Fuzzy Portfolio Allocation Models Through a New Risk Measure and Fuzzy Sharpe Ratio

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***Abstract*—A new portfolio risk measure that is the uncertainty of portfolio fuzzy return is introduced in this paper. Beyond the well-known Sharpe ratio (i.e., the *reward-to-variability* ratio) in modern portfolio theory, we initiate the so-called *fuzzy Sharpe ratio* in the fuzzy modeling context. In addition to the introduction of the new risk measure, we also put forward the *reward-to-uncertainty* ratio to assess the portfolio performance in fuzzy modeling. Corresponding to two approaches based on T_M and T_W fuzzy arithmetic, two portfolio optimization models are formulated in which the uncertainty of portfolio fuzzy returns is minimized, while the fuzzy Sharpe ratio is maximized. These models are solved by the fuzzy approach or by the genetic algorithm (GA). Solutions of the two proposed models are shown to be dominant in terms of portfolio return uncertainty compared with those of the conventional mean-variance optimization (MVO) model used prevalently in the financial literature. In terms of portfolio performance evaluated by the fuzzy Sharpe ratio and the *reward-to-uncertainty* ratio, the model using T_W fuzzy arithmetic results in higher performance portfolios than those obtained by both the MVO and the fuzzy model, which employs T_M fuzzy arithmetic. We also find that using the fuzzy approach for solving multiobjective problems appears to achieve more optimal solutions than using GA, although GA can offer a series of well-diversified portfolio solutions diagrammed in a Pareto frontier.**

***Index Terms*—Fuzzy return, fuzzy Sharpe ratio, genetic algorithm (GA), portfolio optimization, return uncertainty.**

I. Introduction

Modern portfolio theory has been inspired by the Markowitz's [1] pioneering work, which originally initiated the mean-variance optimization (MVO) model. Despite the prevalence of the model, providing accurate expectation of returns and accurate covariance of returns between each pair of stocks is a nontrivial mission [2]. In the increasingly dynamic economic environment, especially the ever-changing stock markets, estimation of the covariance matrix seems to be unreliable and faces big challenges. Returns based roughly on historical data are somewhat unconvincing since they are *ex-ante* rather than *ex-post*. The more dynamic the financial situations, the stock returns tend to be more uncertain and vague. There is an imperative to consider returns as fuzzy random variables in portfolio optimization.

Chen and Huang [3] considered the uncertainty of future returns and risk and presented them in triangular fuzzy numbers and solved the optimal asset allocation by a fuzzy optimization. The portfolio solutions obtained were argued to be more reasonable and suitable in the imprecise financial environment. Hasuike *et al.* [4] modeled uncertain expected returns as fuzzy random variables and proposed several random fuzzy nonlinear portfolio selection models. An efficient solving approach involving parametric convex programming problem is constructed to find a global optimal solution for portfolio optimizations. The proposed method was found to be more flexible and adaptable compared with the models of Carlsson *et al.* [5] and Vercher *et al.* [6].

Li and Xu [7] estimated future returns of securities in fuzzy random variables by combining together statistical techniques and experts' judgment and experience. A new portfolio selection model is built up that considers the random factors and fuzzy information simultaneously. Solutions of the portfolio optimization can generate an efficient frontier according to investors' optimism degree. Huang [8] used entropy of security return fuzzy variables as a risk measure and introduced two types of credibility-based fuzzy mean-entropy models. Entropy computation does not require symmetric membership functions of fuzzy return, and nonmetric data can be adopted for the calculation. Deployment of the models, via a hybrid intelligent algorithm, seeks to minimize entropy to obtain less uncertain portfolio return. Security returns were also treated as triangular, trapezoidal, and Gaussian fuzzy random variables in [9]. The authors created the value-at-risk-based fuzzy portfolio selection models (VaR-FPSM) and designed an improved particle swarm optimization algorithm on the basis of fuzzy simulation to search for the approximate optimal solutions. The VaR-FPSM was found more acceptable to general investors, and its solutions showed dominant performance compared with other existing approaches.

A class of linear programming problems with interval coefficients in both the objective functions and constraints was studied in [10], wherein uncertain returns of assets in a financial market were taken as intervals, and the traditional portfolio semiabsolute deviation measure of risk was generalized to the interval case. A new portfolio selection model was thus initiated that can be transformed to the studied solvable linear interval programming model. On the other hand, Li *et al.* [11] used

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the skewness of fuzzy random variables based on the credibility theory. These authors then constructed the mean-variance-skewness fuzzy portfolio selection models and solved them by the fuzzy simulation and genetic algorithm (GA). Huang [12], in another approach, commenced a new risk measure for portfolio optimization. Neural network is used to calculate the expected value and the change value of optimal portfolios. Other papers, e.g., [13]–[21], etc., also benefit from the advantages of fuzzy random variables in information uncertainty handling to develop fuzzy portfolio optimization models.

Although various fuzzy optimization models have been proposed, there have not been efficient and comprehensive methods to evaluate portfolio risk and also to assess portfolio performance in the fuzzy modeling context. The traditional volatility risk and the well-known Sharpe ratio [22] can be useful in probabilistic modeling but not in the fuzzy possibilistic context. In this paper, we introduce the "fuzzy Sharpe ratio" as well as a new risk measure that is the portfolio fuzzy return uncertainty in the fuzzy modeling environment. We first model asset returns by fuzzy random variables. Portfolio fuzzy returns and (fuzzy) variance are calculated using two approaches based on the strongest t-norm $\mathrm{T_M}$ and the weakest t-norm $\mathrm{T_W}$. Both approaches result in portfolio return represented by fuzzy numbers. Regarding covariance of returns between a pair of stocks, covariances calculated based on the strongest t-norm $\mathrm{T_M}$ fuzzy arithmetic are crisp values, while those based on the weakest t-norm $\mathrm{T_W}$ fuzzy arithmetic are in fuzzy numbers. The uncertainty measure of the portfolio fuzzy return is regarded as a new portfolio risk measure. Corresponding to two approaches for portfolio return and variance calculation, two models are proposed for portfolio allocation, including not only the prevalent variance risk but the newly defined risk measure as well. In particular, the models are to maximize the fuzzy Sharpe ratio and minimize the new risk measure, i.e., return uncertainty. Associated with the new risk measure, a new portfolio performance ratio that is the "reward-to-uncertainty" ratio is conceptualized in fuzzy numbers. Like the "reward-to-variability," i.e., the conventional Sharpe ratio, the higher the "reward-to-uncertainty" ratio, the higher the portfolio performance is.

The rest of this paper is organized as follows. Section II presents some fuzzy background, ranking of fuzzy numbers, and two approaches to calculate the expected values and covariance of fuzzy random variables. Section III is devoted to constructing portfolio optimization models with new uncertainty risk measure and the fuzzy Sharpe ratio. Experimental results are reported in Section IV, and concluding remarks are represented in Section V.

II. Fuzzy Random Variables

A. Relevant Fuzzy Set Concepts and Ranking Fuzzy Numbers

A fuzzy number A is characterized for each $x \in \mathbb{R}$ by the canonical form [23]

$$f_A(x) = \begin{cases} f_A^L(x), & a \le x \le b \\ \omega, & b \le x \le c \\ f_A^R(x), & c \le x \le d \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where $\omega \in (0, 1]$ is a constant; $a, b, c, d \in \mathbb{R}$; $a \le b \le c \le d$; $f_A^L : [a, b] \to [0, \omega]$ is a increasing real-valued function; and $f_A^R : [c, d] \to [0, \omega]$ is a real-valued decreasing function. The most widely used is *trapezoidal fuzzy numbers* denoted by $A = (a, b, c, d; \omega)$ whose membership functions are piecewise linear:

$$f_A(x) = \begin{cases} \omega\,(x-a)\,/\,(b-a)\,, & a \le x \le b \\ \omega, & b \le x \le c \\ \omega\,(d-x)\,/\,(d-c)\,, & c \le x \le d \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

A normal LR fuzzy set $A_{\mathrm{LR}} = (m, s, l, r)_{\mathrm{LR}}$ is defined with the membership function [24]

$$f_{A_{\mathrm{LR}}}(x) = \begin{cases} L\,(m-s-x/l)\,, & m-s-l \le x < m-s \\ 1, & m-s \le x < m+s \\ R\,(x-s-m/r)\,, & m+s \le x < m+s+r \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

where L, R are fixed left-continuous and nonincreasing functions $L, R : [0,1] \to]\,0,1]$ with $R(0) = L(0) = 1$ and $R(1) = L(1) = 0$, and $m \in \mathbb{R}, s, l, r \ge 0$.

If $s = 0$ and $L = R$ and both are in the form

$$T(x) = \begin{cases} 1-x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

then the LR fuzzy number is called an LR triangular fuzzy number and denoted $A_{\mathrm{LR}} = (m, l, r)_{\mathrm{LR}}$. Note the difference in notations of triangular fuzzy numbers in canonical form $A = (a, b, d)$ and in the LR form $A_{\mathrm{LR}} = (m, l, r)_{\mathrm{LR}}$; a and m are both the center, b and d are lower and upper bounds, and l and r are the left and right spreads from the centre.

The efficient fuzzy number ranking paradigm that we used in [25] is also utilized in this study. The principle is based on the centroid formulas of fuzzy numbers proposed by Wang *et al.* [26]. Denote $g_A^L(y) : [0, \omega] \to [a, b]$ and $g_A^R(y) : [0, \omega] \to [a, d]$ are the inverse functions of $f_A^L(x)$ and $f_A^R(x)$ in (1), respectively. In the case of trapezoidal fuzzy number, the functions $g_A^L(y)$ and $g_A^R(y)$ can be analytically expressed as

$$g_A^L(y) = a + (b-a)\,y/\omega, 0 \le y \le \omega$$
$$g_A^R(y) = d + (d-c)\,y/\omega, 0 \le y \le \omega.$$

The Wang *et al.* [26] centroid formulas based on the general canonical form of a trapezoidal fuzzy number A are as follows:

$$\bar{x}_0(A) = \frac{\int_{-\infty}^{+\infty} x f_A(x)dx}{\int_{-\infty}^{+\infty} f_A(x)dx} = \frac{\int_a^b x f_A^L(x)dx + \int_b^c (x\omega)\,dx + \int_c^d x f_A^R(x)dx}{\int_a^b f_A^L(x)dx + \int_b^c (\omega)\,dx + \int_c^d f_A^R(x)dx}$$

and

$$\bar{y}_0(A) = \frac{\int_0^\omega y\left(g_A^R(y) - g_A^L(y)\right)dy}{\int_0^\omega \left(g_A^R(y) - g_A^L(y)\right)dy} \tag{5}$$

where the numerator $\int_0^\omega y\left(g_A^R(y)-g_A^L(y)\right)dy$ represents the weighted average of the area, while the denominator $\int_0^\omega \left(g_A^R(y)-g_A^L(y)\right)dy$ is the area of the trapezoid.

For trapezoidal fuzzy numbers $A=[a,b,c,d;\omega]$, the Wang *et al.*'s [26] centroid approach leads to $\bar{x}_0(A)=\frac{1}{3}[a+b+c+d-\frac{dc-ab}{(d+c)-(a+b)}]$ and $\bar{y}_0(A)=\frac{1}{3}\omega[1+\frac{c-b}{(d+c)-(a+b)}]$.

Likewise, with normal triangular fuzzy numbers $A=(a,b,d)$, which is the special case of normal trapezoidal fuzzy numbers with $b=c$, centroids can be determined by

$$\bar{x}_0(A)=\frac{1}{3}(a+b+d) \quad \text{and} \quad \bar{y}_0(A)=\frac{1}{3}. \tag{6}$$

The above formulas are applied to the centroids of the normal LR triangular fuzzy number, i.e., $A=(m,l,r)$, as follows:

$$\bar{x}_0(A)=\frac{1}{3}[(m-l)+m+(m+r)]=\frac{1}{3}(3*m-l+r) \text{ and}$$
$$\bar{y}_0(A)=\frac{1}{3}. \tag{7}$$

Centroids on the horizontal axis are used as a basis to rank fuzzy numbers. If horizontal coordinates $\bar{x}_0$ of all fuzzy numbers are completely equal, then the vertical centroid coordinates $\bar{y}_0$ will be applied, although this situation seldom occurs in practice. This is consistent with [27] that the representative location on the horizontal axis is more important than the average height in comparing fuzzy numbers.

B. Expected Value and Covariance of Fuzzy Random Variables

The fuzzy random variable concept first introduced by Kwakernaak [28]. It was afterward carefully studied and developed by Puri and Ralescu [29]. Accordingly, fuzzy(-valued) random variable X is defined as a Borel measurable mapping function $X:\Omega\rightarrow\mathcal{F}_0(\mathbb{R}^n)$ such that each α-cut $X^\alpha=\{x\in\mathbb{R}^n: u(x)\geq\alpha\}$ is a nonempty compact convex random sets, where $0<\alpha\leq 1$, $(\Omega,\mathcal{A},\mathbb{P})$ is a probability space, and $\mathcal{F}_0(\mathbb{R}^n)$ denotes all functions (fuzzy subsets of $\mathbb{R}^n$) $u:^n\rightarrow[0,1]$.

Possessing the linear characteristic, the expected values can be derived based on a defined t-norm fuzzy arithmetic. A number of fuzzy t-norms are proposed in the literature [30]. In this paper, we investigate the strongest t-norm $\mathrm{T_M}$ and the weakest t-norm $\mathrm{T_W}$.

Contrasting with the simple expected values calculation, the covariance of fuzzy random variables is, however, more complicated to be determined. Since initiated, there have been a variety of approaches studying covariance of fuzzy random variables. A recent literature review on covariance of fuzzy random variables was presented in [31]. There are two major comprehensions of fuzzy covariance: treating it as either crisp numbers or fuzzy numbers. The crisp numbers approach includes studies of Körner [32], Chiang and Lin [33], Feng *et al.* [34], Carlsson and Fullér [35], Fullér and Majlender [36], and, recently, Kamdem [37]. Alternatively, the fuzzy covariance approach comprises the work by Lee [38], Liu and Kao [39], Wu [40], Hong [41], etc. Tsao [42] used requisite equality constraint introduced by Klir [43] and Klir and Pan [44] to mend the fuzzy covariance calculation algorithms and applied to a portfolio selection application.

In this study, securities returns are represented by LR fuzzy random variables in order to find optimal investment portfolios. Therefore, calculating expected values of fuzzy variable and covariance between a pair of LR fuzzy random variables is a critical job. We investigate two approaches to calculate the expected values and covariance of LR fuzzy random variables.

The first approach follows Körner [32] using the well-known Zadeh extension principle [45], i.e., based on the strongest t-norm $T_M(x,y)=\min(x,y)$. The covariance of this approach is a crisp value. The second approach follows Hong [41] using the weakest t-norm $T_W(x,y)=\min(x,y)$ whenever $\max(x,y)=1$ and $T_W(x,y)=0$ otherwise. The $\mathrm{T_W}$-based multiplication of fuzzy numbers preserves the shape of LR fuzzy numbers. This enables us to derive a fuzzy value for covariance of two LR fuzzy random variables based on $\mathrm{T_W}$ t-norm. The following presents the two mentioned approaches.

Note that, in this paper, to model asset returns, we only employ the normal LR triangular fuzzy numbers by which $L=R$. Henceforth, we will use the notation $X=(m,l,r)$ for simplicity instead of $X_{\mathrm{LR}}=(m,l,r)_{\mathrm{LR}}$ to represent the LR triangular fuzzy number X_{LR}.

