled with T1FS. The steps of our method are listed as follows:

Step 1. Determine α-cuts.

Let $\Gamma = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a set of union of α-levels consisting of all the lower and the upper membership grades of $\tilde{S}^I$ with $0 = \alpha_{m+1} < \alpha_m < \ldots < \alpha_2 < \alpha_1 = 1$.

Step 2. Obtain the lower and the upper crisp sets associated with α-cuts.

Once getting α-cuts, the lower and the upper crisp sets are determined as given in Table I.

$|(\tilde{S}^I)_{\alpha_i}|$ is equal to any integer value in $[q_i, \bar{q}_i]$ in which $|(\underline{S}^I)_{\alpha_i}| = q_i$, $|(\bar{S}^I)_{\alpha_i}| = \bar{q}_i$ for each α-cuts. Based on this definition, a set of possible values is defined as $C_{\alpha_i} = \{q_i/M, (q_i + 1)/M, ..., \bar{q}_i/M\}$ for the relative quantifiers and $C_{\alpha_i} = \{q_i, q_i + 1, ..., \bar{q}_i\}$ for the absolute quantifiers. Once getting the set of possible values, we can compute the values taken by a quantifier for each element of the set of possible values. Finally, we find the minimum and the maximum values taken by a quantifier, denoted by $\min_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond)$ and $\max_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond)$, respectively.

Step 3. Acquire the probability distribution.

After obtaining the α-cuts and the lower and the upper crisp sets, the associated probability distribution is computed as follows:

$$\varphi(\alpha_i) = \alpha_i - \alpha_{i+1} \tag{19}$$

$\varphi(\alpha_i)$ denotes the amount of evidence assigned to level α_i.

Step 4. Compute the degree of truth.

The probabilistic degree of truth for evaluating a type-I quantified sentence is defined as follows:

$$\begin{aligned}\tilde{T} &= (T^L, T^U) \\ &= \sum_{\alpha_i \in \Gamma} \varphi(\alpha_i) \times \left(\min_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond), \max_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond)\right)\end{aligned} \tag{20}$$

in which T^L is the lower bound of $\tilde{T}$ and T^U is the upper bound of $\tilde{T}$. The probabilistic degree of truth is a general case of GD method proposed by Delgado *et al.* [4], [5]. When uncertainty disappears, our probabilistic degree of truth becomes GD method. The following propositions prove that our probabilistic degree of truth for evaluating type-I quantified sentences satisfies the property 1.1–property 1.4.

Proposition 1: The probabilistic degree of truth defined in (20) satisfies the property 1.1.

Proof: If $\tilde{S}^I$ is a crisp set, then $\Gamma = \{1, 0\}$ and $\varphi(\alpha_1) = 1$. C_{α_1} has only one element which is $|\tilde{S}^I|/|Y|$ for the relative quantifiers and $|\tilde{S}^I|$ for the absolute quantifiers. The minimum and the maximum values taken by the relative quantifiers, and the absolute quantifiers become $Q(|\tilde{S}^I|/|Y|)$ and $Q(|\tilde{S}^I|)$, respectively. If we substitute the probability distribution and the minimum and the maximum values taken by quantifier in (20), the probabilistic degree of truth becomes $Q(|\tilde{S}^I|/|Y|)$ for the relative quantifiers and $Q(|\tilde{S}^I|)$ for the absolute quantifiers. This proves that the property 1.1 is satisfied by our probabilistic degree of truth. ∎

Proposition 2: The probabilistic degree of truth defined in (20) satisfies the property 1.2.

Proof: If "exist" is selected as the quantifier and the elements belonging to the set of possible values C_{α_i} for each α_i, $i = k, k+1, ..., m$ are greater than zero, then

$$\left(\min_{c_\diamond \in C_{\alpha_i}} Q\left(c_\diamond\right), \max_{c_\diamond \in C_{\alpha_i}} Q\left(c_\diamond\right)\right) = (1, 1)$$

If the smallest element belonging to the set of possible values C_{α_j} for each α_j, $j = 1, 2, .., k-1$ is equal to zero and the other elements are greater than zero, then

$$\left(\min_{c_\diamond \in C_{\alpha_j}} Q\left(c_\diamond\right), \max_{c_\diamond \in C_{\alpha_j}} Q\left(c_\diamond\right)\right) = (0, 1)$$

If we substitute the probability distribution, the minimum and the maximum values taken by quantifier in (20), the probabilistic degree of truth becomes

$$\begin{aligned}\tilde{T} &= (\alpha_1 - \alpha_2) \times (0, 1) + (\alpha_2 - \alpha_3) \times (0, 1) + ... + (\alpha_{k-1} - \\ &\quad \alpha_k) \times (0, 1) + (\alpha_k - \alpha_{k+1}) \times (1, 1) + ... + (\alpha_m - \alpha_{m+1}) \\ &\quad \times (1, 1) \\ &= (\alpha_k, \alpha_1)\end{aligned}$$

It is easy to see that α_1 is the largest upper membership degree of $\tilde{S}^I$ and α_k is the largest lower membership degree of $\tilde{S}^I$. If maximum is used as fuzzy union in (15), then we shall have $\tilde{T} = (\alpha_k, \alpha_1)$.

If "all" is selected as the quantifier and the element belonging to the set of possible values C_{α_m} for α_m is equal to one (this situation only holds when $q_m = \bar{q}_m = |Y|$), then

$$\left(\min_{c_\diamond \in C_{\alpha_m}} Q\left(c_\diamond\right), \max_{c_\diamond \in C_{\alpha_m}} Q\left(c_\diamond\right)\right) = (1, 1)$$

If at least one element belonging to the set of possible values C_{α_j} for each α_j, j=f, $f+1, ..., m-1$ is equal to one, then

$$\left(\min_{c_\diamond \in C_{\alpha_j}} Q\left(c_\diamond\right), \max_{c_\diamond \in C_{\alpha_j}} Q\left(c_\diamond\right)\right) = (0, 1)$$

If we substitute the probability distribution and the minimum and the maximum values taken by quantifier in (20), the probabilistic degree of truth becomes

$$\begin{aligned}\tilde{T} &= (\alpha_f - \alpha_{f+1}) \times (0, 1) + (\alpha_{f+1} - \alpha_{f+2}) \times (0, 1) + ... + \\ &\quad (\alpha_{m-1} - \alpha_m) \times (0, 1) + (\alpha_m - \alpha_{m+1}) \times (1, 1) \\ &= (\alpha_m, \alpha_f)\end{aligned}$$

It is clearly seen that α_m is the smallest lower membership degree of $\tilde{S}^I$, and α_f is the smallest upper membership degree of $\tilde{S}^I$. If minimum is used as fuzzy intersection in (16), then we shall have $\tilde{T} = (\alpha_m, \alpha_f)$. These two cases prove that the property 1.2 is satisfied by our probabilistic degree of truth. ∎

Proposition 3: The probabilistic degree of truth defined in (20) satisfies the property 1.3.

