Technical Notes

Optimal Control of Backward Doubly Stochastic Systems With Partial Information

Qingfeng Zhu and Yufeng Shi

Abstract—**This technical note is concerned with a class of partial information control problems for backward doubly stochastic systems. By the method of convex variation and duality technique, one sufficient condition (a verification theorem) and one necessary condition for optimality for this type of partial information controls are proved. Then, our theoretical results are applied to study a partial information linear quadratic (LQ) optimal control problem of a backward doubly stochastic system.**

Index Terms—**Backward doubly stochastic differential equation, maximum principle, partial information, stochastic optimal control.**

I. Introduction

In order to provide a probabilistic interpretation for the solutions of a class of semilinear stochastic partial differential equations (SPDEs), Pardoux and Peng [11] introduced a class of backward doubly stochastic differential equation (BDSDE). The authors showed existence and uniqueness for this kind of BDSDE. In fact, they provided a new way to study SPDEs, and along this way many important results for SPDEs have been obtained (see Buckdahn and Ma [3], Wu and Zhang [15], Zhang and Zhao [19]–[21], Zhu and Shi [22], and the references therein).

Recently, the stochastic control problems of BDSDEs have been investigated. Han *et al.* [4] established a necessary condition of optimality (i.e., maximum principle) for this kind of systems and gave an application of the maximum principle to backward doubly stochastic linear quadratic (LQ) optimal control problems. Bahlali and Gherbal [2] proved the necessary and sufficient optimality conditions for the control problems of BDSDEs, where their results were stated in the form of weak stochastic maximum principle and their results were given as well in the global form under additional hypotheses. Zhang and Shi [18] have studied the optimal control problems of fully coupled forward-backward doubly stochastic systems.

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Q. Zhu is with the School of Mathematic and Quantitative Economics, Shandong University of Finance and Economics, Jinan 250014, China and also with the Institute for Financial Studies and School of Mathematics, Shandong University, Jinan 250199, China (e-mail: zhuqf508@sohu.com).

Y. Shi is with the Institute for Financial Studies and School of Mathematics, Shandong University, Jinan 250199, China (e-mail: yfshi@sdu.edu.cn).

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However, in the control problems above, we notice that they all assumed that the information is complete, i.e., decision-makers can completely observe the underlying filtration. This is not reasonable in reality. Generally speaking, decision-makers can only get partial information in many cases. For example, Williams [14] studied a dynamic principal-agent problem with unobservable states and actions. It is well known that there maybe exists so called informal trading such as "insider trading" in the market. That is, the players at the current time t possess extra information of the future developing of the market from t to T that is represented by $\mathcal{F}_{t,T}^B$, as well as the accumulated information $\mathcal{F}_t^W$ from 0 to t. This motivates us to study the optimal control problems of backward doubly stochastic systems under partial information.

In recent years, there have been growing interests on stochastic optimal control problems under partial information. In particular, sometimes an economic model in which there are information gaps among economic agents can be formulated as a partial information optimal control problem (see Øksendal [10], Kohlmann and Xiong [6]). Baghery and Øksendal [1] established some maximum principles of forward stochastic systems within partial information. Huang *et al.* [5] investigated some partial information control problems of backward stochastic systems. Meng [9], Wang and Wu [13] and Xiao and Wang [16] studied the partial information stochastic optimal control problems of forward-backward stochastic systems.

To the best of our knowledge, there are few results devoted to study stochastic control problems for doubly stochastic systems under partial information. In the doubly stochastic systems, $\mathcal{F}_t \doteq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B,\ \forall\, t \in [0,T]$, that is, $\mathcal{F}_t$ is the collection of an increasing filtration $\mathcal{F}_t^W$ and a decreasing filtration $\mathcal{F}_{t,T}^B$, where there is only one filtration that is available to observe. The unobservable filtration can be considered as some extra information that can not commonly be detected in practice, but is available to some insiders in the market, such as in some incomplete or unfair financial securities markets. Therefore this technical note is devoted to investigate a type of backward doubly stochastic control systems with partial information. In this technical note, we shall consider the partial information stochastic control problems for backward doubly stochastic systems. By the method of convex variation and duality technique, we derive the maximum principles under some suitable conditions. Our results are a partial extension to optimal control of partial information backward stochastic systems (see Huang *et al.* [5]) and full information backward doubly stochastic systems (see Han *et al.* [4] and Bahlali and Gherbal [2]).

It is well known that LQ control is one of the most important classes of optimal control, and the solution of this problem has had a profound impact on many engineering applications (see Lim and Zhou [7], [8]). Our theoretical results are applied to study a partial information LQ optimal control problem of a backward doubly stochastic system. The key point to solving them is to get some observable optimal controls by explicitly computing the filtering estimates of the corresponding adjoint equations. Combining the filtering equations for BDSDEs with the stochastic control theory, we obtain the explicit and observable controls.