1) $\mathrm{T_M}$-Based Approach to the Expected Value and Covariance of Fuzzy Random Variables: According to Körner [32], an LR fuzzy set $X=(m,l,r)$ can be represented by $X=m+r.A_R-l.A_L$, where A_L and A_R are the fuzzy sets with membership functions $R.1_{[0,1]}$ and $L.1_{[0,1]}$. The linearity characteristic of the expected value allows us to derive

$$\bar{X}=\bar{m}+A_R.\bar{r}-A_L.\bar{l}.$$

Thus, the expected value $\bar{X}$ has a set representation of an LR fuzzy set

$$\bar{X}=\left(\bar{m},\bar{l},\bar{r}\right). \tag{8}$$

Denote $X_t=(m_{X_t},l_{X_t},r_{X_t})$, $t=1,\ldots,T$, are samples of the fuzzy random variable X; the expected value $\bar{X}$ is derived from (8) as follows:

$$\bar{X}=\left(\frac{1}{T}\sum_{t=1}^{T}m_{X_t},\frac{1}{T}\sum_{t=1}^{T}l_{X_t},\frac{1}{T}\sum_{t=1}^{T}r_{X_t}\right). \tag{9}$$

In accordance with the definition of the expected value above, Körner [32] also defined the covariance of two LR fuzzy random variables X,Y as

$$\begin{aligned}\boldsymbol{\sigma}_{X,Y} &= \mathrm{cov}\,(m_X,m_Y)\\ &\quad+\mathrm{cov}\,(s_X,s_Y)+\mathrm{cov}\,(l_X,l_Y).\|A_L\|_2^2\\ &\quad+\mathrm{cov}\,(r_X,r_Y).\|A_R\|_2^2\\ &\quad+\mathrm{cov}\,(s_X,l_Y)+\mathrm{cov}\,(s_X,l_Y)\\ &\quad-\mathrm{cov}\,(m_X,l_Y)-\mathrm{cov}\,(m_Y,l_X).\|A_L\|_1\\ &\quad+\mathrm{cov}\,(s_X,r_Y)+\mathrm{cov}\,(s_Y,r_X)\\ &\quad-\mathrm{cov}\,(m_X,r_Y)-\mathrm{cov}\,(m_Y,r_X).\|A_R\|_1\end{aligned}$$

where $\|A_L\|_2^2 = \frac{1}{2}\int_0^1 \left(L^{(-1)}(\alpha)\right)^2 d\alpha$, $\|A_R\|_2^2 = \frac{1}{2}\int_0^1 (R^{(-1)}(\alpha))^2 d\alpha$, $\|A_L\|_1 = \frac{1}{2}\int_0^1 L^{(-1)}(\alpha)d\alpha$, and $\|A_R\|_1 = \frac{1}{2}\int_0^1 L^{(-1)}(\alpha)d\alpha$.

When $s = 0$ (fuzzy number), then the covariance is

$$\begin{aligned}\boldsymbol{\sigma}_{X,Y} &= \text{cov}\,(m_X, m_Y) + \text{cov}\,(l_X, l_Y)\,.\|A_L\|_2^2\\&\quad + \text{cov}\,(r_X, r_Y)\,.\|A_R\|_2^2\\&\quad - \text{cov}\,(m_X, l_Y) + \text{cov}\,(m_Y, l_X)\,.\|A_L\|_1\\&\quad - (\text{cov}\,(m_X, r_Y) + \text{cov}\,(m_Y, r_X))\,.\|A_R\|_1.\end{aligned}$$

If $s = 0$ and $L = R$ and both have the formula shown in (4), then the fuzzy number is called a triangular fuzzy number, and we obtain $\|A_L\|_2^2 = \|A_R\|_2^2 = \frac{1}{6}$ and $\|A_L\|_1 = \|A_R\|_1 = \frac{1}{4}$. The formula for covariance between X, Y is, hence, given by

$$\begin{aligned}\boldsymbol{\sigma_{X,Y}} &= \text{cov}\,(m_X, m_Y) + \frac{1}{6}[\text{cov}\,(l_X, l_Y) + \text{cov}\,(r_X, r_Y)]\\&\quad - \frac{1}{4}[\text{cov}\,(m_X, l_Y) + \text{cov}\,(m_Y, l_X) + \text{cov}\,(m_X, r_Y)\\&\quad + \text{cov}\,(m_Y, r_X)].\end{aligned}\tag{10}$$

2) $\mathrm{T_W}$-Based Approach to the Expected Value and Covariance of Fuzzy Random Variables: It is well known that $\mathrm{T_W}$-based addition and multiplication preserve the shape of LR fuzzy numbers. We herein utilize $\mathrm{T_W}$-based fuzzy arithmetic to calculate the covariance between fuzzy random variables. The weakest t-norm $\mathrm{T_W}$-based addition and multiplication are described as follows [41]:

$$\begin{aligned}X + Y &= (m_X, l_X, r_X) + (m_Y, l_Y, r_Y)\\&= (m_X + m_Y, \max(l_X, l_Y), \max(r_X, r_Y))\end{aligned}\tag{11}$$

$$X * Y = \begin{cases}(m_X m_Y, \max(l_X m_Y, l_Y m_X), \max(r_X m_Y, r_Y m_X)) & m_X, m_Y > 0\\(m_X m_Y, \max(r_X m_Y, r_Y m_X), \max(l_X m_Y, l_Y m_X)) & m_X, m_Y < 0\\(m_X m_Y, \max(l_X m_Y, -r_Y m_X), \max(r_X m_Y, -l_Y m_X)) & m_X < 0, m_Y > 0\\(0, l_X m_Y, r_X m_Y), & m_X = 0, m_Y > 0\\(0, -r_X m_Y, -l_X m_Y), & m_X = 0, m_Y < 0\\(0, 0, 0), & m_X = m_Y = 0\end{cases}\tag{12}$$

Denote $X_t = (m_{X_t}, l_{X_t}, r_{X_t})$, $t = 1, \ldots, T$, are samples of the fuzzy random variable X; the expected value of X is defined as $\bar{X} = \frac{1}{T} \odot \sum_{t=1}^{T} X_t$, where $\odot$ is the notation of the constant multiplication operation based on the $\mathrm{T_W}$ t-norm. Thus, from (11), the expected value $\bar{X}$ is computed

$$\bar{X} = \left(\frac{1}{T}\sum_{t=1}^{T} m_{X_t}, \max_{1\le t\le T} l_{X_t}, \max_{1\le t\le T} r_{X_t}\right).\tag{13}$$

The covariance between two fuzzy random variables X and Y is conventionally defined by

$$\sigma_{X,Y} = \frac{\sum_{t=1}^{T}\left(X_t - \bar{X}\right)\left(Y_t - \bar{Y}\right)}{T}.$$

Based on the $\mathrm{T_W}$ t-norm, we can derive that

$$X_t - \bar{X} = \left(m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t}, \max_{1\le t\le T} l_{X_t}, \max_{1\le t\le T} r_{X_t}\right).$$

Similarly, for the fuzzy random variable Y with its samples $Y_t = (m_{Y_t}, l_{Y_t}, r_{Y_t})$, $t = 1, \ldots, T$, we have

$$Y_t - \bar{Y} = \left(m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t}, \max_{1\le t\le T} l_{Y_t}, \max_{1\le t\le T} r_{Y_t}\right).$$

Applying the formulation (12) to calculate $(X_t - \bar{X})(Y_t - \bar{Y})$, we divide the multiplication into several circumstances. First denote $(m_t, l_t, r_t) = (X_t - \bar{X})(Y_t - \bar{Y})$. In all cases, the centre value m_t is always specified: $m_t = (m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t}).(m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t})$. The values of left and right widths l_t and r_t however are changeable in different cases as follows:

Case 1:

$$m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t} > 0 \text{ and } m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t} > 0$$

$$l_t = \max\left(\max_{1\le t\le T} l_{X_t} * \left(m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t}\right),\ \max_{1\le t\le T} l_{Y_t} * \left(m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t}\right)\right)$$

$$r_t = \max\left(\max_{1\le t\le T} r_{X_t} * \left(m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t}\right),\ \max_{1\le t\le T} r_{Y_t} * \left(m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t}\right)\right).$$

Case 2:

$$m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t} < 0 \text{ and } m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t} < 0$$

$$l_t = \max\left(\max_{1\le t\le T} r_{X_t} * \left(m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t}\right),\ \max_{1\le t\le T} r_{Y_t} * \left(m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t}\right)\right)$$

$$r_t = \max\left(\max_{1\le t\le T} l_{X_t} * \left(m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t}\right),\ \max_{1\le t\le T} l_{Y_t} * \left(m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t}\right)\right).$$

Case 3:

$$m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t} < 0 \text{ and } m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t} > 0$$

$$l_t = \max\left(\max_{1\le t\le T} l_{X_t} * \left(m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t}\right),\right.$$
$$\left. - \max_{1\le t\le T} r_{Y_t} * \left(m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t}\right)\right)$$

$$r_t = \max\left(\max_{1\le t\le T} r_{X_t} * \left(m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t}\right),\right.$$
$$\left. - \max_{1\le t\le T} l_{Y_t} * \left(m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t}\right)\right).$$

Case 4:

$$m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t} = 0 \text{ and } m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t} > 0$$

$$l_t = \max_{1\le t\le T} l_{X_t} * \left(m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t}\right)$$

$$r_t = \max_{1\le t\le T} r_{X_t} * \left(m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t}\right).$$

Case 5:

$$m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t} = 0 \text{ and } m_{Y_t} - \frac{1}{T} m_{Y_t} < 0$$

$$l_t = -\max_{1\le t\le T} r_{X_t} * \left(m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t}\right)$$

$$r_t = -\max_{1\le t\le T} l_{X_t} * \left(m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t}\right).$$

Case 6:

$$m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t} = 0 \text{ and } m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t} = 0$$

$$l_t = 0, r_t = 0.$$

For the cases $m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t} > 0$ and $m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t} < 0$, $m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t} > 0$ and $m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t} = 0$, and $m_{X_t} - \frac{1}{T}\sum_{t=1}^{T} m_{X_t} < 0$ and $m_{Y_t} - \frac{1}{T}\sum_{t=1}^{T} m_{Y_t} = 0$, we exchange the roles of $\left(X_t - \bar{X}\right)$ and $\left(Y_t - \bar{Y}\right)$ in Cases 3–5, respectively.

Applying the addition operation expressed in (11), we have

$$\sum_{t=1}^{T}\left(X_t - \bar{X}\right)\left(Y_t - \bar{Y}\right) = \left(\sum_{t=1}^{T} m_t, \max_{1\le t\le T} l_t, \max_{1\le t\le T} r_t\right).$$

Then, the fuzzy covariance between X and Y based on the $\mathrm{T_W}$ t-norm is formulated

$$\boldsymbol{\sigma}_{\boldsymbol{X},\boldsymbol{Y}} = \frac{\sum_{t=1}^{T}\left(X_t - \bar{X}\right)\left(Y_t - \bar{Y}\right)}{T}$$
$$= \left(\frac{\sum_{t=1}^{T} m_t}{T}, \frac{\max_{1\le t\le T} l_t}{T}, \frac{\max_{1\le t\le T} r_t}{T}\right). \quad (14)$$

It is undoubtedly true from (14) that the computation of covariance of fuzzy random variables preserves the LR triangular shape of the variables.

III. Portfolio Optimization Models With Return Uncertainty

The asset returns are modeled by normal triangular fuzzy random variables, and we derive two portfolio optimization models corresponding with two above approaches to estimating expected value and covariance: $\mathrm{T_M}$-based and $\mathrm{T_W}$-based fuzzy arithmetic. Assume fuzzy that return of the asset X_i at time t is $X_{it} = (m_{X_{it}}, l_{X_{it}}, r_{X_{it}})$. Since return and covariance of fuzzy asset returns has been devised in Section II, we are able to express portfolio fuzzy return and (fuzzy) variance of portfolio by the two approaches that are henceforth named $\mathrm{T_M}$-based and $\mathrm{T_W}$-based approaches. Both approaches derive portfolio fuzzy returns, but the $\mathrm{T_M}$-based results in portfolio crisp variance, while the $\mathrm{T_W}$-based offers portfolio fuzzy variance.

A. $\mathrm{T_M}$-*Based Approach*

Using the expected value definition derived in (9), we can calculate the expected return of the fuzzy random variable X_i representing stock ith as

$$\overline{X_i} = \left(\frac{1}{T}\sum_{t=1}^{T} m_{X_{it}}, \frac{1}{T}\sum_{t=1}^{T} l_{X_{it}}, \frac{1}{T}\sum_{t=1}^{T} r_{X_{it}}\right). \quad (15)$$

Then, the return of the portfolio is

$$r_p = \sum_{i=1}^{n} w_i \overline{X_i}$$
$$= \left(\sum_{i=1}^{n} w_i \frac{1}{T}\sum_{t=1}^{T} m_{X_{it}}, \sum_{i=1}^{n} w_i \frac{1}{T}\sum_{t=1}^{T} l_{X_{it}}, \sum_{i=1}^{n} w_i \frac{1}{T}\sum_{t=1}^{T} r_{X_{it}}\right). \quad (16)$$

Using the covariance defined in (10), the portfolio crisp variance is traditionally defined

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}. \quad (17)$$

Portfolio risk is simply

$$\sigma_p = \sqrt{\sigma_p^2} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}}. \quad (18)$$

As the covariance of assets is crisp, the portfolio risk is thus also a crisp number.

B. $\mathrm{T_W}$-Based Approach

Follow the weakest t-norm ($\mathrm{T_W}$) fuzzy arithmetic in (13), the expected value of X_i is

$$\overline{X_i} = \left(\frac{1}{T} \sum_{t=1}^{T} m_{X_{it}}, \max_{1\le t\le T} l_{X_{it}}, \max_{1\le t\le T} r_{X_{it}} \right). \tag{19}$$

It is well known that when w_i is a nonnegative scalar, we have

$$w_i \overline{X_i} = \left(w_i \frac{1}{T} \sum_{t=1}^{T} m_{X_{it}}, w_i \max_{1\le t\le T} l_{X_{it}}, w_i \max_{1\le t\le T} r_{X_{it}} \right). \tag{20}$$

Then, the return of the portfolio is

$$\begin{aligned} r_p &= \sum_{i=1}^{n} w_i \overline{X_i} \\ &= \left(\begin{array}{l} \sum_{i=1}^{n} w_i \frac{1}{T} \sum_{t=1}^{T} m_{X_{it}}, \max_{1\le i\le n} w_i \max_{1\le t\le T} l_{X_{it}}, \\ \max_{1\le i\le n} w_i \max_{1\le t\le T} r_{X_{it}} \end{array} \right). \end{aligned} \tag{21}$$

Adopting the fuzzy covariance between assets ith and jth, which is denoted by $\sigma_{ij} = \left(m_{\sigma_{ij}}, l_{\sigma_{ij}}, r_{\sigma_{ij}}\right)$, as presented in (14), the portfolio fuzzy variance is

$$\begin{aligned} \sigma_p^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \\ &= \left(\begin{array}{l} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j m_{\sigma_{ij}}, \max_{1\le i,j\le n} w_i w_j l_{\sigma_{ij}}, \\ \max_{1\le i,j\le n} w_i w_j r_{\sigma_{ij}} \end{array} \right). \end{aligned} \tag{22}$$

As $\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j m_{\sigma_{ij}}$ is a nonnegative number, from (12), portfolio fuzzy risk can be derived

$$\begin{aligned} \sigma_p &= \sqrt{\sigma_p^2} \\ &= \left(\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j m_{\sigma_{ij}}}, \frac{\max_{1\le i,j\le n} w_i w_j l_{\sigma_{ij}}}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j m_{\sigma_{ij}}}}, \right. \\ &\qquad \left. \frac{\max_{1\le i,j\le n} w_i w_j r_{\sigma_{ij}}}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j m_{\sigma_{ij}}}} \right) \end{aligned} \tag{23}$$

if $\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j m_{\sigma_{ij}}}$ is positive. Otherwise, $\sigma_p = (0, 0, 0)$ if $\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j m_{\sigma_{ij}}}$ is zero, but it seldom occurs in practice because $\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j m_{\sigma_{ij}}}$ is actually the variance of portfolio return in the modern portfolio theory.

Note that in the $\mathrm{T_W}$ approach, portfolio variance preserves the LR triangular shape and so does the portfolio risk.

