Minimizing Bypass Transportation Expenses in Linear Multistate Consecutively-Connected Systems

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***Abstract*—Many continuous transportation systems can be represented as multi-state linear consecutively connected systems consisting of $N+1$ linearly ordered nodes. Some of these nodes contain statistically independent multistate elements with different characteristics. Each element j can provide a connection between the node to which it belongs and X_j next nodes, where X_j is a discrete random variable with known probability mass function. If the system contains nodes not connected with any previous node, then gaps exist that require bypass transportation solutions associated with considerable expenses. An algorithm based on the universal generating function method is suggested for evaluating the expected value of these expenses. A problem of finding the multi-state element allocation that minimizes the expected bypass transportation expenses is formulated and solved. Illustrative examples are presented.**

***Index Terms*—Linear consecutively-connected system, multi-state elements, bypass transportation, universal generating function.**

ACRONYMS AND ABBREVIATIONS

ME	Multistate Element
LCCS	Multistate Linear Consecutively-connected System
UGF	Universal Generating Function (u-function)
pmf	Probability Mass Function
GA	Genetic Algorithm
BTE	Bypass Transportation Expenses
BTS	Bypass Transportation System

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NOTATION

e	random BTE
E	expected value of e
e^*	Maximum allowed value of e
N	number of nodes in LCCS that can contain MEs
X_i	random state of multi-state element i
G_j	random number of the most remote node directly connected with node j
$\boldsymbol{G}_j$	vector of realizations of G_j
$\boldsymbol{P}_j$	vector of probabilities of different realizations of G_j
K_j	number of different realizations of G_j
$g_{j,h}$	h-th realization of G_j
$p_{j,h}$	$\Pr\{G_j = g_{j,h}\}$
Λ_i	set of nodes $\{1,\ldots,i\}$
T_j	random number of the most remote node to which a connection from nodes belonging to Λ_j exists
$t_{j,f}$	f-th realization of T_j
$Q_{j,f}$	$\Pr\{T_j = t_{j,f}\}$
$u_j(z)$	u-function representing pmf of G_j
$U_j(z)$	u-function representing pmf of T_j
ψ	operator obtaining the expected value of random variable with pmf represented by a u-function $U_N(z)$
ξ	operator obtaining the probability that random variable with pmf represented by a u-function $U_N(z)$ does not exceed a specified value
$\underset{f}{\otimes}$	composition operator over two u-functions
$c_{j,f}$	counter representing the BTE caused by the gaps in subsystem Λ_i at state f
$1(A)$	$= \begin{cases} 1 & A \text{ is True} \\ 0, & A \text{ is False} \end{cases}$
D	ME sequencing function where $D(j)$ is the number of ME located at node j
$\boldsymbol{d}$	integer string representing the function D
L_j	cost of flow reloading from LCCS to BTS at node j
U_j	cost of flow reloading from BTS to LCCS at node j
Y_j	cost of flow transportation from node j to node $j+1$ by the BTS

Fig. 1. Examples of LCCS with bypass transportation.

I. INTRODUCTION

THE linear multistate consecutively-connected system (LCCS) consists of $N+1$ consecutively ordered nodes. The first N nodes contain statistically independent multistate elements (MEs) with different characteristics. Each element j can provide a connection between the node to which it belongs and X_j next nodes, where X_j is an integer random variable with a known probability mass function. Note that the special case of a ME is a binary element in which X_j can take only two different values: $X_j > 0$ corresponding to the working state, and $X_j = 0$ corresponding to total failure. The LCCS fails if its first and last nodes are disconnected.

The LCCS was first introduced by Hwang & Yao [1] as a generalization of the binary linear consecutive-k-out-of-n:F system first studied by Kontoleon [2], and Chiang and Niu [3]; and the linear consecutively-connected system with 2-state elements, studied by Shanthikumar [4], [5]. Algorithms for LCCS reliability evaluation were developed by Hwang & Yao [1], Kossow & Preuss [6], Zuo & Liang [7], and Levitin [8]. A comprehensive survey of research on consecutive-k-out-of-n:F and related systems can be found in [9].

This paper considers the expenses associated with failures that cause disconnections in LCCS. The introduction of the bypass transportation expense model was motivated by the following example.

Consider a flow transfer system with pressure generating units (pumps) located at consecutive nodes (Fig. 1). Each unit located at node j provides pressure that is enough to transfer the flow to X_j next nodes (thus node j is considered to be connected with X_j next nodes). If nodes not connected with any previous node exist, the flow cannot be transferred to these nodes by the system, and a bypass transportation system (BTS) should be applied. In the existing LCCS, the normal transportation costs are much lower than the costs of bypass transportation (see for example the comparison of pipeline and track transportation in [10]–[13]). Thus, using the BTS is associated with considerable expenses that depend on the number of the gaps (reloading the transported flow to and from the BTS) as well as on the length of the gaps (flow transportation along a gap).