The rest of this technical note is organized as follows. In Section II, we state our partial information optimal control problems and the main assumptions. In Section III, we obtain the sufficient maximum principle of the stochastic optimal control problem under partial information. Section IV is devoted to the necessary optimality conditions. In Section V, we give an LQ control problem as example to show the applications of our theoretical results.

II. Statement of the Problems

Let $(\Omega, \mathcal{F}, P)$ be a completed probability space, $\{W(t)\}_{t\geq 0}$ and $\{B(t)\}_{t\geq 0}$ be two mutually independent standard Brownian motions, with value respectively in $\mathbb{R}^d$ and $\mathbb{R}^l$, defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0,T]$, we define $\mathcal{F}_t^W \doteq \sigma\{W(r);\ \ 0 \leq r \leq t\} \vee \mathcal{N}$, $\mathcal{F}_{t,T}^B \doteq \sigma\{B(T) - B(r);\ t \leq r \leq T\} \vee \mathcal{N}$ and $\mathcal{F}_t \doteq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$, $\forall\, t \in [0,T]$. Note that the collection $\{\mathcal{F}_t, t \in [0,T]\}$ is neither increasing nor decreasing, and it does not constitute a classical filtration.

We denote $M^2(0,T;\mathbb{R}^n)$ the space of (class of $dP \otimes dt$ a.e equal) all $\mathcal{F}_t$-measurable n-dimensional processes φ with norm of $\|\varphi\|_M \doteq [\mathbb{E}\int_0^T |\varphi(t)|^2\, dt]^{(1/2)} < \infty$. Obviously $M^2(0,T;\mathbb{R}^n)$ is a Hilbert space. Let $L^2(\Omega, \mathcal{F}_T, P;\mathbb{R}^n)$ denote the space of all $\mathcal{F}_T$-measurable $\mathbb{R}^n$-valued random variable ξ satisfying $\mathbb{E}|\xi|^2 < \infty$. We use the usual inner product $\langle\cdot,\cdot\rangle$ and Euclidean norm $|\cdot|$ in $\mathbb{R}^n$, $\mathbb{R}^{n\times l}$ and $\mathbb{R}^{n\times d}$. $*$ appearing in the superscripts denotes the transpose of a matrix. All the equalities and inequalities mentioned in this technical note are in the sense of $dt \times dP$ almost surely on $[0,T] \times \Omega$.

Definition 2.1: A stochastic process $X = \{X(t); t \geq 0\}$ is called $\mathcal{F}_t$-progressively measurable, if for any $t \geq 0$, X on $\Omega \times [0,t]$ is measurable with respect to $(\mathcal{F}_t^W \times \mathcal{B}([0,t])) \vee (\mathcal{F}_{t,T}^B \times \mathcal{B}([t,T]))$.

Under this framework, we consider the following backward doubly stochastic control system:

$$\begin{cases} -dy(t) = f\left(t, y(t), z(t), v(t)\right) dt - z(t) \overrightarrow{d} W(t) \\ \qquad\qquad + g\left(t, y(t), z(t), v(t)\right) \overleftarrow{d} B(t), \\ y(T) = \xi \end{cases} \tag{1}$$

where $\xi \in L^2(\Omega, \mathcal{F}_T, P;\mathbb{R}^n)$, $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^k \to \mathbb{R}^n$, $g : [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^k \to \mathbb{R}^{n\times l}$. Note that the integral with respect to $\{B(t)\}$ is a "backward Itô integral", in which the integrand takes values at the right end points of the subintervals in the Riemann type sum, and the integral with respect to $\{W(t)\}$ is a standard forward Itô integral. These two types of integrals are particular cases of the Itô-Sokorohod integral (see [11] for details).

We assume that

(H1) f and g are continuously differentiable with respect to (y, z, v).

(H2) The norm of f_y, f_z, f_v, g_y, g_v are bounded by $c > 0$, and the norm of g_z is bounded by $\alpha \in (0,1)$.

Let $\mathcal{E}_t$ is a sub-sigma algebra of $\mathcal{F}_t$ at time t, i.e., $\mathcal{E}_t \subset \mathcal{F}_t$. It is remarkable that the sub-sigma algebra is very general. For example, we could have $\mathcal{E}_t = \mathcal{F}_t^W = \sigma\{W(r); 0 \leq r \leq t\} \vee \mathcal{N}$ representing the information available to the controller at time t. And the extra noise $\{B(t)\}$ can be considered as some extra information that can not be detected in practice, such as in a derivative security market, but is available to the insiders of the investors. We say a control variable $v(\cdot) : \Omega \times [0,T] \to U$ is admissible, if it is $\mathcal{E}_t$-adapted and satisfies $[\mathbb{E}\int_0^T |v(s)|^2\, ds]^{(1/2)} < \infty$. We denote by $\mathcal{U}_{ad}$ the collection of all admissible controls. For convenience, let U be a nonempty convex subset of $\mathbb{R}^k$. Since U is convex, $\mathcal{U}_{ad}$ is also a convex set.