C. Optimization Models

To evaluate the uncertainty of fuzzy numbers, similar to Enea and Piazza [46] and Nguyen and Gordon-Brown [25], we use the uncertainty measure that is a natural generalization of the Hartley function for the fuzzy sets. The uncertainty of fuzzy number A is defined as $U(A) = \int_0^1 \log\left[1 + \mu\left({}^{\alpha}A\right)\right] d\alpha$, where α-cut ${}^{\alpha}A$ is measurable, and the Lebesgue-integrable function $\mu\left({}^{\alpha}A\right)$ is the uncertainty measure of ${}^{\alpha}A$. If $A = (a, b, d)$ is a triangular fuzzy number, then the uncertainty is $U(A) = \left[-\frac{1}{u-l}\left[(1+u-l) - \alpha(u-l)\right].\ln(1+u-l) - \alpha\right]_0^1$. Thus, $U(A) = -1 + \frac{1+d-a}{d-a}\ln(1+d-a)$. If A is denoted as an LR fuzzy number $A = (m, l, r)$, where m is the centre, l is the left spread, and r is the right spread, then the above formula is reorganized

$$U(A) = -1 + \frac{1+r+l}{r+l}\ln(1+r+l). \tag{24}$$

It is well noted that the uncertainty $U(A)$ is always a positive value. The portfolio fuzzy returns are not certain but ranging within an interval from the lower bound to upper bound of the fuzzy return. The bigger the gap between the values in the worst and best situations, the more uncertain the portfolio returns that the investor has to bear. It is thus natural to regard the uncertainty of fuzzy return as a new risk criterion in portfolio optimization.

Associated with the introduction of the new risk criterion, i.e., fuzzy return uncertainty, we also propose a new portfolio performance ratio called "reward-to-uncertainty," denoted as I_U

$$I_U = \frac{r_p}{U_p} = \frac{\left(m_{r_p}, l_{r_p}, r_{r_p}\right)}{U_p} = \left(\frac{m_{r_p}}{U_p}, \frac{l_{r_p}}{U_p}, \frac{r_{r_p}}{U_p}\right). \tag{25}$$

Definition (25) shows that I_U preserves the triangular shape of r_p because r_p is divided by the crisp scalar U_p. Higher return and lower uncertainty portfolios are desired. This is analogous to the maximization of the reward-to-uncertainty. Thus, this ratio sounds alike the renowned Sharpe ratio. The same interpretation with the Sharpe ratio is understood herein, the higher the index I_U, the higher is the portfolio performance regarding uncertainty.

On the other hand, the Sharpe ratio represents the portfolio performance in the conventional Markowitz model's context. In the context of fuzzy return, we initiate the fuzzy Sharpe ratio to account for more uncertain information.

1) If portfolio variance is evaluated using the $\mathrm{T_M}$-based operations (crisp values), then the fuzzy Sharpe ratio is simply an LR triangular fuzzy number

$$S_1 = \frac{r_p}{\sigma_p} = \left(\frac{m_{r_p}}{\sigma_p}, \frac{l_{r_p}}{\sigma_p}, \frac{r_{r_p}}{\sigma_p}\right). \tag{26}$$

The fuzzy Sharpe ratio S_1 preserves the triangular shape because the portfolio risk σ_p is a crisp number in the $\mathrm{T_M}$ approach.

2) If the portfolio variance is calculated using the $\mathrm{T_W}$-based operations (fuzzy values), then the fuzzy Sharpe ratio is computed by the fuzzy division

$$S_2 = \frac{r_p}{\sigma_p} = \frac{\left(m_{r_p}, l_{r_p}, r_{r_p}\right)}{\left(m_{\sigma_p}, l_{\sigma_p}, r_{\sigma_p}\right)}. \tag{27}$$

Since the portfolio variance σ_p is calculated based on the weakest t-norm $\mathrm{T_W}$, this fuzzy Sharpe ratio should also be computed by the $\mathrm{T_W}$-based division. Because the element m_{σ_p} is always positive for a portfolio of risky assets, we present herein the $\mathrm{T_W}$-based division X/Y of two triangular LR fuzzy numbers $X = (m_X, l_X, r_X)$ and $Y = (m_Y, l_Y, r_Y)$, where $m_Y > 0$ only.

Case 1: $m_X > 0$

If $\min((m_X - l_X)/m_Y, m_X/(m_Y + r_Y)) \le z \le m_X/m_Y$, and $z \neq 0$

$$\begin{aligned}(X/Y)(z) &= \sup_{z=x/y} T_W\,(X(x), Y(y)) \\ &= \max\left(X\left(m_Y z\right), Y\left(m_X/z\right)\right) \\ &= \max\left(L\left(\tfrac{m_X - m_Y z}{l_X}\right), L\left(\tfrac{m_X/z - m_Y}{r_Y}\right)\right) \\ &= L\left[(m_X/m_Y - z)/(\max(l_X, z r_Y)/b)\right] \\ &= 1 - \left[(m_X/m_Y - z)/(\max(l_X, z r_Y)/b)\right].\end{aligned}$$

Otherwise, $(X/Y)(z) = 0$.

When $m_X/m_Y \le z \le \max((m_X + r_X)/m_Y, m_X/(m_Y - l_Y))$, similarly to the above, we have

$$(X/Y)(z) = R\left[(z - m_X/m_Y)/(\max(r_X, l_Y z)/m_Y)\right]$$

if $z \le \max((m_X + r_X)/m_Y, m_X/(m_Y - l_Y))$. Otherwise, $(X/Y)(z) = 0$.

Then, for Case 1, i.e., $m_X > 0$, we can summarize that

if $\min((m_X - l_X)/m_Y, m_X/(m_Y + l_Y)) \le z \le m_X/m_Y$, then $(X/Y)(z) = 1 - [(m_X/m_Y - z)/(\max(l_X, l_Y z)/m_Y)]$. Else if $\max((m_X + r_X)/m_Y, m_X/(m_Y - r_Y)) \ge z \ge m_X/m_Y$, then $(X/Y)(z) = 1 - [(z - m_X/m_Y)/(\max(r_X, r_Y z)/m_Y)]$.

$$\text{Otherwise, } (X/Y)(z) = 0. \tag{28}$$

Case 2: $m_X = 0$

$$(X/Y)(z) = (0, l_X/m_Y, r_X/m_Y)_{\mathrm{LR}} = (0, l_X/m_Y, r_X/m_Y)$$

$$(X/Y)(z) = \begin{cases} 1 + z/(l_X/m_Y), & -l_X/m_Y \le z \le 0 \\ 1 - z/(r_X/m_Y), & 0 < z \le r_X/m_Y \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

Case 3: $m_X < 0$

If $\min((m_X - l_X)/m_Y, m_X/(m_Y + l_Y)) \le z \le m_X/m_Y$, then $(X/Y)(z) = 1 - [(m_X/m_Y - z)/(\max(l_X, l_Y z)/m_Y)]$. Else if $\max((m_X + r_X)/m_Y, m_X/(m_Y - r_Y)) \ge z \ge m_X/m_Y$, then $(X/Y)(z) = 1 - [(z - m_X/m_Y)/(\max(r_X, r_Y z)/m_Y)]$.

$$\text{Otherwise, } (X/Y)(z) = 0. \tag{30}$$

Applying the formulas for the $\mathrm{T_W}$-based division X/Y presented above, we are able to calculate the fuzzy Sharpe ratio S_2.

Rational investors always attempt to increase portfolio return and reduce portfolio risks or uncertainty. Investors conventionally attempt to maximize the Sharpe ratio in order to obtain the high-performance portfolio. By introducing the new concept of the portfolio return uncertainty U_p, investor's efforts are to seek out portfolios with higher fuzzy Sharpe ratio (S_1 or S_2) and lower return uncertainty U_p. In particular, maximizing the fuzzy Sharpe ratio and minimizing portfolio return uncertainty U_p is the principle to formulate portfolio selection models in this paper. These fuzzy multiobjective programming problems are presented below corresponding to the two approaches of calculating expected values and covariance of fuzzy random variables:

Problem 1: Strongest t-norm $\mathrm{T_M}$ approach

Max S_1
Min U_p
s.t. $0 \le w \le 1$ and $\sum w = 1$.

Problem 2: Weakest t-norm $\mathrm{T_W}$ approach

Max S_2
Min U_p
s.t. $0 \le w \le 1$ and $\sum w = 1$.

Note that U_p is a crisp number, but S_1 and S_2 are both fuzzy numbers. The above two problems are thus the fuzzy multiple objective optimization problems. Solving these problems requires a consistent evaluation of fuzzy objective numbers S_1 and S_2. We apply the ranking paradigm based on the x-centroid coordinate described in Section II for evaluating fuzzy numbers.

Centroid of the LR triangular fuzzy number S_1

$$\text{centroid}\,(S_1) = \frac{1}{3}\left(3 * \frac{m_{r_p}}{\sigma_p} - \frac{l_{r_p}}{\sigma_p} + \frac{r_{r_p}}{\sigma_p}\right). \tag{31}$$

To evaluate S_2, as it is not a triangular fuzzy number in general circumstances, we need to calculate its centroids according to the more general formula (5):

$$\text{centroid}\,(S_2) = \frac{\int_{-\infty}^{+\infty} x f_A(x)dx}{\int_{-\infty}^{+\infty} f_A(x)dx} = \frac{\int_{lb}^{ub} x f_A(x)dx}{\int_{lb}^{ub} f_A(x)dx}. \tag{32}$$

Depending on the value of m_X, the lower bound lb and upper bound ub are determined differently:

If $m_X > 0$: $lb = \min((m_X - l_X)/m_Y, m_X/(m_Y + l_Y))$, $ub = \max((m_X + r_X)/m_Y, m_X/(m_Y - r_Y))$
Else if $m_X = 0$: $lb = -l_X/m_Y$, $ub = r_X/m_Y$
Else: $lb = \min((m_X - l_X)/m_Y, m_X/(m_Y + l_Y))$, $ub = \max((m_X + r_X)/m_Y, m_X/(m_Y - r_Y))$

Problems 1 and 2 are now can be transformed to the following crisp multiobjective optimization problems:

Problem 1′: Strongest t-norm $\mathrm{T_M}$ approach

Max centroid (S_1)
Max $-U_p$
s.t. $0 \le w \le 1$ and $\sum w = 1$.

Problem 2′: Weakest t-norm $\mathrm{T_W}$ approach

Max centroid (S_2)
Max $-U_p$
s.t. $0 \le w \le 1$ and $\sum w = 1$.

The two multiobjective models above can be solved with the same computing procedure. The only difference is in the centroid formula of the fuzzy Sharpe ratios S_1 and S_2. The fuzzy number S_1, which is used in the $\mathrm{T_M}$ approach, is an LR triangular so that its centroid can be simply calculated via its three parameters (31). The centroid of S_2 in the TW approach, however, requires integral computation, and it thus consumes more time if its shape is more complicated (32). In some cases, S_2 may possess the triangular shape; then, its computational complexity is as simple as S_1. Various methods can be employed to solve these above problems. In this paper, we describe two methods using fuzzy approach and using GA.

1) Fuzzy Approach to Solving Multiobjective Optimization: In order to solve Problems $1'$ and $2'$ above, we consider the more general multiple-objective optimization problems

$$\text{Max } F_i(w),\ i = 1, \ldots, k \ \text{ s.t. } W = \{w \in R^n | aw \le b, w \ge 0\} \tag{33}$$

where a is an $m \times n$ matrix, and b is an m-vector.

The following demonstrates a method that is related to fuzzy mathematics for solving the above problem. The solving methodology is the max–min approach that utilizes the fuzzy decision concept of Bellman and Zadeh [47] and the membership function suggested by Zimmermann [48]. By introducing the auxiliary value γ, which is also called the satisfaction level, the solution of (33) is equivalent to that of the following conventional single-objective programming problem:

Problem 3:

$$\text{Max } \gamma$$
$$\text{s.t. } \frac{F_i(w) - F_i^{\min}}{F_i^{\max} - F_i^{\min}} \ge \gamma, i = 1, \ldots, k$$
$$W = \{w \in R^n | aw \le b, w \ge 0\}$$

where $F_i^{\max} = F_i(w^*) = \max_{w \in W} F_i(w)$, $F_i^{\min} = \min\limits_{\substack{j=1,\ldots,k \\ j \ne i}} \left(F_i\left(w_j^*\right)\right), i = 1, \ldots, k.$

Let us assign $F_1(w) = \text{centroid}(S_1(w))$ or $F_1(w) = \text{centroid}(S_2(w))$ depending on whether the problem to be solved is Problems $1'$ or $2'$ and $F_2(w) = -U_p(w)$; then, Problem 3 can be used to solve Problems $1'$ and $2'$ with $k = 2$. More specifically, with these two-objective problems, we can present the solving method in detail as follows. Assume that w_1^* is the solution of the following problem: Max $F_1(w) = \text{centroid}(S_1(w))$ s.t. $0 \le w \le 1$ and $\sum w = 1$ or of the problem: Max $F_1(w) = \text{centroid}(S_2(w))$ s.t. $0 \le w \le 1$ and $\sum w = 1$, and assume that w_2^* is the solution of the problem: Max $F_2(w) = -U_p(w)$ s.t. $0 \le w \le 1$ and $\sum w = 1$; we have that $F_1^{\max} = \text{centroid}(S_1(w_1^*))$ and $F_1^{\min} = \text{centroid}(S_1(w_2^*))$, while $F_2^{\max} = -U_p(w_2^*)$ and $F_2^{\min} = -U_p(w_1^*)$. Fig. 1 demonstrates how these values are organized in the solvable process, where w^* is the final solution of the Problem 3, and γ is the satisfaction level.

Problem 3 is now reassembled as follows:

Problem $3'$:

$$\text{Max } \gamma$$
$$\text{s.t. } \frac{F_1(w) - F_1^{\min}}{F_1^{\max} - F_1^{\min}} \ge \gamma,\ \frac{F_2(w) - F_2^{\min}}{F_2^{\max} - F_2^{\min}} \ge \gamma,\ 0 \le w \le 1$$
$$\sum w = 1.$$

Problem $3'$ can be simply solved by a standard optimization package.

2) Genetic Algorithm Approach to Solving Multiobjective Optimization: A GA [49]–[51] is an unorthodox search or optimization technique operated on a population of n artificial individuals. Individuals are characterized by chromosomes (or genomes) S_k, $k = \{1, \ldots, n\}$. The chromosome is a string of symbols, which are called genes, $S_k = (S_{k1}, \ldots, S_{kN})$, and N is a string length. Individuals are evaluated via calculation of a fitness function. To evolve through successive generations, GA performs three basic genetic operators: selection, crossover, and mutation. Through chromosomes' evolution, GA searches for the best solution(s) in the sense of the given fitness function. Application of GA to solve multiobjective problems has been found widely in the literature. In this study, we use the multiobjective optimization "gamultiobj" function that is implemented in the MATLAB Global Optimization Toolbox to solve the problems. The gamultiobj function employs a variant of the Nondominated Sorting Genetic Algorithm-II. The gamultiobj function uses a controlled elitist GA that compromises fitness values of individuals and diversity of the population. A distance function is deployed to maintain the diversity of population for convergence to an optimal Pareto front [52].

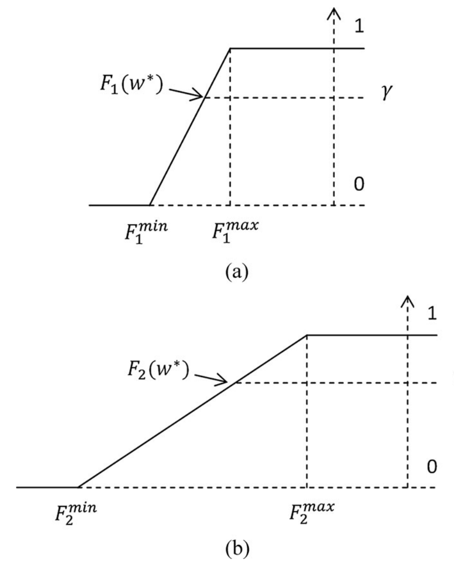

Fig. 1. Membership functions to solve two-objective problems.