Proof: If $Q \subseteq Q'$ then $(\min_{c_\diamond \in C_{\alpha_i}} Q\left(c_\diamond\right), \max_{c_\diamond \in C_{\alpha_i}} Q\left(c_\diamond\right)) \leq (\min_{c_\diamond \in C_{\alpha_i}} Q'\left(c_\diamond\right), \max_{c_\diamond \in C_{\alpha_i}} Q'\left(c_\diamond\right))$ for each α_i. Therefore, $\tilde{T}(Q\ Y's\ are/have\ \tilde{S}^I) \leq \tilde{T}(Q'\ Y's\ are/have\ \tilde{S}^I)$ always holds. ∎

Proposition 4: The probabilistic degree of truth defined in (20) satisfies the property 1.4.

Proof: All the possible values are computed in the probabilistic degree of truth, and then the minimum and the maximum values taken by a quantifier are found. Thus, the probabilistic degree of truth allows us to use any kind of quantifier including the coherent and the non-coherent family of quantifiers on linguistic summarization. ∎

The FOU of an IT2FS can be represented by the union of embedded T1FSs [68]. From this point of view, the probabilistic degree of truth should be represented by the union of the degree of truths of all the type-I quantified sentences "$Q\ Y's\ are/have\ S^{e_t}$".

Theorem 2: Let "$Q\ Y's\ are/have\ \tilde{S}^I$" be a type-I quantified sentence, S^{e_t} be an embedded T1FS of $\tilde{S}^I$ and "$Q\ Y's\ are/have\ S^{e_t}$" ($t = 1, 2, ..., \infty$) be a type-I quantified sentence with the degree of truth (T^{e_t}). The union of all the degree of truths is equal to the probabilistic degree of truth.

$$\tilde{T} = \bigcup_{t=1}^{\infty} T^{e_t} \tag{21}$$

Proof: Let S^{e_t} be an embedded T1FS and $\Gamma_t = \left\{\alpha_1, \alpha_2, ..., \alpha_{m_t}\right\}$ be a set of union of α-levels consisting of all the lower and the upper membership grades of $\tilde{S}^I$ and the membership grades of S^{e_t} with $0 = \alpha_{m_t+1} < \alpha_{m_t} < \ldots < \alpha_2 < \alpha_1 = 1$ for $t = 1, 2, ..., \infty$. It is easy to see that the set of α-levels changes according to the embedded T1FSs S^{e_t}. We know that $(\underline{S}^I)_{\alpha_i} \subseteq (S^{e_t})_{\alpha_i} \subseteq (\bar{S}^I)_{\alpha_i}$ always holds for all $t = 1, 2, ..., \infty$ and each α_i. Therefore, $\min_{c_\diamond \in C_{\alpha_i}} Q\left(c_\diamond\right) \leq Q(c_t) \leq \max_{c_\diamond \in C_{\alpha_i}} Q\left(c_\diamond\right)$ in which $c_t = |(S^{e_t})_{\alpha_i}|$ for the absolute quantifiers and $c_t = |(S^{e_t})_{\alpha_i}|/|Y|$ for the relative quantifiers always holds. It is clear to see that T^L and T^U are constant even if the probability distribution is varied depending on α_i; but, T^{e_t} changes in the interval whose lower bound is T^L and upper bound is T^U, which is equivalently expressed as $T^L \leq T^{e_t} \leq T^U$ for all T^{e_t}, $t = 1, 2, ...\infty$. Since there exist infinite numbers of embedded S^{e_t}, we have infinite numbers of

TABLE II
LOWER AND UPPER CRISP SETS FOR EXAMPLE 1

α	$(\underline{S}^I)_{\alpha_i}$	$(\bar{S}^I)_{\alpha_i}$
1	$\{y_2\}$	$\{y_2\}$
0.8	$\{y_2\}$	$\{y_1, y_2\}$
0.6	$\{y_1, y_2\}$	$\{y_1, y_2\}$

T^{e_t} in $[T^L, T^U]$. Therefore, the union of the degree of truths of all the type-I quantified sentences "Q $Y's$ $are/have$ S^{e_t}", $(t = 1, 2, ..., \infty)$ is equal to the probabilistic degree of truth. ∎

Example 1: Let $\tilde{S}^I$ be the set of young people such that

$$\tilde{S}^I = \left\{ \frac{[0.6, 0.8]}{y_1}, \frac{[1, 1]}{y_2}, \frac{[0, 0]}{y_3}, \frac{[0, 0]}{y_4}, \frac{[0, 0]}{y_5}, \frac{[0, 0]}{y_6} \right\}$$

and "about one out of three" is a noncoherent quantifier defined as

$$Q\left(c_\diamond\right) = \begin{cases} 6c_\diamond - 1, & 1/6 \le c_\diamond \le 1/3 \\ 3 - 6c_\diamond, & 1/3 \le c_\diamond \le 1/2 \\ 0, & \text{otherwise} \end{cases}$$

the degree of truth for the linguistic summary "about one out of three people are young" is computed as follows.

Step 1. Determine α-cuts.

The set of union of α-levels consisting of all the lower and the upper membership grades is $\Gamma = \{1, 0.8, 0.6\}$.

Step 2. Obtain the lower and the upper crisp sets associated with α-cuts.

The lower and the upper crisp sets associated with α-cuts are given in Table II.

Once getting the lower and the upper crisp sets, the set of possible values for each α-cuts are found as $C_1 = \{1/6\}$, $C_{0.8} = \{1/6, 2/6\}$, $C_{0.6} = \{2/6\}$ and $C_0 = \{1\}$ since $|Y| = 6$. When $\alpha_1 = 1$, there exist only one possible value such that $c_\diamond = 1/6$ that makes $\min\{Q(1/6)\} = 0$ and $\max\{Q(1/6)\} = 0$. For $\alpha_2 = 0.8$, there exist two possible values such that $c_\diamond = 1/6$ and $c_\diamond = 2/6$ which makes $\min\{Q(1/6), Q(2/6)\} = 0$ and $\max\{Q(1/6), Q(2/6)\} = 1$. Similarly, when $\alpha_3 = 0.6$, there is only one possible value such that $c_\diamond = 2/6$ which makes $\min\{Q(2/6)\} = 1$ and $\max\{Q(2/6)\} = 1$.

Step 3. Acquire the probability distribution.

The associated probability distribution is found as $\varphi(\alpha_1) = (1 - 0.8) = 0.2$, $\varphi(\alpha_2) = (0.8 - 0.6) = 0.2$, and $\varphi(\alpha_3) = (0.6 - 0) = 0.6$.

Step 4. Compute the degree of truth.