Under the above hypotheses (H1) and (H2), for every $v(\cdot) \in \mathcal{U}_{ad}$, the state equation (1) admits a unique strong solution $(y, z) \in M^2(0,T;\mathbb{R}^n) \times M^2(0,T;\mathbb{R}^{n\times d})$ (see Pardoux and Peng [11]).

The cost functional is the following type:

$$J\left(v(\cdot)\right) = E\left[\int_0^T l\left(t, y(t), z(t), v(t)\right) dt + \Phi\left(y(0)\right)\right] \tag{2}$$

where $l : [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^k \to \mathbb{R}, \Phi : \mathbb{R}^n \to \mathbb{R}$, satisfying the condition

$$\mathbb{E}\left[\int_0^T \left|l\left(t, y(t), z(t), v(t)\right)\right| dt + \left|\Phi\left(y(0)\right)\right|\right] < \infty. \tag{3}$$

We also assume

(H3) l is continuously differentiable in (y, z, v), its partial derivatives are continuous in (y, z, v) and bounded by $c(1 + |y| + |z| + |v|)$;

(H4) Φ is continuously differentiable and Φ_y is bounded by $c(1 + |y|)$.

Our partial information optimal control problem is to minimize the cost functional (2) over $v(\cdot) \in \mathcal{U}_{ad}$ subject to (1), i.e., to find $u(\cdot) \in \mathcal{U}_{ad}$ satisfying

$$J\left(u(\cdot)\right) = \min_{v(\cdot)\in\mathcal{U}_{ad}} J\left(v(\cdot)\right). \tag{4}$$

(1) is called the state equation, the solution $(y(t), z(t))$ corresponding to $u(\cdot)$ is called the optimal trajectory, and $(y(t), z(t), u(t))$ is called an admissible triple.

III. A Partial Information Sufficient Maximum Principle

In this section, we investigate a sufficient maximum principle for the partial information optimal control problem (1)–(4). We first introduce the following adjoint forward doubly stochastic differential equation (SDE) of the system (1) corresponding to the admissible triple $(y(t), z(t), v(t))$:

$$\begin{cases} dp(t) = \left[f_y^*(t)p(t) + g_y^*(t)q(t) - l_y(t)\right] dt - q(t) \overleftarrow{d} B(t) \\ \qquad\qquad + \left[f_z^*(t)p(t) + g_z^*(t)q(t) - l_z(t)\right] \overrightarrow{d} W(t), \\ p(0) = -\Phi_y\left(y(0)\right). \end{cases} \tag{5}$$

Under (H1)-(H4), by the result on forward-backward doubly stochastic differential equations (FBDSDE) (see Peng and Shi [12]), it is not difficult to know that (5) admits a unique solution $(p(t), q(t)) \in M^2(0,T;\mathbb{R}^n) \times M^2(0,T;\mathbb{R}^{n\times l})$.

We define the Hamiltonian function $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \to \mathbb{R}$ as follows:

$$\begin{aligned} H(t, y, z, v, p, q) = -\langle p(t), f(t, y, z, v)\rangle + l(t, y, z, v) \\ - \langle q(t), g(t, y, z, v)\rangle. \end{aligned} \tag{6}$$

Denote $H(t) \equiv H(t, y(t), z(t), v(t), p(t), q(t))$ and its derivatives, then the adjoint (5) can be rewritten as the following stochastic Hamiltonian type:

$$\begin{cases} dp(t) = -H_y(t)dt - H_z(t) \overrightarrow{d} W(t) - q(t) \overleftarrow{d} B(t), \\ p(0) = -\Phi_y\left(y(0)\right). \end{cases} \tag{7}$$

Let $(\tilde{y}(t), \tilde{z}(t), \tilde{u}(t))$ be a triple satisfying (1) and suppose there exists a solution $(\tilde{p}(t), \tilde{q}(t))$ of the corresponding adjoint forward doubly SDE (5). We assume that

(H5) for arbitrary admissible triple $(y^{(v)}(t), z^{(v)}(t), v(t))$ satisfying (1), we have

$$\mathbb{E}\int_0^T \left\langle \tilde{q}(t), y^{(v)}(t) - \tilde{y}(t)\right\rangle^2 dt < \infty, \tag{8}$$

$$\mathbb{E}\int_0^T \left\langle \tilde{p}(t), z^{(v)}(t) - \tilde{z}(t)\right\rangle^2 dt < \infty, \tag{9}$$

$$\mathbb{E}\int_0^T \left\langle y^{(v)}(t)-\tilde{y}(t), H_z(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t),\tilde{p}(t),\tilde{q}(t))\right\rangle^2 dt<\infty, \tag{10}$$

$$\mathbb{E}\int_0^T \left\langle \tilde{p}(t), g\left(t, y^{(v)}(t), z^{(v)}(t), v(t)\right) -g(t,\tilde{y}(t),\tilde{z}(t),\tilde{v}(t))\right\rangle^2 dt < \infty, \tag{11}$$

$$\mathbb{E}\int_0^T \left|H_u\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t),\tilde{p}(t),\tilde{q}(t)\right)\right|^2 dt < \infty. \tag{12}$$

(H6) for all $t \in [0, T]$, $H(t, y, z, v, \tilde{p}(t), \tilde{q}(t))$ is convex in (y, z, v), and $\Phi(y)$ is convex in y.

