# Fuzzy Chance-Constrained Multiobjective Portfolio Selection Model

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***Abstract*—This paper addresses the problem of portfolio selection with fuzzy parameters from a perspective of chance-constrained multiobjective programming. The key financial criteria used here are conventional, namely, return, risk, and liquidity; however, we use short- and long-term variants of return rather than a single measure of an investor's expectations in respect thereof. The proposed model aims to achieve the maximal return (short term as well as long term) and liquidity of the portfolio. It does so at a credibility, which is no less than the confidence levels defined by the investor. Further, to capture uncertain behavior of the financial markets more realistically, fuzzy parameters used here are such as those characterized by general functional forms. To solve the problem, we rely on a specially developed algorithm that hybridizes fuzzy simulation and real-coded genetic algorithm. Numerical experiments are included to showcase the applicability and efficiency of the model in a real investment environment.**

***Index Terms*—Chance-constrained programming, credibility measure, fuzzy portfolio selection, multiobjective optimization, real-coded genetic algorithm (RCGA).**

## I. Introduction

The portfolio selection problem deals with how to obtain a portfolio that meets investor preferences regarding return and risk. Markowitz [1] provided the first mathematical formulation to the portfolio selection problem in 1952. In Markowitz's mean-variance formulation [1], portfolio return is measured by the expected value of the weighted mean of the returns of the assets, and risk is measured by the variance of the portfolio return. In most of the existing portfolio selection models, an investor's choices are mapped with respect only to return and risk; however, in practice, investor preferences transcend these twin criteria of portfolio performance. For example, the investor may like to distinguish between short- and long-term returns; otherwise, he/she may be more concerned about liquidity of the portfolio. The multiple criteria in investor's mind may or may not be equally weighted. Such an understanding of investor behavior is immensely useful in modeling the portfolio selection problem because a deficit on account of some criteria may be allowed to be more than compensated by portfolio performance on the other criteria, resulting in greater overall satisfaction for investors. Thus, the real-world investment modeling is largely based on multicriteria decision making. Accordingly, several researchers have proposed multicriteria portfolio selection models [2]–[13].

Another aspect of conventional modeling of the portfolio selection problem is that it is based on the assumption of perfect information. Such a situation is more of a theoretical ideal rather than descriptive of reality. Real-world investment environment is characterized by incomplete information; thus, decisions are made under uncertainty. In fact, information often is not only incomplete but is masked by vague and ambiguous expressions such as "high risk," "low profit," and "low liquidity as well." In view of this vagueness and ambiguity, researchers have taken recourse to fuzzy set theory [14], [15] while modeling the portfolio selection problem in order to capture investor preferences that are subjective in nature.

Considering that asset returns are more realizable as fuzzy variables rather than random variables, many fuzzy portfolio selection models have been developed using possibility measure. Carlsson *et al.* [16] developed a fuzzy mean-variance model. Inuiguchi and Ramik [17] proposed two portfolio selection models, namely, a necessity maximization model and a modality constrained programming model. Inuiguchi and Tanino [18] proposed a minimax regret model for portfolio selection. Tanaka and Guo [19] developed a center-spread model assuming that asset returns are exponential fuzzy variables. Based on interval-valued possibilistic mean and variance, Zhang *et al.* [20], [21] presented an extended mean-variance model. A survey of research on possibilistic portfolio selection is provided in the work of Wang and Zhu [22]. The recent work in the field has been reported in [3], [13], and [23]–[31].

It is worthy to point out that although the possibility measure is widely used in the literature, but it has certain limitations. One limitation is that a fuzzy event that is assigned the maximum possibility value of 1 may still fail to occur. Second, two fuzzy events with different chances of occurrence may have the same possibility value. In view of limitations such as these, possibility values of the asset characteristics may be of little information to the investor. The primary reason for these limitations is that the possibility measure is not self-dual. To overcome this, Liu and Liu [32] proposed a credibility measure that is self-dual. Based on their work, several researchers have developed fuzzy portfolio selection models using credibility measure. Huang [33] proposed a fuzzy mean-variance model within the framework of credibility theory, which was further extended to mean-semivariance model [34] and mean-variance-skewness model [35]. Huang [36] proposed a new definition of risk for

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portfolio selection in fuzzy environment and used it to develop a portfolio selection model. Huang [37] also constructed mean-entropy models for fuzzy portfolio selection using entropy as a measure of risk. Additionally, Huang [38] used proportion entropy in credibilistic mean-variance and mean-semivariance diversification models for fuzzy portfolio selection. Li *et al.* [39] proposed an expected regret minimization model that minimizes the expected value of the distance between the maximum return and the obtained return associated with each portfolio within the framework of credibility theory. Wang *et al.* [40] proposed a portfolio selection model based on fuzzy value-at-risk. Recently, Gupta *et al.* [41] proposed a multiobjective credibilistic model with fuzzy chance constraints of the portfolio selection problem. A survey on credibilistic portfolio selection based primarily on return and risk is provided in the works of Huang [42] and Li *et al.* [43]. Portfolio selection models incorporating liquidity, which is an important parameter apart from return and risk, to measure asset performance are discussed in [4], [6]–[8], and [41].

Another way of developing credibilistic models is to rely on chance-constrained programming. It may be noted that Charnes and Cooper [44] introduced the chance-constrained programming in 1959 for handling stochastic variables. Chance-constrained programming requires that the objectives should be achieved with stochastic constraints held at least $\rho$ time, where $\rho$ is an appropriate safety margin provided by the investor. The examples of portfolio selection modeling with stochastic parameters can be found in [45]–[47]. Liu [48] developed several general forms of fuzzy chance-constrained programming that could also be used to develop credibilistic measures for portfolio selection.

In 2006, Huang [49] proposed the idea of credibility-based chance-constrained portfolio selection. The idea of Huang [49] is that the investor can preset a confidence level expressed by a credibility value and then try to achieve the maximal investment return. One of the advantages of such a portfolio selection idea is that it is usually easy for an investor to preset a confidence level he/she wishes to accept, and it is easy for an investor to adjust his/her confidence levels in order to pursue higher investment return or to avoid a potentially big loss. Another advantage of Huang's portfolio selection idea is that the model can provide the information on the achievement value of the objective function at a desired confidence level. In Huang's approach, the model can provide the investor with the recognizable maximum investment return. Despite these advantages, the chance-constrained portfolio selection models proposed by Huang [49] have some limitations in terms of applicability in real-world portfolio selection. The models presented in [49] considered only the basic objective of return, i.e., the models do not consider traditional return-risk tradeoff of portfolio selection. Further, these models considered capital budget as the only constraint to obtain optimal portfolios. In real-world portfolio selection, the investor may also have other objectives for portfolio selection. For example, the investor may have different aims toward short-term and long-term returns. In addition, liquidity is another very important factor that needs consideration while making investment decision. Based on the above discussion and to the best of our knowledge, there is a perceptible gap in research on fuzzy chance-constrained portfolio selection. Thus, the present research is motivated by the desire to fill this gap and realize the benefits of the chance-constrained programming technique for real-world multicriteria portfolio selection. Our paper extends the work of Huang [49] in significant and substantive ways. We extend Huang's model [49] by considering multiple objectives and multiple constraints. Further, the proposed research also overcomes the limitations of the model proposed by Gupta *et al.* [41] that generates portfolios which are optimal to the extent of achieving the highest credibility values for the objective functions and, thus, cannot provide the information on the achievement values of the objective functions.

We propose a new credibility-based fuzzy chance-constrained multiobjective portfolio selection model that helps the investor to attain the maximal short-term return, long-term return, and liquidity of the portfolio at credibility not less than the given confidence levels. Rather than a fixed preset severity level of the loss, we profile portfolio risk with the help of a risk curve that represents all likely losses of the portfolio return and corresponding chances of their occurrences. The risk curve needs to be contained below the confidence curve of the investor to build safe portfolios. The portfolio risk is represented in the model by using credibility-based fuzzy chance constraints. Instead of a single measure of portfolio return that does not discriminate investor preferences for short- versus long-term returns, we use two measures. One, short-term return, i.e., the average performance of the asset during a 12-month period, and long-term return, i.e., the average performance of the asset during a 36-month period. Let the fuzzy parameters be characterized by general functional forms with a view to effectively capturing the uncertainty of behavior of the financial markets. However, as we make our parameters amenable to general functional forms, we confront computational difficulty in the conversion of the fuzzy portfolio selection model into its crisp equivalent. To account for this, we rely on a specially developed algorithm that hybridizes fuzzy simulation and real-coded genetic algorithm (RCGA) to solve the model. Numerical experiments conducted using the proposed model adequately show the flexibility in generating safe portfolios corresponding to values of the model parameters, namely, weights of the objectives and predefined confidence levels. In addition, the obtained portfolios are consistent with the investor preferences.

Our proposals for presenting a new portfolio selection model in fuzzy environment may be summarized as follows: 1) Compared with the fuzzy chance-constrained single-objective portfolio selection model presented in [49], we propose a fuzzy chance-constrained multiobjective portfolio selection model. The multiobjective optimization approach results in portfolio constructions having better tradeoffs among the objectives considered; 2) the proposed fuzzy chance-constrained multiobjective portfolio selection model apart from return also considers other important measures of asset characteristics, namely, risk and liquidity. It is worth mentioning here that the models presented in [49] do not consider traditional return-risk tradeoff of portfolio selection; 3) unlike the models presented in [49], which are constrained by only the capital budget constraint, the proposed portfolio selection model is constrained by several other realistic constraints, namely, a cardinality constraint and diversification constraints that apply lower and upper bounds on investments in individual assets; 4) unlike the model presented in [41] that generates portfolios which are optimal to the extent of achieving highest credibility values for the objective functions, the proposed portfolio selection model can provide the information on the achievement values of the objective functions at a desired confidence level; 5) the proposed model can generate portfolios which are consistent with the investor preferences captured by using importance weights of the objectives and predefined confidence levels; and 6) the model can generate safe portfolios by containing risk curve of the portfolio below the corresponding confidence curve of the investor representing his/her tolerance toward portfolio risk.

The organization of remaining of this paper is as follows. Some relevant basic definitions and notation necessary for better understanding of the paper are provided in Section II. Section III presents the credibility-based chance-constrained multiobjective portfolio selection model. This section also presents details of the specially developed algorithm that hybridizes fuzzy simulation and RCGA to solve the model. In Section IV, to test-run the proposed model, we used 36-month data series corresponding to 20 different assets listed on the National Stock Exchange (NSE), Mumbai, India. A detailed discussion of the results obtained is also provided in this section. Further, we provide a comparison of the proposed model with some selected existing portfolio selection models. Finally, we conclude this paper in Section V.

## II. Preliminaries

### A. Credibility Fundamentals

Generally speaking, credibility theory [32], [50] is defined as a branch of mathematics which relies on credibility measure to study the behavior of fuzzy events. Credibility measure is defined from five axioms on which the credibility theory is based on. According to Liu [50], the following five axioms should hold to ensure that the number $Cr\{A\}$ (which represents the credibility that event $A$ will occur) has certain mathematical properties.

1) Credibility measure of the nonempty set is always 1, i.e., $Cr\{\Theta\} = 1$.
2) Credibility measure is nondecreasing, i.e., whenever $A \subset B$, we have $Cr\{A\} \leq Cr\{B\}$.
3) Credibility measure is self-dual, i.e., for any event $A \in P(\Theta)$, we have $Cr\{A\} + Cr\{A^c\} = 1$.
4) $Cr\{\cup_i A_i\} \wedge 0.5 = \sup_i Cr\{A_i\}$ for any $A_i$ with $Cr\{A_i\} \leq 0.5$.
5) Let $\Theta_1, \Theta_2, \ldots, \Theta_n$ be the nonempty sets corresponding to which $Cr_1, Cr_2, \ldots, Cr_n$ satisfy the above defined four axioms, respectively, and let $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$. Then, we have $Cr\{(\theta_1, \theta_2, \ldots, \theta_n)\} = Cr_1\{\theta_1\} \wedge Cr_2\{\theta_2\} \wedge \cdots \wedge Cr_n\{\theta_n\}$ for each $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$, where '$\wedge$' represents minimal operator.

