MIMO Systems With Quantized Covariance Feedback

Rajesh T. Krishnamachari, Mahesh K. Varanasi, *Fellow, IEEE*, and Kaniska Mohanty

***Abstract*—A unified approach is developed for the study of multi-input, multi-output (MIMO) systems under fast fading with quantized covariance feedback. In such systems, the receiver computes, using perfect channel state information (CSI), the covariance matrices to be adopted by the transmitter corresponding to the current channel realization, and feeds back a quantized version of this information to the transmitter using finite, say N_f, bits per channel realization. Our analysis is based on a geometric framework we developed in a companion paper for manifolds of positive semi-definite (covariance) matrices with various trace and rank constraints. That analysis applies to MIMO systems in a unified manner to optimal as well as various suboptimal precoding methods for obtaining input covariances. These include, for example, the capacity achieving spatio-temporal water-filling strategy, the incorporation of just spatial water-filling and reduced-rank beamforming with power allocation in both cases, as well as MIMO systems with antenna selection. For a given system strategy, including the one that achieves capacity, the gap between the communication rates in the perfect and quantized covariance cases is shown to be $O(2^{-N_f/N})$, where N is the dimension of the particular Pn manifold used for quantization. This dimension depends on the precoding strategy adopted and strongly determines how quickly the communication rate with quantized covariance feedback approaches that with perfect CSI at the transmitter (CSIT). Our results show how the choice of covariance quantization manifold can be tailored to the available feedback rate in MIMO system design.**

***Index Terms*—Antenna selection, capacity, fast fading, limited feedback, MIMO, quantization.**

I. Introduction

MULTI-INPUT multi-output (MIMO) systems with finite-rate feedback from a channel-aware receiver to the transmitter have been widely studied (cf. [1] and the comprehensive list of references therein). Under a fast fading model derived from block fading and interleaving, the receiver quantizes some aspect of the CSI using a fixed finite-sized codebook and feeds back once for each block the index of the chosen codeword to the transmitter. The key problem here is to quantify the distortion in the feedback information and obtain a bound on its impact on the achievable rate of the system relative to its perfect CSIT counterpart. Rank-one beamforming, involving transmission along the singular vector of the channel matrix corresponding to the maximum singular value, under limited feedback was considered early on by [2], [3] using ergodic capacity as the performance criterion. This analysis has since been generalized to multi-rank beamforming with equal power allocation in several papers including [4]–[8] (and references therein) in pursuit of perfect CSIT communication rates with small quantization codebooks [4], [5], [7]. The principal ideas in these works lie in the identification of the Grassmann manifold $G^{\mathbb{C}}_{n,k}$ (the space of all k-dimensional subspaces within $\mathbb{C}^n$) as the quantization space, the establishment of the relation between the chordal distance metric with an upper bound on capacity loss, the generation and use of random code books and estimation of the tradeoff between rate loss and the feedback rate. In spite of the impressive advances made on this subject in the last decade [1], there are several limitations of the analyses therein of Grassmannian quantization and its impact on achievable rate. The dependence on specific properties of Grassmann manifold (like chordal distance), the need for the channel matrix to be Rayleigh-faded, and perhaps most significantly, the absence of consideration of spatial or joint spatio-temporal power-allocation, and the collapse of $G^{\mathbb{C}}_{n,k}$ at $k = n$ to a single point, all limit insight derivable from these feedback analyses.

As circuit technology improves, the complexity of designing and placing multiple radio-frequency chains and providing higher feedback rates is mitigated. This motivates the consideration of MIMO systems with improved feedback schemes. Since the optimal, capacity-achieving input covariance matrices for a MIMO channel under Gaussian noise are well-known to be given by a spatio-temporal water-filling procedure [9], the natural and fundamental problem is to evaluate the achievable ergodic rate when the optimal covariances are fed back to the transmitter via a limited number of bits per channel realization. We analyze this and the sub-optimal strategy of space-only water-filling (obtained in [10] for the fixed MIMO channel) both with and without further rank constraints, using our recent results in [11] on the quantization of manifolds of covariance matrices. In particular, we determine the impact of quantized feedback on the achievable rate relative to the ideal CSIT case, and consequently, provide design guidelines on the choice of

Manuscript received July 25, 2012; revised March 01, 2013 and July 16, 2013; accepted August 20, 2013. Date of publication November 20, 2013; date of current version December 27, 2013. The associate editor coordinating the review of this manuscript and approving it for publication was Prof. David Love. This work was supported in part by NSF Grant CCF-0728955 and KAU Grant 4 – 15 1432/HICI. This work appeared in part in the *Proceedings of the IEEE International Symposium on Information Theory*, Austin, TX, 2010, and in the *Proceedings of the EEE International Symposium Information Theory*, St. Petersburg, Russia, Aug. 2011.

R. T. Krishnamachari was with the Department of Electrical, Computer, and Energy Engineering, University of Colorado, Boulder, CO 80309-0425 USA. He now employed in the financial services industry in New York, NY (e-mail: Rajesh.Krishnamachari@Colorado.EDU).

M. K. Varanasi and K. Mohanty are with the Department of Electrical, Computer, and Energy Engineering, University of Colorado, Boulder, CO 80309-0425 USA (e-mail: varanasi@colorado.edu; Kaniska.Mohanty@Colorado.EDU).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TSP.2013.2292036

covariance quantization manifold as a function of the available feedback rate.

In contrast to the well researched problem of reduced-rank beamforming-only feedback (without spatial or temporal power allocation) – to the best of our knowledge – there is no paper in the literature that obtains analytical performance results for approaching capacity under quantized feedback via the optimal (capacity achieving) spatio-temporal water-filling, and only one paper [12] that analyzes theoretically the case of feedback of the spatial water-filling covariance that would be optimal under an additional short-term power constraint (STPC)[1]. Besides analyzing the fundamental question of approaching the ergodic capacity of the MIMO system with quantized feedback with sufficiently high feedback rate, we also provide in this paper an improved analysis for the space-only water-filling strategy relative to that found in [12]. Reduced rank versions of these two strategies are also analyzed in pursuit of high achievable rates when the feedback rate is limited. The key reason for why all of these strategies can now be analyzed is that the technical challenges that precluded a convenient analysis of the Pn manifolds, the set of positive semi-definite matrices under rank and trace constraints, have recently been addressed by the authors in the companion paper [11]. Moreover, while standard tools based on random vector quantization cannot be directly applied to covariance matrix feedback problems since even for an independent, identically distributed (i.i.d.) Rayleigh fading channel the probability density function (p.d.f.) of the spatio-temporal or spatial water-filling matrices are not known to be uniformly distributed over the Pn manifold, the results on sphere-packing and random vector quantization of Pn manifolds developed in [11] that are applicable for sources on those Pn manifolds with general distribution can now be employed to rigorously analyze the problems considered herein. Further, by formulating counterparts to the commonly employed Grassmannian objects on Pn manifolds, a formal setting is constructed on which future analyses of feedback of joint beamforming and power control strategies can base themselves upon.

In particular, we analyze under a general quantized covariance framework, the capacity achieving strategy as well as the strategy that is rate-optimal under the STPC and their rank reduced versions by mapping these problems into one that involves quantization of positive semi-definite matrices with appropriate rank and trace equality and inequality constraints. The ergodic capacity is analyzed and the achievable rates under finite rate feedback is bounded relative to the ideal CSIT limits. Moreover, this bounding enables us to analyze separately the degradation caused by finite-rate quantization and the benefit of precoding strategies. Consequently, the rate (including capacity) loss for a channel under an arbitrary probability distribution is shown to have a $O\left(2^{-\frac{N_f}{N}}\right)$ behavior, where N_f is the number of feedback bits spent by the receiver in quantization per block and N is the dimension of the underlying manifold that is quantized.

[1]Such a strategy, while capacity optimal for the fixed MIMO channel [10], is however not capacity achieving under fading because it forgoes the advantage of temporal water-filling which can be considerable under low SNR scenarios [9].

Other than [12] and the conference versions of this paper in [13], [14], past works on covariance feedback include [15]–[18] but they deal with different issues than does this paper. For example, the early work of [15] addresses the numerical optimization (via Lloyd's algorithm) of covariance quantization codebook to maximize achievable rate and [16] considers the problem of partial (but perfect) knowledge of eigenvectors and power allocation parameters, while more recently, [17] addresses the design of quantization codebook for the feedback of channel second-order statistics (in particular, the covariance matrix of the channel coefficients) with separate vector quantization (via Lloyd's algorithm) of the dominant eigenvectors and scalar quantization of the corresponding eigenvalues. The work in [18] deals with differential quantization of the sequence of channel Gram matrices that exploits time correlation in fading channels. In contrast, this paper considers random (or sphere packing) codebooks that jointly quantize beamformers and power allocation parameters, and analytically estimate, up to first order, the achievable rates with quantized covariance feedback in a channel distribution-independent fashion, and consider fast (i.i.d. across time) fading channels while dealing with a variety of communication schemes in a unified manner.