Theorem 3.1: (Partial information sufficient maximum principle). Assume that (H5) and (H6) are satisfied. Moreover, the following partial information maximum condition holds:

$$\mathbb{E}\left[H\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t),\tilde{p}(t),\tilde{q}(t)\right)|\mathcal{E}_t\right] = \min_{v\in\mathcal{U}_{ad}} \mathbb{E}\left[H\left(t,\tilde{y}(t),\tilde{z}(t),v,\tilde{p}(t),\tilde{q}(t)\right)|\mathcal{E}_t\right]. \tag{13}$$

Then $\tilde{u}(t)$ is an optimal control of the partial information backward doubly stochastic optimal problem (1)–(4).

Proof: Let $(y(t), z(t), v(t)) = (y^{(v)}(t), z^{(v)}(t), v(t))$ be an arbitrary triple satisfying (1). According to the definition of the cost function (2), we have

$$\begin{aligned} J\left(v(\cdot)\right) - J(\tilde{u}(\cdot)) &= \mathbb{E}\int_0^T \left[l\left(t,y(t),z(t),v(t)\right) - l\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t)\right)\right] dt \\ &\quad + \mathbb{E}\left[\Phi\left(y(0)\right) - \Phi\left(\tilde{y}(0)\right)\right] \\ &= \mathbf{I}_1 + \mathbf{I}_2 \end{aligned} \tag{14}$$

where

$$\begin{aligned} \mathbf{I}_1 &= \mathbb{E}\int_0^T \left[l\left(t,y(t),z(t),v(t)\right) - l\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t)\right)\right] dt, \\ \mathbf{I}_2 &= \mathbb{E}\left[\Phi\left(y(0)\right) - \Phi\left(\tilde{y}(0)\right)\right]. \end{aligned}$$

Now applying Itô's formula to $\langle \tilde{p}(t), y(t) - \tilde{y}(t)\rangle$ on $[0, T]$, we get

$$\begin{aligned} &\langle \tilde{p}(T), y(T) - \tilde{y}(T)\rangle - \langle \tilde{p}(0), y(0) - \tilde{y}(0)\rangle \\ &= \langle \Phi_y\left(\tilde{y}(0)\right), y(0) - \tilde{y}(0)\rangle \\ &= \int_0^T \langle z(t) - \tilde{z}(t), -H_z\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t),\tilde{p}(t),\tilde{q}(t)\right)\rangle dt \\ &\quad - \int_0^T \langle \tilde{q}(t), g\left(t,y(t),z(t),v(t)\right) - g\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t)\right)\rangle dt \\ &\quad + \int_0^T \langle y(t) - \tilde{y}(t), -H_y\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t),\tilde{p}(t),\tilde{q}(t)\right)\rangle dt \\ &\quad + \int_0^T \left\langle y(t) - \tilde{y}(t), -H_z\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t),\tilde{p}(t),\tilde{q}(t)\right) \overrightarrow{d}W(t)\right\rangle \\ &\quad - \int_0^T \left\langle y(t) - \tilde{y}(t), \tilde{q}(t) \overleftarrow{d}B(t)\right\rangle \\ &\quad - \int_0^T \langle \tilde{p}(t), f\left(t,y(t),z(t),v(t)\right) - f\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t)\right)\rangle dt \\ &\quad - \int_0^T \left\langle \tilde{p}(t), \left(g\left(t,y(t),z(t),v(t)\right) - g\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t)\right)\right) \overleftarrow{d}B(t)\right\rangle \\ &\quad + \int_0^T \left\langle \tilde{p}(t), \left(z(t) - \tilde{z}(t)\right) \overrightarrow{d}W(t)\right\rangle \end{aligned} \tag{15}$$

where we claim that $y(T) - \tilde{y}(T) = \xi - \xi = 0$, and $\tilde{p}(0) = -\Phi_y(y(0))$.