The GA calibration is presented in the following. The population is initialized randomly that satisfies the bounds and linear constraints of the weights (see Problems $1'$ and $2'$). The population size is 15 times larger than the number of variables, and the maximum number of generations is equal to 200 times of the number of variables. Two fitness functions are also two objectives of the models. The first fitness function is the centroid of the fuzzy Sharpe ratio of the portfolios (S_1 or S_2), while the second is the portfolio fuzzy uncertainty U_p. The algorithm terminates when it reaches the maximum number of generations specified. The selection function employed herein is the tournament function that selects each parent by choosing two individuals at random, i.e., tournament size equals 2, and then choosing the best individual out of that set to be a parent. Weak individuals have a smaller chance to be selected if the tournament size is large. Crossover forms a new individual for the next generation by combining two individuals or parents. There are

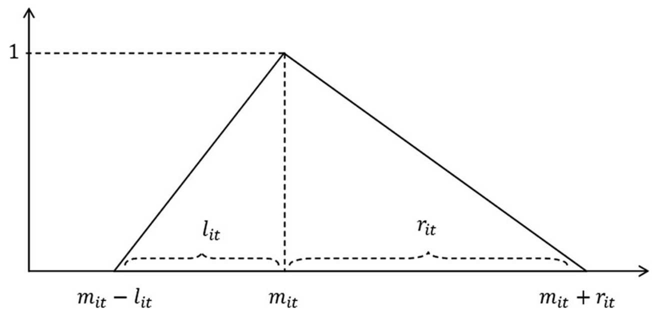

Fig. 2. Stock returns modeled by LR triangular fuzzy numbers.

several functions that can be implemented. We herein deploy the "intermediate" method built in MATLAB. The function creates children by a random weighted average of the parents through a single parameter Ratio:

$$child1 = parent1 + rand * Ratio * (parent2 - parent1)$$

where $parent1$ and $parent2$ are two parents selected to be crossovered, $rand$ is the function generating $U(0, 1)$ random numbers, and $child1$ is the resulting child after crossovering. The Ratio selected as the default is 1.0; therefore, the children produced are within the hypercube defined by the parents' locations at opposite vertices. Mutation functions make small random changes in the genes of individuals in the populations that provide more genetic diversity and enable the GA to search in a broader space of solutions. To solve constraint programming problems, we select the "adaptive feasible" method implemented in MATLAB. This method randomly generates directions that are adaptive with respect to the last successful or unsuccessful generation. Along each direction, a step length is chosen so that linear constraints and bounds are satisfied. For the multiobjective problem settings, the "distancecrowding" is selected as the distance measure function, while the Pareto front population fraction is 0.35, which keeps the most fit population down to the specified fraction in order to maintain a diverse population.

IV. Experimental Results

The experiments are carried out with stock return datasets downloaded from the Datastream database. Two datasets are experimented with the daily dataset of U.S. stocks in the BBC Global 30 index and the weekly dataset comprising German stocks in the Euro Stoxx 50 index. For the stock ith at time point tth, the closing price (denoted as $Price_{it}$), lower price (Low_Price_{it}), and higher price ($High\_Price_{it}$) are obtained. Medium, left spread, and right spread of the LR triangular fuzzy number (m_{it}, l_{it}, r_{it}) representing the fuzzy return of the stock ith (Fig. 2) at time point tth is determined by the following formulas:

$$m_{it} - l_{it} = \ln \frac{Low\_Price_{it}}{Price_{i(t-1)}}, \quad m_{it} = \ln \frac{Price_{it}}{Price_{i(t-1)}}$$

$$m_{it} + r_{it} = \ln \frac{High\_Price_{it}}{Price_{i(t-1)}}. \tag{34}$$

It is obvious that the lower price is less than the price and that both are less than the higher price. Formulas (34) thus ensure that l_{it} and r_{it} are all nonnegative values.

After modeling returns of all stocks by fuzzy random variables following (34), we are able to derive the fuzzy expected return for each stock and the covariance matrices by two approaches: $\mathrm{T_M}$- and $\mathrm{T_W}$-based fuzzy arithmetic. With a selected stock, comparing (15) with (19), we see that using $\mathrm{T_M}$ fuzzy arithmetic produces fuzzy expected returns with left and right widths smaller than those of using $\mathrm{T_W}$ fuzzy arithmetic, although the centre values are the same and equal to the conventional statistical expected return. In other words, stock returns in the $\mathrm{T_W}$-based approach are more uncertain compared with those in the $\mathrm{T_M}$ calculation.

For comparisons, the traditional MVO approach proposed by Markowitz [37] is also executed. The model is to maximize the portfolio return subject to a certain portfolio risk or else minimize portfolio risk subject to a certain portfolio return. Maximizing the Sharpe ratio is an alternative to acquire the high-performance portfolio. Assume the crisp expected return of the stock ith is μ_i and statistical covariance between stock ith and stock jth is cov_{ij}; then, the Markowitz portfolio return and risk are, respectively, $\sum_{i=1}^n w_i \mu_i$ and $\sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \mathrm{cov}_{ij}}$, where w_i is the money allocation weight for the stock ith and n is the number of stocks. Assume that S is the pseudo Sharpe ratio, where the risk-free rate is zero, calculated by the division of the portfolio return on the portfolio risk: $S = \frac{\sum_{i=1}^n w_i \mu_i}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j cov_{ij}}}$; we deploy herein the MVO as follows:

$$\text{Max } S \text{ s.t. } 0 \le w \le 1 \text{ and } \sum w = 1. \tag{35}$$

A. U.S. Dataset Experiment

Stock returns of ten U.S. constituents included in the BBC Global 30 index, i.e., Apple, AT&T, Berkshire Hathaway, E. I. Du Pont de Nemours, Exxon Mobil, General Electric, Johnson & Johnson, Procter & Gamble, Southern Co, and Wal-Mart Stores, are used in this experiment. The time period selected spans from January 1, 2008 to December 30, 2011, comprising 1044 data samples.

1) $\mathrm{T_M}$-Based Approach: In the $\mathrm{T_M}$ approach, expected returns of stocks are computed by (15) and reported in Table I.

The fuzzy portfolio return r_p is then obtained using (16). The $\mathrm{T_M}$-based crisp symmetric covariance matrix of fuzzy stock returns is constructed by computing covariance of each pair of stocks using (8). In this specific experiment, we observe that the $\mathrm{T_M}$ covariance is larger than that of the normal statistical covariance measure, although it is theoretically not anticipated from (8). For example, the variance of the Apple stock calculated using the strongest t-norm $\mathrm{T_M}$ is 7.0691×10^{-4} compared with 5.7870×10^{-4} of the normal statistical calculation. Or else the covariance between Apple and AT&T is 2.2692×10^{-4} by the normal covariance matrix, but it is 2.7958×10^{-4} by the strongest t-norm approach.

TABLE I
FUZZY U.S. STOCK DAILY EXPECTED RETURNS BY THE T_M-BASED ESTIMATION

Stocks	Fuzzy returns
Apple	$(0.6851 \times 10^{-3}, 0.0149, 0.0136)$
AT & T	$(-0.3046 \times 10^{-3}, 0.0116, 0.0115)$
Berkshire Hathaway	$(-0.2013 \times 10^{-3}, 0.012, 0.0101)$
Du Pont de Nemours	$(0.036 \times 10^{-3}, 0.0151, 0.0142)$
Exxon Mobil	$(-0.0959 \times 10^{-3}, 0.0119, 0.0105)$
General Electric	$(-0.6968 \times 10^{-3}, 0.0159, 0.0162)$
Johnson & Johnson	$(-0.0162 \times 10^{-3}, 0.0083, 0.0077)$
Procter & Gamble	$(-0.0918 \times 10^{-3}, 0.0096, 0.0085)$
Southern Co.	$(0.1703 \times 10^{-3}, 0.0091, 0.0078)$
Wal-Mart Stores	$(0.2193 \times 10^{-3}, 0.0095, 0.0092)$

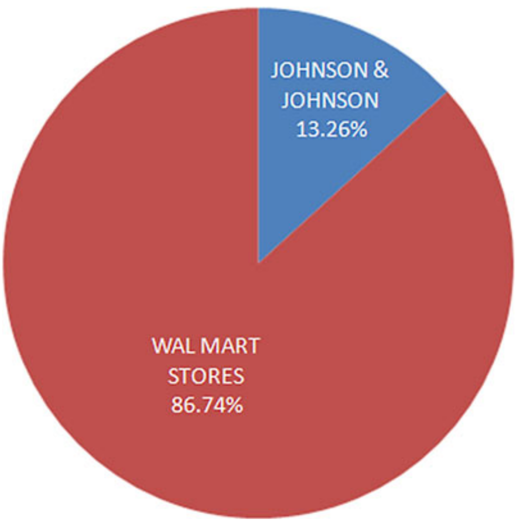

Fig. 3. T_M-based U.S. portfolio solved by fuzzy approach.

Portfolio crisp variance and its risk (standard deviation) are computed, respectively, by (17) and (18). The fuzzy Sharpe ratio is calculated as an LR triangular fuzzy number S_1 (26) and the new risk measure, i.e., fuzzy return uncertainty, is calculated using (24). As S_1 is a triangular fuzzy number, its centroid is derived simply by (31). When the centroid of the fuzzy Sharpe ratio and portfolio return uncertainty computations are completed, we solve Problem 1$'$ by the two approaches, i.e., fuzzy and GA as represented in Section III. Running Problem 1$'$ by the fuzzy approach, we obtain the portfolio solutions depicted in Fig. 3 with the satisfaction level $\gamma = 0.7951$.

With this solution, the portfolio fuzzy return is $(1.8809 \times 10^{-4}, 0.0094, 0.0090)$ as demonstrated in Fig. 4, the crisp risk is minimized to 0.0143, the portfolio fuzzy Sharpe ratio is (0.0132, 0.6538, 0.6289), as in Fig. 5, with its centroid is at 0.0049, return uncertainty obtained at 0.0091, and the fuzzy "reward-to-uncertainty" ratio is reported to be (0.0206, 1.0256, 0.9866), as in Fig. 6.

Instead of solving Problem 1$'$ by the fuzzy approach, now we alternatively solve this problem by the GA approach to see if GA may be more efficient. GA solver results in a series of solutions diagrammed as a Pareto front in Fig. 7. The population of 150 individuals is established. The GA runs and terminates when the number of generations exceeds the maximum number of 2000.

The coordinates of return uncertainty and centroid of fuzzy Sharpe ratio derived from the solution obtained by the fuzzy approach (red circle) are also depicted in this Pareto front for comparisons. We found that the single solution of the fuzzy

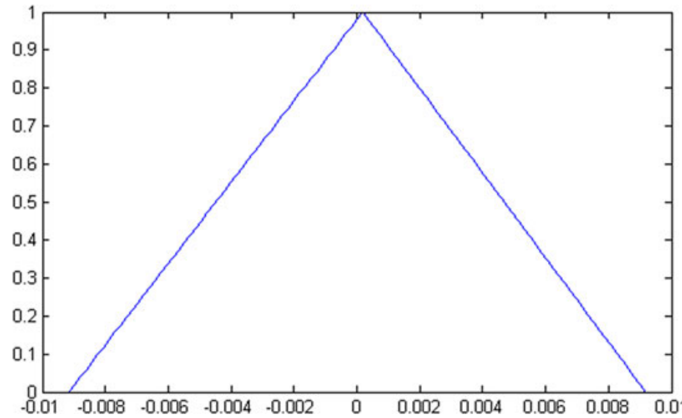

Fig. 4. T_M-based U.S. portfolio fuzzy return.

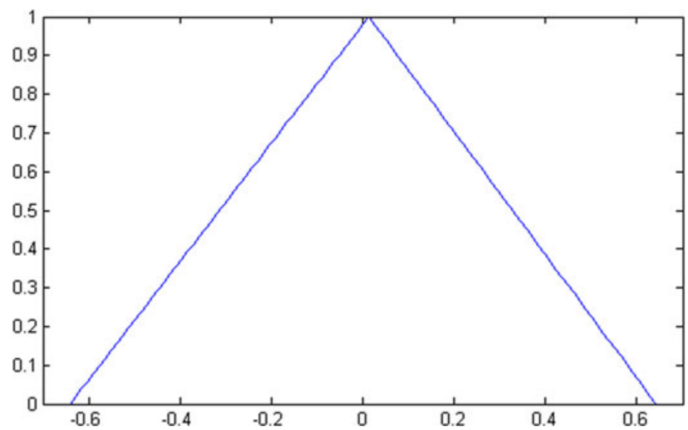

Fig. 5. T_M-based U.S. fuzzy Sharpe ratio.

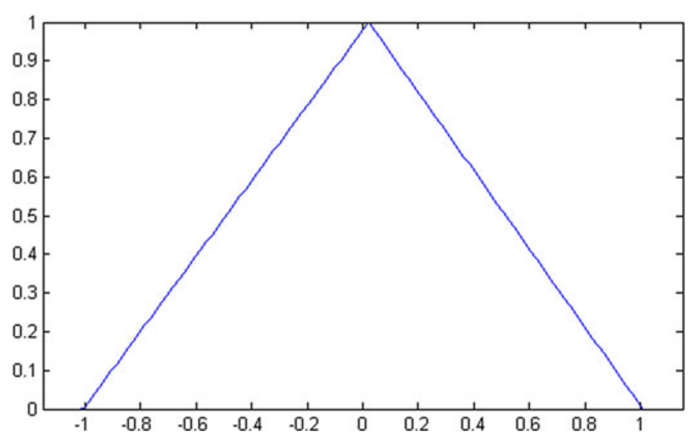

Fig. 6. T_M-based reward-to-uncertainty ratio.

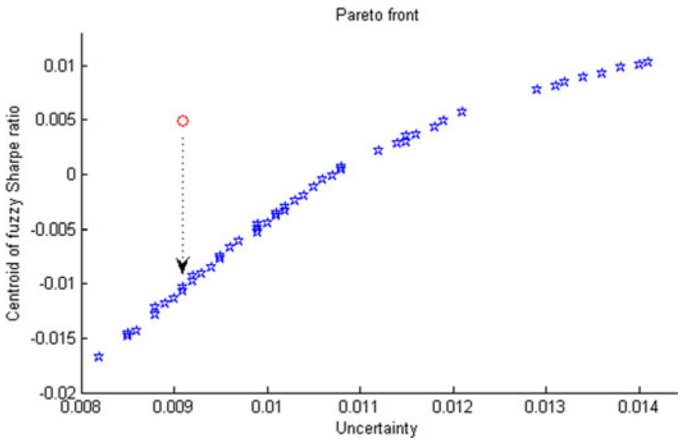

Fig. 7. $\mathrm{T_M}$-based U.S. portfolio Pareto front obtained by GA approach.

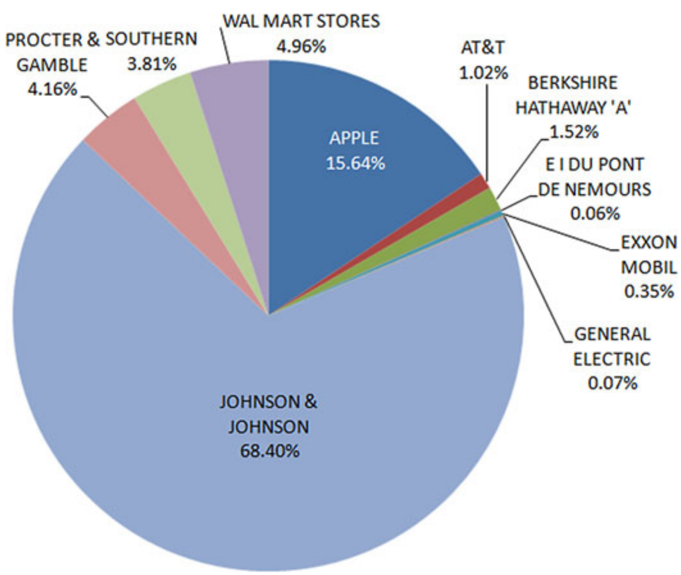

Fig. 8. $\mathrm{T_M}$-based portfolio solved by GA approach.

approach is much superior compared with the GA Pareto front, although the fuzzy approach takes much less time to run than the GA approach. Among the series of GA solutions, the solution having the closest uncertainty value to that of the fuzzy approach (the solution at the end of the arrow in Fig. 7) is selected to investigate. This portfolio solution is illustrated in Fig. 8.