Here, $\Theta$ is a nonempty set, $P(\Theta)$ is the power set of $\Theta$, i.e., collection of all subsets of $\Theta$, and each element in $P(\Theta)$ is called an event. It may be noted that the credibility measure of the empty set is always 0, i.e., $Cr\{\emptyset\} = 0$, and it takes values between 0 and 1. Traditionally, a fuzzy variable is characterized using a membership function [14]. Like a random variable that is defined as a measurable function on a probability space, Liu [50] defined fuzzy variable as a function on a credibility space. Following are the some basic definitions as introduced in [50].

*Definition 2.1:* A function from a credibility space $(\Theta, P(\Theta), Cr)$ to a set of real numbers is called a fuzzy variable.

Based on the above definition, an $n$-dimensional fuzzy vector is a function from a credibility space to the set of $n$-dimensional real vectors.

*Definition 2.2:* Let $f : \Re^n \to \Re$ be a function, and let $\xi_1, \xi_2, \ldots, \xi_n$ be the fuzzy variables defined on the credibility space $(\Theta, P(\Theta), Cr)$; then, $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is a fuzzy variable which is defined as $\xi(\theta) = f(\xi_1(\theta), \xi_2(\theta), \ldots, \xi_n(\theta))$ for any $\theta \in \Theta$.

It must be noted that it is possible to define fuzzy variables on different credibility spaces. In such situations, $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is a fuzzy variable which is defined on the product credibility space $(\Theta, P(\Theta), Cr)$ as $\xi(\theta_1, \theta_2, \ldots, \theta_n) = f(\xi_1(\theta_1), \xi_2(\theta_2), \ldots, \xi_n(\theta_n))$ for any $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$.

*Definition 2.3:* Let $\xi$ be a fuzzy variable having the membership function $\mu$, and let $u, r$ be real numbers; then, credibility of a fuzzy event, which is characterized by $\xi \geq r$, is expressed as

$$Cr\{\xi \geq r\} = \frac{1}{2}\left(\sup_{u \geq r} \mu(u) + 1 - \sup_{u < r} \mu(u)\right).$$

As is known in the literature, the credibility measure may be defined as the average of possibility measure and necessity measure and is expressed as

$$Cr\{\xi \geq r\} = \frac{1}{2}(Pos\{\xi \geq r\} + Nes\{\xi \geq r\}),$$

where $Pos\{\xi \geq r\} = (\sup_{u \geq r} \mu(u))$, and $Nes\{\xi \geq r\} = 1 - \sup_{u<r} \mu(u)$.

*Definition 2.4:* Let $\xi_1, \xi_2, \ldots, \xi_n$ be fuzzy variables having membership functions $\mu_1, \mu_2, \ldots, \mu_n$, and let $u_1, u_2, \ldots, u_n$ be real numbers. Let $f : \Re^n \to \Re$; then, the credibility of the fuzzy event characterized by $f(\xi_1, \xi_2, \ldots, \xi_n) \geq 0$ is expressed as

$$\begin{aligned} &Cr\{f(\xi_1, \xi_2, \ldots, \xi_n) \geq 0\} \\ &= \frac{1}{2}\left(\sup_{u_1, u_2, \ldots, u_n \in R}\left\{\min_{1 \leq i \leq n} \mu_{\xi_i}(u_i) | f(u_1, u_2, \ldots, u_n) \geq 0\right\}\right. \\ &\quad \left. + 1 - \sup_{u_1, u_2, \ldots, u_n \in R}\left\{\min_{1 \leq i \leq n} \mu_{\xi_i}(u_i) | f(u_1, u_2, \ldots, u_n) < 0\right\}\right). \end{aligned}$$

### B. Risk Curve and Confidence Curve

Risk curve is the loci of all the points of probable losses of return. Such a conceptualization of risk is a definite improvement over the conventional conceptualization where risk is stated in terms of one fixed preset level of the loss. It may be noted that investors are realistically concerned about each likely loss and its corresponding chance of occurrence. Thus, the portfolio risk, instead of being set at a specific threshold level, is better represented using a curve rather than a single point. In what

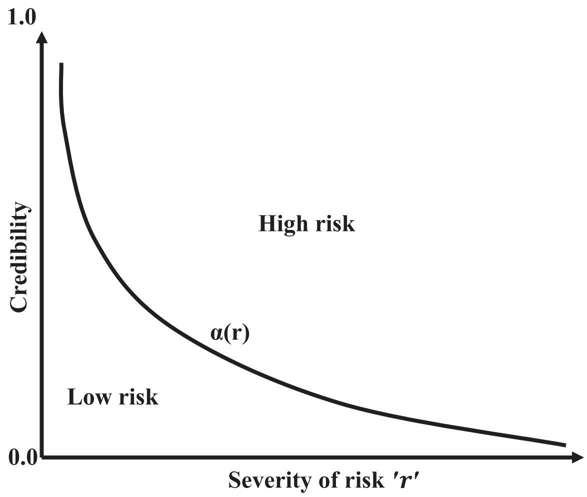


Fig. 1. Confidence curve and investment risk.

follows next, we present the definitions of the risk curve and the confidence curve as introduced in [36].

*Definition 2.5:* Let $\xi$ represents fuzzy return of the portfolio, and $b$ represents the target return; then, the risk curve is expressed as

$$f(r) = Cr\{b - \xi \geq r\}, \quad \forall\, r \geq 0$$

where $(b - \xi)$ represents potential loss, and $r$ represents severity level of the loss for portfolio returns. The greater the indicator $r$, the severer is the portfolio loss, i.e., $(b - \xi)$. The risk curve, i.e., $f(r)$, represents all likely losses of the portfolio and corresponding credibility levels of their occurrences. It is worthy to point out that if the investor is only concerned about one fixed preset level, i.e., $(b - \xi) \geq r_0$, where $r_0$ is the fixed preset severity level of the loss, then the credibility of its occurrence is expressed as $Cr\{(b - \xi) \geq r_0\}$.

For any given severity loss indicator $r$, investors can express their maximal tolerance toward chances of occurrence of all potential losses being greater than or equal to $r$, using a curve called the confidence curve.

*Definition 2.6:* The confidence curve $(\alpha(r))$ is the curve which provides the investor's tolerable credibility levels of all likely losses corresponding to any given $r$.

It must be noted that realistically at lower severity level, i.e., $r$, the investor can tolerate a comparatively high occurrence credibility of the loss. In contrast, when $r$ is high, then the investor may tolerate only a low occurrence credibility of the loss. In view of such investor preferences, the area below the confidence curve, i.e., $\alpha(r)$, is regarded as the low-risk area, whereas the area above the confidence curve, i.e., $\alpha(r)$, is regarded as the high-risk area (see Fig. 1) [36]. It is plausible that different investors may have different confidence curves. However, it is more likely that a typical confidence curve corresponds to investor behavior that the severer the loss, the lower the tolerance of occurrence chance of the loss.

Further, let $\xi$ represents the fuzzy return of the portfolio, and $\alpha(r)$ represents the investor's confidence curve; then, the portfolio is said to be safe if the risk curve of the portfolio is completely below the investor's confidence curve, i.e.,

$$Cr\{(b - \xi) \geq r\} \leq \alpha(r), \quad \forall\, r \geq 0.$$

It may be noted that the above expression represents a fuzzy chance constraint for every $(r, \alpha(r))$ combination.

## III. Research Methodology

### A. Portfolio Selection Model

Here, we formulate the problem of portfolio selection with fuzzy parameters from a perspective of chance-constrained multiobjective programming under the assumption that investors allocate their wealth among $n$ different assets offering fuzzy returns. The parameters and variables used to formulate the mathematical model are described as follows:

| | |
|---|---|
| $\lambda_i$ | Fuzzy short-term return of the $i$th asset. |
| $\gamma_i$ | Fuzzy long-term return of the $i$th asset. |
| $\beta_i$ | Fuzzy turnover rate of the $i$th asset. |
| $r_1$ | Maximal target of the expected short-term return of the portfolio at a credibility, which is no less than the given confidence level $\delta_1$. |
| $r_2$ | Maximal target of the expected long-term return of the portfolio at a credibility, which is no less than the given confidence level $\delta_2$. |
| $r_3$ | Maximal target of the expected liquidity of the portfolio at a credibility, which is no less than the given confidence level $\delta_3$. |
| $b$ | Target return of the portfolio. |
| $r$ | Severity level of the loss. |
| $\alpha(r)$ | Investor's confidence curve corresponding to a given $r$. |
| $u_i$ | Maximal proportion of the capital budget that can be allocated to the $i$th asset. |
| $l_i$ | Minimal proportion of the capital budget that can be allocated to the $i$th asset. |
| $h$ | Number of assets that investor wishes to held in the portfolio. |
| $x_i$ | Proportion of the capital budget that is invested in the $i$th asset. |
| $y_i$ | Binary variable representing whether the $i$th asset is selected in the portfolio or not, i.e., |

$$y_i = \begin{cases} 1, & \text{if } i\text{th asset is selected in the portfolio} \\ 0, & \text{otherwise.} \end{cases}$$

*1) Objective Functions:*

a) *Short-term return:* The maximal short-term return of the portfolio that the investor can obtain at a credibility, which is no less than the given confidence level $\delta_1$, is expressed as

$$\text{Max}\{r_1 | Cr\{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \geq r_1\} \geq \delta_1\}.$$

b) *Long-term return:* The maximal long-term return of the portfolio that the investor can obtain at a credibility, which is no less than the given confidence level $\delta_2$, is expressed as

$$\text{Max}\{r_2 | Cr\{\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n \geq r_2\} \geq \delta_2\}.$$

c) *Liquidity:* The frequency of trading of an asset is a measure of its liquidity. This frequency is referred to as "turnover" and measure with the help of a ratio between the average stock traded at the market and the tradable stock (shares held by public) of that asset. Because of incomplete information about trading volumes and tradable stock, the turnover rates are, at best, vague estimates. In order to handle uncertainty, liquidity has been considered in fuzzy form [4], [6]–[8]. The maximal liquidity of the portfolio that the investor can obtain at a credibility, which is no less than the given confidence level $\delta_3$, is expressed as

$$\text{Max}\{r_3 | Cr\{\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n \geq r_3\} \geq \delta_3\}.$$

*2) Model Constraints:*

a) *Risk:* To capture the portfolio risk, we use the risk curve of the portfolio versus investor's confidence curve based on simulated long-term return of the portfolio and is expressed as

$$Cr\{b - (\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n) \geq r\} \leq \alpha(r) \quad \forall\, r \geq 0.$$

b) *Capital budget constraint on the proportions of the assets:*

$$\sum_{i=1}^{n} x_i = 1.$$

c) *Maximal proportion of the capital budget that may be allocated to an asset:*

$$x_i \leq u_i y_i, \quad i = 1, 2, \ldots, n.$$

d) *Minimal proportion of the capital budget that may be allocated to an asset:*

$$x_i \geq l_i y_i, \quad i = 1, 2, \ldots, n.$$

Allocation of the investor's budget among various assets is an important consideration in portfolio diversification. The decision regarding the maximal and the minimal proportions of the capital budget that an investor may allocate to an asset depends on a number of factors, namely, price/value relative of the asset, lot size, historical data of trading volume, etc. Since investors differ in their interpretation of the available information, they may allocate the same overall capital budget differently. Generally, the proportions must be realistic, that is neither too large so as to defy diversification nor too small to allow meaningful investment. The constraints corresponding to lower bounds $l_i$ and upper bounds $u_i$ on the investment in individual assets $(0 \leq l_i, u_i \leq 1, l_i \leq u_i, \forall i)$ are included for the purpose.

e) *Number of assets held in the portfolio (Cardinality constraint):*

$$\sum_{i=1}^{n} y_i = h.$$

f) *No short selling of assets:*

$$x_i \geq 0, \quad i = 1, 2, \ldots, n.$$

*3) Decision Problem:* The credibility-based fuzzy chance-constrained multiobjective portfolio selection model is formulated as follows:

$$\text{(P1)} \quad \text{Max } \{r_1, r_2, r_3\}$$

subject to

$$Cr\{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \geq r_1\} \geq \delta_1 \tag{1}$$

$$Cr\{\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n \geq r_2\} \geq \delta_2 \tag{2}$$

$$Cr\{\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n \geq r_3\} \geq \delta_3 \tag{3}$$

$$Cr\{b - (\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n) \geq r\} \leq \alpha(r) \quad \forall r \geq 0 \tag{4}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{5}$$

$$\sum_{i=1}^{n} y_i = h \tag{6}$$

$$x_i \leq u_i y_i, \quad i = 1, 2, \ldots, n \tag{7}$$

$$x_i \geq l_i y_i, \quad i = 1, 2, \ldots, n \tag{8}$$

$$x_i \geq 0, \quad i = 1, 2, \ldots, n \tag{9}$$

$$y_i \in \{0, 1\}, \quad i = 1, 2, \ldots, n. \tag{10}$$

It may be noted that while solving the model $(P1)$, the risk constraint (4) is replaced by fuzzy chance constraints corresponding to randomly generated $(r, \alpha(r))$ combinations. Further, the model $(P1)$ is equivalent to the following maximax form:

$$\underset{x_1, x_2, \ldots, x_n}{\text{Max}} \left[ \underset{r_1}{\text{Max}}\, r_1, \underset{r_2}{\text{Max}}\, r_2, \underset{r_3}{\text{Max}}\, r_3 \right]$$

subject to

Constraints (1)–(10).