To model the systems described above, we assume that $N+1$ nodes are allocated along a line. If node j contains a ME that provides its connection with X_j next nodes, the integer random variable $G_j = \min(j + X_j, N+1)$ represents the index of the most remote node connected with node j. Not all nodes must contain ME. The case when node j contains no ME is equivalent to the case when it contains a single-state ME with $X_j \equiv 0$, which gives $G_j \equiv j$. For any given location of ME j and the given pmf of X_j, one can easily obtain the pmf of G_j.

Having the cost of reloading the flow from LCCS to BTS at any node, the cost of the flow transportation between any pair of nodes, and the cost of the reloading the flow from BTS to LCCS at any node, one can evaluate the expected BTE using an algorithm suggested in this paper. If the gap between any node $j < N+1$ and the last node $N+1$ exists, then the bypass transportation always includes reloading from BTS to LCCS at the last node. Therefore the final reloading cost U_{N+1} should always be included into the cost of flow transportation from any node to the last node.

Section 2 of the paper presents a formal formulation of the BTE evaluation problem. Section 3 describes the algorithm suggested for evaluating the expected BTE. Section 4 presents illustrative examples of determining the expected BTE. Section 5 formulates the BTE minimization problem, presents an algorithm for its solving, and examples of the optimal element sequencing. Section 6 concludes.

II. THE MODEL OF LCCS

The state of each node $j \in \{1, \ldots, N\}$ is characterized by the random integer variable $G_j = \min(j + X_j, N+1) \in \{j, \ldots, N+1\}$. The pmf of any G_j is defined by two vectors $\boldsymbol{G}_j = \{g_{j,1}, \ldots, g_{j,K_j}\}$ and $\boldsymbol{P}_j = \{p_{j,1}, \ldots, p_{j,K_j}\}$, where g_{jh} is the h-th realization of G_j, and p_{jh} is the probability of the realization $p_{j,h} = \Pr\{G_j = g_{j,h}\}$, $\sum_{h=1}^{K_j} p_{j,h} = 1$.

Let Λ_i be the set of consecutive nodes $\{1, \ldots, i\}$. The most remote node that is connected with at least one of the nodes belonging to the set Λ_i is $T_i = \max_{1 \le j \le i}(G_j)$. It can be seen that $T_i = \max\{T_{i-1}, G_i\}$ for any $i \ge 1$, where $T_0 = G_0 \equiv 1$ by definition. If $T_i = i$, node $i+1$ is not connected with any one of the previous nodes (belonging to the set Λ_i), and a gap exists. The system has no gaps and succeeds to transfer the flow from node 1 to node $N+1$ iff $T_i > i$ for $i = 1, \ldots, N$.

When $T_{i-1} > i-1$, and $T_i = i$, the gap starts at node i, and the flow should be loaded into the bypass transportation system at this node. In addition, the flow should be transferred from node i to at least node $i+1$. Thus, the cost $L_i + Y_i$ should be added to the total BTE. When $T_{i-1} = i-1$, and $T_i > i$, the gap ends at node i, and the flow should be downloaded from the BTS to LCCS. Thus, the cost U_i should be added to the total BTE. When $T_{i-1} = i-1$, and $T_i = i$, the gap continues at node i, and only the transportation cost Y_i should be added to the total BTE.

Thus, the random total BTE can be obtained as

$$\begin{aligned} e &= \sum_{i=1}^{N} [Y_i \cdot 1(T_i = i) \\ &\quad + L_i \cdot 1(T_i = i) \cdot 1(T_{i-1} > i-1) \\ &\quad + U_i \cdot 1(T_i > i) \cdot 1(T_{i-1} = i-1)] \\ &= \sum_{i=1}^{N} [Y_i \cdot 1(\max_{1 \le j \le i}(G_j) = i) \\ &\quad + L_i \cdot 1(\max_{1 \le j \le i}(G_j) = i) \cdot 1(\max_{1 \le j \le i-1}(G_j) > i-1) \\ &\quad + U_i \cdot 1(\max_{1 \le j, f \le i-1}(G_j) > i) \cdot 1(\max_{1 \le j \le i-1}(G_j) = i-1)]. \end{aligned} \tag{1}$$

For any combination of realizations of the random variables $G_1 = g_{1,h_1}, \ldots, G_N = g_{N,h_N}$, one can obtain the corresponding realization of BTE $e(g_{1,h_1}, \ldots, g_{N,h_N})$ according to

(1). Having the pmf of any G_j, one can obtain the probability of each combination $(g_{j,h_1},\ldots,g_{j,h_N})$ as $\prod_{k=1}^{N} p_{k,h_k}$. Thus, the expected BTE can be obtained as

$$E=\sum_{h_1=1}^{K_1}\sum_{h_2=1}^{K_2}\cdots\sum_{h_N=1}^{K_N} e(g_{i,h_1},\ldots,g_{N,h_N})\prod_{k=1}^{N} p_{k,h_k}. \tag{2}$$

The expected BTE evaluation problem is to obtain E defined in (2) given the pmf of G_j for $1\le j\le N$. The following section describes the recursive algorithm for evaluating the expected BTE of the LCCS.

