A Linear-Quadratic Optimal Control Problem of Forward-Backward Stochastic Differential Equations With Partial Information

Guangchen Wang, *Member, IEEE*, Zhen Wu, and Jie Xiong

***Abstract*—This paper studies a linear-quadratic optimal control problem derived by forward-backward stochastic differential equations, where the drift coefficient of the observation equation is linear with respect to the state x, and the observation noise is correlated with the state noise, in the sense that the cross-variation of the state and the observation is nonzero. A backward separation approach is introduced. Combining it with variational method and stochastic filtering, two optimality conditions and a feedback representation of optimal control are derived. Closed-form optimal solutions are obtained in some particular cases. As an application of the optimality conditions, a generalized recursive utility problem from financial markets is solved explicitly.**

***Index Terms*—Closed-form solution, correlated state and observation noises, forward-backward stochastic differential equation (FBSDE), linear-quadratic optimal control, partial information, recursive utility.**

I. Introduction

OPTIMAL control of forward-backward stochastic differential equations (FBSDEs, for short)[1] with partial information is often encountered in finance and economics, for example, stochastic recursive utility with quadratic performance functional, mean-variance portfolio selection. Due to this, in this paper we study a linear-quadratic (LQ, for short) optimal control problem of FBSDEs: Find an admissible control u such that

$$J[v]=\frac{1}{2}\mathbb{E}\left\{\int_0^T\Big[L_t\left(x_t^v\right)^2+O_t\left(y_t^v\right)^2+R_tv_t^2+2l_tx_t^v+2o_ty_t^v+2r_tv_t\Big]dt+M\left(x_T^v\right)^2+2mx_T^v+N\left(y_0^v\right)^2+2ny_0^v\right\}$$

is minimized, subject to the state equation

$$\begin{cases} dx_t^v=\left(a_tx_t^v+b_tv_t+\bar{b}_t\right)dt+c_tdw_t+\bar{c}_td\bar{w}_t\\ -dy_t^v=\left(A_tx_t^v+B_ty_t^v+C_tz_t^v+\bar{C}_t\bar{z}_t^v+D_tv_t+\bar{D}_t\right)dt\\ \qquad -z_t^vdw_t-\bar{z}_t^vd\bar{w}_t\\ x_0^v=e_0,\quad y_T^v=Fx_T^v+G\end{cases}$$

and the observation equation

$$\begin{cases} dY_t^v=\left(f_tx_t^v+g_t\right)dt+h_tdw_t\\ Y_0^v=0\end{cases}$$

where some assumptions about admissible controls, noises, and coefficients of the state and observation equations will be specified in Section II. Such a class of optimal control problems has attracted more and more attention. Let us now recall briefly some latest developments which are closely related to the LQ problem.

Separation principle plays an important role in decoupling state estimate and optimal control in some cases. However, since mean square error of state estimate depends on control in general, the separation principle is usually invalid to deal with stochastic optimal control problems with partial information. See e.g., Wonham [21], Bensoussan [1], [2], Elliott, Aggoun, and Moore [6] and the references therein for some detailed accounts.

Motivated by this fact, Wang and Wu [16] proposed a backward separation idea, which decouples optimal control and state estimate by first (formally) deducing optimal control and then computing optimal filtering. The backward separation idea is applicable to a broad class of nonlinear control systems, for example, controlled FBSDEs. In [22], Wu studied an optimal nonlinear control (NLC, for short) problem derived by FBSDEs. Without loss of generality, we state the 1-dimensional case of [22].

Problem (NLC): Let $(\Omega,\mathscr{F}^{w,Y},(\mathscr{F}_t^{w,Y})_{0\le t\le T},\mathbb{P})$ be a complete filtered probability space, on which an $\mathscr{F}_t^{w,Y}$-adapted standard Brownian motion (w_t,Y_t) is defined in $\mathbb{R}^2$ with $\mathscr{F}_t^{w,Y}$

Manuscript received July 3, 2013; revised February 12, 2014 and August 11, 2014; accepted February 26, 2015. Date of publication March 11, 2015; date of current version October 26, 2015. This work was supported by the National Natural Science Foundation of China (11371228), the National Natural Science Fund for Excellent Young Scholars of China (61422305), the Research Fund for the Taishan Scholar Project of Shandong Province of China, the Program for New Century Excellent Talents in University of China (NCET-12-0338), the Natural Science Fund for Distinguished Young Scholars of Shandong Province of China (JQ201418), the Postdoctoral Science Foundation of China (2013M540540), the National Natural Science Foundation of China (11221061 and 61174092), 111 project (B12023), the National Natural Science Fund for Distinguished Young Scholars of China (11125102), the Chang Jiang Scholar Program of Chinese Education Ministry, and the FDCT 076/2012/A3. Recommended by Associate Editor Z. Wang.

G. Wang is with the School of Control Science and Engineering, Shandong University, Jinan 250061, China (e-mail: wguangchen@sdu.edu.cn).

Z. Wu is with the School of Mathematics, Shandong University, Jinan 250100, China (e-mail: wuzhen@sdu.edu.cn).

J. Xiong is with the Department of Mathematics, University of Macau, Taipa, Macau (e-mail: jiexiong@umac.mo).

Digital Object Identifier 10.1109/TAC.2015.2411871

[1]As pointed out by a reviewer, the term "forward-backward stochastic differential equation" is not quite correct, mainly due to the fact that its calculus obeys forward integration rules, and the solution method is forward-backward in time. However, since the term has been used widely, we will continue to use "FBSDE" as an abbreviation of the term.

being its natural filtration. The aim is to minimize $J[v]$ subject to the cost functional

$$J[v] = \mathbb{E}^v \left[\int_0^T l\left(t, x_t^v, y_t^v, z_t^v, v_t\right) dt + \phi\left(x_T^v\right) + \gamma\left(y_0^v\right) \right]$$

the state equation

$$\begin{cases} dx_t^v = b\left(t, x_t^v, v_t\right) dt + \sigma\left(t, x_t^v, v_t\right) dw_t \\ -dy_t^v = g\left(t, x_t^v, y_t^v, z_t^v, v_t\right) dt - z_t^v dw_t \\ x_0^v = x_0, \quad y_T^v = f\left(x_T^v\right) \end{cases}$$

the observation equation

$$\begin{cases} dY_t = h(t, x_t^v) dt + d\tilde{w}_t^v \\ Y_0 = 0 \end{cases}$$

and the set of admissible controls

$$\mathcal{V}_{ad} = \Big\{ v | v_t \text{ is a } \sigma\{Y_s; 0 \le s \le t\}\text{-adapted process with values in a non-empty convex subset of } \mathbb{R} \text{ and satisfies } \sup_{0 \le t \le T} \mathbb{E} v_t^8 < +\infty \Big\}.$$

Here the formulation implies that the determination of v_t will depend on $\sigma\{Y_s; 0 \le s \le t\}$, however, Y_t is independent of v_t. The function h is uniformly bounded, and b, σ, g, f, h, l, ϕ, and γ satisfy some usual assumptions [22]. $\mathbb{E}^v$ is expectation under $\mathbb{P}^v$, which is defined by a change of probability

$$\frac{d\mathbb{P}^v}{d\mathbb{P}} = \lambda_t^v \text{ on } \mathscr{F}_t^{w,Y}$$

with a martingale

$$\lambda_t^v = \exp\left\{ \int_0^t (s, x_s^v) \, dY_s - \frac{1}{2} \int_0^t \left| h\left(s, x_s^v\right) \right|^2 ds \right\}.$$

Then it follows that for any $p \in \mathbb{R}$:

$$\sup_{0 \le t \le T} \mathbb{E}\left(\lambda_t^v\right)^p < +\infty.$$

Moreover, $(w_t, \tilde{w}_t^v)$ is an $\mathbb{R}^2$-valued standard Brownian motion on $(\Omega, \mathscr{F}^{w,Y}, (\mathscr{F}_t^{w,Y})_{0 \le t \le T}, \mathbb{P}^v)$.

Problem (NLC) covers some mathematical models arising from the fields of stochastic recursive utility, convex risk measure, leader-follower game, and so on. Making use of classical techniques in the case of full information, a necessary condition for optimality of Problem (NLC) is denoted by the conditional expectation of a function with respect to the observable filtration. Furthermore, some filter estimates of adjoint processes were obtained, which were used to describe optimal control. Along this line, there are a few works including Wang and Wu [17], Shi and Wu [14], Xiao [23]. Concerning optimal control of stochastic differential equations (SDEs, for short) with partial observation, see e.g., Bensoussan [2], Tang [15], Wang, Zhang and Zhang [20], Zhang [30], Zhou [31] for more information.

Note that the drift coefficient h of the observation equations in [14], [15], [17], [20], [22], [23], [30], [31] is uniformly bounded with respect to (t, x, v). The assumption simplifies the computations of this class of problems. However, it excludes some important applications. Very recently, Wang, Wu and Xiong [18] improved Problem (NLC). They assumed that h grows linearly with respect to x, but the diffusion coefficient σ is uniformly bounded in (t, x, v); moreover, the set of admissible controls is

$$\mathcal{V}'_{ad} = \Big\{ v | v_t \text{ is a } \sigma\{Y_s; 0 \le s \le t\}\text{-adapted process with values in a non-empty convex subset of } \mathbb{R}, \text{ and } v_t \text{ is bounded by } 1 + \sup_{0 \le s \le t} |Y_s| \Big\}.$$

With the assumptions and smooth conditions about coefficients, [18, Lemma 3.2] proved that the Radon Nikodym derivative λ_t^v satisfies

$$\mathbb{E}\left(\lambda_t^v\right)^p < +\infty, \qquad 0 \le t \le T$$

for a $p > 1$. Combining high-order moment estimates of adjoint processes of (x, y, z, λ) with an approximation method by bounded and smooth functions, a necessary condition for optimality was established.

Keeping the above developments in mind, we now turn back to the LQ problem. Because the cross-variation of x_t^v and Y_t^v is $\int_0^t c_s h_s ds$, and the drift coefficient of the observation equation is linear with respect to the state x, the LQ problem is different from the setups of [1], [2], [14], [15], [17], [18], [22], [23], [30], [31], and the references therein. In addition, it seems that the approximation method with Girsanov's transformation introduced in [18] is not available to deal with the LQ problem. One possible reason is as follows. Similar to [18], the Radon Nikodym derivative is p-integrable, for a $p > 1$. Moreover, the adjoint equation is an FBSDE with unbounded stochastic coefficients (for more information, see also (2.4) in [18] with h being linear in x). In general, it is difficult to study the solvability and high-order moment estimate of the FBSDE. Then the relevant variational inequality cannot be derived under the probability deduced by the Radon Nikodym derivative. Thus, a certain new approach is desired to develop to solve the LQ problem.

With regard to optimal control with noisy observation, it is natural to determine the control v by the observation Y^v, but Y^v depends on v via the state x^v, in other words, v has an effect on Y^v. The circular dependence between Y^v and v results in an intrinsic difficulty to study such a class of problems. In order to overcome the difficulty, we first separate the state $(x^v, y^v, z^v, \bar{z}^v)$ and the observation Y^v into

$$(x^v, y^v, z^v, \bar{z}^v) = (x^0, y^0, z^0, \bar{z}^0) + (x^1, y^1, z^1, \bar{z}^1)$$

and

$$Y^v = Y^0 + Y^1$$

where $(x^0, y^0, z^0, \bar{z}^0)$ and Y^0 are independent of the control v. Let v_t be adapted to $\sigma\{Y_s^v; 0 \le s \le t\}$ and $\sigma\{Y_s^0; 0 \le s \le t\}$. Finally, combining the backward separation idea with filtering for FBSDEs, the LQ problem is solved. Note that the formulation of admissible controls is inspired by [1]–[3], where a conventional separation principle is used to study optimal control of SDEs and stochastic distributed parameter systems.

Other developments related to the LQ problem were made by Hu and Øksendal [7], Huang, Wang and Xiong [9], Meng [12], Øksendal and Sulem [13], Wang, Wu and Xiong [19], etc., where $\mathscr{G}_t$ available to controller at time t is not a noisy observation of the state x up to time t, but an abstract subfiltration of the full information. Since $\mathscr{G}_t$ is independent of the state and the control, Girsanov's transformation as used in Problem (NLC) is unnecessary. Thus, the current work as well as [14], [17], [18], [22], [23] is distinguished from [7], [9], [12], [13], [19].