## B. Hybrid Intelligent Algorithm

The credibility-based fuzzy chance-constrained multiobjective model $(P1)$ of the portfolio selection problem involves fuzzy parameters. When the membership functions of fuzzy parameters takes general form other than linear, triangular, or trapezoidal, the conversion of fuzzy problem into its crisp equivalent is computationally difficult. Therefore, a hybrid intelligent algorithm (HIA) that hybridizes the fuzzy simulation and an RCGA is proposed as a solution method for such problems involving general cases of fuzzy functional forms.

*1) Fuzzy Simulation:* The fuzzy simulation technique is discussed in detail by Liu [48]. In the proposed HIA, we used fuzzy simulation to compute the maximal short-term return $(r_1)$, maximal long-term return $(r_2)$, and maximal liquidity $(r_3)$ values of the uncertain objective functions that may be obtained at a credibility, which is no less than the given confidence levels $\delta_1, \delta_2$, and $\delta_3$, respectively. It is also used to handle the uncertain system constraints.

Assuming that $x = (x_1, x_2, \ldots, x_n)$ is a decision vector and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ is a fuzzy vector, let $F(x, \xi)$ represent a fuzzy quantity. Moreover, let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ represent the membership function vector of $\xi$. In order to solve the decision model $(P1)$, we require to treat the following two types of uncertain functions.

1) *The uncertain objective functions of the type:*

$$U_1 : x \to \text{Max}\,\{f \mid Cr\{F(x, \xi) \geq f\} \geq \sigma\}$$

where $0 \leq \sigma \leq 1$ is the predefined confidence level given by the investor, and $f$ is the $\sigma$-level value of the uncertain function that investor wishes to maximize. The following algorithm is used in this paper for fuzzy simulation process to compute $U_1$:

*Step 1:* Let $j = 1$.

*Step 2:* Generate randomly $u_{ij}$ in the $j$th simulation run using the $\varepsilon$-level sets of fuzzy variables $\xi_i$ in such a manner that $\mu_{ij}(u_{ij}) \geq \varepsilon, i = 1, 2, \ldots, n, j = 1, 2, \ldots, N$, where $\varepsilon$ and $N$ are sufficiently small positive number and large number, respectively.

*Step 3:* Set $u_j = (u_{1j}, u_{2j}, \ldots, u_{nj})$, and $\mu(u_j) = \mu_{1j}(u_{1j}) \wedge \mu_{2j}(u_{2j}) \wedge \cdots \wedge \mu_{nj}(u_{nj})$.

*Step 4:* $j \leftarrow j + 1$. If $j \leq N$, then go to Step 2; otherwise, go to Step 5.

*Step 5:* For any real number $f$, we define

$$L(f) = \frac{1}{2}\left(\max_{1 \leq j \leq N}\{\mu(u_j) | F(x, u_j) \geq f\} + \min_{1 \leq j \leq N}\{1 - \mu(u_j) | F(x, u_j) < f\}\right).$$

*Step 6:* Determine the maximal value $f$ in such a manner that $L(f) \geq \sigma$ holds.

*Step 7:* Return $f$.

2) *The uncertain system constraints of the type:*

$$U_2 : x \to Cr\{G(x, \xi) \geq r\}$$

where $G(x, \xi) = b - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n)$ and $r \geq 0$ is provided by the investor. The following algorithm is used in this paper for fuzzy simulation process to compute $U_2$:

*Step 1:* Let $j = 1$.

*Step 2:* Generate randomly $u_{ij}$ in the $j$th simulation run using the $\varepsilon$-level sets of fuzzy variables $\xi_i$ in such a manner that $\mu_{ij}(u_{ij}) \geq \varepsilon, i = 1, 2, \ldots, n, j = 1, 2, \ldots, N$, where $\varepsilon$ and $N$ are sufficiently small positive number and large number, respectively.

*Step 3:* Set $u_j = (u_{1j}, u_{2j}, \ldots, u_{nj})$, and $\mu(u_j) = \mu_{1j}(u_{1j}) \wedge \mu_{2j}(u_{2j}) \wedge \cdots \wedge \mu_{nj}(u_{nj})$.

*Step 4:* $j \leftarrow j + 1$. If $j \leq N$, then go to Step 2; otherwise, go to Step 5.

*Step 5:* Return the credibility $Cr\{G(x, \xi) \geq r\}$ as follows:

$$Cr\{G(x, \xi) \geq r\} = \frac{1}{2}\left(\max_{1 \leq j \leq N}\{\mu(u_j) | G(x, u_j) \geq r\} + \min_{1 \leq j \leq N}\{1 - \mu(u_j) | G(x, u_j) < r\}\right).$$

*2) Real-Coded Genetic Algorithm:* In 1975, Holland proposed genetic algorithm (GA) [51]. Since then, it has been widely used to solve problems in different application areas. In this paper, to solve the model $(P1)$, we use RCGA in which the encoding of chromosomes is based on real numbers. In what follows next, we present the details of RCGA used in this paper.

a) *Chromosome encoding:* Here, for the purpose of encoding, we set the length of the chromosome to $n$, i.e., the number of available assets. Further, we use the chromosome $Ch_k$ to represent the solution vector $x = (x_1, x_2, \ldots, x_n)$ which is encoded as an array, as follows:

$$Ch_k = X_k[i] = x_i, i = 1, 2, \ldots, n, k = 1, 2, \ldots, popsize$$

where *popsize* represents the number of chromosomes initialized in order to constitute population of one generation. In the proposed encoding method, the ID number of the asset is represented using the position of the gene $x_i, i = 1, 2, \ldots, n$, and the corresponding proportion of the capital budget that goes into the portfolio is represented using the gene value. For details of the initialization algorithm, see [41].

b) *Fitness evaluation:* The fitness evaluation function is constructed in such a manner that it must be taking care of all the objective functions and make rational tradeoffs among them. We first apply fuzzy simulation technique to the model $(P1)$ to calculate $r_1, r_2, r_3$ for given confidence levels $\delta_1, \delta_2, \delta_3$, respectively, as well as to compute $Cr\{b - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n) \geq r\}$. Thus, constraints (1)–(3) of model $(P1)$ are satisfied by chromosomes generated in the previous step. The portfolio risk constraint (4) is the only constraint of the model $(P1)$ which is not incorporated in the chromosome design. Note that the portfolio risk constraint (4) corresponds to $N'$ constraints, resulting from the randomly generated pairs $(r_j, \alpha(r_j)), j = 1, 2, \ldots, N'$. Therefore, the fitness function is designed in such a manner that these $N'$ constraints are incorporated into the RCGA process by levying a penalty $P$ for the infeasible chromosomes. If there is no violation of the constraints corresponding to portfolio risk, the penalty parameter $P$ will be zero and positive, otherwise. For $j = 1, 2, \ldots, N'$, let

$$f_j(x, r_j) = Cr\{b - (\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n) \geq r_j\} - \alpha(r_j). \quad (11)$$

The penalty levied on the infeasible chromosomes is taken as

$$P_j = \begin{cases} 10^2 * f_j(x, r_j), & \text{if } f_j(x, r_j) > 0 \\ & \text{for } j = 1, 2, \ldots, N' \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the net penalty is given as

$$P = \sum_{j=1}^{N'} P_j.$$

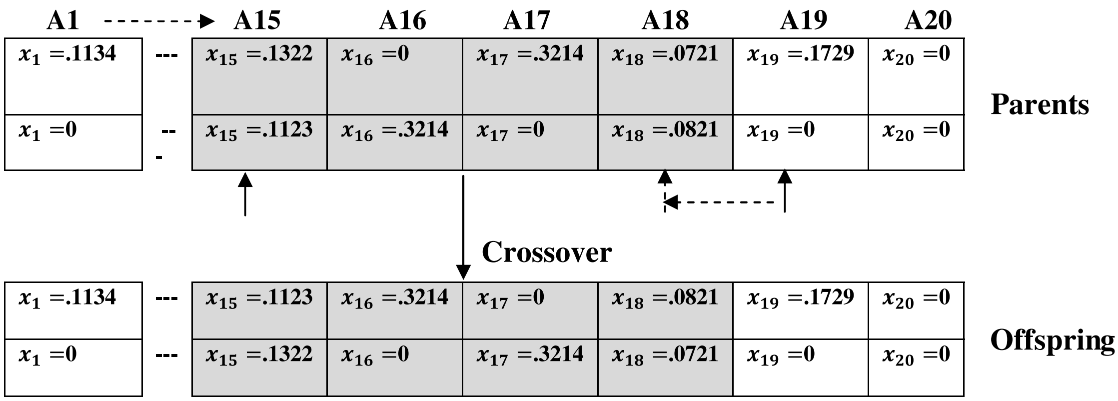


Fig. 2. Shrinking crossover operation.

For the model $(P1)$, which is a multiobjective programming problem, the weighted sum approach that combines multiple objective functions into a scalar fitness function is the simplest method. Note that the constant weight approach and the random weight approach are the commonly used approaches to assign weights to the multiple objective functions. We use the constant weight approach since it provides flexibility to incorporate investor preferences with respect to the multiple objective functions. Therefore, we design the fitness function, i.e., $fit_k$ corresponding to chromosome $Ch_k, k = 1, 2, \ldots, popsize$, as the weighted sum of the objective functions of model $(P1)$ with penalty parameter for treating infeasible chromosomes as follows:

$$fit_k = w_1 r_1 + w_2 r_2 + w_3 r_3 - P$$

where $w_j, j = 1, 2, 3$, represents the weight of the $j$th objective function. Note that values of $r_1, r_2, r_3$ used in the fitness evaluation function are approximated values obtained using the fuzzy simulation process. We now find the solution chromosome $Ch_k$ corresponding to the best found (maximum value) fitness of the function $fit_k$.

c) *Elitism:* Generally, elite-preserving operator is employed to preserve and use previously found best solution in subsequent generations. Note that the main advantage of using elitism is that the statistics of the population-best solutions may not degrade with subsequent generations in an elitist GA. The elite count $(t)$ specifies the number of individuals that are guaranteed to be carried forward to the next generation without performing selection, crossover, and mutation operations. Here, we use $t = 4$ to retain the four most fit individuals of the current population for the population comprising the next generation.

d) *Selection:* The purpose of selection operator is to select those individuals which, on average, are more fit than others to pass on their genes to the next generation. Here, we make use of four-player tournament selection as a selection mechanism. In this selection mechanism, first, four individuals are selected randomly, and then, the one with the highest fitness is selected from the four selected individuals for the parent population. As we already have four members of the next generation resulting from performing elitism, therefore, we need only $popsize - 4$ members of the next generation. For details on the four-player tournament selection algorithm to generate the remaining $popsize - 4$ chromosomes for the parent population, see [41].

e) *Crossover:* Using the crossover operation, the two selected parent chromosomes for mating pool reproduce two child chromosomes (offspring). The chance that the two selected chromosomes will crossover is represented using the crossover probability $p_c \in (0, 1)$. A random number between 0 and 1 is generated for each potential crossover. Note that the number of selected chromosomes must be an even number; therefore, if the number of selected chromosomes is odd, then the above procedure is repeated until one more chromosome is selected. We use shrinking crossover (SX) operator [52] because other standard crossover operators have a good chance of violating cardinality constraint (6) of the model $(P1)$. In SX, it is required to have equal number of selected assets between and including the two crossover points in both the selected parents; therefore, the second crossover point moves leftward until this condition gets satisfied. Once the condition holds, then the gene values of the parent chromosomes are exchanged to produce offspring. For details of the algorithm used for the SX operation, see [41].