The degree of truth for the linguistic summary "about one out of three people are young" is computed using (20) as

$$\begin{aligned} \tilde{T} &= (1-0.8)\times(0,0)+(0.8-0.6)\times(0,1)+(0.6-0)\times(1,1) \\ &= (0.6, 0.8) \end{aligned}$$

E. A Probabilistic Degree of Truth for Evaluation of Type-II Quantified Sentences

In this subsection, we propose a probabilistic degree of truth for evaluating type-II quantified sentences in which the summarizers are labeled with IT2FSs and the quantifier is labeled with T1FS. The steps of the probabilistic degree of truth are listed as follows.

TABLE III
LOWER AND UPPER CRISP SETS

α	$(\underline{S}_g^I)_{\alpha_i}$	$(\bar{S}_g^I)_{\alpha_i}$	$(\underline{S}^I)_{\alpha_i}$	$(\bar{S}^I)_{\alpha_i}$
α_1	$(\underline{S}_g^I)_{\alpha_1}$	$(\bar{S}_g^I)_{\alpha_1}$	$(\underline{S}^I)_{\alpha_1}$	$(\bar{S}^I)_{\alpha_1}$
α_2	$(\underline{S}_g^I)_{\alpha_2}$	$(\bar{S}_g^I)_{\alpha_2}$	$(\underline{S}^I)_{\alpha_2}$	$(\bar{S}^I)_{\alpha_2}$
...	...	...	...	...
α_{m-1}	$(\underline{S}_g^I)_{\alpha_{m-1}}$	$(\bar{S}_g^I)_{\alpha_{m-1}}$	$(\underline{S}^I)_{\alpha_{m-1}}$	$(\bar{S}^I)_{\alpha_{m-1}}$
α_m	$(\underline{S}_g^I)_{\alpha_m}$	$(\bar{S}_g^I)_{\alpha_m}$	$(\underline{S}^I)_{\alpha_m}$	$(\bar{S}^I)_{\alpha_m}$

Step 1. Determine α-cuts.

Let $\Gamma = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ be a set of union of α-levels consisting of all the lower and the upper membership grades of $\tilde{S}^I$ and $\tilde{S}_g^I$ with $0 = \alpha_{m+1} < \alpha_m < \ldots < \alpha_2 < \alpha_1 = 1$.

Step 2. Obtain the lower and the upper crisp sets associated with α-cuts.

Once getting α-cuts, the lower and the upper crisp sets are obtained in Table III.

The important point here is to find the number of elements of $\tilde{S}_g^I$ and $\tilde{S}^I$. We know that $\tilde{S}_g^I$ and $\tilde{S}^I$ have infinite numbers of embedded T1FSs. Let S^{e_t} be an embedded T1FS of $\tilde{S}^I$. The crisp set $(S^{e_t})_{\alpha_i}$ obtained by α-cut is a subset of power set $(\bar{S}^I)_{\alpha_i}$, which includes the crisp set $(\underline{S}^I)_{\alpha_i}$. Similarly, the crisp set $(S_g^{e_j})_{\alpha_i}$ obtained by α-cut is a subset of power set $(\bar{S}_g^I)_{\alpha_i}$, which includes the crisp set $(\underline{S}_g^I)_{\alpha_i}$. Therefore, $|(\tilde{S}_g^I)_{\alpha_i} \cap (\tilde{S}^I)_{\alpha_i}|/|(\tilde{S}_g^I)_{\alpha_i}|$ can be equal to a number of elements, constituting the set of possible values such that

$$C_{\alpha_i} = \left\{ \begin{array}{l} \dfrac{\left|(S_g^{e_1})_{\alpha_i} \cap (S^{e_1})_{\alpha_i}\right|}{\left|(S_g^{e_1})\right|_{\alpha_i}}, \ldots, \dfrac{\left|\left(S_g^{e_\mathcal{J}}\right)_{\alpha_i} \cap (S^{e_1})_{\alpha_i}\right|}{\left|\left(S_g^{e_\mathcal{J}}\right)\right|_{\alpha_i}}, \\ \dfrac{\left|(S_g^{e_1})_{\alpha_i} \cap (S^{e_2})_{\alpha_i}\right|}{\left|(S_g^{e_1})\right|_{\alpha_i}}, \ldots, \dfrac{\left|\left(S_g^{e_\mathcal{J}}\right)_{\alpha_i} \cap (S^{e_2})_{\alpha_i}\right|}{\left|\left(S_g^{e_\mathcal{J}}\right)\right|_{\alpha_i}} \\ , \ldots, \\ \dfrac{\left|(S_g^{e_1})_{\alpha_i} \cap (S^{e_\mathcal{T}})_{\alpha_i}\right|}{\left|(S_g^{e_1})\right|_{\alpha_i}}, \ldots, \dfrac{\left|\left(S_g^{e_\mathcal{J}}\right)_{\alpha_i} \cap (S^{e_\mathcal{T}})_{\alpha_i}\right|}{\left|\left(S_g^{e_\mathcal{J}}\right)\right|_{\alpha_i}} \end{array} \right\}$$

in which

$$(S_g^{e_j})_{\alpha_i} \in \wp((\bar{S}_g^I)_{\alpha_i}) \wedge (S_g^{e_j})_{\alpha_i} \supseteq (\underline{S}_g^I)_{\alpha_i}$$

$$(S^{e_t})_{\alpha_i} \in \wp((\bar{S}^I)_{\alpha_i}) \wedge (S^{e_t})_{\alpha_i} \supseteq (\underline{S}^I)_{\alpha_i}$$

$j = 1, 2, ..., \mathcal{J}$, $t = 1, 2, ..., \mathcal{T}$ and $\wp$ is the symbol of power set.

After getting the set of possible values, we compute the values taken by a quantifier for each element of the set of possible values. Finally, we find the minimum and the maximum values taken by a quantifier, denoted by $\min\limits_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond)$ and $\max\limits_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond)$, respectively.

Step 3. Acquire the probability distribution.

After obtaining the α-cuts and the lower and the upper crisp sets, the associated probability distribution is computed as follows:

$$\varphi(\alpha_i) = \alpha_i - \alpha_{i+1} \tag{22}$$

$\varphi(\alpha_i)$ denotes the amount of evidence assigned to level α_i.

Step 4. Compute the degree of truth.

A type-II quantified sentence is evaluated by our probabilistic degree of truth defined as follows:

$$\tilde{T} = (T^L, T^U) = \sum_{\alpha_i \in \Gamma} \varphi(\alpha_i) \times \left(\min_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond), \max_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond) \right) \tag{23}$$

If uncertainty disappears, our probabilistic degree of truth becomes GD method proposed by Delgado *et al.* [4], [5] for evaluating type-II quantified sentence. The following propositions prove that the probabilistic degree of truth satisfies the property 2.1–property 2.5.