The rest of this paper is organized as follows. In Section II, we formulate an LQ optimal control problem of FBSDEs with partial information, and provide some preliminary results. In Section III, we give two optimality conditions and a feedback representation of optimal control. In Section IV, we study some special cases of the LQ problem. In Section V, we address a generalized utility problem in mathematical finance. Section VI lists some concluding remarks.

II. Problem Formulation and Preliminary

A. Notation

In this paper, we let $\mathbb{R}^m$ be an m-dimensional Euclidean space, and $T > 0$ be a fixed time horizon. Let $(\Omega, \mathscr{F}^{w,\bar{w}}, (\mathscr{F}_t^{w,\bar{w}})_{0\le t\le T}, \mathbb{P})$ be a complete filtered probability space, on which an $\mathscr{F}_t^{w,\bar{w}}$-adapted standard Brownian motion $(w_t, \bar{w}_t)$ is defined in $\mathbb{R}^2$ with $\mathscr{F}_t^{w,\bar{w}}$ being its natural filtration. If $x : \Omega \to \mathbb{R}$ is an $\mathscr{F}_T^{w,\bar{w}}$-measurable, square-integrable random variable, we write $x \in \mathscr{L}^2_{\mathscr{F}_T^{w,\bar{w}}}(\mathbb{R})$. If $x : [0,T] \times \Omega \to \mathbb{R}$ is an $\mathscr{F}_t^{w,\bar{w}}$-adapted, square-integrable process, we write $x \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}(0,T;\mathbb{R})$. Similar notations are adopted for other processes, Euclidean spaces and filtrations.

Next, we consider the decomposition of the LQ optimal control problem.

B. Problem Formulation

Define the processes $(x^0, y^0, z^0, \bar{z}^0)$ and Y^0 by

$$\begin{cases} dx_t^0 = a_t x_t^0 dt + c_t dw_t + \bar{c}_t d\bar{w}_t \\ -dy_t^0 = \left(A_t x_t^0 + B_t y_t^0 + C_t z_t^0 + \bar{C}_t \bar{z}_t^0\right) dt \\ \qquad - z_t^0 dw_t - \bar{z}_t^0 d\bar{w}_t \\ x_0^0 = e_0, \quad y_T^0 = F x_T^0 \end{cases} \tag{1}$$

and

$$\begin{cases} dY_t^0 = f_t x_t^0 dt + h_t dw_t \\ Y_0^0 = 0. \end{cases} \tag{2}$$

Let $v \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}(0,T;\mathbb{R})$ be a control process. Define $(x^1, y^1, z^1, \bar{z}^1)$ and Y^1 by

$$\begin{cases} \dot{x}_t^1 = a_t x_t^1 + b_t v_t + \bar{b}_t \\ -dy_t^1 = \left(A_t x_t^1 + B_t y_t^1 + C_t z_t^1 + \bar{C}_t \bar{z}_t^1 \right. \\ \qquad \left. + D_t v_t + \bar{D}_t\right) dt - z_t^1 dw_t - \bar{z}_t^1 d\bar{w}_t \\ x_0^1 = 0, \quad y_T^1 = F x_T^1 + G \end{cases} \tag{3}$$

and

$$\begin{cases} \dot{Y}_t^1 = f_t x_t^1 + g_t \\ Y_0^1 = 0. \end{cases} \tag{4}$$

Assumption 1: The coefficients a_t, b_t, $\bar{b}_t$, c_t, $\bar{c}_t$, f_t, g_t, h_t, $1/h_t$, A_t, B_t, C_t, $\bar{C}_t$, D_t, and $\bar{D}_t$ are uniformly bounded, deterministic functions. e_0 and F are constants, and $G \in \mathscr{L}^2_{\mathscr{F}_T^{w,\bar{w}}}(\mathbb{R})$.

It is easy to see that (1)–(4) admit unique solutions under Assumption 1, respectively. See e.g., [5], [11], [27]. If we define

$$\begin{aligned} &x_t^v = x_t^0 + x_t^1, \quad y_t^v = y_t^0 + y_t^1, \quad z_t^v = z_t^0 + z_t^1 \\ &\bar{z}_t^v = \bar{z}_t^0 + \bar{z}_t^1, \quad Y_t^v = Y_t^0 + Y_t^1 \end{aligned} \tag{5}$$

it follows from Itô's formula and (1)–(5) that $(x^v, y^v, z^v, \bar{z}^v)$ and Y^v are the unique solutions of:

$$\begin{cases} dx_t^v = \left(a_t x_t^v + b_t v_t + \bar{b}_t\right) dt + c_t dw_t + \bar{c}_t d\bar{w}_t \\ -dy_t^v = \left(A_t x_t^v + B_t y_t^v + C_t z_t^v + \bar{C}_t \bar{z}_t^v \right. \\ \qquad \left. + D_t v_t + \bar{D}_t\right) dt - z_t^v dw_t - \bar{z}_t^v d\bar{w}_t \\ x_0^v = e_0, \quad y_T^v = F x_T^v + G \end{cases} \tag{6}$$

and

$$\begin{cases} dY_t^v = \left(f_t x_t^v + g_t\right) dt + h_t dw_t \\ Y_0^v = 0 \end{cases} \tag{7}$$

where the superscript of $(x^v, y^v, z^v, \bar{z}^v)$ and Y^v emphasizes their dependence on v. In this paper, we say $(x^v, y^v, z^v, \bar{z}^v)$ is the state related to v, and Y^v is the corresponding observation.

Let

$$\mathscr{F}_t^{Y^v} = \sigma\left\{Y_s^v; 0 \le s \le t\right\} \text{ and } \mathscr{F}_t^{Y^0} = \sigma\left\{Y_s^0; 0 \le s \le t\right\}.$$

Then a natural definition of admissible control is $v \in \mathscr{L}^2_{\mathscr{F}^{Y^v}}(0,T;\mathbb{R})$. It implies that we want to determine the control by the observation. However, the circular dependence of the control on the observation leads to an immediate difficulty in determining an optimal control. This is the main reason that (6) and (7) are split in two parts. We now give a definition of admissible control. Set

$$\mathscr{U}_{ad}^0 = \left\{ v | v_t \text{ is an } \mathscr{F}_t^{Y^0}\text{-adapted process with values in } \mathbb{R} \right. \\ \left. \text{such that } \mathbb{E} \sup_{0\le t\le T} v_t^2 < +\infty \right\}.$$

Definition 2.1: A control v is called admissible, if $v \in \mathscr{U}_{ad}^0$ is $\mathscr{F}_t^{Y^v}$-adapted. The set of all admissible controls is denoted by $\mathscr{U}_{ad}$.

The cost functional is of the form

$$\begin{aligned} J[v] = \frac{1}{2}\mathbb{E}\Bigg\{ \int_0^T &\Big[L_t \left(x_t^v\right)^2 + O_t \left(y_t^v\right)^2 + R_t v_t^2 \\ &+ 2 l_t x_t^v + 2 o_t y_t^v + 2 r_t v_t\Big] dt \\ &+ M \left(x_T^v\right)^2 + 2 m x_T^v + N \left(y_0^v\right)^2 + 2 n y_0^v \Bigg\} \end{aligned} \tag{8}$$

where the coefficients satisfy

Assumption 2: $L_t \ge 0$, $O_t \ge 0$, $R_t \ge 0$, l_t, o_t, and r_t are uniformly bounded, deterministic functions. $M \ge 0$, $N \ge 0$, m and n are constants.

Then the LQ control (LQC, for short) problem can be restated as follows.

Problem (LQC): Seek a $u \in \mathscr{U}_{ad}$ such that

$$J[u] = \inf_{v \in \mathscr{U}_{ad}} J[v]$$

subject to (6) and (7).

If such an identity holds, we call u an optimal control, $(x^u, y^u, z^u, \bar{z}^u)$ an optimal state, and Y^u an optimal observation. When there is no confusion, we adopt the notation $(x, y, z, \bar{z}) = (x^u, y^u, z^u, \bar{z}^u)$ and $Y = Y^u$. Our objective is to establish optimality conditions and closed-form optimal solutions in certain particular cases.

C. Preliminary Results

Lemma 2.1: For any $v \in \mathscr{U}_{ad}$, $\mathscr{F}_t^{Y^v} = \mathscr{F}_t^{Y^0}$.

Proof: For any $v \in \mathscr{U}_{ad}$, since v_t is $\mathscr{F}_t^{Y^0}$-adapted, then it follows from (3) that x_t^1 is $\mathscr{F}_t^{Y^0}$-adapted, so is Y_t^1. Then $Y_t^v = Y_t^0 + Y_t^1$ is $\mathscr{F}_t^{Y^0}$-adapted, i.e., $\mathscr{F}_t^{Y^v} \subseteq \mathscr{F}_t^{Y^0}$. In a similar way, we get $\mathscr{F}_t^{Y^0} \subseteq \mathscr{F}_t^{Y^v}$ via the equality $Y_t^0 = Y_t^v - Y_t^1$. The proof is thus complete. ■

The following estimates describe the continuity of state with respect to control, which are derived by Itô's formula and Gronwall's inequality. See also [5], [27] for similar arguments.

Lemma 2.2: Let Assumption 1 hold. For any $v_i \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}(0, T; \mathbb{R})$, let $(x^{v_i}, y^{v_i}, z^{v_i}, \bar{z}^{v_i})$ be the solution of (6) corresponding to v_i $(i = 1, 2)$. Then there is a constant $C_0 > 0$ such that

$$\begin{aligned}
&\sup_{0 \le t \le T} \mathbb{E}\,|x_t^{v_1} - x_t^{v_2}|^2 \\
&\quad \le C_0 \mathbb{E} \int_0^T |v_{1t} - v_{2t}|^2 dt \\
&\sup_{0 \le t \le T} \mathbb{E}\,|y_t^{v_1} - y_t^{v_2}|^2 \\
&\quad \le C_0 \left[\mathbb{E}\,|x_T^{v_1} - x_T^{v_2}|^2 + \int_0^T \sup_{0 \le s \le t} \mathbb{E}\,|x_s^{v_1} - x_s^{v_2}|^2\, dt \right. \\
&\qquad \left. + \mathbb{E} \int_0^T |v_{1t} - v_{2t}|^2 dt \right].
\end{aligned}$$

Lemma 2.3: Under Assumptions 1 and 2, we have

$$\inf_{v' \in \mathscr{U}_{ad}} J[v'] = \inf_{v \in \mathscr{U}_{ad}^0} J[v].$$

Proof: From Definition 2.1, we have $\mathscr{U}_{ad} \subseteq \mathscr{U}_{ad}^0$, and thus

$$\inf_{v' \in \mathscr{U}_{ad}} J[v'] \ge \inf_{v \in \mathscr{U}_{ad}^0} J[v].$$

In what follows, we prove that the reverse inequality holds by three steps.

Step 1: $\mathscr{U}_{ad}$ is dense in $\mathscr{U}_{ad}^0$ under the metric of $\mathscr{L}^2_{\mathscr{F}^{Y^0}}(0, T; \mathbb{R})$.