III. BTE Evaluation Algorithm

Any combination of states of the system MEs produces specific values of T_j $(j=1,\ldots,N)$, and the corresponding distribution of gaps in the system, for which the value of the bypass transportation cost can be easily obtained. Having the probability of any possible combination of the ME states and the corresponding realization of random BTE e, one can obtain the expected value of e using (2). The universal generating function technique applied in this paper allows one to obtain the expected BTE by using straightforward consecutive algebraic procedures instead of complicated combinatorial algorithms. This technique consecutively updates the values of T_j and e, and calculates the corresponding ME state combination probabilities by aggregating system elements one by one.

A. Universal Generating Function

The procedure used in this paper for the expected BTE evaluation is based on the universal z-transform (also called u-function or universal generating function) technique, which was introduced in [14], and which proved to be very effective for the reliability evaluation of different types of multi-state systems [15].

The u-function of a discrete random variable Y is defined as a polynomial

$$u_y(z)=\sum_{k=1}^{K} q_k z^{y_k}, \tag{3}$$

where the variable Y has K possible realizations, and q_k is the probability that Y is equal to y_k.

To obtain the pmf of a function $f(Y,D)$ of two statistically independent random variables Y and D represented by two different u-functions $u_y(z)$ and $u_d(z)$, the following composition operator is used.

$$\begin{aligned} u_y(z)\underset{f}{\otimes}u_d(z)&=\sum_{k=1}^{K} q_k z^{y_k}\underset{f}{\otimes}\sum_{h=1}^{H} p_h z^{d_h}\\ &=\sum_{k=1}^{K}\sum_{h=1}^{H} q_k p_h z^{f(y_k,d_h)}. \end{aligned} \tag{4}$$

The resulting u-function relates any possible realization of the function $f(y_{\mathrm{k}},d_{\mathrm{h}})$ with the probability of the realization $\Pr\{Y=y_k\cap D=d_h\}=q_k p_h$.

B. Application of the UGF Technique to Finding the Expected BTE

To evaluate the expected BTE, we use two u-functions:

$$u_j(z)=\sum_{h=1}^{K_j} p_{j,h} z^{g_{j,h}}, \tag{5}$$

$$\text{and}\qquad U_j(z)=\sum_{f=1}^{K_j} Q_{j,f} z^{t_{j,f}}. \tag{6}$$

These functions represent the pmf of random variables G_j. and T_j respectively. Here $p_{j,h}=\Pr\{G_j=g_{j,h}\}$, $Q_{j,f}=\Pr\{T_j=t_{j,f}\}$, and K_j is the number of realizations of T_j and G_j. $K_j\le N+2-j$ because T_j can take values from j to $N+1$.

Having u-functions (5) representing the pmf of G_j for any j, and the recursive equation $T_j=\max\{T_{j-1},G_j\}$ for $j\ge 1$, we can determine the u-functions $U_j(z)$ using the recursive procedure

$$\begin{aligned} U_0(z)&=z^1,\\ U_j(z)&=U_{j-1}(z)\underset{\max}{\otimes}u_j(z)\\ =\sum_{h=1}^{K_j} Q_{j,h} z^{t_{j,f}}&=\sum_{f=1}^{K_{j-1}}\sum_{h=1}^{K_j} Q_{j-1,f}p_{j,h} z^{\max(t_{j-1,f},g_{j,h})} \end{aligned} \tag{7}$$

for $j=1,\ldots,N$.

To summarize the bypass transportation expenses, a counter $c_{j,f}$ should be incorporated into the u-function $U_j(z)$ so that

$$U_j(z)=\sum_{f=1}^{K_j} Q_{j,f} z^{t_{j,f},c_{j,f}}, \tag{8}$$

and the recursive procedure (7) should be modified as

$$\begin{aligned} U_0(z)&=z^{1,0}\\ U_j(z)&=U_{j-1}(z)\underset{\leftrightarrow}{\otimes}u_j(z)\\ =\sum_{f=1}^{K_{j-1}}\sum_{h=1}^{K_j} &Q_{j-1,f}p_{j,h} z^{\max(t_{j-1,f},g_{j,h}),c_{j,f}} \text{ for } j=1,\ldots,N. \end{aligned} \tag{9}$$

Here the counter is updated according the rule

$$\begin{aligned} c_{j,f}&=\eta\left(c_{j-1,f},\max(t_{j-1,f},g_{j,h}),\right), \text{ where}\\ \eta(a,b)&=\begin{cases} a-Y_j & \text{if } a<0 \text{ and } b=j\\ -a-Y_j-L_j & \text{if } a\ge 0 \text{ and } b=j\\ -a+U_j & \text{if } a<0 \text{ and } b>j\\ a & \text{if } a\ge 0 \text{ and } b>j. \end{cases} \end{aligned} \tag{10}$$

This counter updating rule allows one to calculate the total BTE, and indicate the existence of a gap at node j simultaneously. Indeed, according to (10), the counter $c_{j,f}$ is always non-negative when the path from node j exists, i.e. $t_j=\max(t_{j-1,f},g_{j,h})>j$, and is always negative otherwise. The absolute value of the counter $c_{j,f}$ is equal to the BTE needed to transmit the flow from node 1 to node $j+1$ in state f of subsystem Λ_j. If the previous value of the counter is not negative, and $t_j>j$, then no gap exists, and the counter does not change. If the previous value of the counter is not negative, and $t_j=j$, then a new gap begins, and the counter value increases by L_j+Y_j. Then the counter's value changes from positive to negative to indicate the gap. If the previous value of the counter is negative, and $t_j = j$, then the gap continues, and the absolute value of the counter increases by Y_j, whereas the counter remains negative. If the previous value of counter is negative, and $t_j > j$, then the gap ends, and the absolute value of the counter increases by U_j, whereas the counter changes from negative to positive to indicate that there is no gap at node j.