Proposition 5: The probabilistic degree of truth defined in (23) satisfies the property 2.1.

Proof: If $\tilde{S}_g^I$ and $\tilde{S}^I$ are crisp sets then $\Gamma = \{1, 0\}$. In this case, it is easy to see that $(\min_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond), \max_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond)) = Q(|\tilde{S}_g^I \cap \tilde{S}^I|/|\tilde{S}_g^I|)$ and the associated probability distribution is $\varphi(1) = 1 - 0 = 1$. If we substitute the probability distribution, the minimum and the maximum values taken by quantifier in (23), the probabilistic degree of truth becomes $Q(|\tilde{S}_g^I \cap \tilde{S}^I|/|\tilde{S}_g^I|)$. This proves that the property 2.1 is satisfied by the probabilistic degree of truth. ■

Proposition 6: The probabilistic degree of truth defined in (23) satisfies the property 2.2.

Proof: Let $\tilde{S}_g^I = Y$ and $|(\tilde{S}^I)_{\alpha_i}|$ be equal to any integer value in $[q_i, \bar{q}_i]$ in which $|(\underline{S}^I)_{\alpha_i}| = q_i$, $|(\bar{S}^I)_{\alpha_i}| = \bar{q}_i$ for each α-cuts. Based on this definition, the set of possible values is defined as $C_{\alpha_i} = \{q_i/M, (q_i+1)/M, ..., \bar{q}_i/M\}$ for the relative quantifiers and $C_{\alpha_i} = \{q_i, q_i+1, ..., \bar{q}_i\}$ for the absolute quantifiers. Therefore, (23) is the general case of (20), which could be used to evaluate type-I quantified sentences. ■

Proposition 7: The probabilistic degree of truth defined in (23) satisfies the property 2.3.

Proof: When $\tilde{S}_g^I$ be a normal IT2FS and $\tilde{S}_g^I \cap \tilde{S}^I = \emptyset$, it is obvious that $C_{\alpha_i} = \{0\}$ for each α_i that makes $(\min_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond), \max_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond)) = Q(0)$. Since the sum of the probability distribution is equal to one, $\tilde{T} = Q(0)$. ■

Proposition 8: The probabilistic degree of truth defined in (23) satisfies the property 2.4.

Proof: If "exist" is selected as the quantifier and the elements belonging to the set of possible values C_{α_i} for each α_i, $i = k, k+1, ..., m$ are greater than zero, then $(\min_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond), \max_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond)) = (1, 1)$. If the smallest element belonging to the set of possible values C_{α_j} for each α_j, $j = l, l-1, .., k-1$ is equal to zero and the other elements are greater than zero, then $(\min_{c_\diamond \in C_{\alpha_j}} Q(c_\diamond), \max_{c_\diamond \in C_{\alpha_j}} Q(c_\diamond)) = (0, 1)$. If we substitute the probability distribution, the minimum and the maximum values taken by quantifier in (23), the probabilistic degree of truth becomes

$$\begin{aligned}\tilde{T} &= (\alpha_l - \alpha_{l-1}) \times (0, 1) + (\alpha_{l-1} - \alpha_{l-2}) \times (0, 1) + ...+ \\ &\quad (\alpha_{k-1} - \alpha_k) \times (0, 1) + (\alpha_k - \alpha_{k+1}) \times (1, 1) + ...+ \\ &\quad (\alpha_m - \alpha_{m+1}) \times (1, 1) \\ &= (\alpha_k, \alpha_l)\end{aligned}$$

It is easy to see that α_l is the largest upper membership degree of $\tilde{S}_g^I \cap \tilde{S}^I$, and α_k is the largest lower membership degree of $\tilde{S}_g^I \cap \tilde{S}^I$. If maximum is used as fuzzy union and minimum is used as fuzzy intersection in (18), then we shall have $\tilde{T} = (\alpha_k, \alpha_l)$. ■

Proposition 9: The probabilistic degree of truth defined in (23) satisfies the property 2.5.

Proof: All the possible values are computed in the probabilistic degree of truth, and then the minimum and the maximum values taken by a quantifier are found. Thus, the probabilistic degree of truth allows us to use any kind of quantifier including the coherent and the noncoherent family of quantifiers on linguistic summarization. ■

With a similar idea in Theorem 2, the probabilistic degree of truth should be represented by the union of degree of truths of all the type-II quantified sentences "$Q\ S_g^{e_j}\ Y's\ are/have\ S^{e_t}$".

Theorem 3: Let "$Q\ \tilde{S}_g^I\ Y's\ are/have\ \tilde{S}^I$" be a type-II quantified sentence, $S_g^{e_j}$ be a normal embedded T1FS of $\tilde{S}_g^I$, S^{e_t} be an embedded T1FS of $\tilde{S}^I$ and "$Q\ S_g^{e_j}\ Y's\ are/have\ S^{e_t}$" be a type-II quantified sentences with the degree of truth $(T^{e_{jt}})$. The union of all the degree of truths is equal to the probabilistic degree of truth.

$$\tilde{T} = \bigcup_{j,t=1}^{\infty} T^{e_{jt}} \tag{24}$$

Proof: Let $S_g^{e_j}$ be a normal embedded T1FS, S^{e_t} be an embedded T1FS and $\Gamma_{jt} = \{\alpha_1, \alpha_2, ..., \alpha_{m_{jt}}\}$ be a set of union of α-levels consisting of not only all the lower and the upper membership grades of $\tilde{S}_g^I$ and $\tilde{S}^I$ but also the membership grades of $S_g^{e_j}$ and S^{e_t} with $0 = \alpha_{m_{jt}+1} < \alpha_{m_{jt}} < \ldots < \alpha_2 < \alpha_1 = 1$. It is easy to see that the set of α-levels changes according to the embedded T1FSs $S_g^{e_j}$ and S^{e_t}. We know that $(\underline{S}_g^I)_{\alpha_i} \subseteq (S_g^{e_j})_{\alpha_i} \subseteq (\bar{S}_g^I)_{\alpha_i}$ and $(\underline{S}^I)_{\alpha_i} \subseteq (S^{e_t})_{\alpha_i} \subseteq (\bar{S}^I)_{\alpha_i}$ always holds for all $j, t = 1, 2, ..., \infty$ and each α_i. Therefore, $\min_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond) \leq Q(c_{jt}) \leq \max_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond)$ in which $c_{jt} = |(S_g^{e_j})_{\alpha_i} \cap (S^{e_t})_{\alpha_i}|/|(S_g^{e_j})_{\alpha_i}|$ always holds. It is easy to see that T^L and T^U are constant even if the probability distribution is varied depending on α_i; but, $T^{e_{jt}}$ changes in the interval whose lower bound is T^L and upper bound is T^U, which is equivalently expressed as $T^L \leq T^{e_{jt}} \leq T^U$ for all $T^{e_{jt}}$, $j, t = 1, 2, ..., \infty$. Since there exist infinite numbers of embedded $S_g^{e_j}$ and infinite numbers of embedded S^{e_t}, we have infinite numbers of $T^{e_{jt}}$ in $[T^L, T^U]$. Therefore, the union of all the degree of truths is equal to the probabilistic degree of truth. ■

The application of the probabilistic degree of truth for type-II quantified sentence is illustrated by the following example.