For any $v \in \mathscr{U}_{ad}^0$, define

$$v_{k,t} = \begin{cases} v_0, & \text{for } 0 \le t \le \delta_k \\ \frac{1}{\delta_k} \int_{(i-1)\delta_k}^{i\delta_k} v_s ds, & \text{for } i\delta_k < t \le (i+1)\delta_k \end{cases}$$

where $v_0 \in \mathbb{R}$, i, k are natural numbers, $1 \le i \le k-1$, and $\delta_k = T/k$. Then $v_{k,t}$ is $\mathscr{F}_{i\delta_k}^{Y^0}$-adapted for any $i\delta_k < t \le (i+1)\delta_k$, and for any k

$$\sup_{0 \le t \le T} |v_{k,t}| \le |v_0| + \sup_{0 \le t \le T} |v_t|.$$

Thus, $v_k \in \mathscr{U}_{ad}^0$. Let $(x^{v_k}, y^{v_k}, z^{v_k}, \bar{z}^{v_k})$ and Y^{v_k} be the trajectories of (6) and (7) corresponding to v_k. Similar to [1], [3], from (3), (4) and the last equality of (5), we verify that v_k is adapted to $\mathscr{F}_t^{Y^0}$ and $\mathscr{F}_t^{Y^{v_k}}$, and $\mathscr{F}_t^{Y^0} = \mathscr{F}_t^{Y^{v_k}}$. Then v_k belongs to $\mathscr{U}_{ad}$, and thus, (6) has a unique solution $(x^{v_k}, y^{v_k}, z^{v_k}, \bar{z}^{v_k}) \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}(0, T; \mathbb{R}^4)$. On the other hand, $v_k \to v$ in probability when $k \to +\infty$. Using again the integrability condition of v in $\mathscr{U}_{ad}^0$, we derive $v_k \to v$ as $k \to +\infty$ in $\mathscr{L}^2_{\mathscr{F}^{Y^0}}(0, T; \mathbb{R})$, i.e., $\mathscr{U}_{ad}$ is dense in $\mathscr{U}_{ad}^0$.

Step 2: $\lim_{k \to +\infty} J[v_k] = J[v]$, where v, v_k and $(x^{v_k}, y^{v_k}, z^{v_k}, \bar{z}^{v_k})$ are defined as in Step 1.

By (8) and Hölder's inequality, it yields

$$\begin{aligned}
&2\,|J[v_k] - J[v]| \\
&\le \sqrt{\mathbb{E} \int_0^T |L_t\,(x_t^{v_k} + x_t^v) + 2l_t|^2\, dt} \sqrt{\mathbb{E} \int_0^T |x_t^{v_k} - x_t^v|^2\, dt} \\
&\quad + \sqrt{\mathbb{E} \int_0^T |O_t\,(y_t^{v_k} + y_t^v) + 2o_t|^2\, dt} \sqrt{\mathbb{E} \int_0^T |y_t^{v_k} - y_t^v|^2\, dt} \\
&\quad + \sqrt{\mathbb{E} \int_0^T |R_t(v_{kt} + v_t) + 2r_t|^2\, dt} \sqrt{\mathbb{E} \int_0^T |v_{kt} - v_t|^2 dt} \\
&\quad + \sqrt{\mathbb{E}\,|M\,(x_T^{v_k} + x_T^v) + 2m|^2} \sqrt{\mathbb{E}\,|x_T^{v_k} - x_T^v|^2} \\
&\quad + \sqrt{\mathbb{E}\,|N\,(y_0^{v_k} + y_0^v) + 2n|^2} \sqrt{\mathbb{E}\,|y_0^{v_k} - y_0^v|^2}.
\end{aligned}$$

Then Lemma 2.2 implies that $J[v_k] \to J[v]$ when k goes to $+\infty$.

Step 3: $\inf_{v' \in \mathscr{U}_{ad}} J[v'] \le \inf_{v \in \mathscr{U}_{ad}^0} J[v]$.

Since $v_k \in \mathscr{U}_{ad}$, then

$$\inf_{v' \in \mathscr{U}_{ad}} J[v'] \le J[v_k]$$

and consequently, $\inf_{v' \in \mathscr{U}_{ad}} J[v'] \le J[v]$ by sending $k \to +\infty$. Due to the arbitrariness of v, the desired inequality holds. Thus, the proof is complete. ■

III. Optimal Solution of Problem (LQC)

A. Optimality Condition

We first establish a necessary condition and then a sufficient condition for optimality of Problem (LQC). According to Lemma 2.3, it suffices to study the optimality of $J[v]$ over $\mathscr{U}_{ad}^0$. In addition, $\mathscr{U}_{ad}^0$ is fixed (i.e., independent of control or state), it is more convenient to get the optimality conditions in $\mathscr{U}_{ad}^0$ by variational method. Note that these results are different from the existing literature, say, [9], [12]–[14], [18], [19], [21], [23], mainly due to the fact that the drift coefficient of the observation equation is linear with respect to the state, and the state noise is correlated to the observation noise.

Theorem 3.1: Let Assumptions 1 and 2 hold. Suppose that u is an optimal control of Problem (LQC), in the sense that

$$\frac{d}{d\varepsilon} J[u + \varepsilon v]|_{\varepsilon=0} = 0 \text{ for any } v \in \mathscr{U}_{ad}$$

and $(x, y, z, \bar{z})$ is the corresponding optimal state. Then the FBSDE

$$\begin{cases} dp_t = (B_t p_t - O_t y_t - o_t)dt + C_t p_t dw_t + \bar{C}_t p_t d\bar{w}_t \\ -dq_t = (a_t q_t - A_t p_t + L_t x_t + l_t)dt - k_t dw_t - \bar{k}_t d\bar{w}_t \\ p_0 = -Ny_0 - n, \quad q_T = -Fp_T + Mx_T + m \end{cases} \tag{9}$$

admits a unique solution $(p, q, k, \bar{k}) \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}(0, T; \mathbb{R}^4)$ such that

$$R_t u_t - D_t \mathbb{E}\left[p_t | \mathscr{F}_t^Y\right] + b_t \mathbb{E}\left[q_t | \mathscr{F}_t^Y\right] + r_t = 0 \tag{10}$$

with

$$\mathscr{F}_t^Y = \sigma\left\{Y_s^u; 0 \le s \le t\right\}.$$

Proof: According to Lemma 2.3, if u is an optimal control of Problem (LQC), then

$$J[u] = \inf_{v \in \mathscr{U}_{ad}^0} J[v].$$

For $v \in \mathscr{U}_{ad}$, we introduce a variational equation

$$\begin{cases} \dot{x}_{1,t} = a_t x_{1,t} + b_t v_t \\ -dy_{1,t} = (A_t x_{1,t} + B_t y_{1,t} + C_t z_{1,t} + \bar{C}_t \bar{z}_{1,t} + D_t v_t)dt \\ \qquad\qquad - z_{1,t} dw_t - \bar{z}_{1,t} d\bar{w}_t \\ x_{1,0} = 0, \quad y_{1,T} = Fx_{1,T} \end{cases}$$

which admits a unique solution $(x_1, y_1, z_1, \bar{z}_1) \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}(0, T; \mathbb{R}^4)$ under Assumption 1. By the optimality of u using the first variation of $J[v]$ with [18, Lemma 2.4], we have

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon} J[u + \varepsilon v]|_{\varepsilon=0} \\ &= \mathbb{E}\left\{\int_0^T [(L_t x_t + l_t)x_{1,t} + (O_t y_t + o_t)y_{1,t} \right. \\ &\qquad + (R_t u_t + r_t)v_t]\, dt \\ &\qquad \left. + (Mx_T + m)x_{1,T} + (Ny_0 + n)y_{1,0}\right\}. \end{aligned} \tag{11}$$

On the other hand, once $(x, y, z, \bar{z})$ is determined, (9) admits a unique solution $(p, q, k, \bar{k}) \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}(0, T; \mathbb{R}^4)$. It follows from Itô's formula in a form of differentiation by parts of the product of two stochastic processes that:

$$\begin{aligned} d(p_t y_{1,t}) = &-[(O_t y_t + o_t)y_{1,t} + (A_t x_{1,t} + D_t v_t)p_t]\, dt \\ &+ p_t(C_t y_{1,t} + z_{1,t})dw_t + p_t(\bar{C}_t y_{1,t} + \bar{z}_{1,t})d\bar{w}_t. \end{aligned}$$

Thus

$$\begin{aligned} &\mathbb{E}\left[Fp_T x_{1,T} + (Ny_0 + n)y_{1,0}\right] \\ &\quad = -\mathbb{E}\int_0^T [(O_t y_t + o_t)y_{1,t} + (A_t x_{1,t} + D_t v_t)p_t]\, dt. \end{aligned} \tag{12}$$

Similarly, by differentiation by parts again

$$\begin{aligned} d(q_t x_{1,t}) = &\left\{[A_t p_t - (L_t x_t + l_t)]\, x_{1,t} + b_t q_t v_t\right\} dt \\ &+ k_t x_{1,t} dw_t + \bar{k}_t x_{1,t} d\bar{w}_t \end{aligned}$$

and then

$$\begin{aligned} &\mathbb{E}\left[(Mx_T + m - Fp_T)x_{1,T}\right] \\ &\quad = \mathbb{E}\int_0^T \left\{[A_t p_t - (L_t x_t + l_t)]\, x_{1,t} + b_t q_t v_t\right\} dt. \end{aligned} \tag{13}$$

Recall that R_t, D_t, b_t, and r_t are by Assumptions 1 and 2 deterministic, and u_t and v_t are $\mathscr{F}_t^{Y^0}$-adapted. Substituting (12) and (13) into (11), we get

$$\begin{aligned} 0 &= \mathbb{E}\int_0^T (R_t u_t - D_t p_t + b_t q_t + r_t)v_t dt \\ &= \mathbb{E}\int_0^T \left(R_t u_t - D_t \mathbb{E}\left[p_t \middle| \mathscr{F}_t^{Y^0}\right] + b_t \mathbb{E}\left[q_t \middle| \mathscr{F}_t^{Y^0}\right] + r_t\right) v_t dt. \end{aligned}$$

Hence

$$R_t u_t - D_t \mathbb{E}\left[p_t \middle| \mathscr{F}_t^{Y^0}\right] + b_t \mathbb{E}\left[q_t \middle| \mathscr{F}_t^{Y^0}\right] + r_t = 0.$$

Furthermore, since $u \in \mathscr{U}_{ad}$, it follows from Lemma 2.1 that $\mathscr{F}_t^{Y^0} = \mathscr{F}_t^Y$, and thus the desired conclusion. ■

We now study the sufficiency of the above result. Introduce an FBSDE with (10)

$$\begin{cases} dx_t = (a_t x_t + b_t u_t + \bar{b}_t)dt + c_t dw_t + \bar{c}_t d\bar{w}_t \\ -dy_t = (A_t x_t + B_t y_t + C_t z_t + \bar{C}_t \bar{z}_t + D_t u_t + \bar{D}_t)dt \\ \qquad\qquad - z_t dw_t - \bar{z}_t d\bar{w}_t \\ dp_t = (B_t p_t - O_t y_t - o_t)dt + C_t p_t dw_t + \bar{C}_t p_t d\bar{w}_t \\ -dq_t = (a_t q_t - A_t p_t + L_t x_t + l_t)dt - k_t dw_t - \bar{k}_t d\bar{w}_t \\ x_0 = e_0, \quad y_T = Fx_T + G \\ p_0 = -Ny_0 - n, \quad q_T = -Fp_T + Mx_T + m \end{cases} \tag{14}$$

which is called a generalized stochastic Hamiltonian system in the field of Pontryagin's maximum principle. If a process $(x, y, z, \bar{z}, p, q, k, \bar{k}) \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}(0, T; \mathbb{R}^8)$ satisfies (14), we call it an (adapted) solution of (14). See e.g., Ma and Yong [11], Yong and Zhou [27] for more details.

Theorem 3.2: Suppose that Assumptions 1 and 2 hold. Let $u \in \mathscr{U}_{ad}$ satisfy

$$R_t u_t - D_t \mathbb{E}\left[p_t \middle| \mathscr{F}_t^Y\right] + b_t \mathbb{E}\left[q_t \middle| \mathscr{F}_t^Y\right] + r_t = 0$$

where $(x, y, z, \bar{z}, p, q, k, \bar{k})$ is a solution to (14). Then u is an optimal control of Problem (LQC).

Proof: For any admissible control v, the total variation is

$$J[v] - J[u] = I + II \tag{15}$$

with $(x, y) = (x^u, y^u)$, the pure quadratic part is

$$\begin{aligned} I = \frac{1}{2}\mathbb{E}&\left\{\int_0^T \left[L_t\, (x_t^v - x_t)^2 + O_t\, (y_t^v - y_t)^2 + R_t (v_t - u_t)^2\right] dt \right. \\ &\left. + M\, (x_T^v - x_T)^2 + N(y_0^v - y_0)^2\right\} \end{aligned}$$

and the quasi-linear part is

$$II=\mathbb{E}\left\{\int_0^T [(L_t x_t + l_t)(x_t^v - x_t) + (O_t y_t + o_t)(y_t^v - y_t) + (R_t u_t + r_t)(v_t - u_t)]\,dt + (Mx_T + m)(x_T^v - x_T) + (Ny+n)(y_0^v - y_0)\right\}. \tag{16}$$

Note that $I \geq 0$ holds for any admissible control v. Then it is enough to prove that $II = 0$.