Fig. 2 shows the SX operation for the selected parents between two positions generated randomly: $pos_1 = 15$ and $pos_2 = 19$. As these two positions do not have equal number of selected assets for both the parents, $pos_2$ is shifted toward left. Finally, two point crossover takes place between $pos_1 = 15$ and $pos_2 = 18$ when the number of selected assets matches for both the parents between and including the above two stated positions.

Note that we need to perform normalization after each crossover operation to ensure that constraint (5) holds, i.e.,

$$X_k[i] = \frac{x_i}{x_1 + x_2 + \cdots + x_n}, \quad k = 5, 6, \ldots, popsize.$$

Constraint (6) is taken care of by SX operation. Constraints (1)–(3) always hold because of fuzzy simulation process, and constraints (9) and (10) always hold because of the

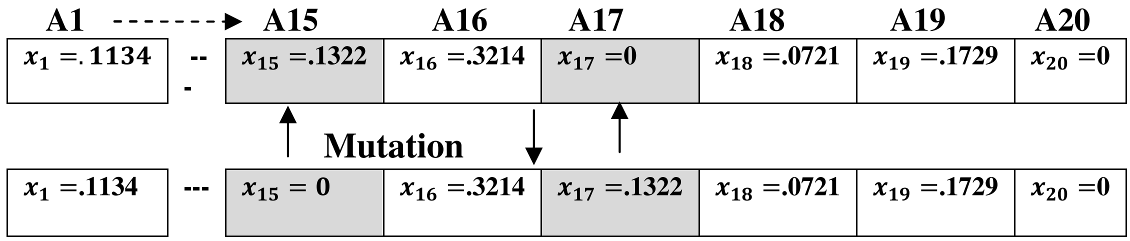


Fig. 3. Swap mutation operation.

design of the chromosomes. However, there are chances that the offspring so generated using the SX operation can still violate constraints (7) and/or (8) as an outcome of performing normalization. Therefore, such offspring must be rejected. Note that only the feasible offspring will substitute the corresponding parents to constitute the population for the next generation; otherwise, both the selected parents will be carried forward to the population for the next generation. All those parents who are not selected for crossover operation will be carried forward to the population for the next generation as is.

f) *Mutation:* The purpose of the mutation operator is to produce spontaneous random changes in chromosomes by varying one or more genes. Note that from the several available mutation operators, we use swap mutation in order to ensure that the capital budget constraint (5) and the cardinality constraint (6) of model $(P1)$ are not violated. With some probability of mutation $p_m \in (0, 1)$, a chromosome is selected for the process of mutation. For details of the algorithm of swap mutation operation, see [41].

Fig. 3. shows the process of swap mutation in a selected chromosome with $pos_1 = 15$ and $pos_2 = 17$. After swap mutation is performed, we obtain $x_{15} = 0$ and $x_{17} = 0.1322$.

Note that the feasibility of the chromosomes obtained after the crossover operation is not disturbed due to swap mutation operation.

*3) Stepwise Description of HIA:* The HIA algorithm is summarized as follows:

*Step 1:* Initialize chromosomes equal to the desired population size.

*Step 2:* Employ fuzzy simulation technique to compute the maximal short-term return $(r_1)$, maximal long-term return $(r_2)$, and maximal liquidity $(r_3)$ values of the uncertain objective functions that can be obtained at a credibility, which is no less than the given confidence levels $\delta_1, \delta_2$, and $\delta_3$, respectively, and the credibility values of uncertain system constraints of the model $(P1)$.

*Step 3:* For given values of $w_1, w_2, w_3$ provided by the investor, compute the fitness of each chromosome using $r_1, r_2, r_3$ and the credibility values of uncertain systems constraints obtained in Step 2, penalizing the infeasible ones corresponding to the violation of fuzzy chance constraints given by (11).

*Step 4:* Perform elitism.

*Step 5:* Select the chromosomes for parent population at any subsequent generation using four-player tournament selection.

*Step 6:* Use crossover and mutation operations to update the parent population.

*Step 7:* For a given number of generations, repeat Steps 2–6.

*Step 8:* Select the best chromosome from all the generations as the solution of the portfolio selection problem.

## IV. Numerical Experiments

To demonstrate the portfolio selection using model $(P1)$ and to test the effectiveness of the designed HIA on the fuzzy input data expressed using both special as well as general functional forms, we present some computational results. For the purpose, the HIA is coded in C++ on a personal computer with Intel Core2Duo CPU, having a speed of 2.8 GHz and a 4-GB RAM. The fuzzy data with respect to short-term return, long-term return, and liquidity for 20 different assets listed on NSE, Mumbai, India, are provided in Table I. We relied on historical data to create the fuzzy estimates. It may be noted that the input data with respect to assets $A1$ to $A17$ are expressed in fuzzy form using trapezoidal possibility distribution represented as $(a, b, c, d)$, where $b$ and $c$ represent center values and $a$ and $d$ represent left end point and right end point, respectively. The input data with respect to assets $A18$ to $A20$ are expressed using general functional forms.

We solve portfolio selection problems using the following confidence curve of the investor:

$$\alpha(r) = \frac{1}{\exp(Kr)} \quad \forall r \geq 0$$

where $K$ is the shape parameter that captures investor's confidence toward portfolio risk. In order to explore different confidence curves of the investor reflecting his tolerance toward portfolio risk, we use three different values of the shape parameter, i.e., $K = 1.3, 2, 3$. For $K = 1.3$, the investor is assumed to have high tolerance toward portfolio risk because the area below the confidence curve is more in comparison with $k = 2, 3$. For $K = 2$, the investor is assumed to have medium tolerance toward portfolio risk because the area below the confidence curve is more than $K = 3$ and less than $k = 1.3$. For $K = 3$, the investor is assumed to have low tolerance toward portfolio risk because the area below the confidence curve is less in comparison with $k = 1.3, 2$. Table II presents the primary attributes of the problems solved.

TABLE I
INPUT DATA

| Asset | Short-term return | Long-term return | Liquidity |
|---|---|---|---|
| A1 | (-0.86, -0.165, 0.465, 0.9) | (-0.38, -0.175, 0.425, 0.78) | (.0065, .0115, .0205, .032) |
| A2 | (-0.51, -0.23, 0.43, 0.75) | (-0.5, -0.203, 0.547, 0.75) | (.0003, .0009, .0027, .0033) |
| A3 | (-0.85, -0.18, 0.54, 0.64) | (-0.383, -0.003, 0.51, 0.65) | (.0003, .00125, .00515, .01145) |
| A4 | (-0.49, -0.125, 0.325, 0.78) | (-0.455, -0.095, 0.625, 0.78) | (.0014, .0017, .0047, .008) |
| A5 | (-0.38, -0.035, 0.235, 0.35) | (-0.265, -0.145, 0.305, 0.482) | (.0002, .000375, .000825, .001) |
| A6 | (-0.44, -0.065, 0.265, 0.56) | (-0.4, -0.03, 0.51, 0.92) | (.000276, .000465, .000855, .0022) |
| A7 | (-0.22, 0.025, 0.535, 0.69) | (-0.38, -0.15, 0.45, 0.69) | (.0008, .00115, .00205, .003) |
| A8 | (-0.66, -0.325, 0.245, 0.57) | (-0.66, -0.275, 0.475, 0.66) | (.000045, .0001625, .0004175, .000546) |
| A9 | (-0.87, 0, 0.72, 0.88) | (-0.75, -0.45, 0.52, 0.88) | (.0011875, .0013875, .0079125, .013) |
| A10 | (-0.68, -0.085, 0.605, 0.94) | (-0.68, -0.22, 0.62, 1.04) | (.0013, .00285, .00735, .013) |
| A11 | (-0.15, -0.05, 0.37, 0.58) | (-0.72, -0.3, 0.3, 0.7) | (.001, .001575, .002625, .003185) |
| A12 | (-0.85, -0.14, 0.4, 0.79) | (-0.74, -0.25, 0.41, 0.79) | (.000845, .0027, .0081, .015) |
| A13 | (-0.55, -0.395, 0.115, 0.44) | (-0.52, -0.275, 0.475, 0.77) | (.0008, .0022, .0082, .010773) |
| A14 | (-0.42, -0.175, 0.215, 0.4) | (-0.76, -0.31, 0.29, 0.86) | (.000256, .0014, .008, .015493) |
| A15 | (-0.5, -0.07, 0.47, 0.8) | (-0.79, -0.18, 0.51, 0.92) | (.000252, .0008, .0038, .00576) |
| A16 | (-0.66, -0.145, 0.425, 0.85) | (-0.72, -0.27, 0.63, 0.85) | (.000691, .00145, .00415, .005058) |
| A17 | (-0.22, -0.025, 0.425, 0.52) | (-0.58, -0.18, 0.66, 0.88) | (.00039, .001065, .003855, .00472) |
| A18 | $\frac{1}{\exp(0.29r)}, r = [4.5, 5.5]$ | $\frac{1}{\exp(0.32r)}, r = [2.2, 3]$ | $\frac{1}{\exp(2r)}, r = [2.4, 3]$ |
| A19 | $\frac{1}{\exp(0.32r)}, r = [4, 5]$ | $\frac{1}{\exp(0.4r)}, r = [2.5, 3.5]$ | $\frac{1}{(350-r)}, r = [4, 6]$ |
| A20 | $\frac{1}{(r+27)}, r = [0.9, 1.2]$ | $\frac{1}{(r+0.39)}, r = [3, 4]$ | $\frac{1}{\exp(8-r)}, r = [0.7, 0.9]$ |

TABLE II
MAIN ATTRIBUTES OF THE PROBLEMS SOLVED USING CONFIDENCE CURVE $\alpha(r) = \frac{1}{\exp(Kr)}\ \forall\, r \geq 0$

| | Cases | | |
|---|---|---|---|
| | Case-I, K=3 | Case-II, K=2 | Case-III, K=1.3 |
| $\alpha(r)$ | $\frac{1}{\exp(3r)}$ | $\frac{1}{\exp(2r)}$ | $\frac{1}{\exp(1.3r)}$ |
| $\delta_1$ | 0.8 | 0.7 | 0.6 |
| $\delta_2$ | 0.8 | 0.7 | 0.6 |
| $\delta_3$ | 0.8 | 0.7 | 0.6 |
| Target return of portfolio($b$) | 0.22 | 0.26 | 0.3 |
| Number of $(r, \alpha(r))$ pairs generated $(N')$ | 40 | 40 | 40 |
| Length of a chromosome $(n)$ | 20 | 20 | 20 |
| $h$ | 7 | 7 | 7 |
| $u_i\,, i = 1, 2, \ldots, 20$ | 0.3 | 0.3 | 0.3 |
| $l_i\,, i = 1, 2, \ldots, 20$ | 0.08 | 0.08 | 0.08 |

### A. *Computational Results and Sensitivity Analysis for Case-I*

In what follows next, we first check the stability of the proposed HIA.