Under the conditions (8)–(12), we can ensure that the stochastic integrals with respect to the Brownian motions have zero expectations. Moreover, by virtue of (15) and convexity of Φ, it instantly follows that:

$$\begin{aligned} \mathbf{I}_2 &= \mathbb{E}\left[\Phi\left(y(0)\right) - \Phi\left(\tilde{y}(0)\right)\right] \\ &\geq \mathbb{E}\left\langle \Phi_y\left(\tilde{y}(0)\right), y(0) - \tilde{y}(0)\right\rangle \\ &= -\mathbb{E}\int_0^T \langle y(t)-\tilde{y}(t), H_y\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t),\tilde{p}(t),\tilde{q}(t)\right)\rangle dt \\ &\quad - \mathbb{E}\int_0^T \langle z(t)-\tilde{z}(t), H_z\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t),\tilde{p}(t),\tilde{q}(t)\right)\rangle dt \\ &\quad - \mathbb{E}\int_0^T \langle \tilde{p}(t), f\left(t,y(t),z(t),v(t)\right) - f\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t)\right)\rangle dt \\ &\quad - \mathbb{E}\int_0^T \langle \tilde{q}(t), g\left(t,y(t),z(t),v(t)\right) - g\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t)\right)\rangle dt \\ &= -\Xi_1 + \Xi_2 + \Xi_3 \end{aligned} \tag{16}$$

where

$$\begin{aligned} \Xi_1 &= \mathbb{E}\int_0^T \langle H_y\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t),\tilde{p}(t),\tilde{q}(t)\right), y(t) - \tilde{y}(t)\rangle dt \\ &\quad + \mathbb{E}\int_0^T \langle H_z\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t),\tilde{p}(t),\tilde{q}(t)\right), z(t) - \tilde{z}(t)\rangle dt, \\ \Xi_2 &= -\mathbb{E}\int_0^T \langle \tilde{p}(t), f\left(t,y(t),z(t),v(t)\right) - f\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t)\right)\rangle dt, \\ \Xi_3 &= -\mathbb{E}\int_0^T \langle \tilde{q}(t), g\left(t,y(t),z(t),v(t)\right) - g\left(t,\tilde{y}(t),\tilde{z}(t),\tilde{u}(t)\right)\rangle dt. \end{aligned}$$

Noting the definition of H and $\mathbf{I}_1$, we have

$$\begin{aligned}\mathbf{I}_1 =& \mathbb{E}\int_0^T \left[l\left(t, y(t), z(t), v(t)\right) - l\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\right] dt \\ =& \mathbb{E}\int_0^T \left[H\left(t, y(t), z(t), v(t), \tilde{p}(t), \tilde{q}(t)\right)\right. \\ &\left. - H\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right)\right] dt \\ &+ \mathbb{E}\int_0^T \left[\langle \tilde{p}(t), f\left(t, y(t), z(t), v(t)\right)\right. \\ &\left. - f\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\rangle\right] dt \\ &+ \mathbb{E}\int_0^T \left[\langle \tilde{q}(t), g\left(t, y(t), z(t), v(t)\right)\right. \\ &\left. - g\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\rangle\right] dt \\ =& \Xi_4 - \Xi_2 - \Xi_3 \end{aligned} \tag{17}$$

where

$$\begin{aligned}\Xi_4 = \mathbb{E}\int_0^T & \left[H\left(t, y(t), z(t), v(t), \tilde{p}(t), \tilde{q}(t)\right)\right. \\ &\left. - H\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right)\right] dt. \end{aligned} \tag{18}$$

Using convexity of $H(t, y, z, v, \tilde{p}(t), \tilde{q}(t))$ with respect to (y, z, v), we obtain

$$\begin{aligned}& H\left(t, y(t), z(t), v(t), \tilde{p}(t), \tilde{q}(t)\right) \\ &\quad - H\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right) \\ &\geq H_y\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right)\left(y(t) - \tilde{y}(t)\right) \\ &\quad + H_z\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right)\left(z(t) - \tilde{z}(t)\right) \\ &\quad + H_u\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right)\left(v(t) - \tilde{u}(t)\right). \end{aligned} \tag{19}$$

Since $v \to \mathbb{E}[H_u(t, \tilde{y}(t), \tilde{z}(t), v, \tilde{p}(t), \tilde{q}(t))|\mathcal{E}_t], v \in \mathcal{U}$ is minimal for $\tilde{u}(t)$ and $v(t), \tilde{u}(t)$ are $\mathcal{E}_t$-measurable, we can get by (12)

$$\begin{aligned}& \mathbb{E}\left[H_u\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right)|\mathcal{E}_t\right]\left(v(t) - \tilde{u}(t)\right) \\ &\quad = \mathbb{E}\left[H_u\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right)\left(v(t) - \tilde{u}(t)\right)|\mathcal{E}_t\right] \\ &\quad \geq 0. \end{aligned} \tag{20}$$

Hence combining (18), (19), and (20), we obtain

$$\begin{aligned}\Xi_4 \geq& \mathbb{E}\int_0^T \left\langle H_y\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right),\right. \\ &\left. y(t) - \tilde{y}(t)\right\rangle dt \\ &+ \mathbb{E}\int_0^T \left\langle H_z\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right),\right. \\ &\left. z(t) - \tilde{z}(t)\right\rangle dt \\ =& \Xi_1. \end{aligned} \tag{21}$$