It follows from Itô's formula that:

$$\begin{aligned} d\left[p_t\left(y_t^v - y_t\right)\right] = &-\{(O_t y_t + o_t)(y_t^v - y_t) \\ &+ p_t\left[A_t\left(x_t^v - x_t\right) + D_t(v_t - u_t)\right]\}\,dt \\ &+ p_t\left[C_t\left(y_t^v - y_t\right) + z_t^v - z_t\right] dw_t \\ &+ p_t\left[\bar{C}_t\left(y_t^v - y_t\right) + \bar{z}_t^v - \bar{z}_t\right] d\bar{w}_t. \end{aligned}$$

Thus, by using differentiation by parts, similar to (12), we get

$$\begin{aligned} &\mathbb{E}\left[Fp_T\left(x_T^v - x_T\right) + (n + Ny_0)\left(y_0^v - y_0\right)\right] \\ &= -\mathbb{E}\int_0^T \{(O_t y_t + o_t)(y_t^v - y_t) \\ &\qquad + p_t\left[A_t\left(x_t^v - x_t\right) + D_t(v_t - u_t)\right]\}\,dt. \end{aligned} \tag{17}$$

Similarly by differentiation by parts, similar to (13), we obtain

$$\begin{aligned} d\left[q_t(x_t^v - x_t)\right] = &\left[(A_t p_t - L_t x_t - l_t)(x_t^v - x_t) + b_t q_t(v_t - u_t)\right] dt \\ &+ k_t(x_t^v - x_t)\,dw_t + \bar{k}_t(x_t^v - x_t)\,d\bar{w}_t \end{aligned}$$

and then

$$\begin{aligned} &\mathbb{E}[(Mx_T - Fp_T + m)(x_T^v - x_T)] \\ &= \mathbb{E}\int_0^T [b_t q_t(v_t - u_t) + (A_t p_t - L_t x_t - l_t)(x_t^v - x_t)]\,dt. \end{aligned} \tag{18}$$

Plugging (17) and (18) into (16) and using Lemma 2.1, we have

$$\begin{aligned} II &= \mathbb{E}\int_0^T (R_t u_t - D_t p_t + b_t q_t + r_t)(v_t - u_t)dt \\ &= \mathbb{E}\int_0^T \left(R_t u_t - D_t \mathbb{E}\left[p_t \middle| \mathscr{F}_t^{Y^0}\right] + b_t \mathbb{E}\left[q_t \middle| \mathscr{F}_t^{Y^0}\right] + r_t\right) \\ &\quad \times (v_t - u_t)dt \\ &= 0. \end{aligned}$$

Then the proof is complete. ∎

Assumption 3: $R_t > 0$ and $1/R_t$ are uniformly bounded, deterministic functions.

Corollary 3.1: Let Assumptions 1, 2, and 3 hold. If u is an optimal control of Problem (LQC), then u is unique.

Proof: Let u and $\tilde{u}$ be both optimal controls of Problem (LQC) with the same optimum, and let $(x, y, z, \bar{z})$ and $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\bar{z}})$ be the corresponding optimal states. It is easy to see that $((x + \tilde{x})/2, (y + \tilde{y})/2, (z + \tilde{z})/2, (\bar{z} + \tilde{\bar{z}})/2)$ is the state corresponding to $(u + \tilde{u})/2$. Then

$$\begin{aligned} 2J[u] &= J[u] + J[\tilde{u}] \\ &= 2J\left[\frac{u + \tilde{u}}{2}\right] \\ &\quad + \mathbb{E}\left\{\int_0^T \left[L_t\left(\frac{x_t - x_t^v}{2}\right)^2 + O_t\left(\frac{y_t - y_t^v}{2}\right)^2 + R_t\left(\frac{u_t - \tilde{u}_t}{2}\right)^2\right] dt \right. \\ &\quad \left. + M\left(\frac{x_T - x_T^v}{2}\right)^2 + N\left(\frac{y_0 - y_0^v}{2}\right)^2\right\} \\ &\geq 2J[u] + \mathbb{E}\int_0^T R_t\left(\frac{u_t - \tilde{u}_t}{2}\right)^2 dt. \end{aligned}$$

Since $R_t > 0$, we have $u = \tilde{u}$. ∎

B. Filtering

If Assumptions 1, 2, and 3 hold, it follows from (10) that:

$$u_t = \frac{1}{R_t}\left(D_t \mathbb{E}\left[p_t \middle| \mathscr{F}_t^Y\right] - b_t \mathbb{E}\left[q_t \middle| \mathscr{F}_t^Y\right] - r_t\right).$$

This shows that it is necessary to compute the optimal filtering of (p_t, q_t) depending on $\mathscr{F}_t^Y$ so as to represent u_t. Furthermore, since (p, q) is closely related to (x, y), we need to analyze the optimal filtering of FBSDEs (6) and (9). The earliest work on filtering for FBSDEs was traced back to Wang and Wu [16], where a Feynman-Kac formula method was used to calculate the filtering of linear FBSDEs. See also Xiao and Wang [24] for the case with random jump. Note that it seems that the Feynman-Kac formula method dose not work here, mainly due to the existence of the control v.

Recall that Y^v governed by (7) is the observation. For any $v \in \mathscr{U}_{ad}$, let

$$\begin{aligned} &\hat{\xi}_t = \mathbb{E}\left[\xi_t \middle| \mathscr{F}_t^{Y^v}\right] \text{ with } \xi_t = x_t^0, x_t^v, y_t^v, z_t^v, \bar{z}_t^v, p_t, q_t, k_t, x_t^v y_t^v \\ &\hat{G} = \mathbb{E}\left[G \middle| \mathscr{F}_T^{Y^v}\right] \text{ and } P_t = \mathbb{E}\left(x_t^v - \hat{x}_t^v\right)^2 \end{aligned} \tag{19}$$

be the optimal filtering and the mean square error of ξ_t, G and $\hat{x}_t^v$, respectively. See e.g., Liptser and Shirayev [10], Elliott, Aggoun and Moore [6] and Xiong [25] for more details.

We state a filtering result of (6), which plays an important role in representing optimal control.

Lemma 3.1: Let Assumption 1 hold. For any $v \in \mathscr{U}_{ad}$, the optimal filtering $(\hat{x}_t^v, \hat{y}_t^v, \hat{z}_t^v, \hat{\bar{z}}_t^v)$ of the solution $(x_t^v, y_t^v, z_t^v, \bar{z}_t^v)$ to (6) with respect to $\mathscr{F}_t^{Y^v}$ satisfies

$$\begin{cases} d\hat{x}_t^v = \left(a_t \hat{x}_t^v + b_t v_t + \bar{b}_t\right) dt + \left(c_t + \dfrac{P_t f_t}{h_t}\right) d\hat{w}_t \\ -d\hat{y}_t^v = \left(A_t \hat{x}_t^v + B_t \hat{y}_t^v + C_t \hat{z}_t^v + \bar{C}_t \hat{\bar{z}}_t^v \right. \\ \qquad\qquad \left. + D_t v_t + \bar{D}_t\right) dt - \hat{Z}_t^v d\hat{w}_t \\ \hat{x}_0^v = e_0, \quad \hat{y}_T^v = F\hat{x}_T^v + \hat{G} \end{cases} \tag{20}$$

where the mean square error P_t of the estimate $\hat{x}_t^v$ is the unique solution of

$$\begin{cases} \dot{P}_t - 2a_t P_t + \left(c_t + \frac{P_t f_t}{h_t}\right)^2 - (c_t + \bar{c}_t)^2 = 0 \\ P_0 = 0 \end{cases} \tag{21}$$

$$\hat{w}_t = \int_0^t \frac{1}{h_s}\left[dY_s - (f_s \hat{x}_s^v + g_s)\,ds\right] = \int_0^t \frac{f_s}{h_s}\left(x_s^v - \hat{x}_s^v\right) ds + w_t \tag{22}$$

is a standard Brownian motion with values in $\mathbb{R}$, and

$$\hat{Z}_t^v = \hat{z}_t^v + \frac{f_t}{h_t}\left(\widehat{x_t^v y_t^v} - \hat{x}_t^v \hat{y}_t^v\right). \tag{23}$$

Proof: Since x_t^1 is $\mathscr{F}_t^{Y^0}$-adapted, it follows from (5) and Lemma 2.1 that

$$\hat{x}_t^v = \mathbb{E}\left[x_t^v \,\middle|\, \mathscr{F}_t^{Y^v}\right] = \mathbb{E}\left[x_t^0 \,\middle|\, \mathscr{F}_t^{Y^0}\right] + x_t^1 = \hat{x}_t^0 + x_t^1.$$

Applying [10, Theorem 12.1] to (2) and the first equation in (1), then (21), (22) and the first equation in (20) are derived. By the way, $\hat{x}_t^v$ can also be derived by virtue of [10, Theorem 8.1]. However, the former is more direct than the later.

The rest is to establish the optimal filtering of $(y^v, z^v, \bar{z}^v)$. The idea is inspired by [20]. For any $v \in \mathscr{U}_{ad}$, the backward stochastic differential equation (BSDE, for short) in (6) admits a unique solution $(y^v, z^v, \bar{z}^v)$. Then

$$y_t^v = y_0^v - \int_0^t \left(A_s x_s^v + B_s y_s^v + C_s z_s^v + \bar{C}_s \bar{z}_s^v + D_s v_s + \bar{D}_s\right) ds + \int_0^t z_s^v dw_s + \int_0^t \bar{z}_s^v d\bar{w}_s \tag{24}$$

and hence, the integral form of the BSDE in (6) can be rewritten as $y_t^v - y_T^v$. Now we say (24) and (7) are the state equation and the observation equation, respectively. Using [10, Theorem 8.1], we get

$$\hat{y}_t^v = y_0^v - \int_0^t \left(A_s \hat{x}_s^v + B_s \hat{y}_s^v + C_s \hat{z}_s^v + \bar{C}_s \hat{\bar{z}}_s^v + D_s v_s + \bar{D}_s\right) ds + \int_0^t \hat{Z}_s^v d\hat{w}_s$$

with $y_0^v = \hat{y}_0^v$. Then it yields

$$\hat{y}_t^v = \hat{y}_T^v + \int_t^T \left(A_s \hat{x}_s^v + B_s \hat{y}_s^v + C_s \hat{z}_s^v + \bar{C}_s \hat{\bar{z}}_s^v + D_s v_s + \bar{D}_s\right) ds - \int_t^T \hat{Z}_s^v d\hat{w}_s.$$

Moreover

$$\hat{y}_T^v = \mathbb{E}\left[F x_T^v + G \,\middle|\, \mathscr{F}_T^{Y^v}\right] = F\hat{x}_T^v + \hat{G}$$

where $\hat{x}_T^v$ is determined by the SDE in (20). Then the proof is complete. ∎

We highlight that $\hat{Z}_t^v$ defined by (23) is a part of solution $(\hat{y}_t^v, \hat{Z}_t^v)$ to the BSDE in (20), which can be computed by the Malliavin derivative of $\hat{y}_t^v$ with respect to $\hat{w}_t$ under some standard conditions [5].

C. Feedback

In this section, we do our best to give a feedback form of the optimal control u. We assume that $O_t = 0$ for simplicity. It implies that the running cost part of (8) does not include the quadratic term of the state y. Similar to (19), let

$$\begin{aligned} \widehat{x_t q_t} &= \mathbb{E}\left[x_t q_t \,\middle|\, \mathscr{F}_t^Y\right], \quad \widehat{x_t^{\mathbf{m}}} = \mathbb{E}\left[x_t^{\mathbf{m}} \,\middle|\, \mathscr{F}_t^Y\right] \\ \widehat{x_t^{\mathbf{m}} p_t} &= \mathbb{E}\left[x_t^{\mathbf{m}} p_t \,\middle|\, \mathscr{F}_t^Y\right], \qquad \mathbf{m} = 1, 2, 3, \ldots \end{aligned}$$

be the optimal filters of $x_t q_t$, $x_t^{\mathbf{m}}$, $x_t^{\mathbf{m}} p_t$ based on the observation Y up to time t, respectively.