*1) Checking for Stability of HIA:* To test the effectiveness of the HIA, we solve the model $(P1)$ for Case-I with different values of the HIA parameters using $w_1 = w_2 = w_3 = 1/3$ and $\delta_1 = \delta_2 = \delta_3 = 0.8$. We executed 20 runs of HIA with respect to each set of the parameter settings and report the best found fitness. Table III presents the parameters used and the corresponding computational results. We use an index called relative error (RE) to compare the results which is defined as

$$\text{RE} = \frac{(\text{Maximal fitness} - \text{Actual fitness})}{\text{Maximal fitness}} \times 100\%$$

where the maximal fitness is the maximum of all the computational results obtained.

It can be seen from Table III that the RE in each case does not exceed 1.5%, which shows that the proposed HIA is effective in terms of the settings of the parameters. Further, we use the parameter setting from Table III with respect to the maximum fitness, i.e., $popsize = 40, p_c = 0.8, p_m = 0.7,$ generations $=$ 3000, and simulation runs $(N) = 4000$ to run the HIA to obtain the optimal portfolio. Corresponding to these parameter settings, we present solution statistics of the 20 runs of HIA in order to show the stability of the solutions obtained. Table IV presents solution statistics for the 20 runs.

In Fig. 4, we show the sensitivity of the maximum fitness attained in each HIA run.

TABLE III
RESULTS CORRESPONDING TO DIFFERENT SETTINGS OF HIA PARAMETERS FOR CASE-I

| $p_c$ | $p_m$ | $popsize$ | Simulation runs (N) | Generations | Fitness | Relative error (%) |
|---|---|---|---|---|---|---|
| 0.5 | 0.4 | 40 | 3000 | 3000 | 0.08077 | 1.4399 |
| 0.6 | 0.5 | 50 | 4000 | 3000 | 0.08130 | 0.7932 |
| 0.7 | 0.3 | 50 | 3000 | 3000 | 0.08122 | 0.8908 |
| 0.7 | 0.5 | 40 | 3000 | 2000 | 0.08145 | 0.6101 |
| 0.8 | 0.7 | 40 | 4000 | 3000 | 0.08195 | 0 |
| 0.8 | 0.4 | 50 | 4000 | 3000 | 0.08143 | 0.6345 |
| 0.9 | 0.5 | 50 | 3000 | 2000 | 0.08150 | 0.5491 |
| 0.9 | 0.8 | 40 | 4000 | 3000 | 0.08099 | 1.1715 |

TABLE IV
SOLUTION STATISTICS FOR 20 HIA RUNS FOR CASE-I

| Best fitness | Average fitness | Standard deviation | Coefficient of variation (%) |
|---|---|---|---|
| 0.08195 | 0.08063 | 0.000785 | 0.973583 |

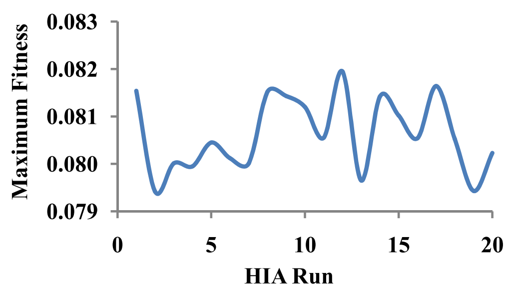


Fig. 4. Maximum fitness versus HIA run for Case-I.

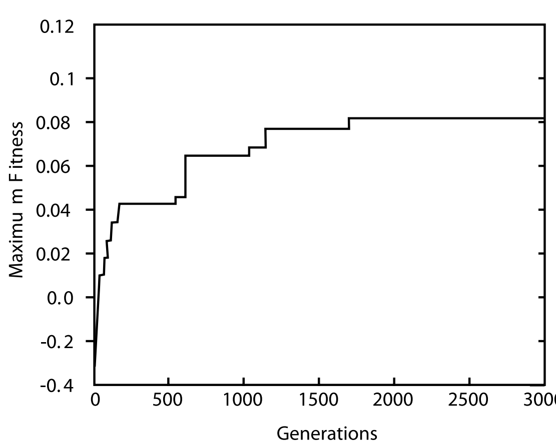


Fig. 5. Maximum fitness versus generations.

TABLE V
SUMMARY RESULT OF PORTFOLIO SELECTION FOR CASE-I

| Short-term return ($r_1$) | Long-term return ($r_2$) | Liquidity ($r_3$) |
|---|---|---|
| 0.08842 | 0.16433 | 0.00310 |

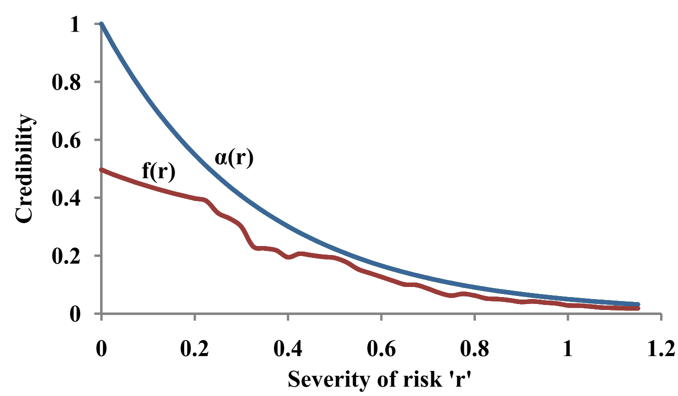


Fig. 6. Confidence curve and risk curve of the obtained portfolio for Case-I.

It may be recalled that each HIA run comprises 3000 generations. In Fig. 5, we show the movement of maximum fitness (out of 40 chromosomes) attained at various generations corresponding to that HIA run which gives the best fitness value reported in Table IV.

*2) Portfolio Selection:* Corresponding to the best found fitness of the HIA runs, as reported in Table IV, we present in Table V the maximum attainment values of the objective functions that the investor may obtain at a credibility, which is no less than the given confidence levels $\delta_1 = 0.80$, $\delta_2 = 0.80$, and $\delta_3 = 0.80$, respectively. Further, the asset allocation is reported in Table VI.

It may be recalled that we desired to invest in seven assets. Therefore, the optimal portfolio must be a combination of those seven assets for which the obtained risk curve $f(r)$ is below the investor's confidence curve $\alpha(r)$, i.e., the low risk area. Note that the risk curve of the combination of assets, which lies above the investor's confidence curve, i.e., the high risk area, will lead to infeasible solutions. Fig. 6 shows the risk curve of the portfolio generated for Case-I with respect to the investible assets listed in Table VI. It can be clearly seen from Fig. 6 that the risk curve of the generated portfolio is completely below the investor's confidence curve. In other words, the combination of these seven assets is safe to invest, i.e., satisfies the investor preferences.

Note that in the above portfolio selection, we assumed that the investor has equal preferences for all the three objectives functions. In what follows next, we present computational results incorporating investor preferences with respect to the three objective functions.

• *Case-I(a):* We consider a hypothetical situation where the investor is aspiring for higher short-term return of the portfolio,

TABLE VI
ASSET ALLOCATION FOR CASE-I

| Assets | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| A1 | A2 | A3 | A4 | A5 | A6 | A7 | A8 | A9 | A10 |
| 0.12895 | 0 | 0 | 0 | 0.08683 | 0 | 0 | 0 | 0.12153 | 0.14377 |
| Assets | | | | | | | | | |
| A11 | A12 | A13 | A14 | A15 | A16 | A17 | A18 | A19 | A20 |
| 0 | 0.12899 | 0 | 0.16520 | 0 | 0 | 0 | 0.22473 | 0 | 0 |

TABLE VII
SUMMARY RESULTS OF PORTFOLIO SELECTION INCORPORATING INVESTOR PREFERENCES FOR CASE-I

| | $w_1$ | $w_2$ | $w_3$ | Short-term return $(r_1)$ | Long-term return $(r_2)$ | Liquidity $(r_3)$ |
|---|---|---|---|---|---|---|
| Case-I(a) | 0.7 | 0.2 | 0.1 | 0.10435 | 0.14567 | 0.00292 |
| Case-I(b) | 0.2 | 0.7 | 0.1 | 0.07043 | 0.18007 | 0.00301 |
| Case-I(c) | 0.1 | 0.2 | 0.7 | 0.08563 | 0.15944 | 0.00324 |

TABLE VIII
ASSET ALLOCATION IN THE OBTAINED PORTFOLIOS INCORPORATING INVESTOR PREFERENCES FOR CASE-I

| | Assets | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | A1 | A2 | A3 | A4 | A5 | A6 | A7 | A8 | A9 | A10 |
| Case-I(a) | 0 | 0 | 0 | 0 | 0.15498 | 0 | 0.11212 | 0 | 0 | 0 |
| Case-I(b) | 0 | 0 | 0.08458 | 0.09214 | 0.11456 | 0 | 0 | 0.12451 | 0 | 0 |
| Case-I(c) | 0 | 0 | 0 | 0.09259 | 0.10521 | 0 | 0.12345 | 0 | 0 | 0 |
| | A11 | A12 | A13 | A14 | A15 | A16 | A17 | A18 | A19 | A20 |
| | 0.09504 | 0 | 0 | 0.09045 | 0 | 0.14243 | 0 | 0.17042 | 0.23456 | 0 |
| | 0.10192 | 0 | 0 | 0 | 0 | 0 | 0 | 0.22564 | 0.25665 | |
| | 0 | 0.13452 | 0 | 0 | 0 | 0 | 0 | 0.18754 | 0.23455 | 0.12214 |

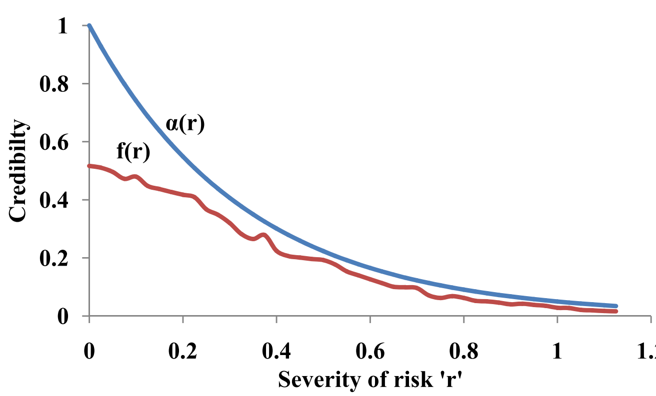


Fig. 7. Confidence curve and risk curve of the obtained portfolio for Case-I(a).

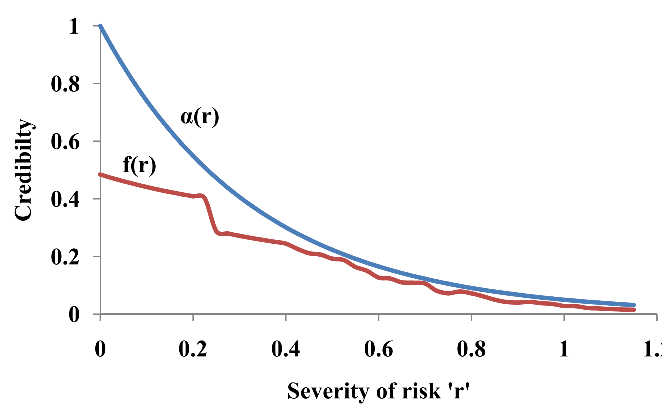


Fig. 8. Confidence curve and risk curve of the obtained portfolio for Case-I(b).

as compared with long-term return and liquidity. Corresponding to $w_1 = 0.7, w_2 = 0.2,$ and $w_3 = 0.1$, we solve the portfolio selection model $(P1)$ using the main attributes reported in Table II. The computational results are reported in Table VII. The asset allocation is reported in Table VIII. In addition, it can be seen from Fig. 7 that the risk curve of the portfolio generated is completely below the investor's confidence curve.