The solutions of the fuzzy approach and GA approach (Figs. 3 and 8, respectively) show some similarities since they both select the Johnson & Johnson and Wal-Mart Stores stocks with major proportions. Although fuzzy return uncertainty (0.0091) and centroid of the fuzzy Sharpe ratio (−0.0103) are dominated by those of the fuzzy approach, i.e., 0.0091 and 0.0049, respectively, the GA solution, however, exhibits a very well-diversified portfolio (see Fig. 8) compared with the two-stock portfolio in Fig. 3.

2 $\mathrm{T_W}$*-Based Approach:* In this approach, the expected returns of stocks are calculated by (19) and assembled in Table II.

TABLE II
FUZZY U.S. STOCK DAILY EXPECTED RETURNS BY THE $\mathrm{T_W}$-BASED CALCULATION

Stocks	Fuzzy returns
Apple	$(0.6851 \times 10^{-3}, 0.2118, 0.1296)$
AT&T	$(-0.3046 \times 10^{-3}, 0.1249, 0.1017)$
Berkshire Hathaway	$(-0.2013 \times 10^{-3}, 0.1475, 0.1329)$
Du Pont de Nemours	$(0.036 \times 10^{-3}, 0.1161, 0.1217)$
Exxon Mobil	$(-0.0959 \times 10^{-3}, 0.1602, 0.1430)$
General Electric	$(-0.6968 \times 10^{-3}, 0.1557, 0.1287)$
Johnson & Johnson	$(-0.0162 \times 10^{-3}, 0.0929, 0.0925)$
Procter Gamble	$(-0.0918 \times 10^{-3}, 0.4338, 0.0895)$
Southern Co.	$(0.1703 \times 10^{-3}, 0.0942, 0.0674)$
Wal-Mart Stores	$(0.2193 \times 10^{-3}, 0.1141, 0.0741)$

Due to the different calculation procedures, i.e., see (15) and (19), it is clear that the expected fuzzy returns obtained in the $\mathrm{T_W}$ approach account for more uncertainty, i.e., wider left and right spreads in the LR triangular fuzzy numbers, compared with those in the $\mathrm{T_M}$ approach (contrast Tables I and II). For instance, the fuzzy return of the Apple stock by the $\mathrm{T_M}$ approach is $(0.6851 \times 10^{-3}, 0.0149, 0.0136)$, while it is $(0.6851 \times 10^{-3}, 0.2118, 0.1296)$ by the TW approach. The fuzzy return of the AT&T stock is $(-0.3046 \times 10^{-3}, 0.0116, 0.0115)$ in the $\mathrm{T_M}$ approach and $(-0.3046 \times 10^{-3}, 0.1249, 0.1017)$ in the $\mathrm{T_W}$ approach.

Fuzzy covariance for each pair of stocks is completed using (14); then, the fuzzy covariance matrix is constructed. The centers of the LR fuzzy covariances are equal to the covariance values that are obtained based on the conventional statistics. For example, the variance of the Apple stock is computed by the LR triangular fuzzy number $(5.7870 \times 10^{-4}, 2.6272 \times 10^{-5}, 1.6080 \times 10^{-5})$, while this value by conventional statistics is 5.7870×10^{-4}. Another example is the covariance between Apple and AT&T is the LR triangular fuzzy number $(2.2692 \times 10^{-4}, 3.0660 \times 10^{-5}, 1.8766 \times 10^{-5})$ by $\mathrm{T_W}$ approach, while it is 2.2692×10^{-4} in conventional covariance.

Formula (21) is used to find the fuzzy portfolio return, while formulas (22) and (23) are applied, respectively, to find the fuzzy portfolio variance and fuzzy portfolio volatility risk. When portfolio return and volatility risk are fuzzy numbers, the fuzzy Sharpe ratio S_2 obtained by (27) is not guaranteed to be an LR triangular fuzzy number. Centroid of this ratio by (32) is more involved to calculate compared with the triangular fuzzy numbers. Solving Problem 2′ by the fuzzy approach, the solution (Fig. 9) is obtained with the satisfaction level reaching $\gamma = 0.7994$.

The corresponding portfolio fuzzy return of the above solution is $(2.0757 \times 10^{-4}, 0.0326, 0.0342)$ as in Fig. 10, portfolio fuzzy risk is $(0.0143, 6.6931 \times 10^{-5}, 7.0181 \times 10^{-5})$ in Fig. 11, the uncertainty is 0.0327, and the reward-to-uncertainty ratio is $(0.0063, 0.9978, 1.0463)$ in Fig. 13.

The fuzzy Sharpe ratio is a fuzzy number by dividing fuzzy return and fuzzy risk as follows:

$S_2(z) = 1 - [(0.0146 - z)/(\max(0.0326, z \times 6.6931 \times 10^{-5})/0.0143)]$ if $-2.2747 \le z \le 0.0146$. Else $S_2(z) = 1 - [(z -$

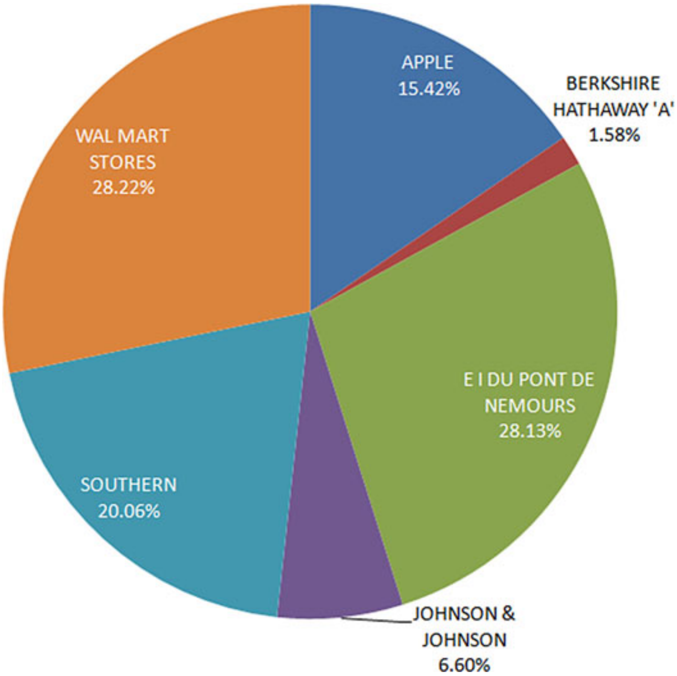

Fig. 9. $\mathrm{T_W}$ -based U.S. portfolio solved by fuzzy approach.

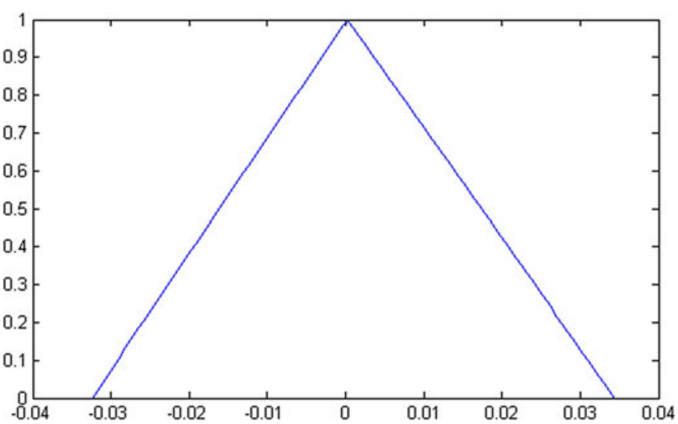

Fig. 10. $\mathrm{T_W}$ -based U.S. portfolio fuzzy return.

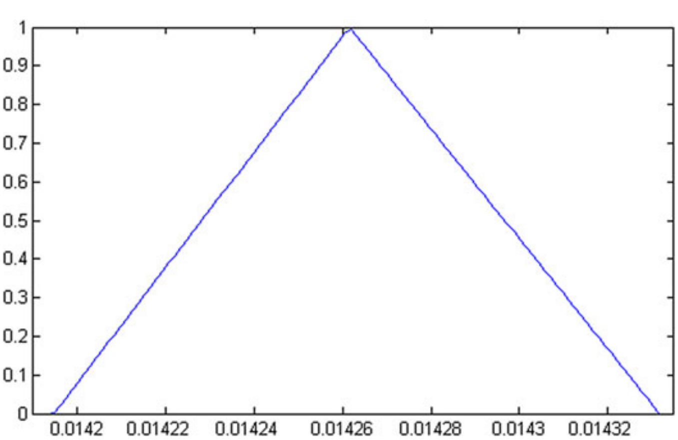

Fig. 11. $\mathrm{T_W}$ -based U.S. portfolio fuzzy risk.

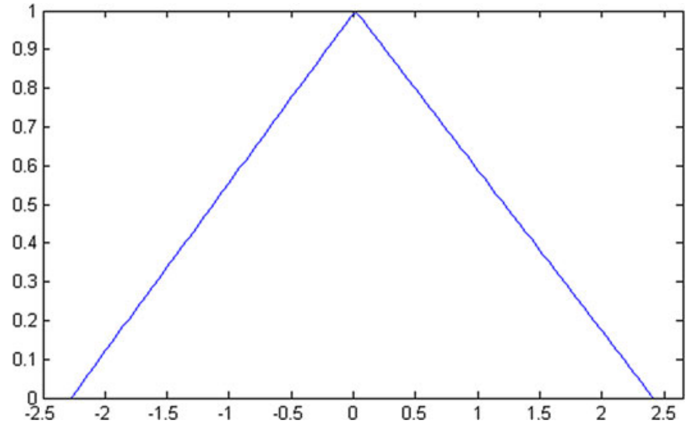

Fig. 12. $\mathrm{T_W}$ -based U.S. fuzzy Sharpe ratio.

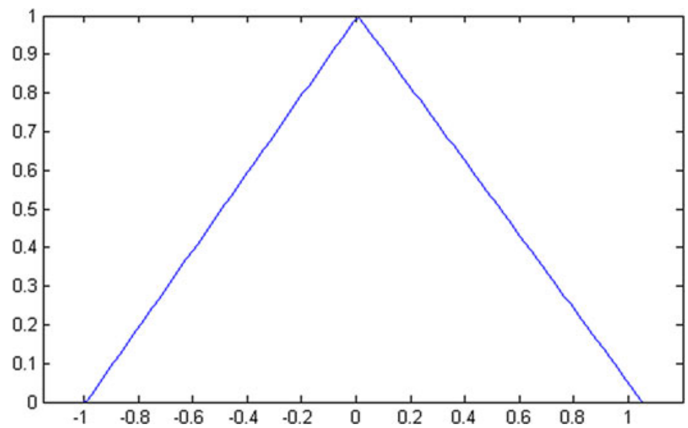

Fig. 13. $\mathrm{T_W}$ -based U.S. reward-to-uncertainty ratio.

$0.0146)/(\max(0.0342, z \times 7.0181 \times 10^{-5})/0.0143)]$ if $0.0146 < z \leq 2.4150$. Otherwise, $S_2(z) = 0$.

However, when z in the above expression ranges from -2.2747 to 0.0146, we see that the expression $\max(0.0326, z \times 6.6931 \times 10^{-5})$ is always equal to 0.0326. The same is applied for the expression $\max(0.0342, z \times 7.0181 \times 10^{-5})$, it is always equal to 0.0342. Thus, in this specific circumstance, the fuzzy number S_2 is a triangular fuzzy number

$$S_2(z) = \begin{cases} 1 - (0.0146 - z)/2.2797, & -2.2747 \leq z \leq 0.0146 \\ 1 - (z - 0.0146)/2.3916, & 0.0146 < z \leq 2.4150 \\ 0, & \text{otherwise.} \end{cases}$$

The graphical presentation of this fuzzy number is in Fig. 12 with its centroid at 0.0516.

Problem $2'$ is now alternatively solved by the GA approach to compare with the fuzzy solving approach. The same GA configuration used in solving Problem $1'$ is applied herein. The GA Pareto front is plotted in Fig. 14.

The portfolio return uncertainty and centroid of the fuzzy Sharpe ratio of the solution obtained by the fuzzy approach (red circle) are also plotted in the Pareto front (see Fig. 14).

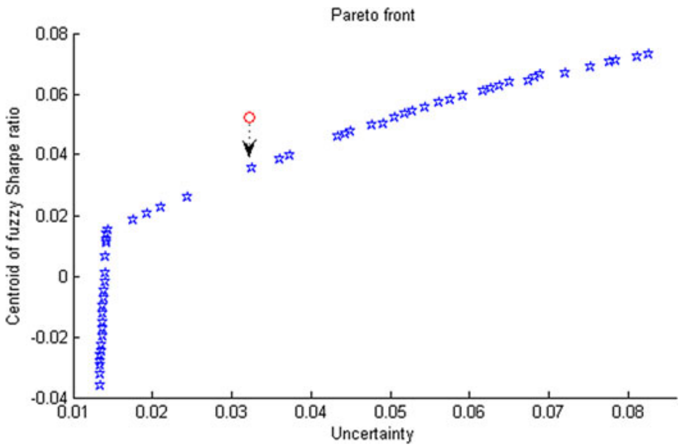

Fig. 14. T_W-based portfolio Pareto front obtained by GA approach.

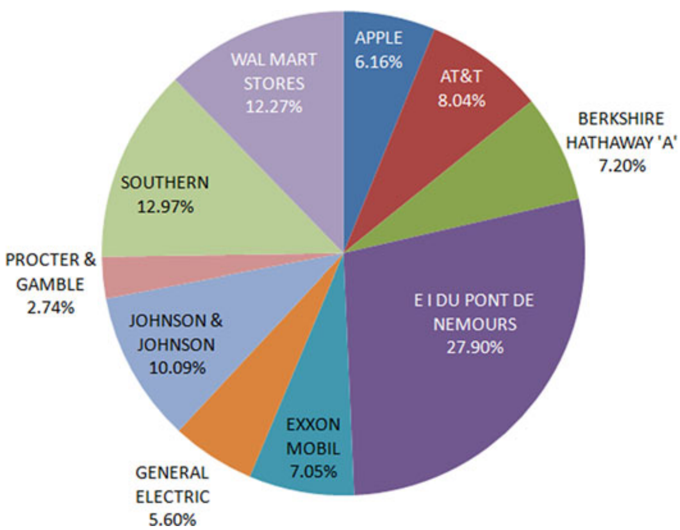

Fig. 15. T_W-based U.S. portfolio solved by GA approach.

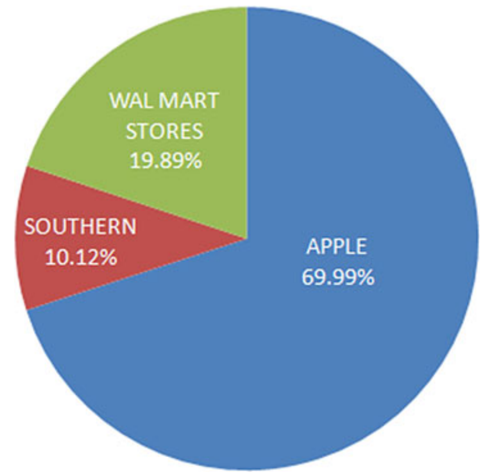

Fig. 16. MVO U.S. portfolio allocation.