The rest of this paper is organized as follows. Section II provides notation, the system model, and identifies the Pn manifolds that should be quantized for MIMO systems under short/long-term power constraints and/or rank-reduced beamforming. Section III employs a perturbation approach to analyze the information rate expression and uses the results on quantization of Pn manifolds from [11] to obtain the tradeoff between achievable rate loss due to quantization versus feedback rate in a very general manner. The achievable rate loss is also characterized succinctly in the form of a power efficiency factor in Section III. Section IV illustrates the application of the unified capacity/rate loss analysis for MIMO architectures that involve the capacity achieving spatio-temporal water-filling and the space-only water-filling strategies as well as their rank constrained counterparts and their comparison with quantized unitary (Grassmannian) precoding. Section V illustrates the versatility of our analysis by applying it to MIMO systems with antenna selection. This is well motivated in systems with fewer RF chains than antennas, which allows for opportunistic channel-dependent antenna selection and precoding strategies (cf. [19] and the references therein). Section VI concludes the paper.

II. System Model and Manifold Classification

Notation

A system with N_t transmit and N_r receive antennas is called a $N_t \times N_r$ system. Define $N_m = \min\{N_t, N_r\}$. If Q is a matrix, $Q^t, Q^H, \mathrm{tr}(Q), \mathrm{rk}(Q)$ represent its transpose, conjugate transpose, trace and rank, respectively. $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of a complex number z, respectively. For a $n \times n$ matrix X, $X > 0$ and $X \geq 0$ indicate that the matrix is positive definite and positive semi-definite, respectively. $\Gamma(z)$ is the usual gamma function given by $\int_0^\infty x^{z-1}e^{-x}\,dx$ for $\Re(z) > 0$. Its two commonly used multivariate generalizations $\Gamma_m(z)$ and $\tilde{\Gamma}_p(z)$ are utilized in this paper and are defined as follows [20]: for

$\Re\{z\} \geq \frac{m-1}{2}$, $\Gamma_m(z) = \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma\left(z - \frac{1}{2}(i-1)\right)$ and for $\Re\{z\} \geq (p-1)$, $\tilde{\Gamma_p}(z) = \pi^{\frac{p(p-1)}{2}} \prod_{i=1}^{p} \Gamma(z-i+1)$. Depending on the context, $|.|$ should be interpreted differently. On a real scalar, it connotes the modulus of the operand; on a complex scalar, it refers to the absolute value; on a matrix, it means the determinant, and on a set, it denotes the cardinality. $\|A\|$ denotes the standard l_2 or Frobenius norm of the matrix A which is $\sqrt{\mathrm{tr}(A^H A)}$. $\mathbb{I}, \mathbb{R}, \mathbb{C}, \mathbb{R}_+, \mathbb{F}$ represent the integer, real, complex, positive real and general fields, respectively. $f(x) = O(g(x))$ implies that there exist positive real numbers B and x_o such that $f(x) \leq B|g(x)|$ for all $x > x_o$. We use the $\lesssim$ in the sense of a *main order inequality*. For two functions $f_1(.)$ and $f_2(.)$ of the feedback rate N_f, $f_1(N_f) \lesssim f_2(N_f)$ means that $\lim_{N_f \to \infty} \frac{f_1(N_f)}{f_2(N_f)} \leq 1$. $\gtrsim$ should be interpreted similarly. N and CN denote the usual Gaussian and circularly symmetric complex Gaussian distributions, respectively. $\log$ denotes the natural logarithm and $\log_2$ logarithm to base 2.

A. MIMO System With Quantized Covariance Feedback

Consider a MIMO fast fading link with N_t transmit and N_r receive antennas, modeled at time m via the input-output equation

$$y(m) = \sqrt{\gamma} H(m) x(m) + n(m). \tag{1}$$

Here, the normalized receiver output at symbol time m, $y(m) \in \mathbb{C}^{N_r}$, $\gamma \in \mathbb{R}_+$, the fading channel matrix $H(m) \in \mathbb{C}^{N_t \times N_r}$, the transmitted signal $x(m) \in \mathbb{C}^{N_t}$, and complex additive noise $n(m) \in \mathbb{C}^{N_r}$ assumed to be white Gaussian, i.e., $n(m) \sim CN(0, I_{N_r})$. We assume $H[.]$ be an i.i.d fading process[2]. However, we do not constrain the probability distribution of $H(m)$ in any way; in particular, it need not be Rayleigh or Rician distributed. The individual elements H_{ij} need not be independent nor are they assumed to be identically distributed. The additive noise process $n[.]$ is taken to be independent and identically distributed (i.i.d.) across time.

Suppose first that the transmitter and receiver have full channel state information (F-CSI), i.e., the entire and perfect knowledge of the fading process $H[.]$. In this case, the transmitter employs precoding to send the sequence $x(m) = P_f(m)\tilde{x}(m)$ where $\tilde{x}[.]$ is a codeword drawn from a random codebook of rate R_{fcsi} with codewords drawn from an i.i.d. sequence of white Gaussian random vectors (of dimension N_t). Without loss of generality, we let the precoding matrix $P_f(H(m)) \in \mathbb{C}^{N_t \times N_t}$ be a function of just $H(m)$ with the function being independent of m[3] such that the power constraint is satisfied, i.e., $E[\mathrm{Tr}(Q_f(H(m))] \leq P$, where $Q_f(H(m))$ is the input covariance at time m defined as $Q_f(H(m)) \triangleq P_f(H(m))P_f^H(H(m))$. The maximum achievable rate (for which the error probability vanishes with increasing blocklength) for this full CSI strategy is then given as

$$R_{\mathrm{fcsi}} = E_H\left[\log\det\left(I + HQ_f(H)H^H\right)\right]$$

where H is a random matrix with a distribution that is the marginal distribution of the fading process $H[.]$.

Next consider the more realistic scenario where only the receiver knows precisely the realization of the process $H[.]$ (through training and channel estimation) from which it can compute the desired input covariance matrices $Q_f(m)$ at time m and employing a delay-free, error-free feedback channel to send a quantized version of $Q_f(m)$, which we denote as $Q_q(m)$, using N_f bits, i.e., using a quantization codebook having 2^{N_f} codewords. In practice, the fast fading channel model of (1) is realized through interleaving from a practical channel that remains approximately constant over the channel coherence time (cf. [21]) and the fed back $Q_q(m)$ would be valid for all symbols in that coherence time. We assume the transmitter employs the sequence of input covariance matrices $Q_q[.]$ in place of their full CSI counterpart, namely the sequence $Q_f[.]$. In this case, the maximum achievable rate under this quantized input covariance strategy is

$$R_{\mathrm{qcsi}} = E_H\left[\log\det\left(I + HQ_q(H)H^H\right)\right].$$

The principal aim of this paper is to analyze the tradeoff between the difference in achievable rate with perfect versus quantized fed back, namely $R_{\mathrm{fcsi}} - R_{\mathrm{qcsi}}$, and the quantization or feedback rate N_f. Of particular interest is the problem of approaching the capacity (with perfect CSIT) with quantized feedback. The capacity achieving strategy (for which $R_{\mathrm{fcsi}} = C_{\mathrm{fcsi}}$) involves at time m, beamforming along the right singular vectors of $H(m)$ together with spatio-temporal water-filling power allocation as specified in [9]. In so doing, it is also of interest to characterize the rate loss due to quantized feedback in the STPC problem where the so-called long-term power constraint (LTPC), $E[\mathrm{Tr}(Q_f(H(m))] \leq P$, is replaced by the more stringent (and hence suboptimal) $\mathrm{Tr}(Q_f(H(m))] \leq P$ for all m. The motivation for considering this suboptimal STPC strategy is, as we will see later in this paper, that its quantized feedback version can outperform the quantized feedback version of the capacity achieving strategy at limited feedback rates. We also analyze rank-constrained covariance feedback strategies that can further boost performance in the limited feedback rate regimes.

B. Pn Manifolds

The first step is to view the feedback matrix $Q_f \in \mathbb{C}^{N_t \times N_t}$ for each coherence time as a point on some manifold. By virtue of being an input covariance matrix, it is clear that Q_f would be non-negative definite. Depending on the system constraints however, we shall classify the space of non-negative matrices into four categories, thus effectively locating Q_f as a point on a manifold with as small a dimension and/or volume [11] as an application would allow.

The first attribute of the manifold to be quantized pertains to a constraint on the trace of Q_f. For the STPC the transmitter would always transmit at full allowed power P at each symbol time. Thus, the object Q_f to be fed back would be a point on a Pn manifold with trace equality constraint $\mathrm{tr}(Q) = P$. On the other hand, achieving the capacity of the MIMO fast fading channel under the LTPC ($E_H\,\mathrm{tr}(Q_f(H)) \leq P$) requires spatio-temporal water filling. While at the optimal solution, denoted $Q_f(m) = Q_o(m)$, this inequality holds with equality, the trace of $Q_o(m)$

[2]The results of this paper also apply to stationary and ergodic fading processes. However, the quantization of correlated fading processes must exploit the structure of correlation (as does [18]) which we do not consider here.