Example 2: Let $\tilde{S}_g^I$ and S^I be two IT2FSs such that

$$\tilde{S}_g^I = \left\{ \frac{[1, 1]}{y_1}, \frac{[0.6, 0.8]}{y_2} \right\}$$

$$\tilde{S}^I = \left\{ \frac{[0.4, 0.7]}{y_1}, \frac{[1, 1]}{y_2} \right\}$$

and the quantifier "all" is defined as

$$Q(c_\diamond) = \begin{cases} 1, & c_\diamond = 1 \\ 0, & \text{otherwise} \end{cases} \tag{25}$$

TABLE IV
LOWER AND UPPER CRISP SETS FOR EXAMPLE 2

α	$(\underline{S}_g^I)_{\alpha_i}$	$(\bar{S}_g^I)_{\alpha_i}$	$(\underline{S}^I)_{\alpha_i}$	$(\bar{S}^I)_{\alpha_i}$
1	$\{y_1\}$	$\{y_1\}$	$\{y_2\}$	$\{y_2\}$
0.8	$\{y_1\}$	$\{y_1, y_2\}$	$\{y_2\}$	$\{y_2\}$
0.7	$\{y_1\}$	$\{y_1, y_2\}$	$\{y_2\}$	$\{y_1, y_2\}$
0.6	$\{y_1, y_2\}$	$\{y_1, y_2\}$	$\{y_2\}$	$\{y_1, y_2\}$
0.4	$\{y_1, y_2\}$	$\{y_1, y_2\}$	$\{y_1, y_2\}$	$\{y_1, y_2\}$

TABLE V
SET OF POSSIBLE VALUES, AND MINIMUM AND MAXIMUM VALUES TAKEN BY QUANTIFIER FOR EXAMPLE 2

α	C_{α_i}	$\min\limits_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond)$	$\max\limits_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond)$
1	$\{0\}$	0	0
0.8	$\{0, 1/2\}$	0	0
0.7	$\{0, 1/2, 1\}$	0	1
0.6	$\{1/2, 1\}$	0	1
0.4	$\{1\}$	1	1

The probabilistic degree of truth for the linguistic summary is computed as follows:

Step 1. Determine α-cuts.

The set of union of α-levels consisting of all the lower and the upper membership grades is $\Gamma = \{1, 0.8, 0.7, 0.6, 0.4\}$.

Step 2. Obtain the lower and the upper crisp sets associated with α-cuts.

The lower and the upper crisp sets associated with α-cuts are given in Table IV.

For $\alpha = 0.8$, the crisp sets of all embedded T1FSs of $\tilde{S}_g^I$, which are obtained by the α-cut are $\{y_1\}$ or $\{y_1, y_2\}$. Similarly, the crisp sets of all embedded T1FSs of $\tilde{S}^I$, which are obtained by the α-cut are $\{y_2\}$. In this condition, there exist two different cases; in the first case, $(S_g^{e_1})_{0.8} = \{y_1\}$ and $(S^{e_1})_{0.8} = \{y_2\}$, which makes $(S_g^{e_1})_{0.8} \cap (S^{e_1})_{0.8} = \emptyset$, and in the second case, $(S_g^{e_2})_{0.8} = \{y_1, y_2\}$ and $(S^{e_1})_{0.8} = \{y_2\}$, which makes $(S_g^{e_2})_{0.8} \cap (S^{e_1})_{0.8} = \{y_2\}$. The set of possible values is $C_{0.8} = \{0, 1/2\}$. The set of possible values, and the minimum and the maximum values taken by the quantifier for each α_i are given in Table V.

Step 3. Acquire the probability distribution.

The associated probability distribution is found as $\varphi(\alpha_1) = (1 - 0.8) = 0.2$, $\varphi(\alpha_2) = (0.8 - 0.7) = 0.1$, $\varphi(\alpha_3) = (0.7 - 0.6) = 0.1$, $\varphi(\alpha_4) = (0.6 - 0.4) = 0.2$, and $\varphi(\alpha_5) = (0.4 - 0) = 0.4$.

Step 4. Compute the degree of truth.

The probabilistic degree of truth for the linguistic summary is computed using (23) as

$$\begin{aligned}\tilde{T} &= 0.2\times(0,0)+0.1\times(0,0)+0.1\times(0,1)+0.2\times(0,1)+0.4\\&\quad\times(1,1)\\&= (0.4, 0.7)\end{aligned}$$

The following two examples compare our probabilistic degree of truth to that of proposed by Wu and Mendel [58].

Example 3a: Let $\tilde{S}_g^I$ and S^I be two IT2FSs such that

$$\tilde{S}_g^I = \left\{\frac{[1,1]}{y_1}, \frac{[0,0]}{y_2}, \frac{[0,0]}{y_3}, \ldots, \frac{[0,0]}{y_{1000}}\right\}$$

$$\tilde{S}^I = \left\{\frac{[0,0]}{y_1}, \frac{[1,1]}{y_2}, \frac{[0,0]}{y_3}, \ldots, \frac{[0,0]}{y_{1000}}\right\}$$

and the quantifier is defined as $Q(c_\diamond) = c_\diamond$.

Since $\tilde{S}_g^I$ and $\tilde{S}^I$ are crisp sets, the probabilistic degree of truth is equal to $Q(|\tilde{S}_g^I \cap \tilde{S}^I|/|\tilde{S}_g^I|) = 0$. The degree of truth proposed by Wu and Mendel [58] is also found as $T = 0$.

Example 3b: Let $\tilde{S}_g^I$ and S^I be two IT2FSs such that

$$\tilde{S}_g^I = \left\{\frac{[1,1]}{y_1}, \frac{[0.08, 0.12]}{y_2}, \frac{[0.08, 0.12]}{y_2}, \ldots, \frac{[0.08, 0.12]}{y_{1000}}\right\}$$

$$\tilde{S}^I = \left\{\frac{[0.08, 0.12]}{y_1}, \frac{[1,1]}{y_2}, \frac{[0.08, 0.12]}{y_3}, \ldots, \frac{[0.08, 0.12]}{y_{1000}}\right\}$$

and the quantifier is defined as $Q(c_\diamond) = c_\diamond$. The probabilistic degree of truth for the linguistic summary is computed as follows.

Step 1. Determine α-cuts.

The set of union of α-levels consisting of all the lower and the upper membership grades is $\Gamma = \{1, 0.12, 0.08\}$.