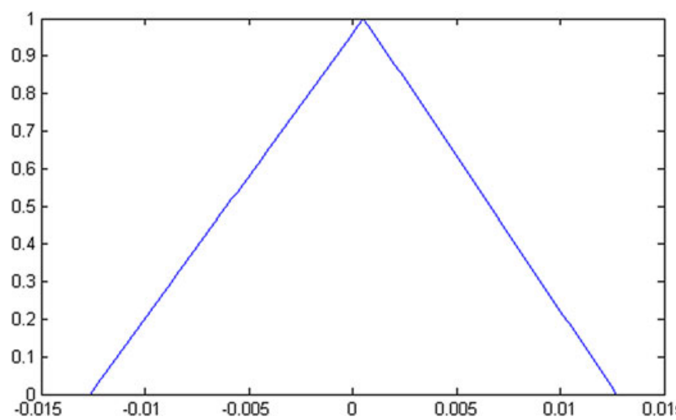

Fig. 17. T_M-based return by MVO solution.

Comparisons display a quite close proximity between the series of solutions of the GA with the fuzzy approach's single solution. The solution of GA with the closest uncertainty with that of the fuzzy approach (the point at the end of the arrow in Fig. 14) is depicted in Fig. 15.

With the above GA solution, the centroid of the fuzzy Sharpe ratio obtained is 0.0355, which is much smaller than 0.0516 of the fuzzy approach solution although the return uncertainty of two solutions is close: 0.0325 and 0.0327. Again, in this T_W approach, we also found solutions of fuzzy-based and GA-based solving approaches are similar when they both select six stocks E. I. Du Pont de Nemours, Southern, Wal-Mart Stores, Apple, Johnson & Johnson, and Berkshire Hathaway. Furthermore, the GA solution also draws a much more diversified portfolio, although inferior in terms of performance, compared with that of the fuzzy approach.

3) MVO Approach: The unique solution achieved from running the MVO (35), which is not well diversified, is displayed in Fig. 16 below. The portfolio return is 5.4033×10^{-4} and portfolio risk is 0.0187. The pseudo Sharpe ratio thus reaches 0.0290.

The MVO solution is closer with that of the T_W-based model than that of the T_M-based model when both problems (i.e., Problems $1'$ and $2'$) are solved by the fuzzy approach (see Figs. 3, 9, and 16). The MVO and T_M-based model have only one common stock (Wal-Mart Stores), while the MVO and T_W-based model have three common stocks (Apple, Southern and Wal-Mart Stores). With the MVO solution, the fuzzy return if calculated based on the strongest t-norm T_M is (5.4033×10^{-4}, 0.0132, 0.0122), while it is (5.4033×10^{-4}, 0.1482, 0.0907) if using the weakest t-norm T_W (see Figs. 17 and 18, respectively). The uncertainty values correspondingly are 0.0126 and 0.1110, which are much larger than those of the two proposed approaches, 0.0091 and 0.0327 in the T_M and T_W approaches, respectively.

It is understandable that the MVO approach does not minimize the uncertainty of the portfolio return, while the proposed approaches do. The differences between the solution obtained by MVO and that of the T_M and T_W approaches pointed out that the uncertainty is distinguished from the normal volatility risk, and it is reasonably regarded as a new risk measure of the

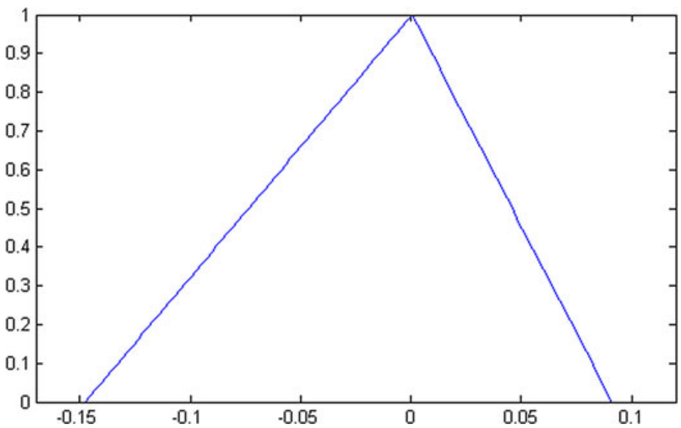

Fig. 18. $\mathrm{T_W}$-based return by MVO solution.

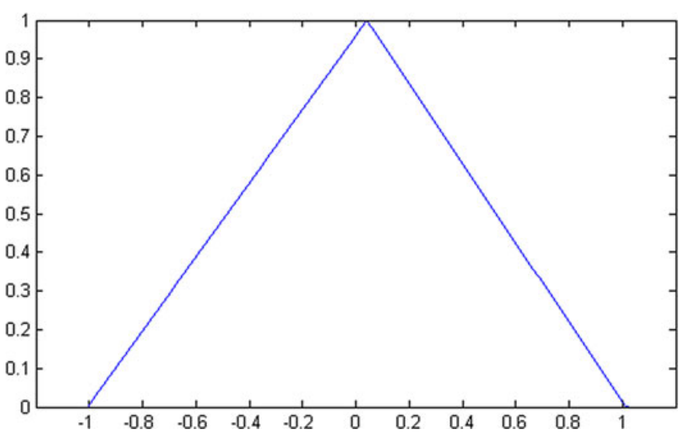

Fig. 19. $\mathrm{T_M}$-based reward-to-uncertainty ratio by MVO solution.

portfolio in the fuzzy environment. Since the covariance measures are different between the traditional statistical convention and the fuzzy $\mathrm{T_M}$ approach, it is not sensible to compare the risk as well as the Sharpe ratio of portfolios among these two approaches. On the other hand, the center value of the fuzzy covariance is equal to the conventional statistical covariance, i.e., see (14), it seems logical to collate the fuzzy Sharpe ratio S_2 and the MVO crisp Sharpe ratio. The centroid of S_2 evaluated by (32) is 0.0516 that is much higher than that of the MVO at 0.0290. Thus, regarding the Sharpe ratio, the $\mathrm{T_W}$ approach is superior to the MVO.

The reward-to-uncertainty ratio of the $\mathrm{T_M}$ approach is (0.0206, 1.0256, 0.9866) with centroid at 0.0076, while that of the $\mathrm{T_W}$ approach is (0.0063, 0.9978, 1.0463) with centroid at 0.0225 (see Figs. 6 and 13, respectively). With the MVO solution, we can derive the reward-to-uncertainty ratios following the $\mathrm{T_M}$ and $\mathrm{T_W}$ methodology as follows: (0.0430, 1.0503, 0.9665) with centroid 0.0150 and (0.0049, 1.3359, 0.8177) with centroid -0.1679 (see Figs. 19 and 20).

Evaluation of these "reward-to-uncertainty" ratios shows that $\mathrm{T_W}$-based solution is better than that of the MVO, and the $\mathrm{T_M}$-based solution is the most dominated by the other two methods. The results suggest using $\mathrm{T_W}$ rather than $\mathrm{T_M}$ in portfolio selection.

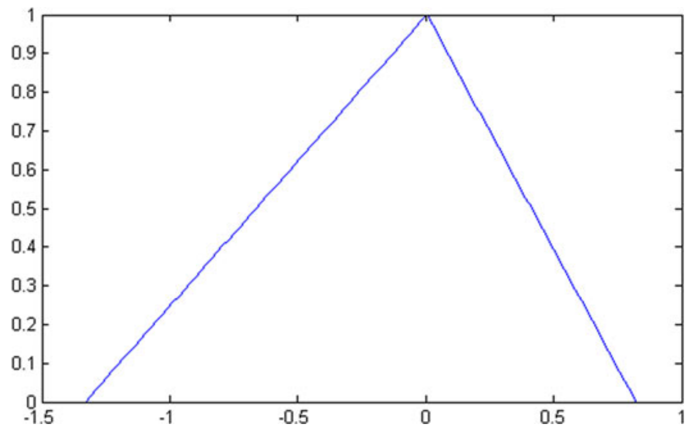

Fig. 20. $\mathrm{T_M}$-based reward-to-uncertainty ratio by MVO solution.

TABLE III
FUZZY GERMAN STOCK WEEKLY EXPECTED RETURNS BY $\mathrm{T_M}$-BASED CALCULATION

Stocks	Fuzzy returns
Allianz	(−0.0027, 0.0154, 0.0157)
BASF SE	(0.0014, 0.0149, 0.0122)
BAYER	(−0.0001, 0.0153, 0.0136)
BMW	(0.0007, 0.0167, 0.0146)
Daimler	(−0.0005, 0.0166, 0.0156)
Deutsche Bank	(−0.0018, 0.0156, 0.0157)
Deutsche Telekom	(−0.0022, 0.0126, 0.0133)
E. ON	(−0.0005, 0.0146, 0.0127)
Munchener Ruck.	(−0.0023, 0.0144, 0.0134)
RWE	(−0.001, 0.0138, 0.0126)
SAP	(0.0005, 0.0144, 0.0136)
Siemens Ltd	(−0.0004, 0.0149, 0.015)
Volkswagen Pref.	(0.0023, 0.021, 0.0146)

B. German Dataset Experiment

The dataset used in this experiment consists of 13 German stocks in the Euro Stoxx 50 index, which represents for blue-chip stocks of supersector leaders in the Eurozone. The German constituents are Allianz, BASF SE, BAYER, BMW, Daimler, Deutsche Bank, Deutsche Telekom, E. ON, Munchener Ruck., RWE, SAP, Siemens Ltd, Volkswagen Pref.. The dataset is sampled weekly from January 1, 2001 to December 30, 2011, comprising 574 data samples.

1) $\mathrm{T_M}$-Based Approach: In the $\mathrm{T_M}$ approach, expected returns of stocks (see Table III) are calculated using (15). The portfolio fuzzy return is obtainable by (16) and construction of the $\mathrm{T_M}$-based crisp covariance matrix of fuzzy stock returns is accomplished using (8).

Similar to the previous experiment, we also detect that the fuzzy covariance values obtained from the $\mathrm{T_M}$ approach are larger than normal statistical covariance obtained by crisp returns. For example, the variance of the Allianz stock by the $\mathrm{T_M}$ approach is 0.0034, while that of the traditional covariance is 0.0031. Another example is the covariance between Allianz with BAYER that is 0.0017 in the $\mathrm{T_M}$ approach, while it is 0.0015 in normal statistical covariance.

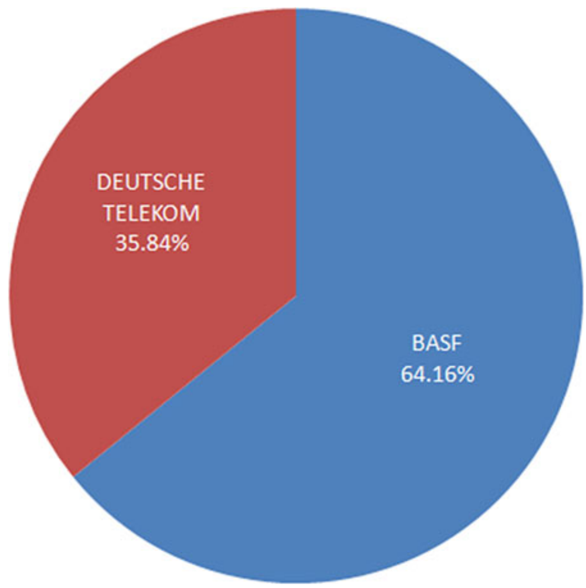

Fig. 21. T_M-based German portfolio allocation solved by fuzzy approach.

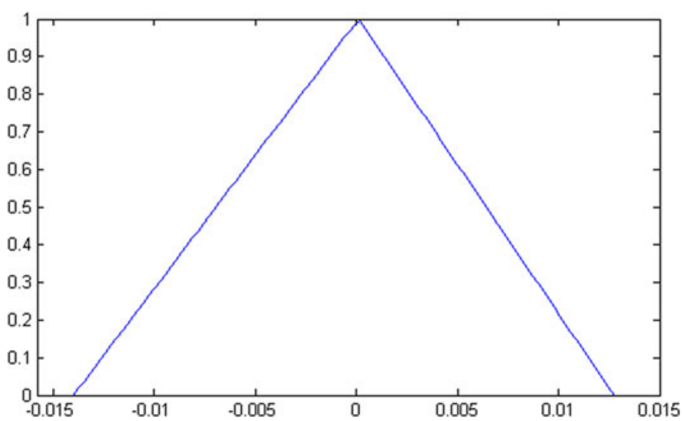

Fig. 22. T_M-based German portfolio fuzzy return.

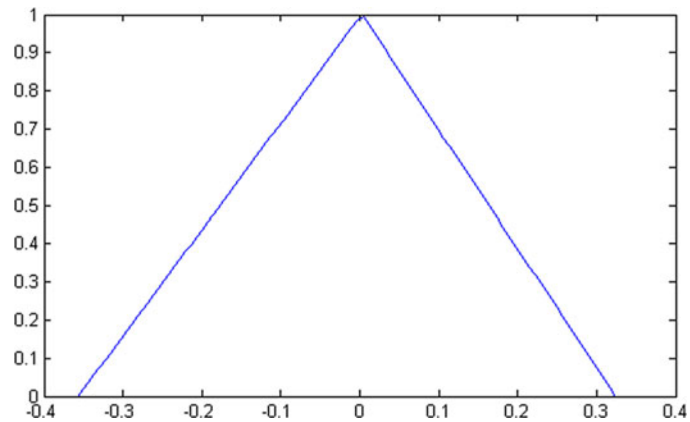

Fig. 23. T_M-based fuzzy Sharpe ratio.

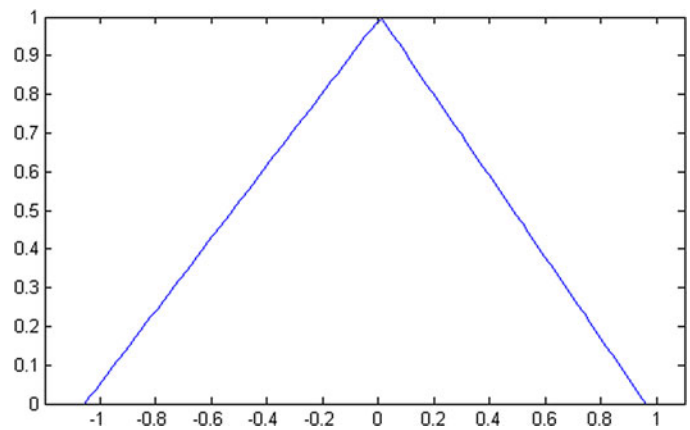

Fig. 24. T_M-based reward-to-uncertainty ratio.

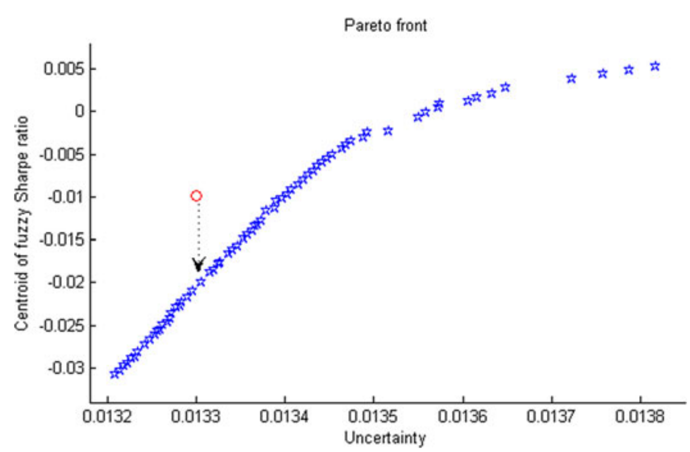

Fig. 25. T_M-based German portfolio Pareto front obtained by GA approach.