[3]The capacity achieving strategy which employs optimal spatio-temporal water-filling satisfies this constraint.

varies with m. Nevertheless, a trace inequality constraint can be shown to hold in this case as shown in Section IV.A.

The second attribute of the Pn manifold to be quantized comes from the nature of constraint on the rank of the input covariance matrix. For example, with a view to improving achievable rate at limited values of the feedback rate under the STPC (or LTPC), it is possible to restrict the rank of the input covariance matrix with little loss in achievable rate (or ergodic capacity) relative to R_{fcsi} (i.e., in the high feedback rate regime) to be some number s that is strictly less than N_t depending on the signal-to-noise ratio (SNR). This is further demonstrated through numerical experiments in Section IV.

Summarizing the above discussion, we note that input covariance matrices can be classified into four different manifolds, denoted below as the Pn-manifolds. These are given by two attributes namely, the trace constraint and the rank constraint. In general, $\mathcal{P}(N_t, *P, *s)$ denotes the manifold of N_t-dimensional positive semi-definite matrices[4] with trace equality or inequality if the first $*$ is $=$ or $\leq$, respectively, and with rank equality or inequality constraint if the second $*$ is $=$ or $\leq$, respectively. We explicitly define one of the four manifolds for clarifying our notational convention,

$$\mathcal{P}(N_t, \leq P, = s) = \{Q \in \mathbb{C}^{N_t \times N_t} \mid \forall x \in \mathbb{C}^{N_t} \\ x^H Q x \geq 0,\ \operatorname{tr}(Q) \leq P,\ \text{ and } \operatorname{rk}(Q) = s\}.$$

For later use, we note that when our results are agnostic to whether we have a trace (rank) equality or inequality, we indicate it by writing $*P$ ($*s$). If the same trace (rank) constraint applies on the Pn manifolds on both sides of an equation, we simply write $*$ for the second (third) parameter of the Pn manifolds.

III. Achievable Rate Loss Versus Feedback Rate Tradeoff

A. Tradeoff Between Capacity Loss and Feedback Rate

In this section, we study the variation of $R_{\text{fcsi}} - R_{\text{qcsi}}$ with respect to the feedback rate N_f. We start by defining the distance between two points P and Q on a Pn manifold as the Frobenius norm of the difference between their lower triangular parts (denoted as $\mathcal{L}(P)$ and $\mathcal{L}(Q)$, resp.) so that

$$d(P,\ Q) \triangleq \|\ \mathcal{L}(P) - \mathcal{L}(Q)\ \|.$$

Consider the following perturbative theorem on the function $f(Q) \triangleq \log\det\ (I + HQH^H)$. This theorem is general in that it is independent of the provenance of Q_1, allowing us to treat a variety of quantized covariance feedback methods in a unified manner.

Theorem 1: If Q_1 and Q_2 lie on a Pn manifold and $d(Q_1, Q_2)$ is small,

$$|f(Q_1) - f(Q_2)| \lesssim \sqrt{2} d(Q_1, Q_2)\ \|H^H (I + HQ_1 H^H)^{-1} H\|.$$

[4]These manifolds are denoted as $\mathcal{P}(N_t,\ \mathbb{C}, *P, *s)$ in [11] since both real and complex-valued covariance matrices are considered there. Here, complex-valued covariance matrices are considered throughout.

Proof: Since a matrix Q on a Pn manifold could potentially be rank-deficient, standard matrix differentiation techniques of [20] do not apply. While a Taylor series can be formally constructed over the Pn manifold following [22], the complicated dependence of the entries of matrix Q over its independent coordinates (denoted as $\operatorname{svec}(Q)$ in [11]) precludes that possibility. Hence, to analyze the $f(Q)$ expression, we adopt a perturbative approach that surprisingly requires only a scalar differentiation.

$$\begin{aligned} &f(Q_1) - f(Q_2) \\ &\quad = \log\det(I + HQ_1H^H) - \log\det(I + HQ_2H^H) \\ &\quad = \log\det(I + HQ_1H^H + sH(Q_2 - Q_1)H^H)\big|_{s=0} \\ &\qquad - \log\det(I + HQ_1H^H + sH(Q_2 - Q_1)H^H)\big|_{s=1} \\ &\quad = -\int_0^1 \frac{d}{ds} \log\det(I + HQ_1H^H + sH(Q_2 - Q_1)H^H)\, ds. \end{aligned}$$

For any operator $A(t)$, we have a general result from [20] that states,

$$\frac{d}{dt} \det A(t) = \operatorname{tr}\left(\frac{dA(t)}{dt} A(t)^{-1}\right) \det A(t),$$

using which we get

$$\begin{aligned} &\frac{d}{ds} \log\det(I + HQ_1H^H + sH(Q_2 - Q_1)H^H) \\ &= \operatorname{tr}\left[H(Q_2 - Q_1)H^H.(I + HQ_1H^H + sH(Q_2 - Q_1)H^H)^{-1}\right] \end{aligned}$$

For sufficiently small $d(Q_1, Q_2)$, we shall approximate the above expression by

$$\begin{aligned} &\frac{d}{ds} \log\det(I + HQ_1H^H + sH(Q_2 - Q_1)H^H) \\ &\qquad \approx \operatorname{tr}\left[(Q_2 - Q_1).\ H^H(I + HQ_1H^H)^{-1}H\right], \end{aligned}$$

This enables us to write

$$f(Q_1) - f(Q_2) \approx \operatorname{tr}\left[(Q_1 - Q_2).\ H^H(I + HQ_1H^H)^{-1}H\right].$$

Using the Cauchy-Schwartz (CS) inequality and the fact that $\|Q_1 - Q_2\| \leq \sqrt{2} d(Q_1, Q_2)$, we get

$$\begin{aligned} &|f(Q_1) - f(Q_2)| \\ &\quad \lesssim \sqrt{\operatorname{tr}\left[(Q_1 - Q_2)^2\right]}\ \sqrt{\operatorname{tr}\left[(H^H(I + HQ_1H^H)^{-1}H)^2\right]} \\ &\quad \leq \sqrt{2} d(Q_1, Q_2)\ \|H^H(I + HQ_1H^H)^{-1}H\|. \end{aligned}$$

■

We can interpret the above result substituting Q_f for Q_1 and Q_q for Q_2. This upper bound, valid for the high feedback rate regime, neatly divides into two separable parts. The first part varies not only, as intuition might suggest, monotonically with the distortion suffered, but is the distance itself. This observation helps explain many of the previous results on the scaling of capacity loss with feedback rate reported earlier in the literature for other kinds of feedback. The second part depends only on the input covariance matrix, formulated according to the system precoding strategy chosen.

A quantization codebook $\mathcal{C}$ of size $K \triangleq 2^{N_f}$ refers to the choice of K points $Q_1, \ldots, Q_K$ on a Pn manifold. A realization Q of any source over that manifold can be quantized by choosing the closest element of $\mathcal{C}$. In this paper, we focus on two codebooks, namely the sphere packing codebook $\mathcal{C}_{sph}$ and the random codebook $\mathcal{C}_{\text{rand}}$ generated via a uniform distribution on the manifold [11]. The rationale for considering sphere packing and random codebooks is that they can be analyzed *and* fundamental insight can be had about what is possible with structured codebooks. This is because random codebook results guarantee the existence of a code that will do at least as well (following Shannon's random coding argument).

Theorem 2: If the transmitter uses the quantized version of the input covariance matrix Q_f in one of the four Pn manifolds $\mathcal{P}(N_t, *P, *s)$ fed back by the receiver using the sphere packing code or a random code with N_f bits per block, one can attain an information rate (in nats per channel use, denoted nats/cu) bounded as

$$R_{\text{qcsi}} \gtrsim R_{\text{fcsi}} - \sqrt{2}e_{\mathcal{C}} \left[E_H\, g(H)\right] 2^{-\frac{N_f}{N}} \tag{2}$$

where $e_{\mathcal{C}}$ depends on the code used for quantization and is given by

$$e_{\mathcal{C}} = \begin{cases} 2\,(c)^{\frac{-1}{N}} & \text{if } \mathcal{C} = \mathcal{C}_{sph}; \\ \frac{\Gamma(\frac{1}{N})}{N} \left(\frac{r}{N+r}\right)^{-r/N} (c)^{\frac{-1}{N}} & \text{if } \mathcal{C} = \mathcal{C}_{\text{rand}}, \end{cases} . \tag{3}$$

where N is the dimension of the Pn manifold[5] and c is the ball volume coefficient[6] chosen for quantization[7], for any $r > \max\{0, \frac{\tau}{6(N+2)}\}$ (τ being the scalar curvature of the manifold) and $g(H) = \|H^H(I + HQ_fH^H)^{-1}H\|$. Further, if a random code is used for quantization, R_{fcsi} and R_{qcsi} (in nats/cu) should be interpreted as averages taken over the ensemble of all random codes.