Step 2. Obtain the lower and the upper crisp sets associated with α-cuts.

The lower and the upper crisp sets associated with α-cuts are given in Table VI.

For $\alpha = 0.12$, we have to consider $2^{1998} \approx 2.87 \times 10^{601}$ cases to construct the set of possible values, which requires a large computational time. Fortunately, we could easily find the minimum and the maximum values taken by the quantifier investigating the three cases. In order to find the minimum value taken by the quantifier, we should consider the following two cases: the first case is $c_\diamond = |(\underline{S}_g^I)_{\alpha_i} \cap (\underline{S}^I)_{\alpha_i}|/|(\underline{S}_g^I)_{\alpha_i}|$ and the second case is $c_\diamond = |(\bar{S}_g^I)_{\alpha_i} \cap (\underline{S}^I)_{\alpha_i}|/|(\bar{S}_g^I)_{\alpha_i}|$ since the quantifier is the member of the coherent family. In the first case, $c_\diamond$ may be equal to zero since it is possible that $(\underline{S}_g^I)_{\alpha_i} \cap (\underline{S}^I)_{\alpha_i} = \emptyset$. If it is not empty set, we should investigate the second case; because, $(\bar{S}_g^I)_{\alpha_i}$ has the largest number of elements among the crisp sets obtained by α-cut of the embedded T1FSs $(S_g^{e_j})_{\alpha_i}$, and $(\underline{S}^I)_{\alpha_i}$ has the smallest number of elements among the crisp sets obtained by α-cut of the embedded T1FSs $(S^{e_t})_{\alpha_i}$. In order to find the maximum value taken by the quantifier, we should focus on the third case, i.e, $c_\diamond = |(\underline{S}_g^I)_{\alpha_i} \cap (\bar{S}^I)_{\alpha_i}|/|(\underline{S}_g^I)_{\alpha_i}|$ since $(\underline{S}_g^I)_{\alpha_i}$ has the smallest number of elements among the crisp sets obtained by α-cut of the embedded T1FSs $(S_g^{e_j})_{\alpha_i}$, and $(\bar{S}^I)_{\alpha_i}$ has the largest number of elements among the crisp sets obtained by α-cut of the embedded T1FSs $(S^{e_t})_{\alpha_i}$. Based on these definitions, we find $\min\limits_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond) = 0$ and $\max\limits_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond) = 1$ for $\alpha = 0.12$. The set of possible values, and the minimum and the maximum values taken by the quantifier for each α_i are given in Table VII.

Step 3. Acquire the probability distribution.

The associated probability distribution is found as $\varphi(\alpha_1) = (1 - 0.12) = 0.88$, $\varphi(\alpha_2) = (0.12 - 0.08) = 0.04$, and $\varphi(\alpha_3) = (0.08 - 0) = 0.08$.

TABLE VI
LOWER AND UPPER CRISP SETS FOR EXAMPLE 3B

α	$(\underline{S}^I_g)_{\alpha_i}$	$(\bar{S}^I_g)_{\alpha_i}$	$(\underline{S}^I)_{\alpha_i}$	$(\bar{S}^I)_{\alpha_i}$
1	$\{y_1\}$	$\{y_1\}$	$\{y_2\}$	$\{y_2\}$
0.12	$\{y_1\}$	$\{y_1, y_2, \ldots, y_{1000}\}$	$\{y_2\}$	$\{y_1, y_2, \ldots, y_{1000}\}$
0.08	$\{y_1, y_2, \ldots, y_{1000}\}$	$\{y_1, y_2, \ldots, y_{1000}\}$	$\{y_1, y_2, \ldots, y_{1000}\}$	$\{y_1, y_2, \ldots, y_{1000}\}$

TABLE VII
SET OF POSSIBLE VALUES, AND MINIMUM AND MAXIMUM VALUES TAKEN BY QUANTIFIER FOR EXAMPLE 3B

α	$\min_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond)$	$\max_{c_\diamond \in C_{\alpha_i}} Q(c_\diamond)$
1	0	0
0.12	0	1
0.08	1	1

Step 4. Compute the degree of truth.

The probabilistic degree of truth of linguistic summary is computed using (23) as

$$\tilde{T} = 0.88 \times (0,0) + 0.04 \times (0,1) + 0.08 \times (1,1) = (0.08,\ 0.12)$$

The same problem in Example 3b is also solved by the degree of truth proposed by Wu and Mendel [58]. The degree of truth is computed as

$$T = \mu_Q\left(\frac{AC(\tilde{S}^I_g \cap \tilde{S}^I)}{AC(\tilde{S}^I_g)}\right) = \mu_Q\left(\frac{100}{100,9}\right)$$
$$= \mu_Q(0.99) = 0.99$$

Even though IT2FSs in Example 3b are disjoint, the degree of truth is extremely high. The reason for this is that the small membership degrees compensate for a few large membership degrees. The degree of truth proposed by Wu and Mendel [58] is therefore very highly affected by small changes in the membership degrees, which is unreliable for the cases like here.

The application of the probabilistic degree of truth to the evaluation of IT2F linguistic summaries is presented in Fig. 1. First, the type of quantified sentences should be designated. If quantified sentence is type-I quantified sentence, then the type of quantifier whether it is relative or absolute should be determined. If quantified sentence is type-II quantified sentence, only a relative quantifier is used.

V. APPLICATION

The approach proposed in this paper was tested and compared to the approach proposed by Wu and Mendel [58] on the time series data of Europe Brent Spot Price (dollars per barrel) from 2003 to 2012 [80]. Even though the developments have taken place in alternative energy sources, fossil fuels, especially oil, still meet the majority demand of the world's energy. Therefore, it is very crucial to identify price trends and anticipate fluctuations on price per barrel of oil to plan oil importation finance for countries whose energy supply

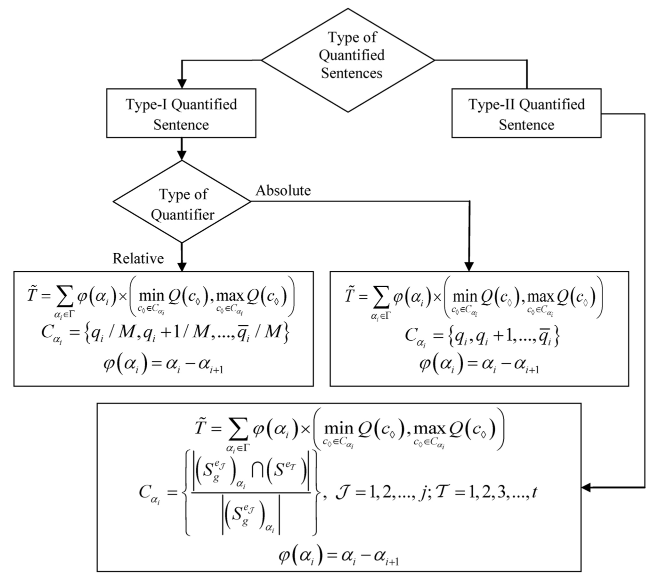

Fig. 1. Flowchart of our generic method for the evaluation of IT2F linguistic summaries.