After applying operator (9), the u-function can contain several terms with the same exponent. These terms correspond to different combinations of realizations of T_{j-1} and G_j that result in the same T_j, and the same value of the counter. These terms can be collected, which gives cumulative probabilities of different gap combinations resulting in the same BTE (see the example in Section V.A).

Finally, the u-function $U_N(z)$ contains terms with counters $c_{N,f}$ representing all the possible realizations of the random BTE, and coefficients $Q_{N,f}$ representing the corresponding probabilities of these realizations. In fact, $U_N(z)$ represents the pmf of the random BTE. Applying the following operator ψ over the u-function $U_N(z)$,

$$\psi(U_N(z)) = \psi\left(\sum_{f=1}^{K} Q_{N,f} z^{t_{N,f}, c_{N,f}}\right) = \sum_{f=1}^{K} |c_{N,f}| \cdot Q_{N,f}, \tag{11}$$

one obtains the expected value of the BTE.

In addition, having the u-function $U_N(z)$, one can obtain the probability that the random BTE does not exceed any desired level e^* by applying the operator ξ over the u-function as in (12).

C. Recursive Algorithm for BTE Evaluation

Based on the considerations above, the following simple consecutive algorithm determines the expected BTE and $\Pr(e \le e^*)$ in LCCS for any given pmf of G_j for $1 \le j \le N$.

1. Determine $u_j(z)$ for $1 \le j \le N$ using $\boldsymbol{G}_j, \boldsymbol{P}_j$, and (5).
2. Assign $U_0(z) = z^{1,0}$.
3. For $j = 1, \ldots, N$, apply $U_j(z) = U_{j-1}(z) \underset{\leftrightarrow}{\otimes} u_j(z)$, collect like terms in $U_j(z)$.
4. Obtain $E = \psi(U_N(z))$ or $\Pr(e \le e^*) = \xi(U_N(z))$.

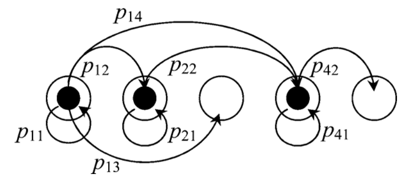

Fig. 2. Example of simple LCCS with five nodes and three MEs.

In the worst case, when the number of different realizations of G_j is $N - j + 2$, and different realizations of T_{j-1} and G_j always produce different values of T_j and c_j, the algorithm above requires $(N + 1)!$ term multiplications. However, in practice, collecting the like terms reduces the algorithm complexity considerably (see the example in the following section).

IV. Illustrative Examples of Determining the Expected BTE

A. Analytical Example

Consider a LCCS consisting of five nodes $(N + 1 = 5)$. The pmf of G_j for the first four nodes are $\boldsymbol{G}_1 = \{1, 2, 3, 4\}$, $\boldsymbol{P}_1 = \{p_{11}, p_{12}, p_{13}, p_{14}\}$; $\boldsymbol{G}_2 = \{2, 4\}$, $\boldsymbol{P}_2 = \{p_{21}, p_{22}\}$; $\boldsymbol{G}_3 = \{3\}$, $\boldsymbol{P}_3 = \{1\}$ (node 3 contains no ME); $\boldsymbol{G}_4 = \{4, 5\}$, $\boldsymbol{P}_4 = \{p_{41}, p_{42}\}$. The system with all the possible inter-node connections is presented in Fig. 2.

The u-functions of the nodes take the form of the first equation shown at the bottom of the next page.

Following step 2 of the procedure, $U_0(z) = z^{1,0}$ is assigned. Then the composition operator is consecutively applied.

For $j = 1$,

$$U_1(z) = u_1(z) = z^{1,0} \underset{\leftrightarrow}{\otimes} (p_{11}z^1 + p_{12}z^2 + p_{13}z^3 + p_{14}z^4)$$
$$= p_{11}z^{1,-L1-Y1} + p_{12}z^{2,0} + p_{13}z^{3,0} + p_{14}z^{4,0}.$$