Proof: In Theorem 1, we can substitute Q_f for Q_1 and $q(Q_f)$ for Q_2 to get a lower bound on $f(q(Q_f))$ as

$$\begin{aligned} &f(q(Q_f)) \\ &\quad \gtrsim f(Q_f) - \sqrt{2}d(Q_f, q(Q_f))\, \|H^H(I + HQ_fH^H)^{-1}H\|. \end{aligned}$$

If the code $\mathcal{C}_{sph}$ code is used, then from Theorem 4 of [11],

$$d(Q_f, q(Q_f)) \leq \triangle_{\max} \lesssim e_{\mathcal{C}}|_{\mathcal{C}_{sph}}\, 2^{-\frac{N_f}{N}}.$$

This holds true for any one of the four Pn manifolds provided we use the corresponding N and c for the calculation of $e_{\mathcal{C}}$ in this expression from [11]. If the random code $\mathcal{C}_{\text{rand}}$ code is used, then we average over the ensemble of all random codes over the particular Pn manifold and use the upper bound in Theorem 5 of [11] to get

$$E_{\mathcal{C}_{\text{rand}}}\, d(Q_f, q(Q_f)) \lesssim e_{\mathcal{C}}|_{\mathcal{C}_{\text{rand}}}\, 2^{-\frac{N_f}{N}}.$$

Note that $R_{\text{fcsi}} \triangleq E_H f(Q_f)$ and $R_{\text{qcsi}} \triangleq E_H f(q(Q_f))$. Since $e_{\mathcal{C}}$ is independent of H, we can average over H to get

$$R_{\text{qcsi}} \gtrsim R_{\text{fcsi}} - \sqrt{2}e_{\mathcal{C}}|_{\mathcal{C}_{\text{rand}}} \left[E_H\, g(H)\right] 2^{-\frac{N_f}{N}}.$$

■

The approximate lower bound of (2) is a key result of this paper and it is applicable for general channel distributions and general MIMO strategies via the specification of the function Q_f as a function of H. In computing the estimate for R_{qcsi} from (2) we need to estimate $E_H\, g(H)$ for which Monte-Carlo simulation is used in all examples in this paper. A simple variant of the above theorem is

Corollary 3: To limit $R_{\text{fcsi}} - R_{\text{qcsi}}$ to X nats/cu while using the code $\mathcal{C}$ over a Pn manifold in the regime of sufficiently high feedback rates (for any distribution of the channel matrix H) it is sufficient that $N_f \gtrsim N \log_2\left(\frac{\sqrt{2}e_{\mathcal{C}}\ E_H g(H)}{X}\right)$, where $e_{\mathcal{C}}$ and $g(H)$ are as defined in Theorem 2.

Note that we consistently use N_f to denote bits of feedback whereas the achievable rates are given in nats/cu throughout this paper. The latter can of course be converted into bits/cu by simply dividing the rate in nats/cu by $\log 2$.

B. An Alternative View: Power Efficiency

To obtain an alternate perspective on R_{qcsi}, we seek a representation of R_{qcsi} as an equivalent R_{fcsi} expression, albeit with reduced power. Mathematically, if the rate $R_{\text{fcsi}}(P)$ reflects the ideal CSIT rate attainable under transmit power P, then we seek to represent R_{qcsi} as $R_{\text{fcsi}}(\mu P)$, where $\mu \in (0, 1]$ serves as a 'power efficiency' indicator reflecting efficiency relative to the ideal CSIT scenario. Given Theorem 2, we expect μ to approach 1 exponentially as $N_f \to \infty$. This is confirmed by

Corollary 4: The power efficiency factor μ is bounded as

$$\mu \gtrsim 1 - d\, 2^{-\frac{N_f}{N}}, \tag{4}$$

where

$$d = \left[\sqrt{2}e_{\mathcal{C}} . \frac{E_H \sqrt{\text{tr}\left[(HH^H)^2(I + HQ_fH^H)^{-2}\right]}}{E_H\ \text{tr}\left[\,(I + HQ_fH^H)^{-1}HQ_fH^H\,\right]}\right], \tag{5}$$

with N and c being the dimension and ball volume coefficient of the Pn manifold chosen for quantizing Q_f.

Proof: For $\mu > 0$, consider the matrix-valued logarithmic function $G(\mu)$ defined using the usual functional calculus as

$$G(\mu) = \text{Log}\,(I + \mu\, HQ_fH^H).$$

Differentiating over the scalar μ, without requiring any matrix differentials, we have

$$G'(\mu) = (I + \mu\, HQ_fH^H)^{-1}HQ_fH^H.$$

[5]It was shown in [11] that $N = 2N_ts - s^2$ for $\mathcal{P}(N_t, \leq P, *s)$ and $N = 2N_ts - s^2 - 1$ for $\mathcal{P}(N_t, = P, *s)$.

[6]It was shown in [11] that $c_{N_t,*P,*s} = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N+2}{2}).\text{Vol}(\mathcal{P}(N_t,*P,*s))}$ where $\text{Vol}(\mathcal{P}(N_t, *P, *s))$ is the volume of the manifold $\mathcal{P}(N_t, *P, *s)$, formulas for which are given in [11, Theorem 2].

[7]The dependence of the dimension and the ball volume coefficient on N_t, s and P are suppressed so that N and c denote those quantities respectively for whatever quantization manifold is under consideration.

Employing Taylor's expansion up to the first order term, we get

$$G(\mu) \approx G(1) + (\mu - 1)\, G'(\mu)|_{\mu=1}$$
$$\Rightarrow \mathrm{Log}\,(I + \mu\, HQ_fH^H) = \mathrm{Log}\,(I + HQ_fH^H) + (\mu - 1)\,(I + HQ_fH^H)^{-1}HQ_fH^H.$$

By taking the trace of these matrices and by taking an expectation over the given H ensemble,

$$E_H\ \mathrm{tr}\ \mathrm{Log}\,(I + \mu\, HQ_fH^H) \approx E_H\ \mathrm{tr}\ \mathrm{Log}\,(I + HQ_fH^H) + (\mu - 1)\, E_H\ \mathrm{tr}\,[\,(I + HQ_fH^H)^{-1}HQ_fH^H\,].$$

Recalling that $\log\,\det\,A = \mathrm{tr}\ \mathrm{Log}\,A$ for a positive definite matrix A, we get

$$E_H\ \log\,\det\,(I + \mu\, HQ_fH^H) \approx E_H\ \log\,\det\,(I + HQ_fH^H) + (\mu - 1)\, E_H\ \mathrm{tr}\,[\,(I + HQ_fH^H)^{-1}HQ_fH^H\,].$$

Since we want to represent R_{qcsi} as $E_H\ \log\,\det\,(I + \mu\, HQ_fH^H)$ and we have defined R_{fcsi} as $E_H\ \log\,\det\,(I + HQ_fH^H)$, we can rewrite the above equation as

$$R_{\text{fcsi}} - R_{\text{qcsi}} \approx (1 - \mu)\, E_H\ \mathrm{tr}\,[\,(I + HQ_fH^H)^{-1}HQ_fH^H\,]. \quad (6)$$

From Theorem 2 above, we can quantize Q_f over an appropriate Pn manifold (of dimension N and ball volume coefficient c) yielding an approximation of the $R_{\text{fcsi}} - R_{\text{qcsi}}$ gap to the first order term as

$$R_{\text{fcsi}} - R_{\text{qcsi}} \lesssim \sqrt{2}e_{\mathcal{C}} \cdot E_H\sqrt{\mathrm{tr}\,[(HH^H)^2(I + HQ_fH^H)^{-2}]}\ 2^{-\frac{N_f}{N}}. \quad (7)$$

By combining the terms in (6) and (7), the final expression for the power efficiency factor is obtained. ■

The constant d of (5) depends on the distribution of the channel matrix H, the precoding strategy chosen to obtain Q_f, and the manifold chosen for quantization. It must as such be computed using Monte-Carlo simulation. As $N_f \rightarrow \infty$, the power efficiency factor μ approaches unity at an exponential rate not slower than as $1 - d.2^{-N_f/N}$.

C. Simulation

To demonstrate the validity of the lower bound on R_{qcsi} in (2) of Theorem 2 at finite N_f, a 2×2 MIMO i.i.d. Rayleigh fading system with an STPC of 5 dB was simulated. In particular, the optimal covariance was quantized over the 4-dimensional $\mathcal{P}(2, = 5\ \text{dB}, \leq 2)$ manifold. A random quantization codebook was used. The lower bound and simulated achievable rates are shown in Fig. 1 (obtained by averaging over 100 realizations of the codebook for each of 100 channel realizations), where it is seen that the lower bound for R_{qcsi} of (2) is valid even for small N_f. As an estimate of R_{qcsi} it is somewhat conservative (recall the use of the CS and $\|Q_1 - Q_2\| \leq \sqrt{2}d(Q_1, Q_2)$ inequalities in proving Theorem 1).