We now state a filtering result of the adjoint equation of Problem (LQC).

Lemma 3.2: Let Assumptions 1, 2 and $O_t = 0$ hold. The optimal filtering of (p_t, q_t, k_t) depending on $\mathscr{F}_t^Y$ satisfies

$$\begin{cases} d\hat{p}_t = (B_t \hat{p}_t - o_t)dt + \left[C_t \hat{p}_t + \dfrac{f_t}{h_t}(\widehat{x_t p_t} - \hat{x}_t \hat{p}_t)\right] d\hat{w}_t \\ -d\hat{q}_t = (a_t \hat{q}_t - A_t \hat{p}_t + L_t \hat{x}_t + l_t)dt - \hat{K}_t d\hat{w}_t \\ \hat{p}_0 = -N y_0 - n, \quad \hat{q}_T = M\hat{x}_T - F\hat{p}_T + m \end{cases} \tag{25}$$

with

$$\hat{K}_t = \hat{k}_t + \frac{f_t}{h_t}(\widehat{x_t q_t} - \hat{x}_t \hat{q}_t)$$

where $(\hat{x}, \hat{y})$, $\hat{w}$ and $\widehat{x^{\mathbf{m}} p}$ satisfy (20) with $v = u$, (22), and

$$\begin{aligned} d\widehat{x_t^{\mathbf{m}} p_t} = &\Big[(\mathbf{m} a_t + B_t)\widehat{x_t^{\mathbf{m}} p_t} - o_t \widehat{x_t^{\mathbf{m}}} \\ &+ \mathbf{m}(b_t u_t + \bar{b}_t + c_t C_t + \bar{c}_t \bar{C}_t)\widehat{x_t^{\mathbf{m}-1} p_t}\Big] dt \\ &+ \Big[\mathbf{m} c_t \widehat{x_t^{\mathbf{m}-1} p_t} + C_t \widehat{x_t^{\mathbf{m}} p_t} \\ &+ \frac{f_t}{h_t}\left(\widehat{x_t^{\mathbf{m}+1} p_t} - \hat{x}_t \widehat{x_t^{\mathbf{m}} p_t}\right)\Big] d\hat{w}_t \\ \widehat{x_0^{\mathbf{m}} p_0} = &-e_0^{\mathbf{m}}(N y_0 + n), \qquad \mathbf{m} = 1, 2, 3, \ldots \end{aligned}$$

respectively.

Proof: Since the proof is similar to Lemma 3.1, we omit it for simplicity. ∎

Theorem 3.3: Let Assumptions 1, 2, 3, and $O_t = 0$ hold. If

$$u_t = \frac{1}{R_t}(D_t \hat{p}_t - b_t \hat{q}_t - r_t)$$

is the optimal control of Problem (LQC), then it can be represented as

$$u_t = \frac{1}{R_t}\left[(D_t - b_t \Sigma_t)\hat{p}_t - b_t \pi_t \hat{x}_t - b_t \theta_t - r_t\right]$$

where $(\hat{x}, \hat{y}, \hat{z}, \hat{\bar{z}})$, $(\hat{p}, \hat{q}, \hat{k})$, π, Σ and θ are the solutions of (20) with $v = u$, (25), (29)–(31), respectively.

Proof: For any $v \in \mathscr{U}_{ad}$, (6) admits a unique solution $(x^v, y^v, z^v, \bar{z}^v)$, and consequently, (9) admits a unique solution $(p, q, k, \bar{k})$. Noting the terminal condition of (9), we set

$$q_t = \pi_t x_t + \Sigma_t p_t + \theta_t \quad \text{with } \pi_T = M, \quad \Sigma_T = -F \text{ and } \theta_T = m \tag{26}$$

where π, Σ, and θ are deterministic and differentiable functions. Substituting (26) into

$$u_t = \frac{1}{R_t}(D_t \hat{p}_t - b_t \hat{q}_t - r_t)$$

and applying Itô's formula to (26), we obtain

$$\begin{aligned} dq_t &= \dot{\pi}_t x_t dt + \pi_t dx_t + \dot{\Sigma}_t p_t dt + \Sigma_t dp_t + \dot{\theta}_t dt \\ &= \Bigg\{ \dot{\pi}_t x_t + \pi_t \Bigg[a_t x_t + \frac{1}{R_t} b_t ((D_t - b_t \Sigma_t)\hat{p}_t \\ &\qquad - b_t \pi_t \hat{x}_t - b_t \theta_t - r_t) + \bar{b}_t \Bigg] \\ &\quad + \dot{\Sigma}_t p_t + \Sigma_t (B_t p_t - o_t) + \dot{\theta}_t \Bigg\} dt \\ &\quad + (c_t \pi_t + C_t p_t \Sigma_t) dw_t + (\bar{c}_t \pi_t + \bar{C}_t p_t \Sigma_t) d\bar{w}_t. \end{aligned}$$

Comparing the above equality with (9), it yields

$$k_t = c_t \pi_t + C_t p_t \Sigma_t, \quad \bar{k}_t = \bar{c}_t \pi_t + \bar{C}_t p_t \Sigma_t$$

and

$$\begin{aligned} &\dot{\pi}_t x_t + \pi_t \left\{ a_t x_t + \frac{1}{R_t} b_t [(D_t - b_t \Sigma_t)\hat{p}_t - b_t \pi_t \hat{x}_t - b_t \theta_t - r_t] + \bar{b}_t \right\} \\ &\qquad + \dot{\Sigma}_t p_t + \Sigma_t (B_t p_t - o_t) + \dot{\theta}_t \\ &\quad = -(a_t q_t - A_t p_t + L_t x_t + l_t) \\ &\quad = -(a_t \pi_t + L_t) x_t - (a_t \Sigma_t - A_t) p_t - a_t \theta_t - l_t. \end{aligned} \tag{27}$$

Taking the conditional expectation $\mathbb{E}[\cdot|\mathscr{F}_t^Y]$ on both sides of (27)

$$\begin{aligned} &\dot{\pi}_t \hat{x}_t + \pi_t \Bigg[a_t \hat{x}_t + \frac{1}{R_t} b_t ((D_t - b_t \Sigma_t)\hat{p}_t - b_t \pi_t \hat{x}_t - b_t \theta_t - r_t) + \bar{b}_t \Bigg] \\ &\qquad + \dot{\Sigma}_t \hat{p}_t + \Sigma_t (B_t \hat{p}_t - o_t) + \dot{\theta}_t \\ &\quad = -(a_t \pi_t + L_t) \hat{x}_t - (a_t \Sigma_t - A_t) \hat{p}_t - a_t \theta_t - l_t. \end{aligned} \tag{28}$$

Comparing the coefficients of $\hat{x}_t$ and $\hat{p}_t$ in (28), we derive

$$\begin{cases} \dot{\pi}_t + 2a_t \pi_t - \frac{1}{R_t} b_t^2 \pi_t^2 + L_t = 0 \\ \pi_T = M \end{cases} \tag{29}$$

$$\begin{cases} \dot{\Sigma}_t + \left(a_t + B_t - \frac{1}{R_t} b_t^2 \pi_t \right) \Sigma_t + \frac{1}{R_t} b_t D_t \pi_t - A_t = 0 \\ \Sigma_T = -F \end{cases} \tag{30}$$

and then

$$\begin{cases} \dot{\theta}_t + \left(a_t - \frac{1}{R_t} b_t^2 \pi_t \right) \theta_t - o_t \Sigma_t - \frac{1}{R_t} b_t r_t \pi_t + \bar{b}_t \pi_t + l_t = 0 \\ \theta_T = m. \end{cases} \tag{31}$$

It is clear that there are unique solutions to them, respectively. Thus we have

$$u_t = \frac{1}{R_t} \left[(D_t - b_t \Sigma_t)\hat{p}_t - b_t \pi_t \hat{x}_t - b_t \theta_t - r_t \right].$$

The proof is then complete. ■

Remark 3.1: The optimal control u in Theorem 3.3 can be represented as the feedback of $(\hat{x}, \hat{y})$ under some additional assumptions, say, $A_t = D_t = 0$.

Remark 3.2: The integrability condition in Definition 2.1 plays an important role in proving a density property of $\mathscr{U}_{ad}$. If $\mathscr{F}_t^{Y^v}$ does not depend on control or state, the integrability condition can be weakened. For example, let $f_t = 0$ in Problem (LQC). It is easy to see that $\mathscr{U}_{ad} = \mathscr{U}_{ad}^0$ via $\mathscr{F}_t^{Y^v} = \mathscr{F}_t^{Y^0} = \sigma\{w_s; 0 \le s \le t\}$, and thus, Lemma 2.3 holds automatically. Then the integrability condition can be relaxed to

$$\mathbb{E} \int_0^T v_s^2 ds < +\infty.$$

Moreover, all the results obtained in Section III hold true under the weakened assumption.

IV. Some Special Cases of Problem (LQC)

This section is devoted to some special cases of Problem (LQC). We adopt Assumptions 1, 2, 3 and notation introduced in Sections II and III unless noted otherwise.

A. Preliminary

For the convenience of the reader, we first cite an existence and uniqueness theorem of FBSDEs. Consider an FBSDE

$$\begin{cases} dx_t = b(\omega, t, x_t, y_t, z_t)dt + \sigma(\omega, t, x_t, y_t, z_t)dw_t \\ -dy_t = f(\omega, t, x_t, y_t, z_t)dt - z_t dw_t \\ x_0 = \Psi(y_0), \quad y_T = \xi \end{cases}$$

where $(\omega, t, x, y, z) \in \Omega \times [0,T] \times \mathbb{R}^3$, $b: \Omega \times [0,T] \times \mathbb{R}^3 \to \mathbb{R}$, $\sigma : \Omega \times [0,T] \times \mathbb{R}^3 \to \mathbb{R}$, $f : \Omega \times [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ and $\Psi : \mathbb{R} \to \mathbb{R}$ are four mappings. Define

$$\vartheta = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } A(t, \vartheta) = \begin{pmatrix} -f \\ b \\ \sigma \end{pmatrix} (\omega, t, \vartheta).$$

Assumption 4:

i) $A(t, \vartheta)$ and $\Psi(x)$ are uniformly Lipschitz with respect to ϑ and x, respectively. For each ϑ, $A(\cdot, \vartheta)$ belongs to $\mathscr{L}^2_{\mathscr{F}^w}(0,T;\mathbb{R}^3)$ with $\mathscr{F}_t^w = \sigma\{w_s; 0 \le s \le t\}$.

ii)

$$\begin{cases} \left(A(t,\vartheta) - A(t,\tilde{\vartheta}) \right) (\vartheta - \tilde{\vartheta}) \le -\beta_1 \check{x}^2 - \beta_2 (\check{y}^2 + \check{z}^2) \\ (\Psi(y) - \Psi(\tilde{y}))(y - \tilde{y}) \le -\mu \check{y}^2 \\ \forall \vartheta = (x,y,z), \quad \tilde{\vartheta} = (\tilde{x}, \tilde{y}, \tilde{z}), \quad \check{x} = x - \tilde{x} \\ \check{y} = y - \tilde{y}, \quad \check{z} = z - \tilde{z} \end{cases}$$

where β_1, β_2, and μ are non-negative constants with $\beta_1 + \beta_2 > 0$ and $\beta_1 + \mu > 0$.

Lemma 4.1: (Yu and Ji [28]) Let Assumption 4 hold. The FBSDE admits a unique solution $(x,y,z) \in \mathscr{L}^2_{\mathscr{F}^w}(0,T;\mathbb{R}^3)$.

See also Yong [26] for the recent development on the theory of FBSDEs.