For $j = 2$,

$$\begin{aligned}
U_2(z) &= U_1(z) \underset{\leftrightarrow}{\otimes} u_2(z) = (p_{11}z^{1,-L1-Y1} + p_{12}z^{2,0} + p_{13}z^{3,0} \\
&\quad + p_{14}z^{4,0}) \underset{\leftrightarrow}{\otimes} (p_{21}z^2 + p_{22}z^4) = p_{11}p_{21}z^{2,-L1-Y1-Y2} \\
&\quad + p_{12}p_{21}z^{2,-L2-Y2} + p_{13}p_{21}z^{3,0} + p_{14}p_{21}z^{4,0} \\
&\quad + p_{11}p_{22}z^{4,L1+Y1+U2} + p_{12}p_{22}z^{4,0} + p_{13}p_{22}z^{4,0} + p_{14}p_{22}z^{4,0} \\
&= p_{11}p_{21}z^{2,-L1-Y1-Y2} + p_{12}p_{21}z^{2,-L2-Y2} + p_{13}p_{21}z^{3,0} \\
&\quad + p_{11}p_{22}z^{4,L1+Y1+U2} + [(p_{12} + p_{13} + p_{14})p_{22} + p_{14}p_{21}]z^{4,0} \\
&= p_{11}p_{21}z^{2,-L1-Y1-Y2} + p_{12}p_{21}z^{2,-L2-Y2} + p_{13}p_{21}z^{3,0} \\
&\quad + p_{11}p_{22}z^{4,L1+Y1+U2} + q_1 z^{4,0},
\end{aligned}$$

where $q_1 = (1 - p_{11})p_{22} + p_{14}p_{21}$.

$$\Pr(e \le e^*) = \xi\left(\sum_{f=1}^{K} Q_{N,f} z^{t_{N,f}, c_{N,f}}\right) = \sum_{f=1}^{K} Q_{N,f} \cdot 1\left(|c_{N,f}| \le e^*\right). \tag{12}$$

$$u_1(\mathrm{z}) = p_{11}z^1 + p_{12}z^2 + p_{13}z^3 + p_{14}z^4, u_2(\mathrm{z}) = p_{21}z^2 + p_{22}z^4, u_3(\mathrm{z}) = z^3,$$
$$u_4(\mathrm{z}) = p_{41}z^4 + p_{42}z^5.$$

Observe that collecting the like terms reduced the length of $U_2(z)$ from eight to five terms.

For $j = 3$, see the second equation at the bottom of the page. For $j = 4$, see the third equation at the bottom of the page. Finally, see the fourth equation at the bottom of the page. After equivalent transforms taking into account that

$$\sum\nolimits_{h=1}^{K_j} p_{j,h} = 1,$$

we get the fifth equation at the bottom of the page.

B. Numerical Example

Consider a LCCS with $N + 1 = 11$ nodes. Nodes 1, 2, 4, 5, 7, 8, 9, 10 contain MEs that provide connection with other nodes. Nodes 3 and 6 contain no MEs, which is expressed by introducing dummy MEs with $\Pr\{X_j = 0\} = 1$. The pmf of X_j for all the MEs, and of G_j for first ten nodes, are presented in Table I. Table II presents the values of the expected BTE obtained for different transportation and reloading costs when $Y_j = Y^*$, $L_j = L^*$, and $U_j = U^*$ for any j. The value of the expected BTE obtained for the system with transportation and reloading costs presented in Table III is $E = 68.22$.

V. Optimal Element Sequencing in LCCS

A. Problem Definition

When LCCS consists of different MEs, its reliability strongly depends on their distribution among the system nodes. Consider a LCCS with N nodes, and M MEs. The MEs distribution can be represented by an allocation function D which maps any node number j into the number of ME $D(j)$ allocated to this node. Observe that, for any $M \leq N$, the allocation problem can be defined as a sequencing problem in which N MEs (among which M MEs are real and $N - M$ MEs are dummy) should be allocated among N nodes. Any dummy element cannot transmit the flow, and thus $X_i \equiv 0$ for $M < i \leq N$ and $G_{D(j)} = j$ for any $D(j) > M$. The optimal element sequencing problem can be formulated as follows.

Find the allocation function D that minimizes the expected value of

$$\begin{aligned} e(D) = \sum_{i=1}^{N} [&Y_i \cdot 1(\max_{1\leq j\leq i}(G_{D(j)}) = i) \\ &+ L_i \cdot 1(\max_{1\leq j\leq i}(G_{D(j)}) = i) \cdot 1(\max_{1\leq j\leq i-1}(G_{D(j)}) > i-1) \\ &+ U_i \cdot 1(\max_{1\leq j\leq i}(G_{D(j)}) > i) \cdot 1(\max_{1\leq j\leq i-1}(G_{D(j)}) = i-1)], \end{aligned} \tag{13}$$

$$\begin{aligned} U_3(z) &= U_2(z) \underset{\leftrightarrow}{\otimes} u_3(z) \\ &= [p_{11}p_{21}z^{2,-L1-Y1-Y2} + p_{12}p_{21}z^{2,-L2-Y2} + p_{13}p_{21}z^{3,0} + p_{11}p_{22}z^{4,L1+Y1+U2} \\ &+ q_1 z^{4,0}] \underset{\leftrightarrow}{\otimes} z^3 = p_{11}p_{21}z^{3,-L1-Y1-Y2-Y3} + p_{12}p_{21}z^{3,-L2-Y2-Y3} \\ &+ p_{13}p_{21}z^{3,-L3-Y3} + p_{11}p_{22}z^{4,L1+Y1+U2} + q_1 z^{4,0}. \end{aligned}$$