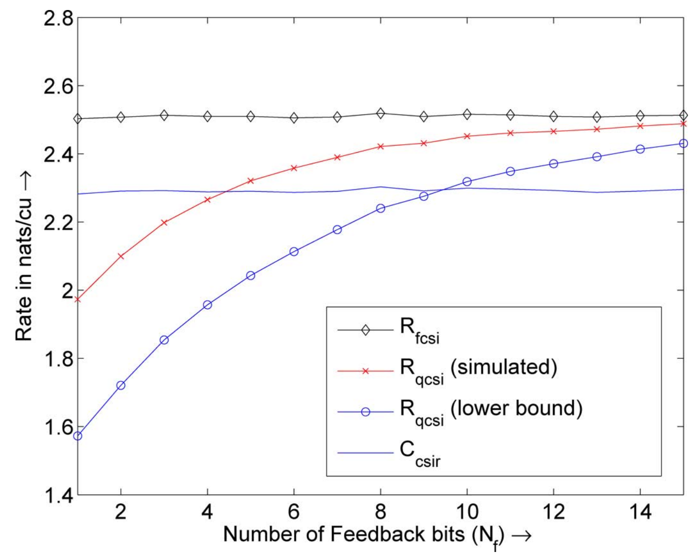

Fig. 1. R_{qcsi} (in nats/cu) versus N_f through both Monte-Carlo simulations and our theoretical estimate. The manifold quantized is $\mathcal{P}(2, \leq 5\ \text{dB}, \leq 2)$.

Nevertheless, in what follows, we refer to the theoretically obtained conservative estimate of R_{qcsi} of Theorem 2 as the achievable rate with quantized CSI even if it is a lower bound on achievable rate (at high N_f).

D. Discussion

The results of Theorem 2 and Corollary 4 establish that the expected loss in achievable rate (including ergodic capacity) due to quantized feedback proceeds as $O\left(2^{-\frac{N_f}{N}}\right)$. The simplicity of the proofs of these results and their generality illustrate the power of the geometric paradigm, especially when it is juxtaposed against the earlier work due to Dabbagh and Love in [12] dealing with the same question for one specific manifold, namely $\mathcal{P}(N_t, = 1, \leq N_m)$. Indeed, the generality of Theorem 2 and Corollary 4 allows for a unified approach for the evaluation of any system precoding strategy for the computation of $Q_f[i]$ at the receiver, thus providing insight into how quantized covariance feedback MIMO systems ought to be designed. In particular, the strong dependence of the rate at which one approaches R_{fcsi} with quantized covariance feedback on the dimension of the quantization manifold points to the employment of optimal choice of $Q_f[i]$ for large N_f and good suboptimal choices of $Q_f[i]$ as objects in smaller dimensional Pn manifolds, leading to better achievable rates with quantized covariance feedback when N_f is limited. This theme is explored in Section IV.

It is notable moreover that in the case of quantization of $\mathcal{P}(N_t, = 1, \leq N_m)$ considered in [12] to approximate the space-only water-filling strategy, the key result therein bounds the rate difference $R_{\text{fcsi}} - R_{\text{qcsi}}$ as $N_m \log\left(1 + \hat{c}\, 2^{-\frac{N_f}{2N_mN_t - 2}}\right)$, and $\hat{c}$ is some positive constant.

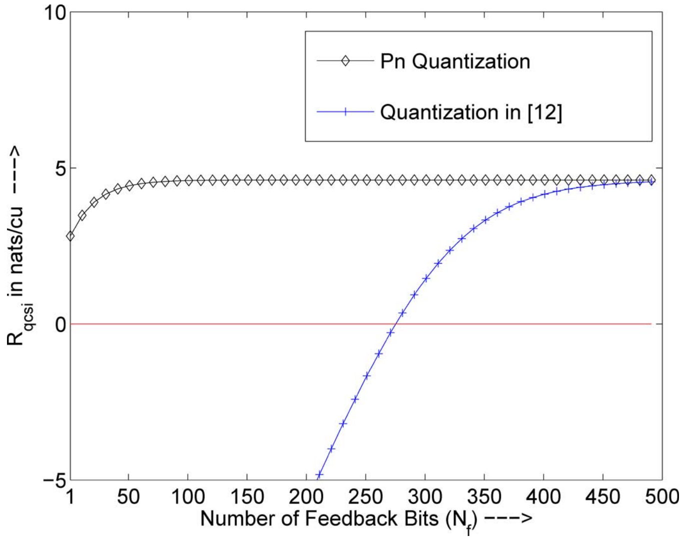

Fig. 2. Comparing (lower bounds on) R_{qcsi} for a 4×4 Rayleigh-faded MIMO system with quantization manifold $\mathcal{P}(4, = 4\ \mathrm{dB}, \leq 4)$ versus the result in [12].

This yields a scaling of $O\left(2^{-\frac{N_f}{2N_m N_t - 2}}\right)$, which is much slower than the $O\left(2^{-\frac{N_f}{2N_m N_t - N_m^2 - 1}}\right)$ obtained by our method through quantization on the same $\mathcal{P}(N_t, = 1, \leq N_m)$ manifold. We also illustrate the benefit in this case of our analysis through a numerical example.

1) Example 1: The bounding technique in [12] can also result in a large value for $\hat{c}$. For example, in a 4×4 i.i.d. Rayleigh faded MIMO system with an STPC of 4 dB, R_{qcsi} is bounded below by $4.57 - 4\ \log(1 + 1255\ 2^{-N_f/30})$ nats/cu in [12] as compared to $4.57 - 1.87 \cdot 2^{-N_f/15}$ nats/cu in this paper's approach. Unfortunately, the large $\hat{c}$ results in the lower bound of [12] for the achievable rate remaining negative – and thus inoperative – even at $N_f = 250$. The contrast between the two answers is illustrated in Fig. 2.

It is also noteworthy that unlike the answer in [12], our solution naturally splits into geometric, source and channel coding considerations. The general theorems on quantization performance on arbitrary manifolds offer intuitive insight into the $2^{-\frac{N_f}{N}}$ scaling finally obtained. Moreover, while the work in [12] does not apply to either the reduced rank cases or the trace inequality scenario as would be necessary in achieving the ergodic capacity of the channel (see Section IV.A), our approach, through a single calculation, yields the answers for all the Pn manifolds and has applicability in approaching the capacity of the channel under a wide range of N_f as is demonstrated in Section IV.

IV. MIMO Systems With Quantized Covariance Feedback

In this section, we apply our Pn framework to study the performance of MIMO systems with finite-rate feedback under various constraints. We start in Section IV.A with the fundamental question of approaching capacity with joint optimal spatio-temporal water-filling and then deal with the sub-optimal choice of input covariance for the STPC in Section IV.B which involves only spatial water-filling. In Section IV.C, we then compare quantized covariance feedback with quantized unitary precoding as analyzed in [7].

A. Approaching Capacity (Spatio-Temporal Water-Filling)

In this section, we analyze the performance of a MIMO system under the LTPC $E_H[\mathrm{tr}(Q)] = P$. If a codeword spans B blocks, then the average power constraint is given by $\frac{1}{B} \sum_{i=1}^{B} |x[i]|^2 = P$, where $x[i]$ is the signal transmitted in the i-th block. The principal advantage of considering the LTPC lies in affording us the choice to use different powers across blocks depending on the channel conditions, thereby achieving the ergodic capacity of the channel. This implies however that the rank and trace of the optimal covariance at time i, denoted as $Q_o[i]$, would both also change with i and the problem is to determine the right Pn manifold to quantize.

Strictly speaking, the rank of $Q_o[i]$ can be any number up to N_t depending on the channel realization which dictates that in order to approach capacity with increasing feedback rate N_f, we must choose the Pn manifold with the rank inequality constraint $\mathrm{rk}(Q_o[i]) \leq N_m$. Denoting the N_m non-zero eigenvalues of the matrix $H[i]H[i]^H$ (formed from the channel matrix in the i-th block $H[i]$) in descending order as $\lambda_i^{[1]} > \cdots > \lambda_i^{[N_m]}$, the water-level ζ in the average power constrained problem is given by

$$\frac{1}{B} \sum_{i=1}^{B} \sum_{j=1}^{N_m} \left(\zeta - \frac{1}{\lambda_i^{[j]}}\right)^+ = P. \tag{8}$$

Forming the $N_t \times N_t$ diagonal matrix $\Gamma[i]$ with $(\Gamma[i])_{jj} = \left(\zeta - \frac{1}{\lambda_i^{[j]}}\right)^+$ for $j \leq N_m$ we get the optimal input covariance matrix $Q_o[i] = V[i]\Gamma[i]V[i]^H$ with $V[i]$ arising from the right singular vectors of $H[i]$. This gives

$$C_{\mathrm{fcsi}} = \lim_{B \longrightarrow \infty} \frac{1}{B} \sum_{i=1}^{B} \log\det(I + H[i]Q_o[i]H[i]^H).$$

While the water level ζ may appear at first to depend on the fading process realization for all time, thereby precluding its causal computation, it is easily seen [9] that it is given by the *off-line* pre-computation of the solution to the channel realization-independent fixed point equation

$$\frac{P}{N_m} = \int_{\zeta^{-1}}^{\infty} \left(\zeta - \frac{1}{\lambda}\right) p(\lambda) d\lambda, \tag{9}$$

with $p(\lambda)$ being the p.d.f. of a typical unordered eigenvalue of the matrix $H[1]H[1]^H$. Hence, $C_{\mathrm{fcsi}} = N_m \int_{\zeta^{-1}}^{\infty} \log(\zeta\lambda) p(\lambda) d\lambda$.