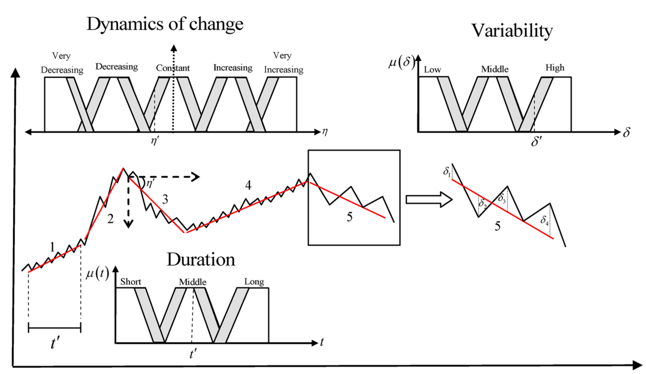

Fig. 2. Three aspect of time series: the dynamics of change, the duration, and the variability of local trends.

security highly rely on oil importation. The main objective of this application is to establish a decision support system that can be used for generating linguistic statements helping the estimation of future behaviors and fluctuations of oil price time series for countries whose economy is highly affected by oil price.

Few studies have been reported on linguistic summarization of time series [41]–[44]. In order to identify the behavior of time series, we have adopted the idea of Kacprzyk *et al.* [42], [43], in which the local trends are found out by means of the concept of a uniform partially linear approximation of time series introduced by Sklansky and Gonzales [81]. (Readers are referred to [42], [43] for more details of the algorithm).

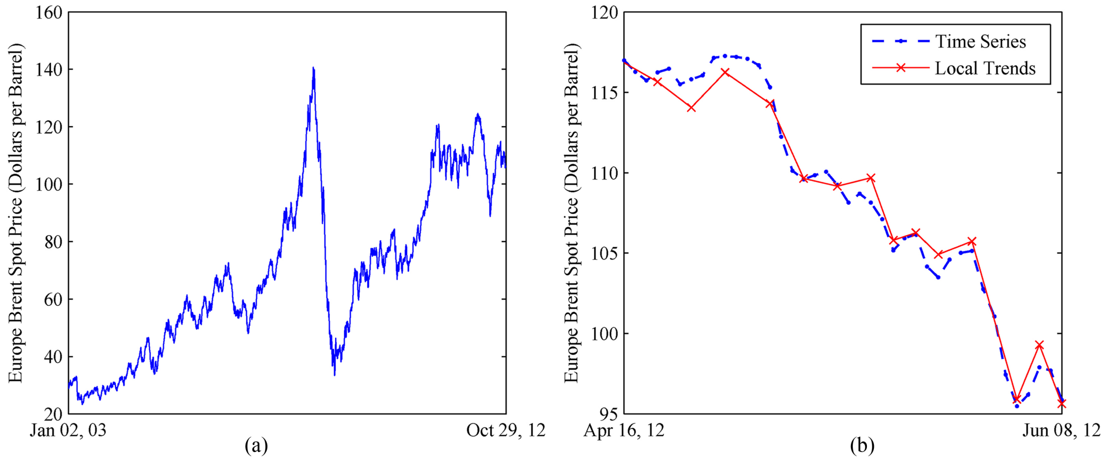

Fig. 3. (a) Time series of Europe Brent Spot Price. (b) Local trends from Apr 16, 2012 to Jun 08, 2012.

Three aspects of time series; dynamics of change, duration and variability have been used for summarizing the behavior of time series after local trends have been extracted. Fig. 2 clearly illustrate how the dynamics of change, the duration and the variability of local trends are used in summarization of time series. The numbered lines (1–5) represents the local trends. The duration of local trend (t'), the dynamics of change (η') and the variability (δ') are explained in the first, the third and the fifth trends, respectively. These three aspect of time series are defined with labels, each of which are modeled by IT2FSs. For example, the duration is expressed by three labels: short, middle and long. The membership functions of t', η' and δ', corresponding to the labels, are used in the evaluation of linguistic summaries.

The time series of Europe Brent Spot Price (dollars per barrel) contains 2537 entries as working days from January 2, 2003 to October 29, 2012, shown in Fig. 3(a), the minimum of which was 23.27 $ and the maximum of which was 140.73 $ [80]. Once implemented the algorithm introduced by Sklansky and Gonzales [81], 644 local trends were extracted. The local trends from Apr 16, 2012 to Jun 08, 2012 are illustrated in Fig. 3(b).

While the longest trend took 16 days, the shortest trends took only 2 days. The angels of trends changed between -83^0 and 80^0. The normalized distance between the points of time series and the partial trends was used to measure the variability (%). The lowest value of variability found as 0% and the highest value of variability found as 10%. After identifying the related parameters, the next step is to construct IT2FSs. In order to construct IT2FSs, we have adopted the idea of Liu and Mendel [82] and Wu *et al.* [83]. IT2FSs used for labeling the dynamics of change, the duration and the variability of local trends have been illustrated in Fig. 4(a)–Fig. 4(c), respectively. "All", "about half", and "most" have been selected as the quantifiers. The quantifier "all" has been modeled as in (25), "most" has been modeled as $Q_{Most}(c_\diamond) = c_\diamond$ and "about half" has been modeled in

$$Q_{About\ half}(c_\diamond) = \begin{cases} 2c_\diamond, & 0 \le c_\diamond \le 0.5 \\ 2(1-c_\diamond), & 1 \ge c_\diamond \ge 0.5 \end{cases} \tag{26}$$

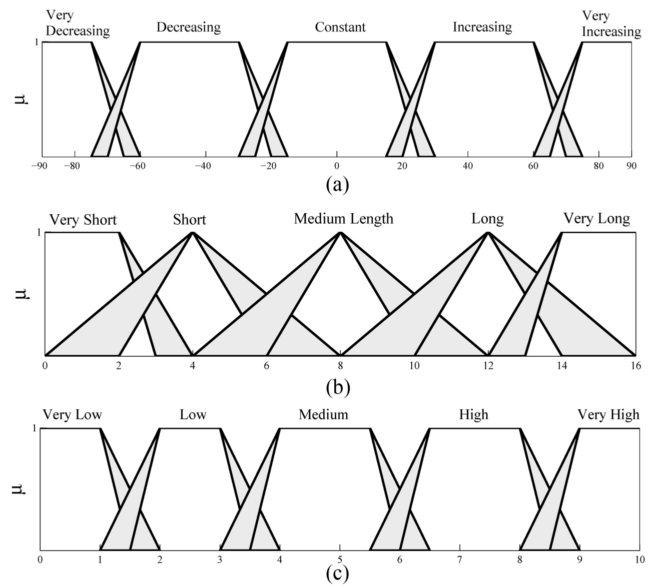

Fig. 4. (a) IT2FSs for the dynamics of change of local trends (Angel). (b) IT2FSs for the duration of local trends (Day). (c) IT2FSs for the variability of local trends (%).