$$\begin{aligned} U_4(z) &= U_3(z) \underset{\leftrightarrow}{\otimes} u_4(z) = [p_{11}p_{21}z^{3,-L1-Y1-Y2-Y3} + p_{12}p_{21}z^{3,-L2-Y2-Y3} \\ &+ p_{13}p_{21}z^{3,-L3-Y3} + p_{11}p_{22}z^{4,L1+Y1+U2} + q_1 z^{4,0}] \underset{\leftrightarrow}{\otimes} (p_{41}z^4 + p_{42}z^5) \\ &= p_{11}p_{21}p_{41}z^{4,-L1-Y1-Y2-Y3-Y4} + p_{12}p_{21}p_{41}z^{4,-L2-Y2-Y3-Y4} \\ &+ p_{13}p_{21}p_{41}z^{4,-L3-Y3-Y4} + p_{11}p_{22}p_{41}z^{4,-L1-Y1-U2-L4-Y4} + q_1 p_{41}z^{4,-L4-Y4} \\ &+ p_{11}p_{21}p_{42}z^{5,L1+Y1+Y2+Y3+U4} + p_{12}p_{21}p_{42}z^{5,L2+Y2+Y3+U4} \\ &+ p_{13}p_{21}p_{42}z^{5,L3+Y3+U4} + p_{11}p_{22}p_{42}z^{5,L1+Y1+U4} + q_1 p_{42}z^{5,0}. \end{aligned}$$

$$\begin{aligned} E &= p_{11}p_{21}p_{41}(L_1 + Y_1 + Y_2 + Y_3 + Y_4) + p_{12}p_{21}p_{41}(L_2 + Y_2 + Y_3 + Y_4) \\ &+ p_{13}p_{21}p_{41}(L_3 + Y_3 + Y_4) + p_{11}p_{22}p_{41}(L_1 + Y_1 + U_2 + L_4 + Y_4) \\ &+ q_1 p_{41}(L_4 + Y_4) + p_{11}p_{21}p_{42}(L_1 + Y_1 + Y_2 + Y_3 + U_4) \\ &+ p_{12}p_{21}p_{42}(L_2 + Y_2 + Y_3 + U_4) + p_{13}p_{21}p_{42}(L_3 + Y_3 + U_4) \\ &+ p_{11}p_{22}p_{42}(L_1 + Y_1 + U_2). \end{aligned}$$

$$\begin{aligned} E &= p_{11}(L_1 + Y_1) + p_{12}p_{21}L_2 + (p_{11} + p_{12})p_{21}Y_2 + p_{11}p_{22}U_2 + p_{13}p_{21}L_3 \\ &+ (1 - p_{14})p_{21}Y_3 + (p_{14}p_{21} + p_{22})p_{41}L_4 + p_{41}Y_4 + (1 - p_{14})p_{21}p_{42}U_4. \end{aligned}$$

where $G_{D(j)} = \min(j + X_{D(j)}, N + 1)$, and the pmf of X_i for $1 \le i \le N$ are given.

Finding the optimal ME sequence in LCCS is a complicated combinatorial optimization problem having $N!$ possible solutions. An exhaustive examination of all these solutions is not realistic, even for a moderate number of MEs, considering reasonable time limitations. As in most combinatorial optimization problems, the quality of a given solution is the only information available during the search for the optimal solution. Therefore, a heuristic search algorithm is needed which uses only estimates of solution quality, and which does not require derivative information to determine the next direction of the search.

Several powerful universal optimization meta-heuristics exist, such as Genetic Algorithm [16], [17], Ant Colony Optimization [18], Tabu Search [19], Variable Neighborhood Descent [20], Great Deluge Algorithm [21], Immune Algorithm [22], and Particle Swarm Algorithm [23], [24]. These meta-heuristics and their hybrid optimization techniques prove to be effective in solving different reliability optimization problems of real size and complexity [25]. The most frequently used meta-heuristic in reliability optimization is the genetic algorithm (GA) based on the simple principle of evolutionary search in the solution space [16], [17], [26]–[29].

It is recognized that GAs have the theoretical property of global convergence [30]. Despite the fact that their convergence reliability and convergence velocity are contradictory, for most practical, moderately sized combinatorial problems, the proper choice of GA parameters allows solutions close enough to the optimal one to be obtained in a short time.

B. Genetic Algorithm

Basic notions of GAs are originally inspired by biological genetics. GAs operate with chromosomal representation of solutions, where crossover, mutation, and selection procedures are applied. Chromosomal representation requires the solution to be coded as a finite length string. Unlike various constructive optimization algorithms that use sophisticated methods to obtain a good singular solution, the GA deals with a set of solutions (population) and tends to manipulate each solution in the simplest manner.

The detailed information on GAs, and recent developments in GA theory and practice, can be found in books [29]–[31]. The steady state version of the GA used in this paper was developed by Whitley [32]. As reported in [33], this version, named GENITOR, outperforms the basic generational GA. The structure of steady state GA is as follows.