Once ζ is obtained, the process of computing the optimal input covariance for time i, namely $Q_o[i]$, is not just causal, but instantaneous. The receiver computes $Q_o[i]$ at time i and quantizes it over an appropriate Pn manifold. Since the minimum value that $1/\lambda_i^{[j]}$ of a positive semi-definite matrix $H[i]H[i]^H$ is bounded below by zero, the maximum value of $\mathrm{tr}(Q_o[i])$ is given by $\mathrm{tr}(Q_o[i]) \leq \sum_{j=1}^{N_m} \zeta = N_m\zeta$. Note that this upper

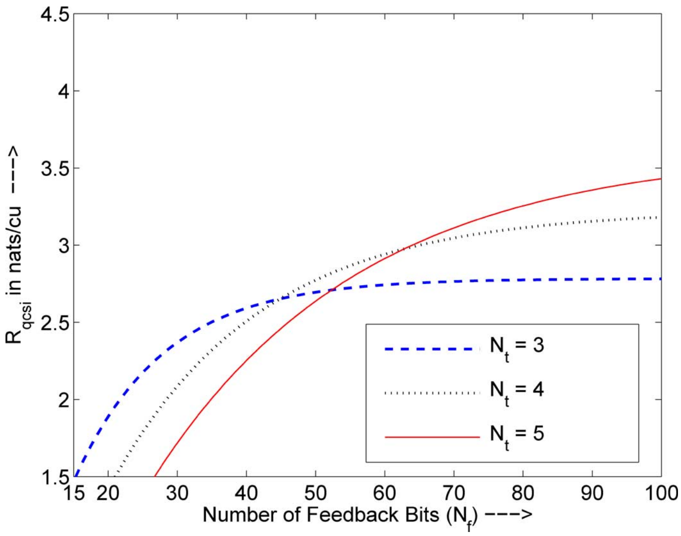

Fig. 3. $R_{\rm qcsi}$ (lower bound) vs N_f for $N_t \times 3$ MIMO systems with $\rm SNR = 2$ dB. The manifold quantized is $\mathcal{P}(N_t, \leq 3\zeta, \leq 3\})$.

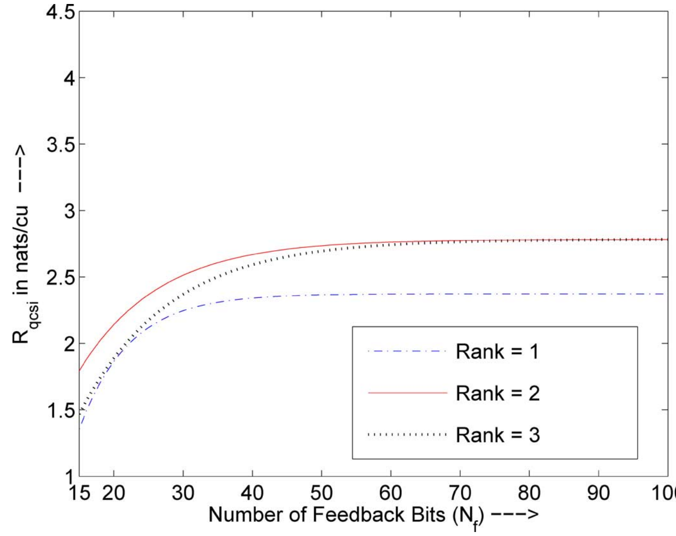

Fig. 4. $R_{\rm qcsi}$ (lower bound) vs N_f for a 3×3 MIMO system with $\rm SNR = 2$ dB. The manifold quantized is $\mathcal{P}(N_t, \leq 3\zeta, \leq s\}), \; s \in \{1, 2, 3\}$.

bound is not loose, i.e., any reduction in its value would result in the possibility that $Q_o[i]$ will not lie on the Pn manifold chosen for quantization. Thus, in order to approach the ergodic capacity with increasing feedback rate, we choose to quantize $\mathcal{P}(N_t, \leq N_m\zeta, \leq N_m)$.

1) Example 2: Let us analyze a 3×3 i.i.d. Rayleigh faded MIMO system under a low average LTPC of 2 dB. We obtain $\zeta = 1.16$, $C_{\rm fcsi} = 2.78$ nats/cu and $C_{\rm csir} = 2.3$ nats/cu (this is the capacity with CSI only at the receiver). Using the sphere-packing code $\mathcal{C}_{sph}$ over the manifold of 3×3 p.s.d. matrices $\mathcal{P}(3, \leq 3.48, \leq 3)$, we obtain $R_{\rm qcsi} \gtrsim 2.78 - 4.18 \cdot 2^{-\frac{N_f}{9}}$.

Continuing our analysis, we fix $N_r = 3$ but increase N_t. As N_f increases, we approach $C_{\rm fcsi}$. Hence it is clear that an $N_t = 5$ system would outperform an $N_t = 3$ system. However, when N_f is limited, using the same number of bits for quantization induces a significantly bigger error in the larger manifold (for larger N_t) and, consequently, a larger loss relative to $C_{\rm fcsi}$ as seen in Fig. 3.

With quantized CSI, when N_f is limited, there can be an advantage, the extent of which depends on SNR, to considering the more general case of $\mathrm{rk}(Q_o[i]) \leq s$ for all i, for some fixed $1 \leq s \leq N_m$ even if the restriction $s < N_m$ means that we compromise on achieving the exact capacity in the large N_f regime. The water-level can still be determined by the fixed-point (9) but the power to transmit beams corresponding to all but the largest s eigenvalues is set to zero (note in this case that the average power used in general is lower than P). The quantization manifold is thus chosen to be $\mathcal{P}(N_t, \leq N_m\zeta, \leq s)$. With reduced rank comes reduced dimension of this manifold and hence a faster approach to the achievable rate under perfect CSIT (even if this is lower than $C_{\rm fcsi}$) resulting in a benefit at limited quantization rates over the full-rank case.

2) Example 3: We consider Example 2 but vary the rank constraint as $\mathrm{rk}(Q_o) \leq s$ for each $1 \leq s \leq N_m$ and plot the resulting $R_{\rm qcsi}$ rates (using $\mathcal{C}_{sph}$). At an SNR of 2 dB, the difference between the $R_{\rm fcsi}$ values for the $\mathrm{rk}(Q_o[i]) \leq 2$ and $\mathrm{rk}(Q_o[i]) \leq 3$ cases is quite small. Since the dimension of the quantization manifold in the two-beam case is smaller however, we get a quicker convergence towards the $C_{\rm fcsi}$ as seen in Fig. 4 enabling the $R_{\rm fcsi}$ for the two-beam case to supersede that of the three-beam case for almost all values of N_f with significant benefits at small N_f.

B. Space-Only Water Filling

We focus on the problem of approaching $C_{\rm fcsi}$ for limited N_f through sup-optimal precoding strategies. By forcing the STPC for example (i.e., $\mathrm{tr}(Q_s[i]) = P, \forall i$), one gets a reduction in the dimension of the quantization manifold by 1. Further reduction in the dimension is possible by identifying the rank(s) of the preponderance of the spatial water-filling matrices for given SNR and N_t and N_r.

1) Example 4: We illustrate the use of a reduced rank Pn manifold under the STPC. Consider the 5×4 i.i.d. Rayleigh faded MIMO system with an SNR of 0 dB under the STPC. It is easily ascertained through a Monte Carlo simulation that the rank of the optimal input covariance matrix (under the STPC) obtained through space-only water-filling alternates between 2 and 3. This allows us to quantize over the manifold of five-dimensional p.s.d. matrices with unit trace and rank no greater than 3, namely, $\mathcal{P}(N_t, = 1, \leq 3)$. The achievable rates $R_{\rm fcsi}$, $R_{\rm qcsi}$ obtained by averaging over random codes $\mathcal{C}_{\rm rand}$ as well as $C_{\rm csir}$ are plotted in Fig. 5. Note that even at large N_f, there is little loss relative to the ideal $R_{\rm fcsi}$.

2) Example 5 (LTPC-STPC Comparison): We seek to compare the performance of LTPC and STPC water-filling strategies under quantized covariance feedback for a 2×2 Rayleigh fading channel with SNR at 2 dB. For a fair comparison, we do not constrain the rank in either case and hence use a random code to quantize over the 2-dimensional covariance matrices $\mathcal{P}(N_t, = 2\ \mathrm{dB}, \leq 2)$ for the STPC case and $\mathcal{P}(N_t, \leq 5.44\ \mathrm{dB}, \leq 2)$ for the LTPC case. The results are plotted in Fig. 6.