A MATLAB code has been developed to generate, evaluate with our probabilistic degree of truth, and rank linguistic summaries. The method given in Appendix A have been used for ranking linguistic summaries, which have a degree of truth with interval number. In total, 450 ($6 \times 3 \times 5 \times 5$) linguistic summaries exist when one qualifier and one summarizer are considered in the type-II quantified linguistic summary form. After computed the degree of truths of all possible linguistic summaries, we compared our approach to Wu and Mendel's approach by filtering top five linguistic summaries for each of the quantifiers, presented in Table VIII.

When compared the results of our approach to those of Wu and Mendel's approach, there are some differences when the quantifier is "about half" and "all". The top five linguistic summaries obtained by our approach for the quantifier "most" are same to the top five linguistic summaries obtained by Wu and Mendel's approach, but the order is different. The

TABLE VIII
LINGUISTIC SUMMARIES OBTAINED BY OUR APPROACH, AND WU AND MENDEL'S APPROACH

Q	Our Approach $(\tilde{T})$		Wu and Mendel's Approach (T)	
Most	Most of very long trends are constant	$\tilde{T}=[1,1]$	Most of very long trends are constant	$T=[1]$
	Most of long trends have low variability	$\tilde{T}=[0.813,0.950]$	Most of long trends are constant	$T=[0.858]$
	Most of long trends are constant	$\tilde{T}=[0.755,0.869]$	Most of long trends have low variability	$T=[0.842]$
	Most of trends with very high variability are constant	$\tilde{T}=[0.684,0.788]$	Most of very long trends have low variability	$T=[0.760]$
	Most of very long trends have low variability	$\tilde{T}=[0.600,0.800]$	Most of trends with very high variability are constant	$T=[0.730]$
About half	About half of trends with low variability are constant	$\tilde{T}=[0.793,0.908]$	About half of trends with very high variability are very short	$T=[0.968]$
	About half of medium length trends are constant	$\tilde{T}=[0.733,0.939]$	About half of medium length trends have low variability	$T=[0.967]$
	About half of medium length trends have low variability	$\tilde{T}=[0.717,0.889]$	About half of trends with low variability are constant	$T=[0.896]$
	About half of very decreasing trends have medium variability	$\tilde{T}=[0.667,0.855]$	About half of very decreasing trends have medium variability	$T=[0.891]$
	About half of short trends have medium variability	$\tilde{T}=[0.638,0.766]$	About half of constant trends are short	$T=[0.853]$
All	All very long trends are constant	$\tilde{T}=[1,1]$	All very long trends are constant	$T=[1]$
	All very long trends have low variability	$\tilde{T}=[0.600,0.800]$		
	All long trends have low variability	$\tilde{T}=[0.250,0.700]$		
	All long trends are constant	$\tilde{T}=[0.100,0.467]$		
	All trends with very high variability are short	$\tilde{T}=[0.050,0.500]$		

main reason of this is that "most" is a coherent quantifier, and the distribution of data does not lead to inconsistency like in Example 3b. However, when the quantifier is "about half", three out of top five linguistic summaries are same, while two out of top five linguistic summaries are different in our approach and Wu and Mendel's approach. The reason for this difference is that, in some cases, scalar cardinality leads to compensate the membership degrees, and degree of truth is equal to a very high value (Readers are referred to [84] p.179 for more details). The linguistic summary "about half of trends with very high variability are very short" with $T = 0.967$ could be given as an example for this situation. When the quantifier is "all", Wu and Mendel's approach generates only one linguistic summary and leads to miss other crucial linguistic summaries since it provides very strict results in the evaluation.

VI. CONCLUSION

In this paper, we have put forward a generic method using a probabilistic degree of truth for evaluating linguistic summaries in the forms of type-I and type-II quantified sentences. We have also extended some properties proposed by Delgado *et al.* [5], which should be fulfilled by any degree of truth to IT2F environment. In the literature, there exist two approaches for evaluating IT2F linguistic summaries: the first one proposed by Niewiadomski *et al.* [64] is not able to model linguistic summaries when the summarizers and the qualifier are labeled with IT2Fs and the second one proposed by Wu and Mendel [58] deals with linguistic summaries in the form of the if-then fuzzy rules in which the antecedent and the consequent are labeled with IT2Fs. The approach proposed by Wu and Mendel [58] was highly affected by the small changes in membership grades of IT2FS, which leads to inconsistency in the evaluation. Furthermore, only if-then fuzzy rule could be used as linguistic summary since it does not properly model noncoherent quantifiers. Our probabilistic degree of truth is the first approach not only capable of dealing with linguistic summaries in the forms of type-I and type-II quantified sentences in which both the summarizers and the qualifiers are labeled with IT2Fs while the quantifier is labeled with T1FS but also allowing us to use any kind of quantifier including coherent and non-coherent family of quantifiers on linguistic summarization. Our probabilistic degree of truth is also the only approach satisfying the given properties; but it requires some computational time since the all cases are considered to obtain the set of possible values. Furthermore, our probabilistic approach can be easily extended to model semi-fuzzy quantifiers that provide the more complex summary form such as "All except five people are blonde" or "All except about 10 % of people are tall", which are not possible to be modeled by the existing approaches since our approach is based on α-cuts.

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APPENDIX A
RANKING OF DEGREE OF TRUTH

Let $\tilde{T}_1 = [T_1^L, T_1^U]$ and $\tilde{T}_2 = [T_2^L, T_2^U]$ be two degree of truths. If the (27) is hold then $\tilde{T}_1 \leqslant \tilde{T}_2$

$$\tilde{T}_1 \leqslant \tilde{T}_2 \leftrightarrow \frac{T_1^L + T_1^U}{2} \leqslant \frac{T_2^L + T_2^U}{2} \wedge \frac{T_1^U - T_1^L}{2} \geq \frac{T_2^U - T_2^L}{2} \quad (27)$$

When degree of truths cannot be compared via (27), the following method proposed by Sengupta *et al.* [85] is introduced:

$$\mu_{(\leqslant)}(\tilde{T}_1, \tilde{T}_2) = \frac{((T_2^L + T_2^U) - (T_1^L + T_1^U))}{((T_1^U - T_1^L) + (T_2^U - T_2^L))} \quad (28)$$

When all the values of $\mu_{(\leqslant)}$ exceeding 1 are reduced to 1 and all the values under 0 are treated as 0.

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