B. An LQ Optimal Control of BSDEs With Partial Information

Example 4.1: Consider the following optimal control problem:

$\inf_{v \in \mathscr{U}_{ad}} J[v]$ with the cost functional

$$J[v] = \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left[O_t (y_t^v)^2 + R_t v_t^2 \right] dt + N (y_0^v)^2 + 2n y_0^v \right\}$$

subject to the state equation

$$\begin{cases} -dy_t^v = \left(B_t y_t^v + C_t z_t^v + \bar{C}_t \bar{z}_t^v + D_t v_t \right) dt - z_t^v dw_t - \bar{z}_t^v d\bar{w}_t \\ y_T^v = G. \end{cases}$$

Suppose that w_t is observable at time t. It can be regarded as the case of (7) with $f_t = g_t = 0$ and $h_t = 1$. Clearly, $\mathscr{F}_t^{Y^v} = \mathscr{F}_t^{Y^0} = \sigma\{w_s; 0 \le s \le t\}$, i.e., $\mathscr{F}_t^{Y^v}$ is given a prior. Just as noted in Remark 3.2, the integrability condition in $\mathscr{U}_{ad}$ can be replaced by

$$\mathbb{E}\int_0^T v_t^2 dt < +\infty.$$

The observation equation looks simple, but the classical separation principle still does not work, then the resulting mathematical deductions are non-trivial in the fields of filtering and control. In order to obtain a unique closed-form solution, many theoretical results such as Theorems 3.1, 3.2, 3.3, Corollary 3.1, Lemmas 3.1, 3.2, 4.1, and Remark 3.2 are used here. It should be emphasized that [9] first proposed the problem with $n = 0$. However, they were unable to completely solve the problem due to the limit of techniques used there. In detail, they were not able to include the important process $\bar{C}\bar{z}$ in the drift term of the state equation. In fact, the solution to this example is one of the motivations of the current paper.

We use four steps to solve the problem.

Step 1—Candidate Optimal Control: According to Theorem 3.1, if u is optimal, then it is necessary to satisfy

$$u_t = \frac{1}{R_t} D_t \mathbb{E}\left[p_t \,\middle|\, \mathscr{F}_t^Y\right]$$

where $\mathscr{F}_t^Y = \sigma\{w_s; 0 \le s \le t\}$, and $(p, y, z, \bar{z}) \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}(0, T; \mathbb{R}^4)$ is the unique solution of the Hamiltonian system

$$\begin{cases} dp_t = (B_t p_t - O_t y_t)dt + C_t p_t dw_t + \bar{C}_t p_t d\bar{w}_t \\ -dy_t = (B_t y_t + C_t z_t + \bar{C}_t \bar{z}_t + D_t u_t)dt \\ \qquad\quad - z_t dw_t - \bar{z}_t d\bar{w}_t \\ p_0 = -N y_0 - n, \quad y_T = G. \end{cases} \tag{32}$$

Using Lemmas 3.1 and 3.2 to (32), we get the optimal filtering $(\hat{p}_t, \hat{y}_t, \hat{z}_t, \hat{\bar{z}}_t)$ of $(p_t, y_t, z_t, \bar{z}_t)$ with respect to $\mathscr{F}_t^Y$, which is governed by

$$\begin{cases} d\hat{p}_t = (B_t \hat{p}_t - O_t \hat{y}_t)dt + C_t \hat{p}_t dw_t \\ -d\hat{y}_t = (B_t \hat{y}_t + C_t \hat{z}_t + \bar{C}_t \hat{\bar{z}}_t + D_t u_t)dt - \hat{z}_t dw_t \\ \hat{p}_0 = -N y_0 - n, \quad \hat{y}_T = \hat{G}. \end{cases} \tag{33}$$

Note that, (33) is not a standard FBSDE because an additional filtering estimate $\hat{\bar{z}}$ is contained. Therefore, its existence and uniqueness is not an immediate result of Lemma 4.1.

Step 2—Existence and Uniqueness of FBSDE (36): We first introduce two differential equations

$$\begin{cases} \dot{\alpha}_t - \left(2B_t + C_t^2 + \bar{C}_t^2\right)\alpha_t - \frac{1}{R_t} D_t^2 \alpha_t^2 + O_t = 0 \\ \alpha_0 = -N \end{cases} \tag{34}$$

and

$$\begin{cases} \dot{\beta}_t - \left(B_t + C_t^2 + \bar{C}_t^2 + \frac{1}{R_t} D_t^2 \alpha_t\right)\beta_t = 0 \\ \beta_0 = -n. \end{cases} \tag{35}$$

In fact, these two equations can be obtained similar to (29)–(31). See also Step 3 of this example for more details.

It is well known that (34) and (35) admit unique solutions, which are uniformly bounded in $[0, T]$, respectively. Next, we introduce a standard FBSDE

$$\begin{cases} d\hat{p}_t = (B_t \hat{p}_t - O_t \hat{y}_t)dt + C_t \hat{p}_t dw_t \\ -d\hat{y}_t = \left[B_t \hat{y}_t + C_t \hat{z}_t + \left(\frac{1}{\alpha_t}\bar{C}_t^2 + \frac{1}{R_t} D_t^2\right)\hat{p}_t\right] dt - \hat{z}_t dw_t \\ \hat{p}_0 = -N y_0 - n, \quad \hat{y}_T = \hat{G} \end{cases} \tag{36}$$

which is subject to an additional assumption condition as follows.

Assumption 5: The solution α of (34) satisfies

$$\frac{1}{\alpha_t}\bar{C}_t^2 + \frac{1}{R_t} D_t^2 \ge 0.$$

Clearly, (36) satisfies Assumption 4. Thus, it follows from Lemma 4.1 that (36) admits a unique solution $(\hat{p}, \hat{y}, \hat{z}) \in \mathscr{L}^2_{\mathscr{F}^Y}(0, T; \mathbb{R}^3)$.

Step 3—Equivalence Between (33) and (36): We first prove that the solution $(\hat{p}, \hat{y}, \hat{z})$ of (36) is a solution of (33). Set

$$u_t = \frac{1}{R_t} D_t \hat{p}_t.$$

Then u is an admissible control, and consequently, (32) admits a unique solution $(p, y, z, \bar{z}) \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}(0, T; \mathbb{R}^4)$. Similar to Theorem 3.3, set

$$p_t = \alpha_t y_t + \beta_t \quad \text{with } \alpha_0 = -N \text{ and } \beta_0 = -n \tag{37}$$

where α_t and β_t are deterministic and differentiable functions. Combining (32) with

$$u_t = \frac{1}{R_t} D_t \hat{p}_t$$

we get by Itô's formula

$$\begin{aligned} dp_t &= \dot{\alpha}_t y_t dt + \alpha_t dy_t + \dot{\beta}_t dt \\ &= \left[\dot{\alpha}_t y_t - \left(B_t y_t + C_t z_t + \bar{C}_t \bar{z}_t + \frac{1}{R_t} D_t^2 \hat{p}_t\right)\alpha_t + \dot{\beta}_t\right] dt \\ &\quad + \alpha_t z_t dw_t + \alpha_t \bar{z}_t d\bar{w}_t. \end{aligned}$$

Comparing the above equality with the SDE in (32), we have

$$\alpha_t z_t = C_t p_t, \quad \alpha_t \bar{z}_t = \bar{C}_t p_t \tag{38}$$

and

$$\dot{\alpha}_t y_t - \left(B_t y_t + C_t z_t + \bar{C}_t \bar{z}_t + \frac{1}{R_t} D_t^2 \hat{p}_t\right)\alpha_t + \dot{\beta}_t = B_t p_t - O_t y_t. \tag{39}$$

Taking $\mathbb{E}[\cdot|\mathscr{F}_t^Y]$ on both sides of (38) and (39), it yields

$$\alpha_t \hat{z}_t = C_t \hat{p}_t, \quad \alpha_t \hat{\bar{z}}_t = \bar{C}_t \hat{p}_t \tag{40}$$

and

$$\dot{\alpha}_t \hat{y}_t - \left(B_t \hat{y}_t + C_t \hat{z}_t + \bar{C}_t \hat{\bar{z}}_t + \frac{1}{R_t} D_t^2 \hat{p}_t\right)\alpha_t + \dot{\beta}_t = B_t \hat{p}_t - O_t \hat{y}_t. \tag{41}$$

According to the second equation of (40), it is easy to see that the solution $(\hat{p}, \hat{y}, \hat{z})$ of (36) solves (33). By the way, (40) and (41) show us how to get (34) and (35).

Next, we prove that for fixed u, if (33) admits a solution $(\hat{p}, \hat{y}, \hat{z}, \hat{\bar{z}}) \in \mathscr{L}^2_{\mathscr{F}^Y}(0, T; \mathbb{R}^4)$, then $(\hat{p}, \hat{y}, \hat{z}, \hat{\bar{z}})$ satisfies (36).

Take

$$u_t = \frac{1}{R_t} D_t \hat{p}_t.$$

Then there exists a unique solution $(p, y, z, \bar{z}) \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}(0, T; \mathbb{R}^4)$ to (32). Similar to the above analysis, it yields

$$\alpha_t \hat{\bar{z}}_t = \bar{C}_t \hat{p}_t$$

where α is the unique solution of (34). Putting

$$\hat{\bar{z}}_t = \frac{1}{\alpha_t} \bar{C}_t \hat{p}_t \text{ and } u_t = \frac{1}{R_t} D_t \hat{p}_t$$

into (33), we arrive at (36), which admits a unique solution $(\hat{p}, \hat{y}, \hat{z}) \in \mathscr{L}^2_{\mathscr{F}^Y}(0, T; \mathbb{R}^3)$. This shows that the solution $(\hat{p}, \hat{y}, \hat{z}, \hat{\bar{z}})$ of (33) is a solution of (36).

Therefore, the existence and uniqueness of (33) is equivalent to those of (36).

Step 4—Optimal Feedback: Define

$$u_t = \frac{1}{R_t} D_t \mathbb{E}\left[p_t \left| \mathscr{F}_t^Y \right.\right] = \frac{1}{R_t} D_t \hat{p}_t \tag{42}$$

where $(\hat{p}, \hat{y}, \hat{z})$ is the unique solution of (33). According to the existence of (33), (32) with (42) admits a unique solution (p, y, z). Theorem 3.2 and Corollary 3.1 imply that (42) is a unique optimal control. Inserting (37) into (42), it leads to

$$u_t = \frac{1}{R_t} D_t (\alpha_t \hat{y}_t + \beta_t) \tag{43}$$

where α, β and $(\hat{p}, \hat{y}, \hat{z})$ are the solutions of (34), (35), and (33).

We summarize the above analysis as follows.

Proposition 4.1: Let Assumptions 1, 2, 3, and 5 hold. Then (42) is the unique optimal control of Example 4.1. Furthermore, (43) is its feedback representation.

We emphasize that Example 4.1 is related to [19], in which the cost functional does not include $2ny_0^v$, moreover, the uniqueness of optimal control was not proved.

C. An LQ Optimal Control of FBSDEs With Partial Information

Example 4.2: Consider a purely quadratic cost optimal control problem as follows:

$\inf_{v \in \mathscr{U}_{ad}} J[v]$ with the cost functional

$$J[v] = \frac{1}{2} \mathbb{E}\left[\int_0^T R_t v_t^2 dt + M(x_T^v)^2 + 2mx_T^v + N(y_0^v)^2 + 2ny_0^v\right]$$

subject to the state equation

$$\begin{cases} dx_t^v = (a_t x_t^v + b_t v_t)\, dt + \bar{c}_t d\bar{w}_t, \\ -dy_t^v = (A_t x_t^v + B_t y_t^v + D_t v_t)\, dt - z_t^v dw_t - \bar{z}_t^v d\bar{w}_t, \\ x_0^v = e_0, \quad y_T^v = F x_T^v + G \end{cases}$$

and the observation equation same to the one in Example 4.1.