The achievable rates in the two cases are given by $R_{\rm qcsi} \gtrsim 1.85 - 1.84 \cdot 2^{-\frac{N_f}{4}}$ for the LTPC case and $R_{\rm qcsi} \gtrsim 1.82 - 1.12 \cdot 2^{-\frac{N_f}{3}}$ for the STPC case. Clearly, the difference between $C_{\rm fcsi}$ (under LTPC) and $R_{\rm fcsi}$ (under STPC) is marginal, but

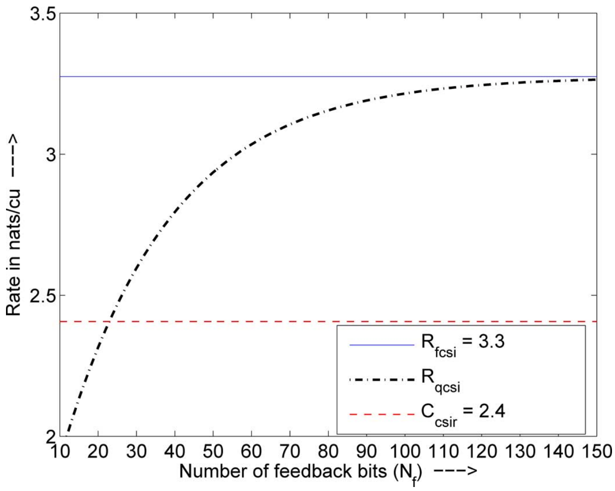

Fig. 5. Comparing (lower bounds on) achievable rates under CSIR, CSIT and quantized feedback (with a random codebook) for a 5×4 Rayleigh-faded channel with quantization over the 20-dimensional $\mathcal{P}(5, =1, \leq 3)$ manifold.

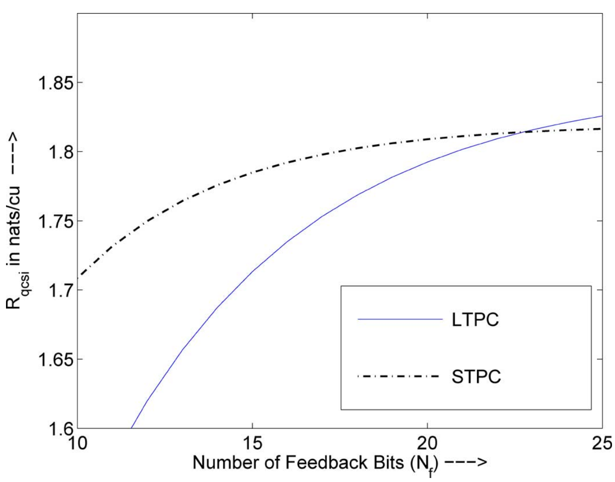

Fig. 6. Lower bounds on achievable rates in a 2×2 MIMO system with quantized covariance feedback under LTPC and STPC for Rayleigh fading.

notice the dominant role in deciding R_{qcsi} in both cases played by the scaling factor $2^{-\frac{N_f}{N}}$. By virtue of having a trace equality in place of a trace inequality, the dimension of the Pn manifold in the STPC case is one smaller than the corresponding value in the LTPC case. This enables R_{qcsi} in the STPC case to again exceed that achieved under LTPC in the limited N_f regime.

3) Example 6: In this example we consider a non-Rayleigh channel distribution. Let the 2×2 channel H have the SVD of UDV^H, where U and V are Haar-distributed over the unitary group and $D \triangleq \text{diag}\{d, d\}$. If d^2 takes the values 0.75 and 7.5 with equal probability, then even the naive LTPC scheme of transmitting only when $d^2 = 7.5$ can outperform the STPC rate for all values of the feedback rate as is illustrated in Fig. 7 at an SNR of 2 dB.

C. Comparison With Unitary Precoding

The notion of the power efficiency factor was also considered by Dai *et al.* [7] in the context of Grassmannian feedback. For a singular value decomposition of the channel matrix $H = UDV^H$, form the covariance matrix as $Q = P_{on} V_s V_s^H$, where

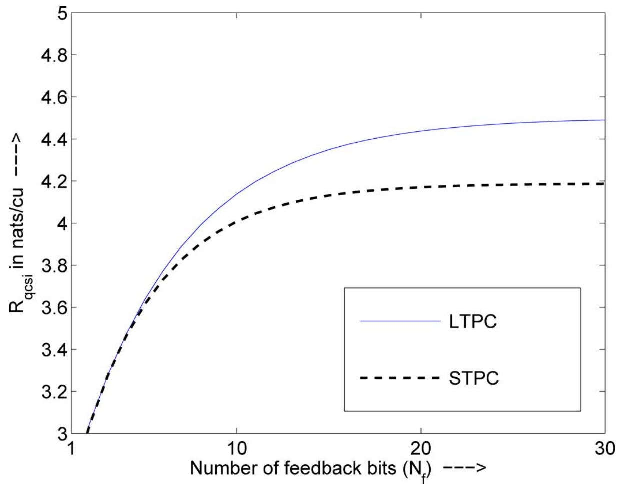

Fig. 7. Lower bounds on achievable rates in a 2×2 MIMO system with quantized covariance feedback under LTPC and STPC for a widely varying non-Rayleigh channel.

V_s represents the s columns of the V matrix corresponding to the s largest singular values of H. P_{on} is a positive real scalar representing the constant power allocated to each beam (or a column of V_s). Under this sub-optimal scheme, the rate achieved under full CSI is seen to be $R_{\text{fcsi}} \triangleq E_H \sum_{i=1}^{s} \log\,(\,1 + P_{on}\,\lambda_i\,)$, where λ_i is the i-th largest eigenvalue of the matrix HH^H. The authors in [7] compute the subspace spanned by the columns of V_s and quantize it under a chordal distance metric using a random codebook with 2^{N_f} entries constructed over the space of all s-dimensional subspaces within $\mathbb{C}^{N_t}$ known as the complex Grassmann manifold $G_{N_t,s}$. Under the assumption of H being Rayleigh-faded, they obtain the approximation $R_{\text{qcsi}} \approx E_H \sum_{i=1}^{s} \log\,(\,1 + \mu\, P_{on}\,\lambda_i\,)$ where

$$\mu = 1 - \left[\frac{1}{s} \frac{\Gamma\left(\frac{2}{N}\right)}{\frac{N}{2}} c^{-\frac{2}{N}} \right] 2^{-\frac{2N_f}{N}},$$

where we denote the dimension of $G_{N_t,s}$ by N and the ball volume coefficient in $G_{N_t,s}$ by c.

1) Example 7: To compare the performance of Pn and Grassmannian feedback schemes, let us consider a 3×3 system under a STPC of $\text{tr}(Q) = 1$ dB. The matrices U and V composed, respectively, of the left and right singular vectors of the channel matrix H, are assumed to be uniformly distributed over the complex Stiefel manifold $V_{3,3}$. The singular values of H are denoted in descending order by d_1, d_2 and d_3. The probability distributions are chosen such that d_1 and d_3 always represent 'good' and 'bad' transmission directions, respectively, whereas d_2 is chosen such that it alternates between these states. In particular, letting $U[a, b]$ denote the uniform distribution over the interval $[a, b]$ we let $d_1 \sim U[10,\ 11]$, and $d_3 \sim U[0.1,\ 0.2]$ and d_2 is $\sim U[9, 10]$ with probability 0.2 and $U[0.2,\ 0.3]$ with probability 0.8.