Portfolio crisp variance and its risk are computed by (17) and (18) in that order. The fuzzy Sharpe ratio is calculated as an LR triangular fuzzy number by (26). When the centroid of the fuzzy Sharpe ratio and portfolio return uncertainty computations are completed by (31) and (24), respectively, we solve Problem 1′ by the two solving approaches: fuzzy and GA. The solution obtained by solving Problem 1′ with the fuzzy approach is depicted in Fig. 21 with the satisfaction level $\gamma = 0.5685$.

The resultant portfolio fuzzy return is $(9.6863 \times 10^{-5}, 0.0141, 0.0126)$ as presented in Fig. 22, while portfolio crisp risk is 0.0393. The fuzzy Sharpe ratio is then (0.0025, 0.3593, 0.3221) in Fig. 23 with the centroid equals -0.0099. The return uncertainty is minimized to 0.0133, while the fuzzy reward-to-uncertainty ratio is reported at (0.0073, 1.0640, 0.9537), as in Fig. 24 with centroid -0.0295.

Alternatively, solving Problem 1′ by the GA approach, we are able to produce the Pareto front in Fig. 25. The population encompasses 195 individuals and the number of generations is of 2600, which is the stopping criterion of the GA.

The coordinates of return uncertainty and centroid of fuzzy Sharpe ratio obtained by the fuzzy approach (red circle) are also depicted in this Pareto front for comparisons between two solving approaches: fuzzy and GA. Again, the single solution

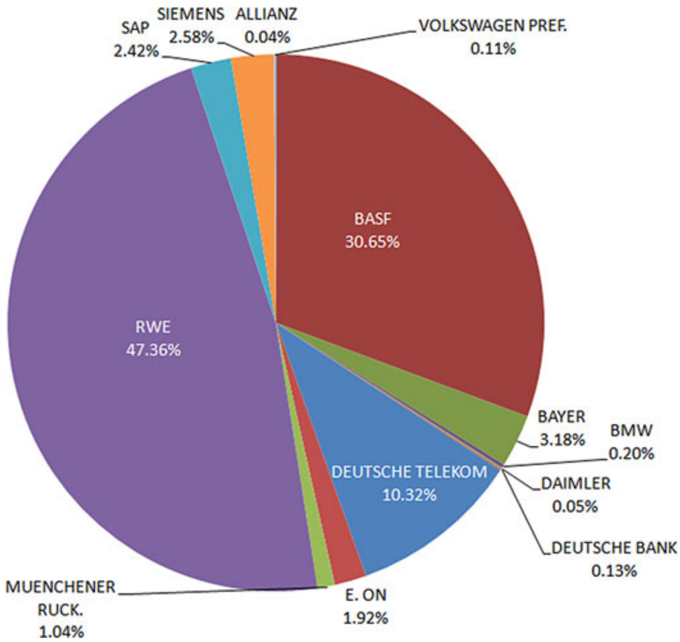

Fig. 26. $\mathrm{T_M}$-based German portfolio solved by GA approach.

of the fuzzy approach is superior compared with the GA Pareto front though the fuzzy approach takes much less time to run than the GA approach. Among the series of GA solutions, the solution having the closest uncertainty value to that of the fuzzy approach (the solution at the end of the arrow in Fig. 25) is presented in Fig. 26.

The solutions of the fuzzy approach and GA approach show some resemblances as they both select the BASF and Deutsche Telekom stocks in high proportions. The difference happens when the GA approach allocates the RWE stock with highest portfolio weight at 47.36%, while the fuzzy approach does not pick this stock at all. Although fuzzy return uncertainty (0.0133) and centroid of fuzzy Sharpe ratio (−0.0210) are dominated by those of the fuzzy solving approach, i.e., 0.0133 and −0.0099, respectively, the GA solution, however, exposes the completely well-diversified portfolio compared with the only two-stock portfolio in Fig. 21.

2) $\mathrm{T_W}$-Based Approach: In this approach, the expected returns of stocks are obtained from (19) and presented in Table IV. Since the $\mathrm{T_M}$ approach uses the expected value operation for left and right spreads of the LR triangular fuzzy numbers, while the $\mathrm{T_W}$ approach uses the maximum operation, we are thus unsurprised to find the more uncertain expected fuzzy returns in the $\mathrm{T_W}$ approach against those of the $\mathrm{T_M}$ approach (see Tables III and IV).

As discussed in the previous experiment, formula (14) shows that the center element of the LR fuzzy covariance in the $\mathrm{T_W}$ approach is identical to the normal statistical covariance. For example, the variance of the Allianz stock is the fuzzy number $(0.0031, 9.8019 \times 10^{-5}, 6.5149 \times 10^{-5})$ in the $\mathrm{T_W}$ approach, while it is just simply 0.0031 in conventional covariance. Or else, in the $\mathrm{T_W}$ approach, the fuzzy covariance between Allianz and BAYER is $(0.0015, 8.8836 \times 10^{-5}, 5.9046 \times 10^{-5})$, while it is 0.0015 in normal covariance.

Formula (21) is used to find the fuzzy portfolio return, while formulae (22) and (23) are applied to obtain the fuzzy portfolio variance and fuzzy portfolio volatility risk. When portfolio return and volatility risk are fuzzy numbers, the fuzzy Sharpe ratio obtained by (27) may be not an LR triangular fuzzy number. Centroid of this ratio by (32) is more involved to work out compared with the triangular fuzzy numbers. Solving Problem $2'$ by the fuzzy approach, the solution (see Fig. 27) is obtained with the satisfaction level at $\gamma = 0.5897$.

The corresponding portfolio fuzzy return is $(-2.7222 \times 10^{-4}, 0.0483, 0.0919)$ in Fig. 28, portfolio fuzzy risk is $(0.0374, 1.5694 \times 10^{-4}, 2.9834 \times 10^{-4})$ in Fig. 29, the uncertainty

TABLE IV
FUZZY GERMAN STOCK WEEKLY EXPECTED RETURNS BY $\mathrm{T_W}$-BASED CALCULATION

Stocks	Fuzzy returns
Allianz	(−0.0027, 0.1665, 0.1107)
BASF SE	(0.0014, 0.0997, 0.0712)
BAYER	(−0.0001, 0.1267, 0.0822)
BMW	(0.0007, 0.1625, 0.1235)
Daimler	(−0.0005, 0.1138, 0.1151)
Deutsche Bank	(−0.0018, 0.1157, 0.0982)
Deutsche Telekom	(−0.0022, 0.1003, 0.1161)
E. ON	(−0.0005, 0.1244, 0.0771)
Munchener Ruck.	(−0.0023, 0.1483, 0.1250)
RWE	(−0.001, 0.0840, 0.0852)
SAP	(0.0005, 0.1842, 0.1198)
Siemens Ltd	(−0.0004, 0.1133, 0.2154)
Volkswagen Pref.	(0.0023, 0.2791, 0.1206)

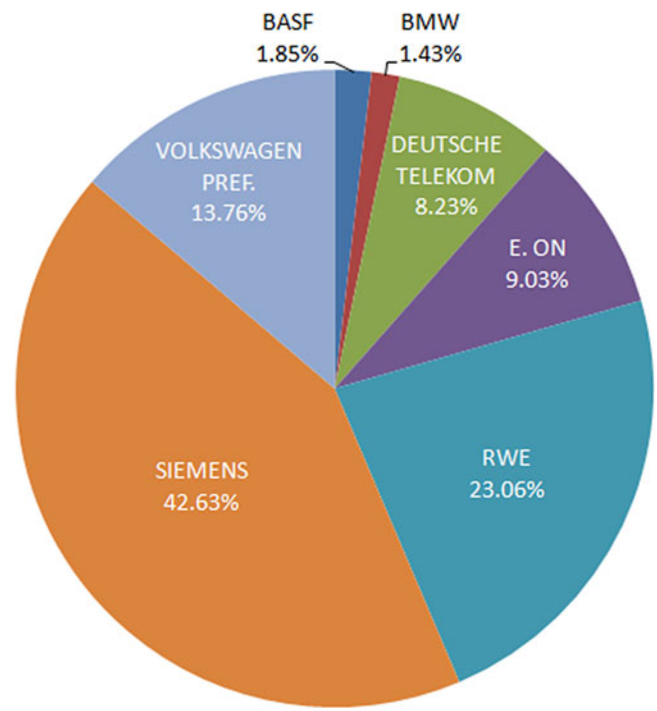

Fig. 27. $\mathrm{T_W}$-based German portfolio solved by fuzzy approach.

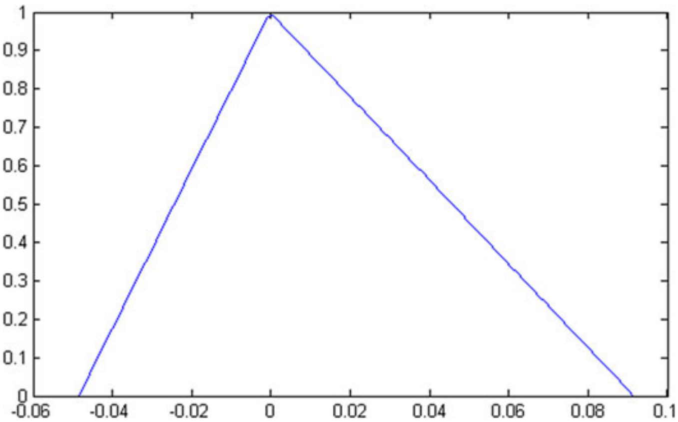

Fig. 28. $\mathrm{T_W}$-based German portfolio fuzzy return.

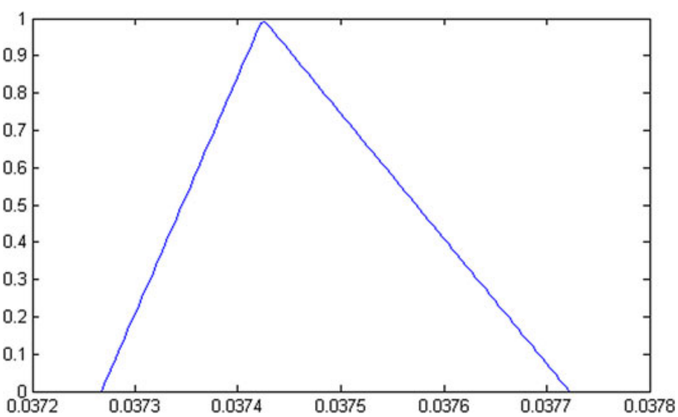

Fig. 29. $\mathrm{T_W}$-based German portfolio fuzzy risk.

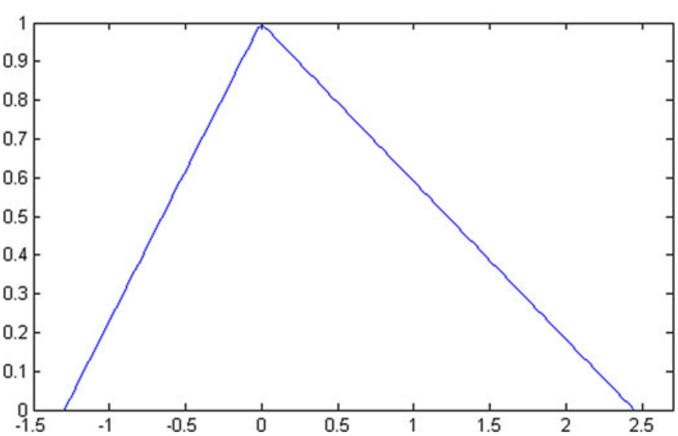

Fig. 30. $\mathrm{T_W}$-based German portfolio fuzzy Sharpe ratio.

is 0.0670, the reward-to-uncertainty ratio is $(-0.0041, 0.7209, 1.3705)$, as in Fig. 31 with centroid at 0.2125, and the fuzzy Sharpe ratio with centroid at 0.3805 is devised in the following formula:

$S_2(z) = 1 - [(-0.0073 - z)/(\max(0.0483, z \times 1.5694 \times 10^{-4})/0.0374)]$ if $-1.2984 \leq z \leq -0.0073$. Else $S_2(z) = 1 - [(z + 0.0073)/(\max(0.0919, z \times 2.9834 \times 10^{-4})/0.0374)]$ if $-0.0073 < z \leq 2.4471$. Otherwise, $S_2(z) = 0$.

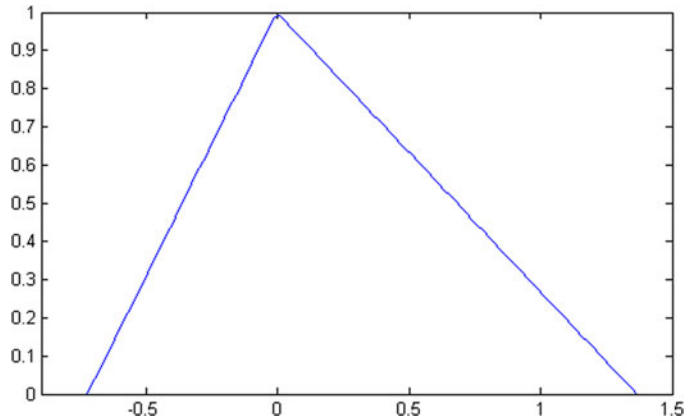

Fig. 31. $\mathrm{T_W}$-based German portfolio reward-to-uncertainty ratio.

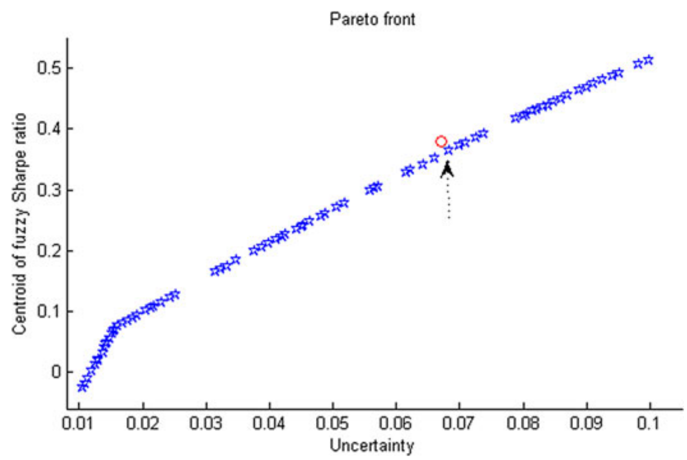

Fig. 32. $\mathrm{T_W}$-based German portfolio Pareto front obtained by GA approach.

When z in the above expression varies in the interval from -1.2984 to -0.0073, we see that the expression $\max(0.0483, z \times 1.5694 \times 10^{-4})$ is always equal to 0.0483. The same is for the expression $\max(0.0919, z * 2.9834 * 10^{-4})$. It is always equal to 0.0919. Thus, in this specific circumstance, the fuzzy number S_2 is a triangular fuzzy number (see Fig. 30):

$$S_2(z) = \begin{cases} 1 - (-0.0073 - z)/1.2914, & -1.2984 \leq z \leq -0.0073 \\ 1 - (z + 0.0073)/2.4572, & -0.0073 < z \leq 2.4471 \\ 0, & \text{otherwise.} \end{cases}$$

The same GA configuration applied to solving Problem 1′ (195 individuals in a population and 2600 generations are used in the evolving process) is used to solve Problem 2′. The GA Pareto front is scattered in Fig. 32.