1. Generate an initial population of N_s randomly constructed solutions (strings), and evaluate their fitness. Unlike the generational GA, the steady state GA performs the evolution search within the same population, improving its average fitness by replacing worst solutions with better ones.

2. Select two solutions randomly, and produce a new solution (offspring) using a crossover procedure that provides inheritance of some basic properties of the parent strings in the offspring. The probability of selecting the solution as a parent is proportional to the rank of this solution. (All the solutions in the population are ranked by increasing order of their fitness). Unlike the fitness-based parent selection scheme, the rank-based scheme reduces GA dependence on the fitness function structure, which is especially important when constrained optimization problems are considered [34].

3. Allow the offspring to mutate. Mutation results in slight changes in the offspring structure, and maintains diversity of solutions. This procedure avoids premature convergence to a local optimum, and facilitates jumps in the solution space. The positive changes in the solution code created by the mutation can be later propagated throughout the population via crossovers.

4. Decode the offspring to obtain the objective function (fitness) values. These values are a measure of quality, which is used in comparing different solutions.

5. Apply a selection procedure that compares the new offspring with the worst solution in the population, and selects the one that is better. The better solution joins the population, and the worse one is discarded. If the population contains equivalent solutions following the selection process, redundancies are eliminated; and, as a result, the population size decreases. Note that each time the new solution has sufficient fitness to enter the population, it alters the pool of prospective parent solutions, and increases the average fitness of the current population. The average fitness increases monotonically (or, in the worst case, does not vary) during each genetic cycle (steps 2–5).

6. Generate new randomly constructed solutions to replenish the population after repeating steps 2–5 for N_{rep} times (or until the population contains a single solution or solutions with equal quality). Run the new genetic cycle (return to step 2). In the beginning of a new genetic cycle, the average fitness can decrease drastically due to inclusion of poor random solutions into the population. These new solutions are necessary to bring into the population new genetic material, which widens the search space, and, like a mutation operator, prevents premature convergence to the local optimum.

7. Terminate the GA after N_c genetic cycles.

The final population contains the best solution achieved. It also contains different near-optimal solutions, which may be of interest in the decision-making process.

C. Solution Representation, and Basic GA Procedures

To apply the genetic algorithm to a specific problem, one must define a solution representation, and a decoding procedure, as well as specific crossover and mutation procedures. In our problem, each solution should be represented by an N length string $\boldsymbol{d}$ of integer numbers ranged from 1 to N such that the j-th position of the string $d(j)$ represents the number of ME located at node j. To provide solution feasibility, each number should appear in the string only once. The order in which the numbers appear determines the ME allocation in LCCS. For each integer string $\boldsymbol{d}$, the random variable G_j for node j is determined as $G_j = \min(j + X_{d(j)}, N + 1)$, and the solution fitness equal to the expected BTE (or to $\Pr(e \le e^*)$) is estimated by applying the algorithm presented in Section III.C.

Crossover and mutation procedures should preserve the feasibility of newly obtained solutions given that parent solutions are feasible. A crossover procedure that was first suggested in [35], and was proven to be highly efficient in [36], is used in this

TABLE I
PMF OF G_j FOR THE NUMERICAL EXAMPLE

No of ME	No of state	1	2	3	4	5
1	***P***	0.3	0.1	0.6	-	-
	X	0	2	4	-	-
	G	1	3	5	-	-
2	***P***	0.2	0.1	0.4	0.3	-
	X	0	1	2	4	-
	G	2	3	4	6	-
3	***P***	1.0	-	-	-	-
	X	0	-	-	-	-
	G	3	-	-	-	-
4	***P***	0.25	0.05	0.4	0.3	-
	X	0	1	2	3	-
	G	4	5	6	7	-
5	***P***	0.08	0.2	0.15	0.45	0.12
	X	0	1	2	3	6
	G	5	6	7	8	11
6	***P***	1.0	-	-	-	-
	X	0	-	-	-	-
	G	6	-	-	-	-
7	***P***	0.3	0.1	0.1	0.5	-
	X	0	1	2	3	-
	G	7	8	9	10	-
8	***P***	0.05	0.25	0.7	-	-
	X	0	1	3	-	-
	G	8	9	11	-	-
9	***P***	0.6	0.4	-	-	-
	X	0	1	-	-	-
	G	9	10	-	-	-
10	***P***	0.25	0.75	-	-	-
	X	0	1	-	-	-
	G	10	11	-	-	-

work. This procedure first copies all the string elements from the first parent to the same positions of the offspring. Then all the offspring elements belonging to the fragment, defined as a set of adjacent positions between two randomly defined sites, are reallocated within this fragment in the order they appear in the second parent. The following is an example of the crossover procedure in which the fragment is marked in bold.

First parent:	1 2 **3 4 5 6 7** 8 9 10.
Second parent:	7 8 **9 2 4 5 1** 3 6 10.
Off-spring:	1 2 **7 4 5 3 6** 8 9 10.

The mutation procedure used in our GA just swaps elements initially located in two randomly chosen positions of the string. This procedure also preserves solution feasibility.