Therefore, it follows from (14)–(16) and (21), that:

$$\begin{aligned}& J\left(v(\cdot)\right) - J\left(u(\cdot)\right) \\ &\quad \geq \Xi_4 - \Xi_2 - \Xi_3 - \Xi_1 + \Xi_2 + \Xi_3 \\ &\quad \geq \Xi_1 - \Xi_2 - \Xi_3 - \Xi_1 + \Xi_2 + \Xi_3 = 0. \end{aligned}$$

Since $\forall\, v(\cdot) \in \mathcal{U}_{ad}$ is arbitrary, we say that $\tilde{u}(\cdot)$ is an optimal control of the partial information backward doubly stochastic optimal problem (1)–(4). ■

IV. A Partial Information Necessary Maximum Principle

In this section, we will prove that if $\tilde{u}(\cdot)$ is a local optimal control of the partial information backward doubly stochastic optimal problem (1)–(4) in some sense, then $\tilde{u}(\cdot)$ satisfies the partial information maximum condition in some local form.

In addition to the assumptions in Section II, more assumptions are given as follows:

(H7) For all t, r such that $0 \leq t < t + r \leq T$, and all bounded $\mathcal{E}_t$-measurable $\alpha = \alpha(\omega)$, the control $\theta(s) := (0, \ldots, \theta^i(s), 0, \ldots, 0) \in U \subset \mathbb{R}^k$, with $\theta^i(s) := \alpha_i \mathbf{1}_{[t,t+r]}(s), s \in [0, T]$ belongs to $\mathcal{U}_{ad}, i = 1, 2, \ldots, k$.

(H8) For all $u(\cdot), \theta(\cdot) \in \mathcal{U}_{ad}$, with $\theta(\cdot)$ bounded, there exists $\delta > 0$ such that $u(\cdot) + \beta\theta(\cdot) \in \mathcal{U}_{ad}$, for all $\beta \in (-\delta, \delta)$.

For given $u(\cdot), \theta(\cdot) \in \mathcal{U}_{ad}$ with $\theta(\cdot)$ bounded, we define the processes $(y^1(t), z^1(t))$ by

$$\begin{aligned} y^1(t) &:= y^{(u,\theta)}(t) := \frac{d}{d\beta} y^{(u+\beta\theta)}(t)|_{\beta=0}, \\ z^1(t) &:= z^{(u,\theta)}(t) := \frac{d}{d\beta} z^{(u+\beta\theta)}(t)|_{\beta=0}. \end{aligned} \tag{22}$$

By the differentiability of the solutions to BDSDEs, it is easy to see that $(y^1(t), z^1(t))$ satisfies the following linear BDSDE

$$\begin{cases} -dy^1(t) = \left[f_y(t)y^1(t) + f_z(t)z^1(t) + f_u(t)v(t)\right] dt \\ \qquad + \left[g_y(t)y^1(t) + g_z(t)z^1(t)\right. \\ \qquad \left. + g_u(t)v(t)\right] \overleftarrow{d} B(t) - z^1(t) \overrightarrow{d} W(t), \\ y^1(T) = 0. \end{cases} \tag{23}$$

Theorem 4.1: (Partial information necessary maximum principle). Let $\tilde{u}(\cdot) \in \mathcal{U}_{ad}$ be a local minimum for cost function $J(v(\cdot))$ (see (2)) in the sense that for all bounded $\theta(\cdot) \in \mathcal{U}_{ad}$, there exists $\delta > 0$ such that $\tilde{u}(\cdot) + \beta\theta(\cdot) \in \mathcal{U}_{ad}$ for all $\beta \in (-\delta, \delta)$ and $h(\beta) := J(\tilde{u}(\cdot) + \beta\theta(\cdot)), \beta \in (-\delta, \delta)$ is minimal at $\beta = 0$.

Suppose there exists a solution $(\tilde{p}(t), \tilde{q}(t))$ for the adjoint forward doubly SDE (7) corresponding to the admissible triple $(\tilde{y}(t), \tilde{z}(t), \tilde{u}(t))$, that is

$$\begin{cases} d\tilde{p}(t) = -\tilde{H}_y(t)dt - \tilde{H}_z(t) \overrightarrow{d} W(t) - \tilde{q}(t) \overleftarrow{d} B(t), \\ \tilde{p}(0) = -\Phi_y\left(y(0)\right) \end{cases} \tag{24}$$

where $\tilde{H}_\beta = H_\beta(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)), \quad \beta = y, z,$ respectively. Moreover, suppose that if $\tilde{y}^1(t) = y^{(\tilde{u},\theta)}(t), \tilde{z}^1(t) = z^{(\tilde{u},\theta)}(t)$, (noting (22), (23)), then

$$\mathbb{E}\int_0^T \left\langle \tilde{q}(t), \tilde{y}^1(t)\right\rangle^2 dt < \infty, \tag{25}$$

$$\mathbb{E}\int_0^T \left\langle \tilde{p}(t), \tilde{z}^1(t)\right\rangle^2 dt < \infty, \tag{26}$$

$$\mathbb{E}\int_0^T \left\langle \tilde{y}^1(t), H_z\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right)\right\rangle^2 dt < \infty, \tag{27}$$

$$\mathbb{E}\int_0^T \left\langle \tilde{p}(t), \tilde{\eta}(t)\left(\tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\right\rangle^2 dt < \infty \tag{28}$$

where

$$\tilde{\eta}(t) = g_y\left(\tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\tilde{y}^1(t) + g_z\left(\tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\tilde{z}^1(t) + g_v\left(\tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\theta(t).$$