- *Case-I(b):* We consider a hypothetical situation where the investor is aspiring for higher long-term return of the portfolio, as compared with short-term return and liquidity. Corresponding to $w_1 = 0.2, w_2 = 0.7,$ and $w_3 = 0.1$, the computational results are reported in Table VII. The asset allocation is reported in Table VIII. In addition, it can be seen from Fig. 8 that the risk curve of the portfolio generated is completely below the investor's confidence curve.
- *Case-I(c):* We consider a hypothetical situation where the investor is aspiring for higher liquidity of the portfolio as

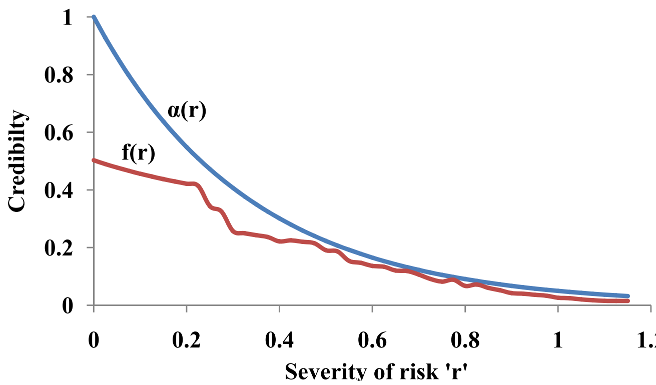


Fig. 9. Confidence curve and risk curve of the obtained portfolio for Case-I(c).

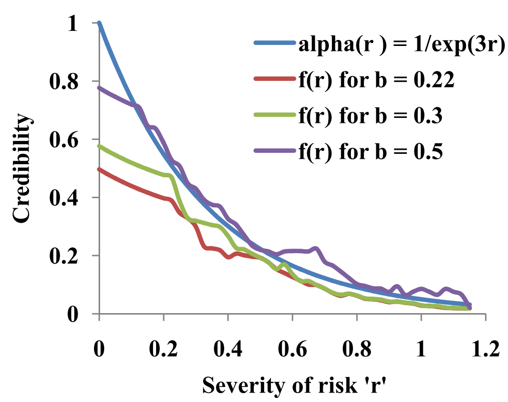


Fig. 10. Effect of changes in parameter $b$ on risk curve for Case-I.

compared with short-term return and long-term return. Corresponding to $w_1 = 0.1, w_2 = 0.2,$ and $w_3 = 0.7$, we solve the portfolio selection model $(P1)$ using the main attributes reported in Table II. The computational results are reported in Table VII. The asset allocation is reported in Table VIII. In addition, it can be seen from Fig. 9 that the risk curve of the portfolio generated is completely below the investor's confidence curve.

*3) Sensitivity Analysis With Respect to Parameter b on Risk Curve:* We perform sensitivity analysis with respect to the parameter $b$. A graphical representation of the sensitivity results is presented in Fig. 10.

It can be clearly seen from Fig. 10 that as $b$ increases, the respective risk curve of the portfolio generated moves toward investor's confidence curve. After a threshold value of $b$ is reached, the portfolio selection problem becomes infeasible as the risk curve of the generated portfolio moves beyond the investor's confidence curve and enters into the high risk area which is undesirable. As mentioned earlier, the portfolio risk constraint (4) of model $(P1)$ is replaced by fuzzy chance constraints with respect to randomly generated pairs $(r_j, \alpha(r_j)),\ j = 1, 2, \ldots, 40$. The infeasibility of the portfolio selection problem that arises is because of the violation of such fuzzy chance constraints.

### B. Computational Results and Sensitivity Analysis for Case-II

*1) Checking for Stability of Hybrid Intelligent Algorithm:* Here, we consider the following parameter settings

TABLE IX
SOLUTION STATISTICS OF 20 HIA RUNS FOR CASE-II

| Best fitness | Average fitness | Standard deviation | Coefficient of variation (%) |
|---|---|---|---|
| 0.10385 | 0.10013 | 0.00168 | 1.67781 |

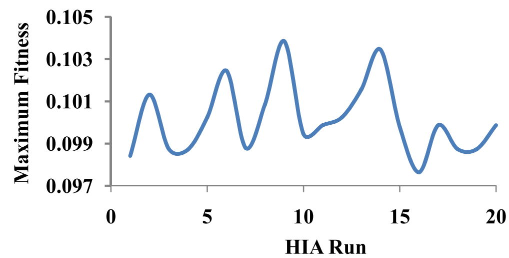


Fig. 11. Maximum fitness versus HIA run for Case-II.

TABLE X
SUMMARY RESULT OF PORTFOLIO SELECTION FOR CASE-II

| Short-term return $(r_1)$ | Long-term return $(r_2)$ | Liquidity $(r_3)$ |
|---|---|---|
| 0.11317 | 0.19512 | 0.00326 |

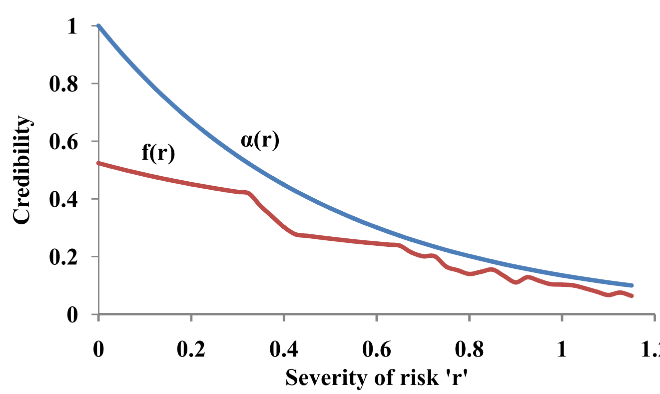


Fig. 12. Confidence curve and risk curve of the obtained portfolio for Case-II.

to run the HIA for Case-II: $popsize = 40, p_c = 0.8, p_m = 0.7$, generations $= 3000$, and simulation runs $(N) = 4000$. Using $w_1 = w_2 = w_3 = 1/3$ and $\delta_1 = \delta_2 = \delta_3 = 0.7$, solution statistics in 20 runs of HIA are reported in Table IX. A graphical representation of the sensitivity of the maximum fitness attained with each HIA run is shown in Fig. 11.

*2) Portfolio Selection:* Corresponding to the best found fitness of the HIA runs, as reported in Table IX, we present the maximum attainment values of the objective functions that the investor may obtain at credibility not less than predefined confidence levels $\delta_1 = 0.70, \delta_2 = 0.70$, and $\delta_3 = 0.70$, respectively, in Table X. Further, the asset allocation is reported in Table XI.

Again, as we desire to invest in seven assets, the optimal portfolio should be a combination of those seven assets for which the obtained risk curve $f(r)$ is below the investor's confidence curve $\alpha(r)$. Fig. 12 gives the risk curve of the portfolio generated. It

TABLE XI
ASSET ALLOCATION FOR CASE-II

| Assets | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| A1 | A2 | A3 | A4 | A5 | A6 | A7 | A8 | A9 | A10 |
| 0 | 0 | 0 | 0 | 0.10214 | 0.10142 | 0.13487 | 0 | 0 | 0.14456 |
| Assets | | | | | | | | | |
| A11 | A12 | A13 | A14 | A15 | A16 | A17 | A18 | A19 | A20 |
| 0 | 0 | 0 | 0.12899 | 0 | 0 | 0 | 0.17456 | 0.21346 | 0 |

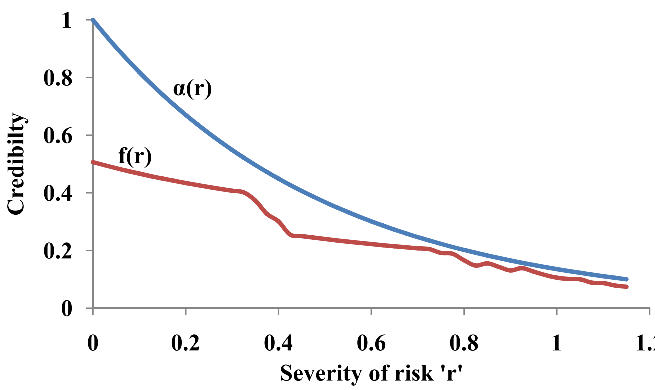


Fig. 13. Confidence curve and risk curve of the obtained portfolio for Case-II(a).

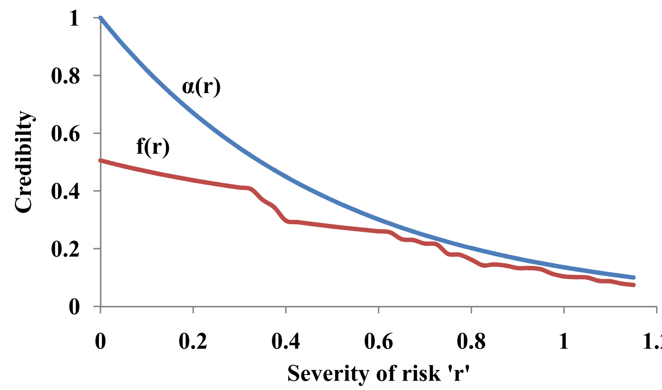


Fig. 14. Confidence curve and risk curve of the obtained portfolio for Case-II(b).

can be seen from Fig. 12 that the risk curve of the portfolio generated is completely below the investor's confidence curve.

Next, we present computational results incorporating investor preferences with respect to the three objective functions.

- *Case-II(a):* We consider a hypothetical situation where the investor is aspiring for higher short-term return of the portfolio, as compared with long-term return and liquidity. Corresponding to $w_1 = 0.7, w_2 = 0.2,$ and $w_3 = 0.1$, we solve the portfolio selection model $(P1)$ using the main attributes reported in Table II for Case-II. The computational results are reported in Table XII. The asset allocation is reported in Table XIII. In addition, it can be seen from Fig. 13 that the risk curve of the portfolio generated is below the investor's confidence curve.
- *Case-II(b):* We consider a hypothetical situation where the investor is aspiring for higher long-term return of the portfolio as compared with short-term return and liquidity. Corresponding to $w_1 = 0.2, w_2 = 0.7,$ and $w_3 = 0.1$, the computational results are reported in Table XII. The asset allocation is reported in Table XIII. In addition, it can be seen from Fig. 14 that the risk curve of the portfolio generated is below the investor's confidence curve.
- *Case-II(c):* We consider a hypothetical situation where the investor is aspiring for higher liquidity of the portfolio as compared with short-term return and long-term return. Corresponding to $w_1 = 0.1, w_2 = 0.2,$ and $w_3 = 0.7$, we solve the portfolio selection model $(P1)$ using the main attributes reported in Table II. The computational results are reported in Table XII. The asset allocation is reported in Table XIII. In addition, it can be seen from Fig. 15 that the risk curve of the portfolio generated is below the investor's confidence curve.

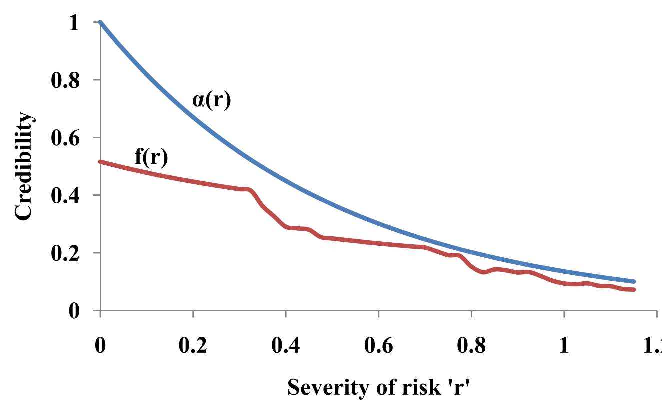


Fig. 15. Confidence curve and risk curve of the obtained portfolio for Case-II(c).