Proposition 4.2: Under Assumptions 1, 2, and 3, Example 4.2 admits a unique optimal control

$$u_t = \frac{1}{R_t}(D_t \hat{p}_t - b_t \hat{q}_t)$$

where $(\hat{x}, \hat{y}, \hat{z}, \hat{p}, \hat{q}, \hat{k})$ is the unique solution of

$$\begin{cases} \dot{\hat{x}}_t = a_t \hat{x}_t + \frac{1}{R_t} b_t D_t \hat{p}_t - \frac{1}{R_t} b_t^2 \hat{q}_t \\ -d\hat{y}_t = (A_t \hat{x}_t + B_t \hat{y}_t + \frac{1}{R_t} D_t^2 \hat{p}_t - \frac{1}{R_t} b_t D_t \hat{q}_t) dt - \hat{z}_t dw_t \\ \hat{x}_0 = e_0, \quad \hat{y}_T = F\hat{x}_T + \hat{G} \end{cases} \tag{44}$$

and

$$\begin{cases} \dot{\hat{p}}_t = B_t \hat{p}_t, \quad \hat{p}_0 = -Ny_0 - n \\ -d\hat{q}_t = (a_t \hat{q}_t - A_t \hat{p}_t) dt - \hat{k}_t dw_t \\ \hat{q}_T = M\hat{x}_T - F\hat{p}_T + m. \end{cases} \tag{45}$$

Furthermore, u can be rewritten as

$$u_t = -\frac{1}{R_t}\left[(D_t - b_t \Sigma_t)(N\hat{y}_0 + n) e^{\int_0^t B_s ds} + b_t \pi_t \hat{x}_t + b_t \theta_t\right]$$

where π, Σ, θ and $(\hat{x}, \hat{y}, \hat{z}, \hat{p}, \hat{q}, \hat{k})$ are the unique solutions of (29)–(31) with $L_t = o_t = r_t = l_t = 0$ and (44) with (45), respectively.

Proof: With the setup, it is easy to see that the coefficients of (6) and (8) satisfy

$$L_t = O_t = \bar{b}_t = c_t = C_t = \bar{C}_t = \bar{D}_t = f_t = l_t = o_t = r_t = 0.$$

We introduce a coupled FBSDE

$$\begin{cases} \dot{\hat{x}}_t = a_t \hat{x}_t + \frac{1}{R_t} b_t D_t \hat{p}_t - \frac{1}{R_t} b_t^2 \hat{q}_t \\ -d\hat{y}_t = \left(A_t \hat{x}_t + B_t \hat{y}_t + \frac{1}{R_t} D_t^2 \hat{p}_t - \frac{1}{R_t} b_t D_t \hat{q}_t\right) dt \\ \qquad - \hat{z}_t dw_t \\ \hat{x}_0 = e_0, \quad \hat{y}_T = F\hat{x}_T + \hat{G} \end{cases}$$

with

$$\begin{cases} \dot{\hat{p}}_t = B_t \hat{p}_t, \quad \hat{p}_0 = -N\hat{y}_0 - n \\ -d\hat{q}_t = (a_t \hat{q}_t - A_t \hat{p}_t) dt - \hat{k}_t dw_t \\ \hat{q}_T = M\hat{x}_T - F\hat{p}_T + m. \end{cases}$$

It follows from Theorem 1 in Yu [29] that the FBSDE admits a unique solution $(\hat{x}, \hat{y}, \hat{z}, \hat{p}, \hat{q}, \hat{k}) \in \mathscr{L}^2_{\mathscr{F}^w}(0, T; \mathbb{R}^6)$. Let

$$u_t = \frac{1}{R_t}(D_t \hat{p}_t - b_t \hat{q}_t).$$

Similar to Example 4.1, Theorem 3.2, Corollary 3.1, and Remark 3.2 imply that the above u is a unique optimal control. The feedback representation of u is an immediate result of Theorem 3.3. Then we omit it for simplicity. ■

From Lemmas 3.1 and 3.2, we know that the filtering equations are so complicated that the analytical solutions of Problem (LQC) are hard to obtain except for some special cases. Therefore, it is highly desirable to study some numerical computations of the optimal controls. We hope to come back to this topic in the future.

V. Application to Mathematical Finance

Stochastic recursive utility (RU, for short) is a generalization of standard additive utility with instantaneous utility depending not only on instantaneous consumption rate, but also on future utility. See e.g., Duffie and Epstein [4]. The reference [5] found that the RU can be denoted as solutions of BSDEs. This section focuses on studying an extended RU problem. Making use of the results in Section III, we obtain a unique and closed-form solution. For convenience, we still adopt the assumptions and notation introduced in Sections II–IV unless noted otherwise.

Consider now a firm whose liability process $\bar{l}_t^v$ is governed by

$$-d\bar{l}_t^v = (b_t v_t - \bar{b}_t)dt + c_t dw_t + \bar{c}_t d\bar{w}_t.$$

Here v is a control strategy of policymaker and may denote the rate of capital injection or withdrawal so as to achieve a certain goal; $\bar{b}_t > 0$ is the expected liability rate; $c_t > 0$ and $\bar{c}_t > 0$ describe the liability risk. Assume that the firm owns an initial investment e_0, and only invests in a money account with compounded interest rate $a_t > 0$. Then the cash balance process x_t^v of the firm is

$$x_t^v = e^{\int_0^t a_s ds}\left(e_0 - \int_0^t e^{-\int_0^s a_r dr} d\bar{l}_s^v\right)$$

whose differential form reads

$$\begin{cases} dx_t^v = \left(a_t x_t^v + b_t v_t - \bar{b}_t\right)dt + c_t dw_t + \bar{c}_t d\bar{w}_t \\ x_0^v = e_0. \end{cases} \tag{46}$$

Suppose that the policymaker can only get information from the stock

$$\begin{cases} dS_t^v = S_t^v\left[\left(f_t x_t^v + g_t + \frac{1}{2}h_t^2\right)dt + h_t dw_t\right] \\ S_0^v = 1. \end{cases}$$

Then $\sigma\{S_s^v; 0 \le s \le t\}$, rather than $\mathscr{F}_t^{w,\bar{w}}$, is the information available to the policymaker at time t. Define

$$Y_t^v = \log S_t^v.$$

It is easy to see

$$\mathscr{F}_t^{Y^v} = \sigma\{Y_s^v; 0 \le s \le t\} = \sigma\{S_s^v; 0 \le s \le t\}.$$

Moreover, it follows from Itô's formula that:

$$\begin{cases} dY_t^v = (f_t x_t^v + g_t)dt + h_t dw_t \\ Y_0^v = 0. \end{cases} \tag{47}$$

The generalized RU problem under consideration here is

Problem (RU): To minimize

$$\tilde{J}[v] = \frac{1}{2}\mathbb{E}\left[\int_0^T R_t(v_t - \tilde{r}_t)^2 dt + M\left(x_T^v - \tilde{m}\right)^2 - 2\tilde{n}y_0^v\right]$$

over $\mathscr{U}_{ad}$, subject to (46), (47) and

$$\begin{cases} -dy_t^v = \tilde{g}\left(t, y_t^v, z_t^v, \bar{z}_t^v, v_t\right)dt - z_t^v dw_t - \bar{z}_t^v d\bar{w}_t \\ y_T^v = x_T^v \end{cases} \tag{48}$$

where $\tilde{g}$ is concave with respect to $(y, z, \bar{z}, v)$ and satisfies some usual conditions for BSDEs. The above performance functional consists of three terms. In detail, the first one measures the difference between v and a benchmark $\tilde{r}$. The second one evaluates the risk of the terminal wealth. The third one describes a generalized RU resulting from the control strategy v. See, e.g., [5] and the references therein.

Problem (RU) is related to [8] and [18]. Note that, the drift term in (47) is linear rather than bounded with respect to x, and the cost functional in [8] does not contain the RU. Then Problem (RU) is distinguished from the drift term of [18] and the RU of [8].

Assumption 6: Let $\tilde{g}(t, y, z, \bar{z}, v) = \tilde{B}_t y + \tilde{D}_t v$. The coefficients $\tilde{B}_t$, $\tilde{D}_t$ and $\tilde{r}_t$ are uniformly bounded, deterministic functions. $\tilde{n} \ge 0$, and $\tilde{m}$ are constants.

Solving (48) with Assumption 6

$$\mathbb{E}\left[y_t^v\right] = \mathbb{E}\left[x_T^v e^{\int_t^T \tilde{B}_s ds} + \int_t^T \tilde{D}_s e^{\int_t^s \tilde{B}_\nu d\nu} v_s ds\right].$$

Then Problem (RU) is equivalent to a classical utility (CU, for short) problem as follows.

Problem (CU): Find a $u \in \mathscr{U}_{ad}$ to minimize

$$J[v] = \frac{1}{2}\mathbb{E}\left[\int_0^T \left(R_t v_t^2 + 2r_t v_t\right)dt + M\left(x_T^v\right)^2 + 2m x_T^v\right] \tag{49}$$

subject to (46) and (47) with

$$r_t = -\tilde{n}\tilde{D}_t e^{\int_0^t \tilde{B}_s ds} - R_t\tilde{r}_t \text{ and } m = -\tilde{n}e^{\int_0^T \tilde{B}_s ds} - M\tilde{m}. \tag{50}$$

Note that r_t is deterministic and bounded, and m is a constant.

Obviously, Problem (CU) is a special case of Problem (LQC), i.e., the coefficients of (6) and (8) satisfy

$$A_t = B_t = C_t = \bar{C}_t = D_t = \bar{D}_t = 0, \quad F = G = 0$$
$$L_t = O_t = l_t = o_t = 0, \quad N = n = 0.$$

According to Theorem 3.1, if u is optimal, then

$$\begin{cases} dx_t = (a_t x_t + b_t u_t - \bar{b}_t)dt + c_t dw_t + \bar{c}_t d\bar{w}_t \\ -dq_t = a_t q_t dt - k_t dw_t - \bar{k}_t d\bar{w}_t \\ x_0 = e_0, \quad q_T = Mx_T + m \end{cases} \tag{51}$$

admits a unique solution $(x, q, k, \bar{k}) \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}(0, T; \mathbb{R}^4)$ such that

$$u_t = -\frac{1}{R_t}\left(b_t\mathbb{E}\left[q_t \left|\mathscr{F}_t^Y\right.\right] + r_t\right).$$

Furthermore, it follows from Theorem 3.3 that the above u is equivalent to:

$$u_t = -\frac{1}{R_t}\left[b_t(\pi_t\hat{x}_t + \theta_t) + r_t\right] \tag{52}$$

where π, θ, and $\hat{x}$ are the unique solutions of

$$\begin{cases} \dot{\pi}_t + 2a_t\pi_t - \frac{1}{R_t}b_t^2\pi_t^2 = 0 \\ \pi_T = M \end{cases}$$
$$\begin{cases} \dot{\theta}_t + \left(a_t - \frac{1}{R_t}b_t^2\pi_t\right)\theta_t - \frac{1}{R_t}b_t r_t\pi_t - \bar{b}_t\pi_t = 0 \\ \theta_T = m \end{cases}$$

and

$$\begin{cases} d\hat{x}_t = \left[\left(a_t - \frac{1}{R_t}b_t^2\pi_t\right)\hat{x}_t - \frac{1}{R_t}b_t(b_t\theta_t + r_t) - \bar{b}_t\right]dt \\ \qquad + \left(c_t + \frac{P_t f_t}{h_t}\right)d\hat{w}_t \\ \hat{x}_0 = e_0 \end{cases} \tag{53}$$

with P, $\hat{w}$, r, and m satisfying (21), (22), and (50), respectively. Applying Itô's formula to $(\hat{x}_t)^2$ with Burkholder-Davis-Gundy inequality, we derive $\mathbb{E}\sup_{0\le t\le T}(\hat{x}_t)^2 < +\infty$. In addition, (52) is adapted to $\mathscr{F}_t^Y$ and to $\mathscr{F}_t^{Y^0}$. Thanks to the estimate and the adaptability above, it is easy to see that (52) is an admissible control, and thus, (51) admits a unique solution $(x, q, k, \bar{k}) \in \mathscr{L}^2_{\mathscr{F}^{w,\bar{w}}}(0,T;\mathbb{R}^4)$ such that $u_t = -1/R_t(b_t\mathbb{E}[q_t|\mathscr{F}_t^Y] + r_t)$. Theorem 3.2 and Corollary 3.1 then imply that (52) is the unique optimal control strategy of Problem (CU) or Problem (RU).