Then $\tilde{u}(\cdot)$ is a stationary point for $\mathbb{E}[H|\mathcal{E}_t]$ in the sense that for almost all $t \in [0,T]$, we have

$$\mathbb{E}\left[H_u\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right)|\mathcal{E}_t\right] = 0. \tag{29}$$

Proof: From the fact that $h(\beta)$ is minimal at $\beta = 0$, we have

$$\begin{aligned} 0 = h'(0) = \mathbb{E}\int_0^T &\Big[l_y\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\tilde{y}^1(t) \\ &+ l_z\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\tilde{z}^1(t) \\ &+ l_v\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\theta(t)\Big]dt \\ + \mathbb{E}&\left\langle \tilde{y}^1(0), \Phi_y\left(y(0)\right)\right\rangle. \end{aligned} \tag{30}$$

Applying Itô's formula to $\langle \tilde{y}^1(t), \tilde{p}(t)\rangle$ on $[0,T]$, we have

$$\begin{aligned} &\mathbb{E}\left\langle \tilde{y}^1(0), \Phi_y\left(y(0)\right)\right\rangle \\ &= -\mathbb{E}\int_0^T \Big[l_y\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\tilde{y}^1(t) \\ &\qquad + l_z\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\tilde{z}^1(t)\Big]dt \\ &\quad + \mathbb{E}\int_0^T \big[-\langle \tilde{p}(t), f_v\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\theta(t)\rangle \\ &\qquad - \langle \tilde{q}(t), g_v\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t)\right)\theta(t)\rangle\big]dt \end{aligned} \tag{31}$$

where the L^2 conditions (25)–(28) ensure that the stochastic integrals with respect to the Brownian motions have zero expectations.

Substituting (31) into (30), we have

$$\mathbb{E}\left[\int_0^T \langle H_v\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right), \theta(t)\rangle\, dt\right] = 0. \tag{32}$$

Fix $t \in [0,T]$ and apply the above to $\theta = (0, \ldots, \theta_i, 0, \ldots, 0)$, where $\theta_i(s) = \alpha_i(\omega)\mathbf{1}_{[t,t+r]}(s), s \in [0,T],\ t + r \le T$ and $\alpha_i = \alpha_i(\omega)$ is bounded $\mathcal{E}_t$-measurable.

Then it follows from (32) that

$$\mathbb{E}\left[\int_t^{t+r} \frac{\partial}{\partial v_i} H\left(s, \tilde{y}(s), \tilde{z}(s), \tilde{u}(s), \tilde{p}(s), \tilde{q}(s)\right)\alpha_i\, ds\right] = 0.$$

Differentiating with respect to r at $r = 0$ gives

$$\mathbb{E}\left[\frac{\partial}{\partial v_i} H\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right)\alpha_i\right] = 0.$$

Since this holds for all bounded $\mathcal{E}_t$-measurable α_i, we conclude that

$$\mathbb{E}\left[\frac{\partial}{\partial v_i} H\left(t, \tilde{y}(t), \tilde{z}(t), \tilde{u}(t), \tilde{p}(t), \tilde{q}(t)\right)|\mathcal{E}_t\right] = 0.$$

■

V. Applications to Backward Doubly Stochastic LQ Problems

In this section we work out an example of partial information backward doubly stochastic linear-quadratic (LQ) optimal control problem to illustrate the application of the theoretical results. For notational simplification, we assume $d = l = 1$ and $U = \mathbb{R}$.

Examples 5.1: Consider the following:

$$\begin{cases} -dy(t) = [A(t)y(t) + C(t)z(t) + D(t)v(t)]\, dt \\ \qquad\qquad - z(t)\overleftarrow{d}W(t) + F(t)\overleftarrow{d}B(t), \\ y(T) = \xi. \end{cases}$$

The cost functional is

$$J\left(v(\cdot)\right) = \frac{1}{2}\mathbb{E}\left[\int_0^T \left(M(t)v^2(t) + N(t)y^2(t)\right)dt + L_0\left(y(0)\right)^2\right]$$

where constant $L_0 \ge 0$. Functions $A(\cdot), C(\cdot), D(\cdot), F(\cdot), M(\cdot), N(\cdot)$ are bounded and deterministic, $M^{-1}(\cdot)$ is also bounded. $W(\cdot), B(\cdot)$ are two mutually independent standard Brownian motions on $(\Omega, \mathcal{F}, P)$. We consider the following partial information 1-dimensional control problem:

$$J\left(u(\cdot)\right) = \inf_{v(\cdot)\in\mathcal{U}_{ad}} J\left(v(\cdot)\right).$$