*3) Sensitivity Analysis With Respect to Parameter $b$ on Risk Curve:* We perform sensitivity analysis with respect to the parameter $b$. A graphical representation of the sensitivity results is shown in Fig. 16.

It may also be seen from Fig. 16 that after a threshold value of $b$ is reached, the portfolio selection problem becomes infeasible as the risk curve of the generated portfolio moves into the high risk area which is undesirable.

### *C. Computational Results and Sensitivity Analysis for Case-III*

*1) Checking for Stability of Hybrid Intelligent Algorithm:* Here, we consider the following parameter settings to run the HIA for Case-III: $popsize = 40, p_c = 0.8, p_m =$

TABLE XII
SUMMARY RESULTS OF PORTFOLIO SELECTION INCORPORATING INVESTOR PREFERENCES FOR CASE-II

| | $w_1$ | $w_2$ | $w_3$ | Short-term return $(r_1)$ | Long-term return $(r_2)$ | Liquidity $(r_3)$ |
|---|---|---|---|---|---|---|
| Case-II(a) | 0.7 | 0.2 | 0.1 | 0.12337 | 0.17192 | 0.00307 |
| Case-II(b) | 0.2 | 0.7 | 0.1 | 0.08557 | 0.20918 | 0.00315 |
| Case-II(c) | 0.1 | 0.2 | 0.7 | 0.09973 | 0.18592 | 0.00336 |

TABLE XIII
ASSET ALLOCATION IN THE OBTAINED PORTFOLIOS INCORPORATING INVESTOR PREFERENCES FOR CASE-II

| | Assets | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | A1 | A2 | A3 | A4 | A5 | A6 | A7 | A8 | A9 | A10 |
| Case-II(a) | 0 | 0 | 0 | 0 | 0.09164 | 0 | 0.12415 | 0 | 0 | 0 |
| Case-II(b) | 0 | 0 | 0.09014 | 0 | 0.10115 | 0.10152 | 0.11151 | 0 | 0 | 0 |
| Case-II(c) | 0 | 0 | 0 | 0.09059 | 0.10741 | 0.13142 | 0.13472 | 0 | 0 | 0 |
| | A11 | A12 | A13 | A14 | A15 | A16 | A17 | A18 | A19 | A20 |
| | 0.15462 | 0 | 0 | 0.09045 | 0 | 0.13416 | 0 | 0.16040 | 0.24458 | 0 |
| | 0 | 0 | 0 | 0 | 0.10192 | 0 | 0 | 0.23252 | 0.26214 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.17456 | 0.22415 | 0.13715 |

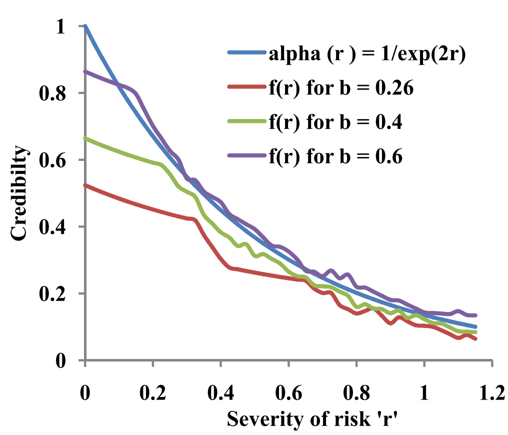


Fig. 16. Effect of changes in parameter $b$ on risk curve for Case-II.

TABLE XIV
SOLUTION STATISTICS OF 20 HIA RUNS FOR CASE-III

| Best fitness | Average fitness | Standard deviation | Coefficient of variation (%) |
|---|---|---|---|
| 0.11012 | 0.10811 | 0.00164 | 1.51697 |

0.7, generations $= 3000$, and simulation runs $(N) = 4000$. For $w_1 = w_2 = w_3 = 1/3$ and $\delta_1 = \delta_2 = \delta_3 = 0.6$, the solution statistics in 20 runs of HIA are reported in Table XIV. A graphical representation of the sensitivity of maximum fitness attained with each HIA run is shown in Fig. 17.

*2) Portfolio Selection:* Corresponding to the best found fitness of the HIA runs as reported in Table XIV, we present the maximum attainment values of the objective functions that the investor may obtain at credibility not less than a predefined confidence levels $\delta_1 = 0.60, \delta_2 = 0.60$, and $\delta_3 = 0.60$, respectively, in Table XV. Further, the asset allocation is reported in Table XVI.

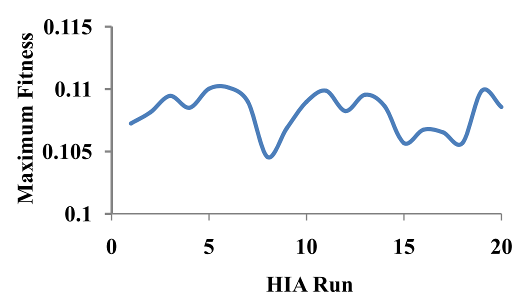


Fig. 17. Maximum fitness versus HIA run for Case-III.

TABLE XV
SUMMARY RESULT OF PORTFOLIO SELECTION FOR CASE-III

| Short-term return $(r_1)$ | Long-term return $(r_2)$ | Liquidity $(r_3)$ |
|---|---|---|
| 0.12639 | 0.21596 | 0.00345 |

Again, as we desire to invest in seven assets, the optimal portfolio should be a combination of those seven assets for which the obtained risk curve $f(r)$ is below the investor's confidence curve $\alpha(r)$. Fig. 18 gives the risk curve of the portfolio generated.

It can be seen from Fig. 18 that the risk curve of the portfolio generated is below the investor's confidence curve.

Next, we present computational results incorporating investor preferences with respect to the three objective functions.

- *Case-III(a):* We consider a hypothetical situation where the investor is aspiring for higher short-term return of the portfolio, as compared, with long-term return and liquidity. Corresponding to $w_1 = 0.7, w_2 = 0.2$, and $w_3 = 0.1$, we solve the portfolio selection model $(P1)$ using the main attributes reported in Table II for Case-III. The computational results are reported in Table XVII. Further, the asset allocation is reported in Table XVIII. In addition, it can be seen from Fig. 19 that

TABLE XVI
ASSET ALLOCATION IN THE OBTAINED PORTFOLIO FOR CASE-III

| Assets | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| A1 | A2 | A3 | A4 | A5 | A6 | A7 | A8 | A9 | A10 |
| 0 | 0 | 0 | 0 | 0.09815 | 0.11214 | 0.12226 | 0 | 0 | 0.13614 |
| Assets | | | | | | | | | |
| A11 | A12 | A13 | A14 | A15 | A16 | A17 | A18 | A19 | A20 |
| 0 | 0 | 0 | 0 | 0.12645 | 0 | 0 | 0.18341 | 0.22145 | 0 |

TABLE XVII
SUMMARY RESULTS OF PORTFOLIO SELECTION INCORPORATING INVESTOR PREFERENCES FOR CASE-III

| | $w_1$ | $w_2$ | $w_3$ | Short-term return $(r_1)$ | Long-term return $(r_2)$ | Liquidity $(r_3)$ |
|---|---|---|---|---|---|---|
| Case-III(a) | 0.7 | 0.2 | 0.1 | 0.13461 | 0.19241 | 0.00325 |
| Case-III(b) | 0.2 | 0.7 | 0.1 | 0.10639 | 0.23505 | 0.00331 |
| Case-III(c) | 0.1 | 0.2 | 0.7 | 0.11412 | 0.20491 | 0.00358 |

TABLE XVIII
ASSET ALLOCATION IN THE OBTAINED PORTFOLIOS INCORPORATING INVESTOR PREFERENCES FOR CASE-III

| | Assets | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | A1 | A2 | A3 | A4 | A5 | A6 | A7 | A8 | A9 | A10 |
| Case-III(a) | 0 | 0 | 0 | 0 | 0.08474 | 0 | 0.13616 | 0 | 0 | 0 |
| Case-III(b) | 0 | 0 | 0.08426 | 0 | 0.12214 | 0.09564 | 0.08477 | 0 | 0 | 0 |
| Case-III(c) | 0 | 0 | 0 | 0.09502 | 0.10421 | 0.12535 | 0 | 0.12644 | 0 | 0 |
| | A11 | A12 | A13 | A14 | A15 | A16 | A17 | A18 | A19 | A20 |
| | 0.14415 | 0 | 00.09245 | 0 | 0 | 0.11284 | 0 | 0.17354 | 0.25612 | 0 |
| | 0 | 0 | 0 | 0 | 0.09315 | 0 | 0 | 0.24456 | 0.27548 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.15879 | 0.23456 | 0.15563 |

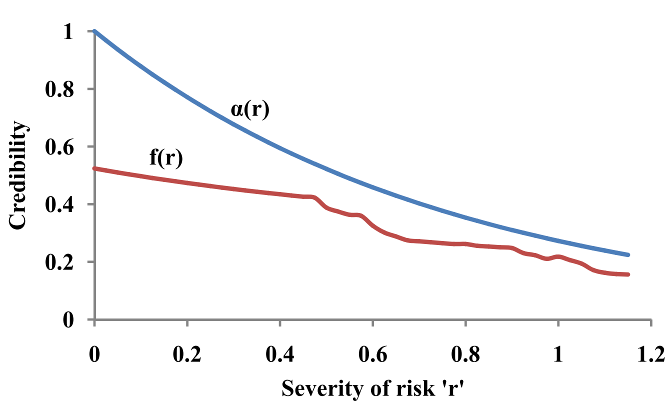


Fig. 18. Confidence curve and risk curve of the obtained portfolio for Case-III.

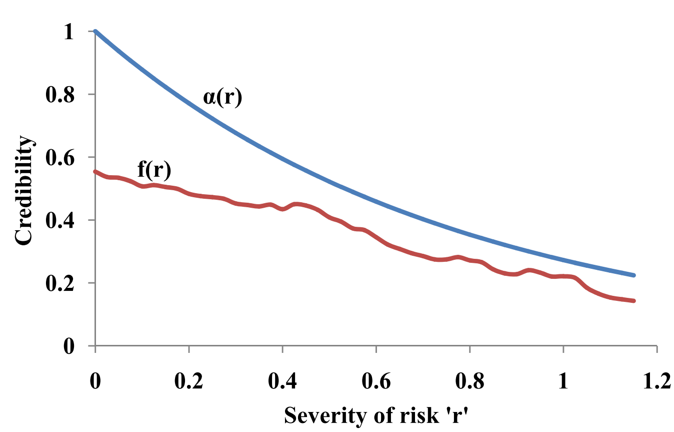


Fig. 19. Confidence curve and risk curve of the obtained portfolio for Case-III(a).

the risk curve of the portfolio generated is below the investor's confidence curve.

• *Case-III(b):* We consider a hypothetical situation where the investor is aspiring for higher long-term return of the portfolio as compared with short-term return and liquidity. Corresponding to $w_1 = 0.2, w_2 = 0.7$, and $w_3 = 0.1$, the computational results are reported in Table XVII. The asset allocation is reported in Table XVIII. In addition, it can be seen from Fig. 20 that the risk curve of the portfolio generated is below the investor's confidence curve.

• *Case-III(c):* We consider a hypothetical situation where the investor is aspiring for higher liquidity as compared with short- and long-term returns. Corresponding to $w_1 = 0.1, w_2 = 0.2$, and $w_3 = 0.7$, we solve the portfolio selection model $(P1)$ using the main attributes reported in Table II. The computational results are reported in Table XVII. The asset allocation

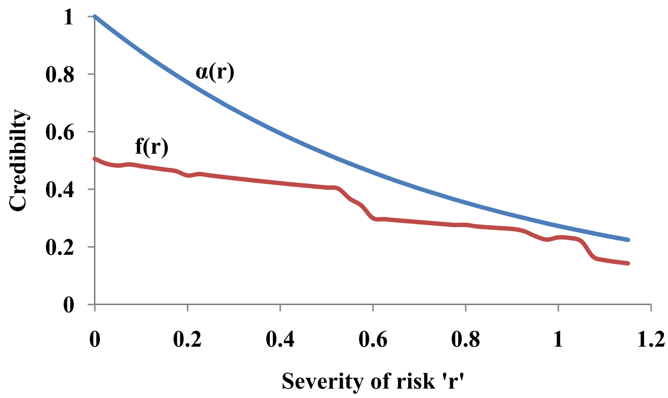


Fig. 20. Confidence curve and risk curve of the obtained portfolio for Case-III(b).