The return uncertainty and centroid of the fuzzy Sharpe ratio of the solution obtained by the fuzzy approach is also plotted in the Pareto front. The series of solutions of the GA compare closely with the fuzzy approach's single solution (see the red

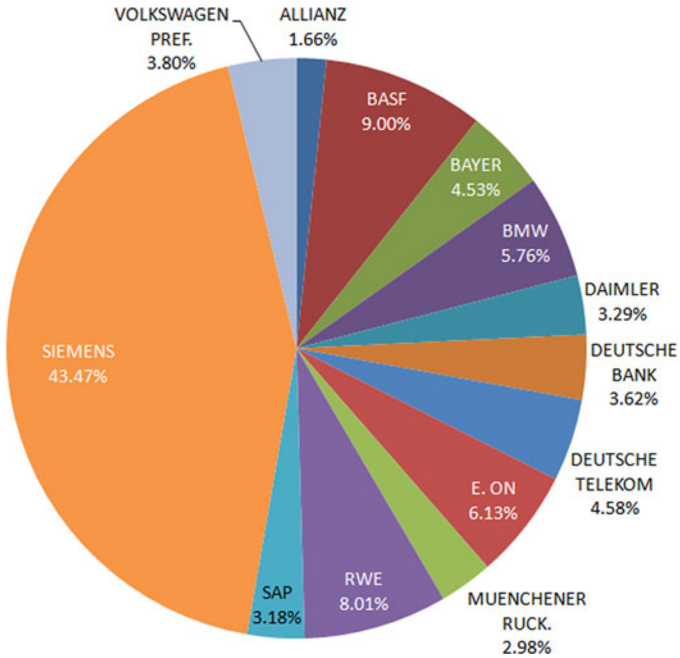

Fig. 33. $\mathrm{T_W}$-based portfolio solved by GA approach.

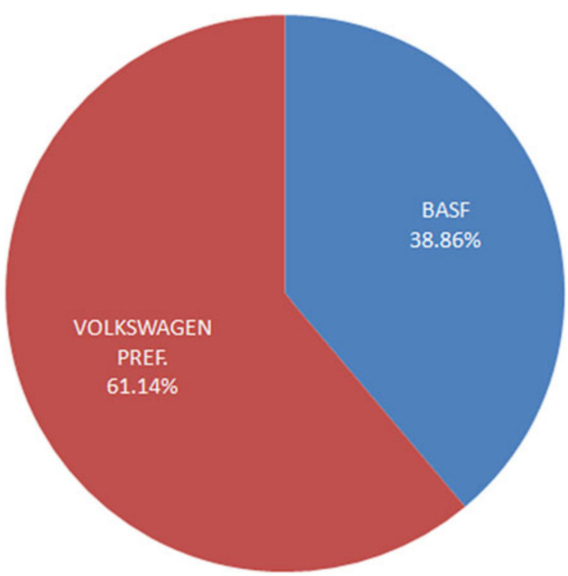

Fig. 34. MVO German portfolio allocation.

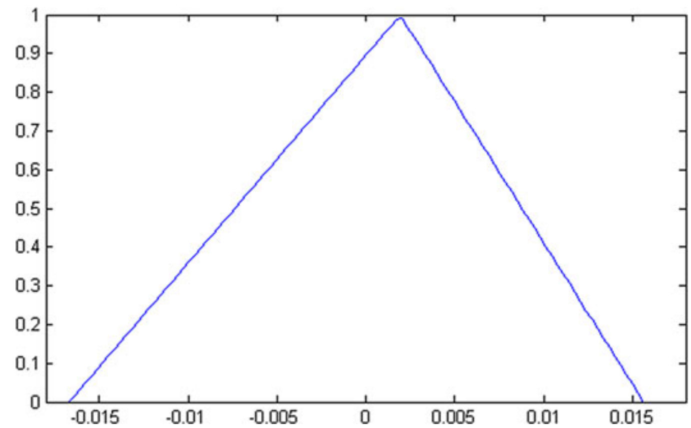

Fig. 35. $\mathrm{T_M}$-based return by MVO solution.

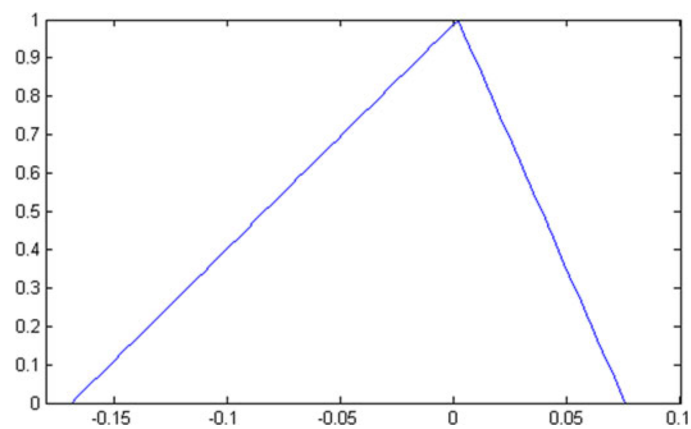

Fig. 36. $\mathrm{T_W}$-based return by MVO solution.

circle in Fig. 32). The solution of GA with the closest uncertainty with that of the fuzzy approach (the point at the end of arrow in Fig. 32) is illustrated in Fig. 33.

Again, using GA approach offers a very well-diversified solution. The resulting centroid of the fuzzy Sharpe ratio is 0.3638 that is smaller than 0.3805 of the fuzzy approach solution, although the return uncertainty of two solutions is approximated: 0.0683 and 0.0670 correspondingly.

3) MVO Approach: The unique solution achieved using the MVO is not well diversified as shown in Fig. 34. The portfolio return is 0.0019 and portfolio risk is 0.0472. The pseudo-Sharpe ratio is, thus, at 0.0410. With the MVO solution, the fuzzy return based on the strongest t-norm $\mathrm{T_M}$ is (0.0019, 0.0187, 0.0137) in Fig. 35, while it is (0.0019, 0.1706, 0.0737) in Fig. 36 using the weakest t-norm $\mathrm{T_W}$.

The uncertainty values 0.0160 and 0.1133 are larger than those of the two proposed approaches: 0.0133 and 0.0670 in the $\mathrm{T_M}$ and $\mathrm{T_W}$ approaches, respectively. This along with the previous experiment emphasizes the distinction between the uncertainty U_p and the normal variance risk σ_p. It is reasonable since the MVO only considers the normal variance risk, while the proposed approaches take into account not only the normal variance risk but the uncertainty of portfolio return in the fuzzy modeling as well. The centroid of the fuzzy Sharpe ratio S_2 evaluated by (32) is 0.3805 is much higher than that of the MVO at 0.0410. In terms of the Sharpe ratio performance, the superiority belongs to the $\mathrm{T_W}$ approach rather than the MVO. This fact reinforces the results of the previous experiment.

The reward-to-uncertainty ratio of the $\mathrm{T_M}$ approach is at (0.0073, 1.0640, 0.9537) and its centroid is at -0.0295, while that of the $\mathrm{T_W}$ approach is (-0.0041, 0.7209, 1.3705) with centroid at 0.2125. With the MVO solution obtained in Fig. 34, we can derive that the reward-to-uncertainty ratio corresponding to the $\mathrm{T_M}$ approach is (0.1206, 1.1650, 0.8565) in Fig. 37 with

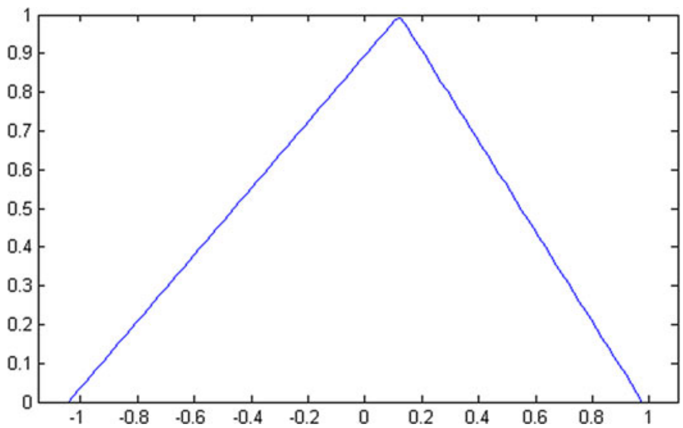

Fig. 37. $\mathrm{T_M}$-based reward-to-uncertainty ratio by MVO solution.

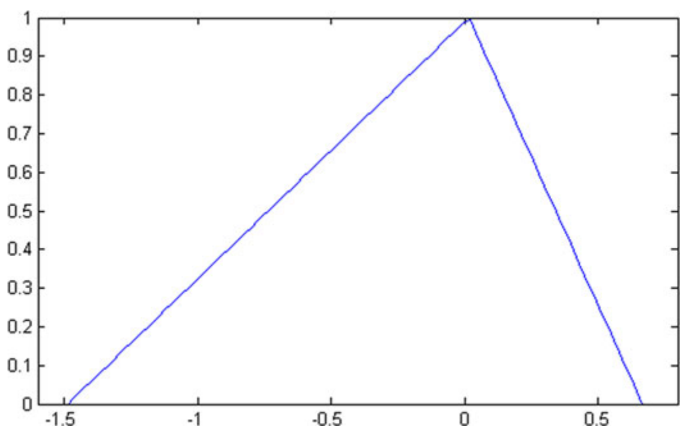

Fig. 38. $\mathrm{T_W}$-based reward-to-uncertainty ratio by MVO solution.

centroid at 0.0178, while that of the $\mathrm{T_W}$ approach is (0.0170, 1.5061, 0.6508) in Fig. 38 with centroid at -0.2680.

Evaluation of these reward-to-uncertainty ratios shows that $\mathrm{T_W}$ is better than MVO and better than $\mathrm{T_M}$. Like the previous experiment, the results in this experiment also advise investors to use $\mathrm{T_W}$ rather than $\mathrm{T_M}$ approach in portfolio selection.

C. Discussions

Through two experiments, we find that the fuzzy returns obtained by the $\mathrm{T_M}$ approach or by the MVO are usually negatively skewed, while those obtained by the $\mathrm{T_W}$ approach are positively skewed. For example, in the U.S. dataset, the $\mathrm{T_M}$-based portfolio fuzzy return is (1.8809×10^{-4}, 0.0094, 0.0090), which is negatively skewed because the left width 0.0094 is greater than the right width 0.0090 (see Fig. 4). On the other hand, the $\mathrm{T_W}$-based portfolio fuzzy return is (2.0757×10^{-4}, 0.0326, 0.0342), which is positively skewed because $0.0326 < 0.0342$ (see Fig. 10). The MVO solution of this dataset, if evaluated by the $\mathrm{T_M}$ t-norm and the $\mathrm{T_W}$ t-norm, results in portfolio fuzzy returns, which are negatively skewed (see Figs. 17 and 18). Similarly, in the German dataset, the $\mathrm{T_M}$-based portfolio fuzzy return is (9.6863×10^{-5}, 0.0141, 0.0126), which is negatively skewed (see Fig. 22). Alternatively, the $\mathrm{T_W}$-based German portfolio fuzzy return is (-2.7222×10^{-4}, 0.0483, 0.0919), which is positively skewed (see Fig. 28). The fuzzy portfolio returns derived from the MVO solution of the German dataset based on the $\mathrm{T_M}$ t-norm and the $\mathrm{T_W}$ t-norm are negatively skewed (see Figs. 35 and 36 respectively).

When the fuzzy returns are positively skewed, they tend to have high performance evaluated by the centroid approach. The skewness characteristic explains why the $\mathrm{T_W}$ approach outperforms both the $\mathrm{T_M}$ approach and MVO in terms of portfolio performance evaluation, i.e., with respect to the fuzzy Sharpe ratio and the reward-to-uncertainty ratio.

In the $\mathrm{T_W}$ approach, the portfolio return and risk are calculated in LR triangular fuzzy numbers. The fuzzy Sharpe ratios are thus not guaranteed to be triangular fuzzy numbers. However, we obtained the fuzzy Sharpe ratios in the triangular shape in two portfolio optimization experiments herein. This happens because the left and right spreads of the portfolio fuzzy risk are so small compared with the left and right spreads of the portfolio fuzzy return. The portfolio fuzzy return and risk are graphed in Figs. 10 and 11, respectively, for the U.S. dataset experiment and Figs. 28 and 29 for the German dataset experiment. The portfolio fuzzy Sharpe ratios are represented in the triangular shape in Figs. 12 and 30, respectively.

In contrast with the fuzzy solving approach, which produces a unique optimal portfolio solution, the GA approach to solving the multiple objective portfolio models can generate a series of solutions to form up the Pareto front. Solutions of GA are completely diversified compared with those of the fuzzy approach. However, GA solutions are all dominated by those of the fuzzy approach. Thus, the fuzzy approach to solving proposed models is recommended in order to obtain high-performance portfolios rather than the GA approach.

Comparing across three methods, i.e., $\mathrm{T_M}$-based, $\mathrm{T_W}$-based, and MVO, it is seen that the $\mathrm{T_W}$ approach always offers more diversified portfolios than the $\mathrm{T_M}$-based and MVO models (assuming the $\mathrm{T_M}$-based and $\mathrm{T_W}$-based approaches are solved by the fuzzy approach). For example, in the U.S. dataset experiment, comparing three solutions of the $\mathrm{T_M}$-based, $\mathrm{T_W}$-based and MVO approaches, the $\mathrm{T_M}$-based chooses only two stocks (see Fig. 3), the MVO selects only three stocks (see Fig. 16), while the $\mathrm{T_W}$-based approach accounts for up to six stocks (see Fig. 9). The same situation occurs in the German dataset experiment. The $\mathrm{T_M}$-based and MVO methods both selects only two stocks (see Figs. 21 and 34), while this number in the $\mathrm{T_W}$-based approach is seven stocks (see Fig. 27).

Although uncertainty values of fuzzy returns obtained by the $\mathrm{T_M}$ approach are smaller than that of the MVO, but the fuzzy returns themselves are smaller than those of the MVO. Therefore, the reward-to-uncertainty ratios in the $\mathrm{T_M}$ approach are usually smaller than those of the MVO. The $\mathrm{T_M}$ approach thus can produce portfolios with low uncertainty returns but the performance of those portfolios regarding the reward-to-uncertainty ratio is inefficient compared with the MVO. With regard to both portfolio performance ratios, i.e., fuzzy Sharpe ratio and reward-to-uncertainty ratio, the $\mathrm{T_W}$ approach is superior to both the $\mathrm{T_M}$ approach and MVO in both the U.S. and German dataset experiments.

V. Conclusion

The volatility risk and the well-known Sharpe ratio have been used commonly in modern portfolio theory. However, there has been no efficient risk measure nor a portfolio performance evaluation ratio in the context of fuzzy modeling for portfolio selection. To deal with this deficiency, through modeling stock returns by the LR triangular fuzzy random variables, we have introduced a new risk measure, i.e., uncertainty of portfolio fuzzy return, and have conceptualized a portfolio performance ratio named *reward-to-uncertainty ratio*. Besides the newly defined reward-to-uncertainty ratio, we also initiated the *fuzzy Sharpe ratio* concept to evaluate the *reward-to-volatility* of portfolios in fuzzy modeling.

Two approaches have been deployed for fuzzy portfolio selection. The $\mathrm{T_M}$ approach calculates the covariance between a pair of assets in crisp numbers, and therefore, the portfolio variance is also a crisp value. Alternatively, in the case of the $\mathrm{T_W}$ approach, the covariance of two assets is represented by fuzzy numbers, and the portfolio variance is thus a fuzzy value that preserves the LR triangular shape. Correspondingly, two multiple objective programming problems for portfolio optimization are derived that include not only the fuzzy Sharpe ratio but the new risk measure as well, i.e., portfolio return uncertainty. Solving these two problems by the fuzzy approach or GA approach has been demonstrated efficiently in the two experiments. GA has been recommended to solve these problems if a series of well-diversified portfolio solutions is needed. However, if investors just want a one-off portfolio with high-performance regardless of portfolio diversification, then the fuzzy approach should be employed. Experimental results have revealed the dominance of the $\mathrm{T_W}$ approach in terms of portfolio return uncertainty, performance and also diversification compared with the $\mathrm{T_M}$ approach and MVO.

Further research directions might be devoted to the higher moments of fuzzy random variables for fuzzy portfolio selection. As asset returns are commonly known to be not normally distributed, consideration of skewness and kurtosis of fuzzy random variables as fuzzy numbers is necessary, although it may be computationally expensive. Other portfolio risk measures like semi-variance, value-at-risk, or expected shortfall, etc., can also be extended to be fuzzy numbers in fuzzy modeling for portfolio selection.

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