D. Examples of Optimal Element Sequencing in LCCS

Consider the optimal allocation of eight MEs among ten positions in an LCCS. This problem is equivalent to sequencing ten MEs, two of which are dummy with $\Pr\{X_j = 0\} = 1$. The pmf of X_j for all the MEs are presented in Table I (elements 3 and 6 are dummy). The best $\min E$ sequence obtained by the GA for the system with transportation and reloading costs presented in Table III is $\boldsymbol{d} = (8, 4, 6, 5, 2, 3, 1, 7, 9, 10)$. The value of BTE obtained for this LCCS can vary from $e = 0$ (no gaps in the system) to $e = 697$ (all the elements are disconnected and $e = L_1 + Y_1 + Y_2 \ldots + Y_{10}$). The expected BTE is $E = 56.456$. This solution is presented in Table IV as solution 1. The solutions maximizing the probabilities $\Pr(e \leq e^*)$ for $e^* = 100$, and $e^* = 200$ are also presented in Table IV as solutions 2, and 3 respectively. See that the increase of the probability $\Pr(e \leq e^*)$ is achieved by the price of increasing the expected BTE. The solution that provides the greatest probability that the system will perform the transportation task without using the BTS (i.e. $\Pr(e = 0)$) is presented in Table IV as solution 4. This solution, however, has the greatest value of the expected BTE, and the lowest values of $\Pr(e < 100)$ and $\Pr(e < 200)$ among the considered optimal solutions. Thus, it can be seen that the tradeoffs exists among different optimization criteria. The probabilities $\Pr(e \leq e^*)$ for the four obtained solutions as functions of e^* are presented in Fig. 3.

The running time for the considered optimization problem on a PC for $N_s = 100$, $N_c = 15$, and $N_{rep} = 2000$ does not exceed 20 seconds.

TABLE II
E FOR DIFFERENT TRANSPORTATION AND RELOADING COSTS

L^*	Y^*	U^*	E
2	50	1	55.024
5	60	3	69.371
2	70	1	76.116
3	80	3	88.890

TABLE III
TRANSPORTATION AND RELOADING COSTS FOR THE NUMERICAL EXAMPLE

Node j	L_j	Y_j	U_j
1	7	50	3
2	6	50	2
3	7	70	4
4	8	80	5
5	7	115	3
6	6	40	2
7	6	40	2
8	7	70	3
9	8	80	5
10	8	95	4

TABLE IV
THE BEST ME SEQUENCES IN LCCS OBTAINED BY THE GA

No	***d***	E	Pr(e<100)	Pr(e<200)	Pr(e=0)
1	8,4,6,5,2,3,1,7,9,10	**56.457**	0.766	0.932	0.563
2	5,1,9,2,7,3,6,8,4,10	64.790	**0.810**	0.929	0.458
3	8,9,7,5,2,3,1,4,6,10	60.570	0.756	**0.936**	0.492
4	8,7,3,5,2,10,1,6,4,9	65.456	0.664	0.909	**0.586**

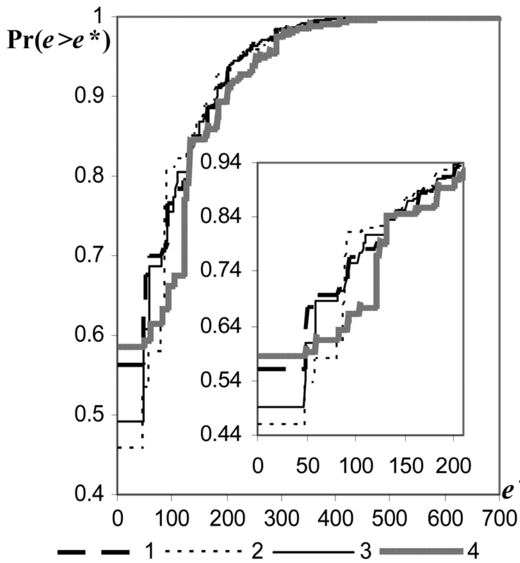

Fig. 3. $\Pr(e \leq e^*)$ as functions of e^* for the solutions presented in Table IV (inner plot presents the same functions for $0 \leq e^* \leq 200$).

VI. Conclusions, and Further Research

This paper introduced the model of bypass transportation expenses (BTE) in linear multistate consecutively-connected systems. The algorithm developed for obtaining the pmf of the random BTE is based on the universal generating function technique.

The computational efficiency of the algorithm allows one to use it in an optimization procedure solving the problem of optimal sequencing of multi-state elements in LCCS. The genetic algorithm is used for finding the optimal ME sequences that minimize the expected BTE or maximize the probability that the BTE does not exceed an allowed level.

The developed optimization procedure can be used for minimizing the energy consumption, costs, and efforts associated with the bypass transportation in real continuous transportation systems.

Further research can be devoted to developing a method of evaluating and minimizing the BTE in multi-phase LCCS that are aimed at providing the flow transportation during several consecutive phases in which the characteristics of multi-state elements vary due to changes of the system functioning conditions. The suggested approach can also be extended to transportation networks with non-linear configurations, and to systems with common cause failures.

The efficiency of different optimization meta-heuristics in solving the optimal element sequencing problem may be compared to select the most efficient one.

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