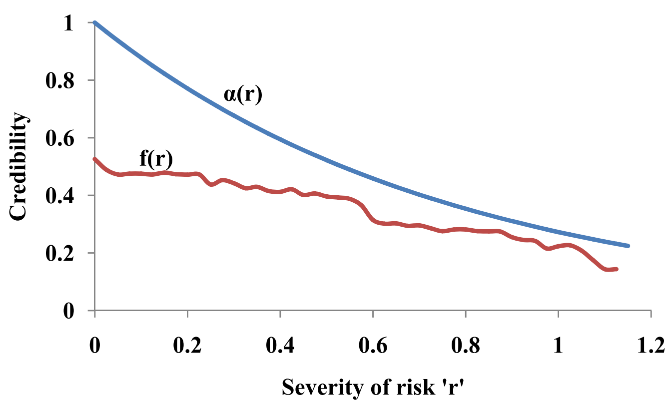


Fig. 21. Confidence curve and risk curve of the obtained portfolio for Case-III(c).

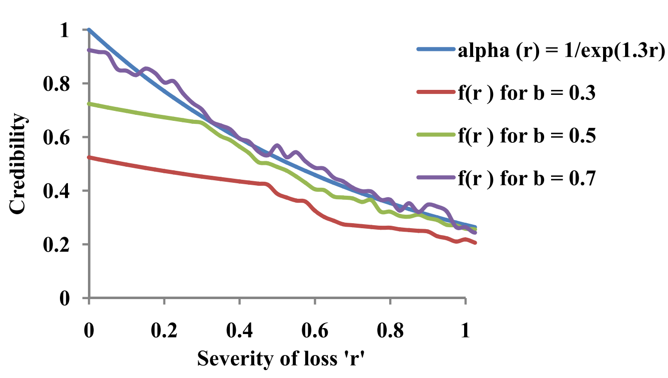


Fig. 22. Effect of changes in parameter $b$ on risk curve for Case-III.

is reported in Table XVIII. In addition, it can be seen from Fig. 21 that the risk curve of the portfolio generated is below the investor's confidence curve.

*3) Sensitivity Analysis With Respect to Parameter b on Risk Curve:* We perform sensitivity analysis with respect to the parameter $b$. A graphical representation of the sensitivity results is shown in Fig. 22.

It may also be seen from Fig. 22 that after a threshold value of $b$ is reached, the portfolio selection problem becomes infeasible as the risk curve of the generated portfolio moves into the high-risk area, which is undesirable.

### D. Discussion of the Computational Results

A comparison of the results listed in Tables V, X, and XV clearly highlights that if the investor has low tolerance toward portfolio risk (Case-I), the attainment values of short-term return, long-term return, and liquidity objective functions are also low. If the investor has medium tolerance toward portfolio risk (Case-II), the attainment values of short-term return, long-term return, and liquidity objective functions are at medium level. If the investor has high tolerance toward portfolio risk (Case-III), the attainment values of short-term return, long-term return, and liquidity objective functions are also high. This relationship is in line with return-risk tradeoff. In addition, the comparison of the results listed in Tables VII, XII, and XVII highlights the same relationship.

Additionally, a comparison of the results listed in Tables V, X, and XV highlights the relationship between the given confidence levels and the attainment values of the three objective functions. If the given confidence levels are high, then the attainment values of the three objective functions are low. If the given confidence levels are medium, then the attainment values of the three objective functions are medium. If the given confidence levels are low, then the attainment values of the three objective functions are high. In order to exemplify, the investor's confidence levels with respect to three objectives functions in Case-I ($\delta_1 = \delta_2 = \delta_3 = 0.8$) are high in comparison with Case-II ($\delta_1 = \delta_2 = \delta_3 = 0.7$) and Case-III ($\delta_1 = \delta_2 = \delta_3 = 0.6$); therefore, the attainment values of three objective functions are low in Case-I, in comparison with the other two cases.

Further, the comparison of the results of portfolio selection incorporating investor preferences listed in Tables VII, XII, and XVII for Case-I, Case-II, and Case-III, respectively, highlights that the attainment values of the three objective functions are in accordance with the investor preferences. In order to exemplify, the investor in Case-I has given the highest preference to short-term return in Case-I(a), in comparison with Case-I(b) and Case-I(c). Therefore, the attainment value of short-term return in Case-I(a) is the highest, in comparison with other two cases.

The computational results presented in Tables V, VII, X, XII, XV, and XVII of the preceding section clearly shows that the proposed model has following advantages that makes it more realistic and flexible in terms of applying to the real-world portfolio selection problem.

1) It can generate solutions which are consistent with the investor preferences.
2) It generates safe portfolio by containing the risk curve of the portfolio completely below the respective confidence curve of the investor.
3) It is more flexible in terms of providing different solutions for specific problem instances corresponding to the values of the model parameters, namely, importance weight of objectives $(w_1, w_2, w_3)$ and given confidence levels $(\delta_1, \delta_2, \delta_3)$.

4) In comparison with the existing credibility-based portfolio selection models, the proposed model helps in achieving the maximal short-term return, long-term return, and liquidity of the portfolio at a credibility, which is no less than the given confidence levels of the investor.

### *E. Comparison of the Proposed Model With Safety-First and Value-At-Risk-Based Portfolio Selection Models*

In order to show the wide applicability of the proposed model, here, we highlight the advantages of the proposed model in comparison with safety-first and value-at-risk-based portfolio selection models. In portfolio selection models based on safety-first criterion, the nonfulfillment probability, which means that the total portfolio return is less than the target value, is considered as the factor for risk management [53], [54]. Now, we show that the use of risk curve of the portfolio versus investor's confidence curve to capture the portfolio risk generalizes the idea of safety-first criterion for portfolio selection. In this paper, the portfolio risk is expressed as

$$Cr\{b - (\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n) \geq r\} \leq \alpha(r),\ \forall\, r \geq 0.$$

In the above expression, when $r$ degenerate to a specific number $r_0$, i.e., the investor is only concerned about one fixed preset level, the confidence curve will correspond to a specific confidence level $\alpha_0$, where $\alpha_0 = \alpha(r_0)$. Then, the above risk constraint reduces to

$$Cr\{(\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n) \leq b - r_0\} \leq \alpha_0. \quad (12)$$

The above expression captures the portfolio risk in a manner similar to the one proposed by Roy [53] and Telser [54]. It may be noted that in [53] and [54], returns of the assets are represented by random variable, whereas in this study, these are represented by using fuzzy variables; therefore, the probability of nonfulfillment is replaced by credibility to make the comparison meaningful. Furthermore, it is worth mentioning that if we solve the model $(P1)$ replacing the risk constraint (4) by the risk constraint (12) (i.e., using the safety-first criterion as the risk measurement), then the obtained portfolio is safe to invest according to risk constraint (12) but may be at risk when judged by using constraint (4). However, if a portfolio is safe to invest obtained using the proposed model (i.e., using the risk constraint (4) for portfolio risk), then it must also be safe when judged by using constraint (12). This is because in risk constraint (4), risk is regarded as a curve instead of a specific value; however, in safety-first models [53], [54], investors are concerned only about one preset bad case.

Next, we show that the use of risk curve of the portfolio versus investor's confidence curve also captures the idea of value-at-risk criterion for portfolio selection. Value-at-risk is the maximum amount that can be lost with a certain confidence level. For this purpose, we compare the proposed model with the models proposed in [40]. Since the credibility measure is self-dual, inequality (12) is equivalent to

$$Cr\{(\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n) \geq b - r_0\} \geq 1 - \alpha_0. \quad (13)$$

The above inequality can be interpreted using the concept of value-at-risk [40]. At the (high) credibility value $1 - \alpha_0$, the portfolio return will be equal to or greater than the low return level $b - r_0$. In other words, at (high) credibility value $1 - \alpha_0$, the portfolio loss compared with the target return of the portfolio will be equal to or less than $r_0$. Alternatively, the loss of the portfolio return compared with the target return of the portfolio will be at least $r_0$ at a credibility level $\alpha_0$. Therefore, we can say that the proposed model captures the concept of value-at-risk. It is worthy mentioning that if a safe portfolio is obtained using the proposed model, then it must also be safe when judged by value-at-risk as the criterion for risk measurement. Further, it may be noted that the value-at-risk only provides information about one preset bad case of the loss that the investor may suffer. However, other bad cases of the loss of portfolio return may also occur which are not captured by value-at-risk. On the other hand, the risk curve represents all the likely losses of portfolio return and the respective chances of their occurrence.

The other advantages of the proposed model in comparison with the models presented in [40] are as follows: 1) Instead of a single measure of portfolio return that does not discriminate investor preferences for short- versus long-term returns, we use two measures. One is short-term return, i.e., the average performance of the asset during a 12-month period, and the other is long-term return, i.e., the average performance of the asset during a 36-month period; 2) the proposed fuzzy chance-constrained multiobjective portfolio selection model apart from return also considers another important measure of asset characteristics, namely, liquidity; 3) unlike the models presented in [40], which are constrained by only the capital budget constraint, the proposed portfolio selection model is constrained by several other realistic constraints, namely, a cardinality constraint and diversification constraints that apply lower and upper bounds on investments in individual assets; 4) the proposed model can generate portfolios which are consistent with the investor preferences captured by using importance weights of the objectives and predefined confidence levels; and 5) the portfolio selection models using credibility measure developed in [40] are based on the assumption that model parameters are characterized using special fuzzy variables such as trapezoidal, triangular, and Gaussian type. In contrast, we propose fuzzy chance-constrained multiobjective portfolio selection model to treat uncertainty of the financial markets more realistically, assuming that the fuzzy parameters used may take any general functional forms. Under this assumption, the conversion of the fuzzy portfolio selection model into the crisp equivalent becomes computationally difficult. To account for this, we rely on a specially developed algorithm that hybridizes fuzzy simulation and RCGA to solve the model.

## V. Conclusion

In this paper, we have addressed the problem of portfolio selection with fuzzy parameters from a perspective of chance-constrained multiobjective programming. The key financial criteria used here are short-term return, long-term return, risk, and liquidity characteristics of the portfolio. The proposed model helps in achieving the maximal return (short-term as well as long-term) and liquidity of the portfolio at a credibility, which is no less than the given confidence levels defined by the investor. The portfolio risk was characterized by a risk curve representing all the likely losses of portfolio return and the respective chances of their occurrence, instead of one fixed preset severity level of the loss. Such a risk curve needed to be contained below the confidence curve of the investor in order to build safe portfolios. We used an HIA under the assumption that the fuzzy parameters of the model may take general functional forms in order to build safe portfolios. We conducted several numerical experiments corresponding to three different confidence curves for portfolio selection. These curves represented the tolerances of the investor toward the chances of occurrence of all the potential losses being equal to or greater than any given severity indicator of the loss of portfolio return. The numerical results clearly highlight that the relationship between investor's tolerance toward portfolio risk and the attainment values of the three objective functions is in line with return-risk tradeoff. Additionally, the results also highlight the relationship between the given confidence levels and the attainment values of the three objective functions. We also performed numerical experiments in order to incorporate investor preferences with respect to the three objective functions. The obtained results are consistent with investor preferences. Further, the portfolios generated are not only safe in the sense of generating the risk curve of the portfolio which is below the corresponding investor's confidence curve, but these portfolios also meet investor's given confidence levels with respect to the three objective functions. We performed sensitivity analysis by testing the obtained solutions for perturbations in the target value of the portfolio return. The results of the numerical analysis clearly shows the efficiency of the HIA and adaptability of the model in different situations.

## ACKNOWLEDGMENT

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