We begin to compute the optimal performance functional. First let us recall (21). Since P is independent of v, the optimal performance functional is rewritten as

$$\begin{aligned} J[u] &= \frac{1}{2}\mathbb{E}\left[\int_0^T \left(R_t u_t^2 + 2r_t u_t\right) dt + M x_T^2 + 2m x_T\right] \\ &= \frac{1}{2}\mathbb{E}\left[\int_0^T \left(R_t u_t^2 + 2r_t u_t\right) dt + M(x_T - \hat{x}_T + \hat{x}_T)^2 + 2m\hat{x}_T\right] \\ &= \mathcal{J}[u] + \frac{1}{2}MP_T \end{aligned} \tag{54}$$

with

$$\mathcal{J}[u] = \frac{1}{2}\mathbb{E}\left[\int_0^T \left(R_t u_t^2 + 2r_t u_t\right) dt + M(\hat{x}_T)^2 + 2m\hat{x}_T\right].$$

Here $\hat{x}$ solves (53), and M and m are constants. Applying Itô's formula to $\theta_t\hat{x}_t$ and $\pi_t(\hat{x}_t)^2$, we get

$$\begin{aligned} d(\theta_t\hat{x}_t) &= \dot{\theta}_t\hat{x}_t dt + \theta_t d\hat{x}_t \\ &= \left\{\left(\frac{1}{R_t}b_t r_t + \bar{b}_t\right)\pi_t\hat{x}_t - \left[\bar{b}_t + \frac{1}{R_t}b_t(b_t\theta_t + r_t)\theta_t\right]\right\} dt \\ &\quad + \theta_t\left(c_t + \frac{P_t f_t}{h_t}\right) d\hat{w}_t \end{aligned} \tag{55}$$

and

$$\begin{aligned} d\left[\pi_t(\hat{x}_t)^2\right] &= \dot{\pi}_t(\hat{x}_t)^2 dt + \pi_t\left[2\hat{x}_t d\hat{x}_t + (d\hat{x}_t)^2\right] \\ &= \pi_t\Bigg\{-2\hat{x}_t\left[\bar{b}_t + \frac{1}{R_t}b_t(b_t\theta_t + r_t)\right] \\ &\qquad - \frac{1}{R_t}b_t^2(\hat{x}_t)^2\pi_t + \left(c_t + \frac{P_t f_t}{h_t}\right)^2\Bigg\} dt \\ &\quad + 2\pi_t\hat{x}_t\left(c_t + \frac{P_t f_t}{h_t}\right) d\hat{w}_t. \end{aligned} \tag{56}$$

Inserting (52), (55), and (56) into (54), we derive

$$\begin{aligned} J[u] &= \frac{1}{2}\int_0^T \left[\left(c_t + \frac{P_t f_t}{h_t}\right)^2 \pi_t - 2\bar{b}_t - \frac{1}{R_t}(b_t\theta_t + r_t)^2\right] dt \\ &\quad + \pi_0 e_0^2 + 2\theta_0 e_0 + \frac{1}{2}MP_T. \end{aligned} \tag{57}$$

Now we conclude the aforementioned discussion with the following

Proposition 5.1: If Assumptions 1, 2, 3, and 6 hold, then (52) and (57) are the unique optimal control strategy and the optimal cost functional of Problem (CU), respectively.

VI. Conclusion

We gave a backward separation approach to solve an LQ optimal control problem of FBSDEs with partial information, where the state noise is correlated to the observation noise, and the drift coefficient of the observation equation is linear with respect to the state x. More specifically, we first use Theorem 3.1 to find all candidate optimal controls, and then use Theorem 3.2 to check whether the candidates ones are optimal or not, finally use Corollary 3.1 to prove uniqueness of optimal control. Combining the approach with stochastic filtering and existence of FBSDEs, unique analytical solutions of optimal control were derived in some detailed cases. Also, a generalized RU problem was explicitly solved. These results obtained in this paper extend the references [9], [14], [15], [17]–[19], [22], [23], [30], [31] within the framework of LQ optimal control and correlated state and observation noise. By the way, the backward separation approach is applicable to study some more complicated LQ optimal control and differential game problems of BSDEs or FBSDEs with partial information. The details will be presented elsewhere.

Note that Example 4.2 and Problem (RU) can also be solved by the following idea: First, we compute optimal filtering of state, and then deduce the corresponding optimal control problem derived by the filtering. It seems that, however, the backward separation approach offers a more convenient and straight way to solve Example 4.2 and Problem (RU).

Acknowledgment

The authors would like to thank five anonymous referees for their constructive comments, which lead to a much better version of this paper.

References

[1] A. Bensoussan, *Stochastic Control by Functional Analysis Methods*. New York, NY, USA: North-Holland Publishing Company, 1982.
[2] A. Bensoussan, *Stochastic Control of Partially Observable Systems*. Cambridge, U.K.: Cambridge Univ. Press, 1992.
[3] A. Bensoussan and M. Viot, "Optimal control of stochastic linear distributed parameter systems," *SIAM J. Control*, vol. 13, pp. 904–926, 1975.
[4] D. Duffie and L. Epstein, "Stochastic differential utility," *Econometrica*, vol. 60, pp. 353–394, 1992.
[5] N. El Karoui, S. Peng, and M. C. Quenez, "Backward stochastic differential equations in finance," *Math. Finance*, vol. 7, pp. 1–71, 1997.
[6] R. J. Elliott, L. Aggoun, and J. B. Moore, *Hidden Markov Models: Estimation and Control*. New York, NY, USA: Springer-Verlag, 1997.
[7] Y. Hu and B. Øksendal, "Partial information linear quadratic control for jump diffusions," *SIAM J. Control Optim.*, vol. 47, pp. 1744–1761, 2008.
[8] J. Huang, G. Wang, and Z. Wu, "Optimal premium policy of an insurance firm: full and partial information," *Insurance Math. Econ.*, vol. 47, pp. 208–215, 2010.
[9] J. Huang, G. Wang, and J. Xiong, "A maximum principle for partial information backward stochastic control problems with applications," *SIAM J. Control Optim.*, vol. 48, pp. 2106–2117, 2009.
[10] R. S. Liptser and A. N. Shiryayev, *Statistics of Random Processes*. New York, NY, USA: Springer-Verlag, 1977.
[11] J. Ma and J. Yong, *Forward-Backward Stochastic Differential Equations and Their Applications, Lecture Notes in Mathematics, 1702*. New York, NY, USA: Springer-Verlag, 1999.
[12] Q. Meng, "A maximum principle for optimal control problem of fully coupled forward-backward stochastic systems with partial information," *Sci. China Ser. A Math.*, vol. 52, pp. 1579–1588, 2009.
[13] B. Øksendal and A. Sulem, "Maximum principles for optimal control of forward-backward stochastic differential equations with jumps," *SIAM J. Control Optim.*, vol. 48, pp. 2845–2976, 2009.

[14] J. Shi and Z. Wu, "The maximum principle for partially observed optimal control of fully coupled forward-backward stochastic system," *J. Optim. Theory Appl.*, vol. 145, pp. 543–578, 2010.

[15] S. Tang, "The maximum principle for partially observed optimal control of stochastic differential equations," *SIAM J. Control Optim.*, vol. 36, pp. 1596–1617, 1998.

[16] G. Wang and Z. Wu, "Kalman-Bucy filtering equations of forward and backward stochastic systems and applications to recursive optimal control problems," *J. Math. Anal. Appl.*, vol. 342, pp. 1280–1296, 2008.

[17] G. Wang and Z. Wu, "The maximum principles for stochastic recursive optimal control problems under partail information," *IEEE Trans. Autom. Control*, vol. 54, pp. 1230–1242, 2009.

[18] G. Wang, Z. Wu, and J. Xiong, "Maximum principles for forward-backward stochastic control systems with correlated state and observation noises," *SIAM J. Control Optim.*, vol. 51, pp. 491–524, 2013.

[19] G. Wang, Z. Wu, and J. Xiong, "Partial information LQ optimal control of backward stochastic differential equations," in *Proc. 10th World Congress Intell. Control Autom.*, Beijing, China, Jul. 2012, pp. 1694–1697.

[20] G. Wang, C. Zhang, and W. Zhang, "Stochastic maximum principle for mean-field type optimal control with partial information," *IEEE Trans. Autom. Control*, vol. 59, pp. 522–528, 2014.

[21] W. M. Wonham, "On the separation theorem of stochastic control," *SIAM J. Control*, vol. 6, pp. 312–326, 1968.

[22] Z. Wu, "A maximum principle for partially observed optimal control of forward-backward stochastic control systems," *Sci. China Ser. F Inf. Sci.*, vol. 53, pp. 1–10, 2010.

[23] H. Xiao, "The maximum principle for partially observed optimal control of forward-backward stochastic systems with random jumps," *J. Syst. Sci. Complex.*, vol. 24, pp. 1083–1099, 2011.

[24] H. Xiao and G. Wang, "The filtering equations of forward-backward stochastic systems with random jumps and applications to partial information stochastic optimal control," *Stoch. Anal. Appl.*, vol. 28, pp. 1003–1019, 2010.

[25] J. Xiong, *An Introduction to Stochastic Filtering Theory*. London, U.K.: Oxford Univ. Press, 2008.

[26] J. Yong, "Forward backward stochastic differential equations with mixed initial and terminal conditions," *Trans. Amer. Math. Soc.*, vol. 362, pp. 1047–1096, 2010.

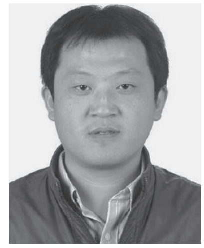
[27] J. Yong and X. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*. New York, NY, USA: Springer-Verlag, 1999.

[28] Z. Yu and S. Ji, "Linear-quadratic nonzero-sum differential game of backward stochastic differential equations," in *Proc. 27th Chinese Control Conf.*, Kunming, China, Jul. 2008, pp. 562–566.

[29] Z. Yu, "Linear-quadratic optimal control and nonzero-sum differential game of forward-backward stochastic system," *Asian J. Control*, vol. 14, pp. 1–13, 2012.

[30] Q. Zhang, "Controlled partially observed diffusions with correlated noise," *Appl. Math. Optim.*, vol. 22, pp. 265–285, 1990.

[31] X. Zhou, "On the necessary conditions of optimal controls for stochastic partial differential equations," *SIAM J. Control Optim.*, vol. 31, pp. 1462–1478, 1993.



Guangchen Wang (M'15) received the B.S. degree in mathematics from Shandong Normal University, Jinan, China, in 2001, and the Ph.D. degree in probability theory and mathematical statistics from the School of Mathematics and System Sciences, Shandong University, Jinan, China, in 2007.

From July 2007 to August 2010, he served as a Lecturer in the School of Mathematical Sciences, Shandong Normal University. He joined the School of Control Science and Engineering, Shandong University as an Associate Professor in September 2010, where he has been a Full Professor since September 2014. His current research interests include stochastic control, nonlinear filtering and mathematical finance.

Zhen Wu received the B.S. degree in control sciences and the Ph.D. degree in applied mathematics from Shandong University, Jinan, China, in 1991 and 1997, respectively.

He was a postdoctoral at the University of Maine, France, from 1998 to 1999 and joined the School of Mathematics, Shandong University as a Lecturer in 1997, where he has been a Full Professor since September 2002. He is an Associate Member of The Abdus Salam International Center for Theoretical Physics, Italy, and also a Visiting Professor in several universities in France, U.K., USA, and Hong Kong. He is currently a Changjiang Scholar of the Education Ministry. He serves as Associate Editor for several international journals including the *SIAM Journal on Control and Optimization, and Statistics & Probability Letters*. His research interests include stochastic control theory, forward and backward stochastic differential equations and mathematical finance.

Jie Xiong received the B.S. degree in mathematics from Peking University, Beijing, China, in 1983, and the Ph.D. degree in statistics from the University of North Carolina at Chapel Hill, USA, in 1992.

He was a Canada Research Chair in Stochastic Processes and Filtering during 2002–2003 and a Humboldt Research Fellow during 2003–2004. He was an Assistant, an Associate, and a Full Professor at the University of Tennessee at Knoxville, USA, until Spring 2014. He is currently a Professor at the University of Macau, China. His research interests includes stochastic filtering and control, mathematical finance, mathematical biology, measured-valued branching processes, and stochastic partial differential equations.