By (6), the Hamiltonian function is given by

$$\begin{aligned} H(t, y, z, v, p, q) = &-p(t)\left[A(t)y + C(t)z + D(t)v\right] \\ &- q(t)F(t) + \frac{1}{2}\left(M(t)v^2(t) + N(t)y^2(t)\right) \end{aligned}$$

where $(p(\cdot), q(\cdot))$ is the solution of the following forward doubly SDE:

$$\begin{cases} dp(t) = [A(t)p(t) + N(t)y(t)]\, dt \\ \qquad\quad + C(t)p(t)\overrightarrow{d}W(t) - q(t)\overleftarrow{d}B(t), \\ p(0) = -L_0 y(0). \end{cases}$$

According to Theorem 4.1, if $u(\cdot)$ is optimal, then

$$u(t) = M^{-1}(t)D(t)\mathbb{E}\left[p(t)|\mathcal{F}_t^W\right].$$

In the rest of this part, we try to get a more explicit and observable representation of $u(\cdot)$. For this, we set

$$\hat{x}(t) = \mathbb{E}\left[x(t)|\mathcal{F}_t^W\right], \quad x = y, z, p.$$

Similar to Lemma 5.4 of [17], the optimal filter $\hat{p}(\cdot)$ of $p(\cdot)$ satisfies

$$\begin{cases} d\hat{p}(t) = [A(t)\hat{p}(t) + N(t)\hat{y}(t)]\, dt + C(t)\hat{p}(t)\overrightarrow{d}W(t), \\ \hat{p}(0) = -L_0\hat{y}(0) \end{cases} \tag{33}$$

where $\hat{y}(\cdot)$ is the solution of

$$\begin{cases} -d\hat{y}(t) = [A(t)\hat{y}(t) + C(t)\hat{z}(t) \\ \qquad\qquad + D^2(t)M^{-1}(t)\hat{p}(t)]\, dt - \hat{z}(t)\overrightarrow{d}W(t), \\ \hat{y}(T) = \xi. \end{cases} \tag{34}$$

The filtering (33) explicitly depends on the backward variable $\hat{y}(\cdot)$, thus (33) and (34) formulate a coupled forward-backward stochastic differential equations (FBSDE) which admits a unique solution. This class of filtering equations are originally found by Huang *et al.* [5] when they studied the partial information control problems of backward stochastic systems. Similar to [5], we put

$$\hat{p}(t) = \varphi(t)\hat{y}(t), \quad \varphi(0) = L_0 \tag{35}$$

where $\varphi(t)$ is a deterministic function defined later on. Apply Itô's formula to $\hat{p}(\cdot)$,

$$d\hat{p}(t) = \Big[\varphi'(t)\hat{y}(t) + \varphi(t)\left(-A(t)\hat{y}(t) - C(t)\hat{z}(t) - D^2(t)M^{-1}(t)\hat{p}(t)\right)\Big]dt + \varphi(t)\hat{z}(t)\overrightarrow{d}W(t). \quad (36)$$

Comparing the drift and diffusion terms of (33) and (36), we have

$$\begin{cases} \varphi'(t)\hat{y}(t) + \varphi(t)(-A(t)\hat{y}(t) - C(t)\hat{z}(t) \\ -D^2(t)M^{-1}(t), \hat{p}(t)) = A(t)\hat{p}(t) + N(t)\hat{y}(t), \\ \varphi(t)\hat{z}(t) = C(t)\hat{p}(t). \end{cases}$$

Then it follows that:

$$\begin{cases} \varphi'(t) - 2A(t)\varphi(t) - C^2(t)\varphi(t) \\ -D^2(t)M^{-1}(t)\varphi^2(t) - N(t) = 0, \\ \varphi(0) = L_0. \end{cases} \quad (37)$$

Proposition 5.2: If all the hypotheses hold, then the optimal control $u(\cdot)$ can be rewritten as

$$u(t) = M^{-1}(t)D(t)\varphi(t)\hat{y}(t) \quad (38)$$

where $\hat{y}(\cdot)$ and $\varphi(\cdot)$ are given by (34) and (37).

VI. Conclusion

In this technical note, we develop the sufficient condition and necessary condition for optimal control problems of partial information BDSDEs. Compared with the existing literature, the contributions of this technical note are: 1) The control systems are formulated as BDSDEs, and the optimal control problems of backward doubly stochastic systems are under partial information. 2) Both the sufficient condition (a verification theorem) and the necessary condition for optimality are proved. 3) The control domain is assumed to be convex in this technical note. When the control domain is not convex, we can also obtain the maximum principle by means of the spike variational technique. We will address this issue in our future publications. 4) The results partly generalize the existing partial information ones (see e.g., [2], [4], and [5]). In addition, since there are many partial information optimization problems in finance and economics, we believe that the results have applications in these areas.

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