With full CSIT and Grassmannian beamforming with s beams we have $R_{\text{fcsi}} = E_H \log \left(1 + \sum_{i=1}^{s} \frac{d_i^2 \text{SNR}}{s}\right)$ which for $s = 1, 2, 3$ turns out to be 4.94, 5.10 and 4.62, thereby justifying the selection of $s = 2$ beams and the use of $G_{3,2}$ as the quantization manifold. Note that Grassmannian quantization fails when $s = 3$ ($R_{\text{fcsi}} = R_{\text{csir}} = 4.62$); and hence we

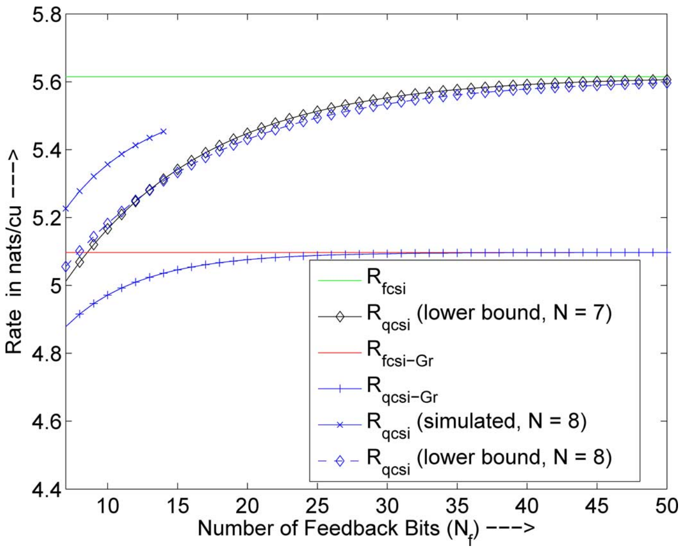

Fig. 8. Comparison of (lower bounds on) achievable rates with quantized Grassmannian (unitary) precoding with quantized covariance precoding (and their ideal counterparts) as a function of feedback rate N_f for a 3×3 non-Rayleigh MIMO system with $\mathrm{SNR} = 1$ dB. The manifolds quantized for unitary precoding is the Grassmannian manifold $G_{3,2}$. For covariance feedback they are the 7-dimensional $\mathcal{P}(3, = 1 \text{ dB}, \leq 2)$ and the 8-dimensional $\mathcal{P}(3, \leq 1 \text{ dB}, \leq 2)$ manifolds. In the latter case, the simulated value of R_{qcsi} is also provided for small N_f.

have chosen the SNR value appropriately to yield $s \neq 3$ as the optimal answer. R_{fcsi} with $s = 2$ is denoted as $R_{\mathrm{fcsi-Gr}}$ in Fig. 8. Using the results from [7], we obtain for $G_{3,2}$ that $N = \dim G_{n,p}|_{n=3,p=2} = 2p(n-p)|_{n=3,p=2} = 4$. Computing the ball volume coefficient as given in [7] and hence μ above, we obtain the Grassmannian power efficiency factor as $\mu = 1 - 0.56\, 2^{-\frac{N_f}{2}}$. This results in the rate achievable under Grassmannian finite-rate feedback to be approximated as

$$R_{\mathrm{qcsi}} \approx E_H \sum_{i=1}^{2} \log \left(1 + \frac{\mathrm{SNR}}{2} d_i^2 (1 - 0.56\, 2^{-\frac{N_f}{2}})\right).$$

For quantization over the Pn manifold, it can be shown that for this problem, the rank of the optimal input covariance matrix obtained via water-filling is constrained to be less than or equal to two with probability 1 and $R_{\mathrm{fcsi}} = 5.61$ nats/cu. Using this fact, we can quantize over the manifold $\mathcal{P}(3, = 1 \text{ dB}, \leq 2)$. Calculating further in a manner similar to the procedure in previous examples, the R_{qcsi} estimate can be obtained using the main result of Theorem 2 as $R_{\mathrm{qcsi}} \gtrsim 5.61 - 1.21 \cdot 2^{-\frac{N_f}{7}}$.

We plot the estimates of achievable rate for Pn and Grassmann quantization in Fig. 8 and notice that Pn quantization outperforms Grassmannian quantization even if the latter converges faster to its ideal value.

V. Antenna Selection

Finally, we further demonstrate the versatility of our analysis of quantized covariance feedback in MIMO systems by tackling MIMO systems with a limited number of RF chains which entails choosing and switching to a subset of the available transmit and/or receive antennas for each channel realization in order to take advantage of having more antennas than RF chains (cf. [19] and the references therein). In particular, let $L_t \leq N_t$ and $L_r \leq N_r$ denote the number of RF chains or antennas selected at the transmitter and receiver, respectively. We apply our general results from Section III to this MIMO system under quantized covariance feedback under the STPC $\mathrm{tr}(Q_f) = P$.

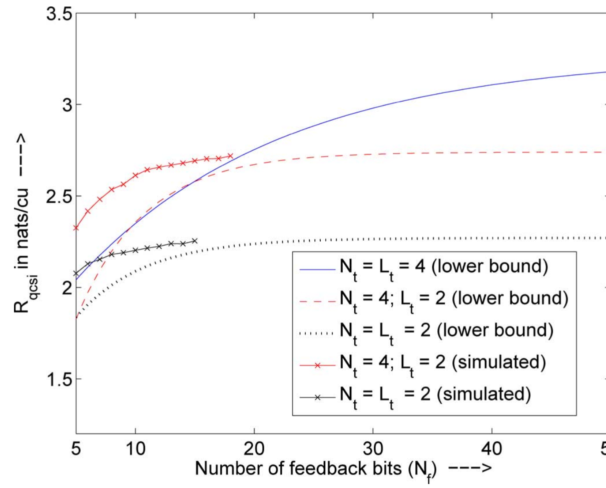

Fig. 9. Achievable rates (simulated and lower bounds) with quantized covariance feedback in a 4×2 MIMO system with transmit antenna selection ($L_t = 2$) and in 2×2 and 4×2 MIMO systems without antenna selection.

To highlight the dependence of the input covariance matrix obtained using the spatial water-filling strategy on the channel realization H, we refer to it in this section as $Q_w(H)$. If $\tilde{H} \subseteq H$ denotes a $L_r \times L_t$ submatrix of the original $N_r \times N_t$ channel matrix H, the optimal submatrix can be chosen by the receiver as

$$\tilde{H}_{AS} = \arg\max_{\tilde{H} \subseteq H} \log \det \left(I + \tilde{H} Q_w(\tilde{H}) \tilde{H}^H\right). \tag{10}$$

The optimal input covariance matrix Q_{AS} is now simply $Q_{AS} = Q_w(\tilde{H}_{AS})$, the output of the water-filling algorithm with $\tilde{H}_{AS}$ as the selected effective channel matrix with $\mathrm{tr}(Q_{AS}) = P$ so that the STPC is satisfied.

The receiver possesses the CSI which now belong to the composite manifold $\mathcal{S} \oplus \mathcal{P}(L_t, = P, \leq L_t)$, where $\mathcal{S}$ reflects all possible choices of L_t transmit antennas within N_t transmit antennas. For simplicity however, we do not jointly quantize over the manifold $\mathcal{S} \oplus \mathcal{P}(L_t, = P, \leq L_t)$. An element $x \in \mathcal{S}$ is represented by $\left\lceil \log_2 \binom{N_t}{L_t} \right\rceil$ bits, and the remaining $N_f^{\mathrm{eff}} \triangleq N_f - \left\lceil \log_2 \binom{N_t}{L_t} \right\rceil$ bits are used to quantize $Q_{AS} \in \mathcal{P}(L_t, = P, \leq L_t)$.

The rate achievable under perfect CSIT is given by $R_{\mathrm{fcsi}} \triangleq E_H \log \det (I + \tilde{H}_{AS} Q_{AS} \tilde{H}_{AS}^H)$. Since, evaluating even the p.d.f. of Q_{AS} is intractable, it is not surprising that the above expression has not been expressed so far in closed form. Under finite-rate feedback, we obtain an information rate of $R_{\mathrm{qcsi}} \triangleq E_H \log \det (I + \tilde{H}_{AS} q(Q_{AS}) \tilde{H}_{AS}^H)$, where $q(Q_{AS})$ is Q_{AS} quantized over a codebook of rate N_f^{eff} over the manifold $\mathcal{P}(L_t, = P, \leq L_t)$.

1) Example 8: In Fig. 9, we see that an antenna selection scheme choosing $L_t = 2$ antennas from $N_t = 4$ antennas in a $N_r = 2$ MIMO system, $\mathrm{tr}(Q_{\mathrm{opt}}) = 4$ dB system always outperforms a 2×2 system in spite of losing 3 bits in conveying the antenna choice. Again, the simulated values of R_{qcsi} show that the lower bound is conservative.

Other precoding strategies and antenna selection algorithms can be similarly handled. For example, it is possible, in principle, to incorporate the LTPC under the above antenna selection rule.

VI. CONCLUSION

The MIMO fast fading channel is studied under limited rate quantized covariance feedback, wherein the receiver computes the desired input covariance matrices and feeds back their quantized versions to the transmitter back using N_f bits per block. The impact that this quantized covariance has on the achievable rate of the system is quantified. The input covariance matrices are located on manifolds comprising non-negative definite matrices classified further by the rank and trace constraints imposed on them. Using previous results by the authors on the maximum distortion of sphere packing codes and expected distortion due to quantization by random codebooks, the difference in information rate relative to perfect CSIT (including capacity) is analyzed, and is shown to be bounded by a constant times $2^{-\frac{N_f}{N}}$, where N is the dimension of the manifold under consideration. By an appropriate choice of the quantization manifold which determines its dimension and hence the rate at which the achievable rate under quantized covariance feedback approaches the CSIT ideal, the ergodic capacity of the MIMO channel can be well-approximated for a wide range of values of N_f. Improving accuracy of communication rate estimate under quantized covariance feedback at small N_f and the design of structured/efficient codebooks for quantizing positive semi-definite matrices are interesting problems for future research.

ACKNOWLEDGMENT

The authors would like to thank Kaniska Mohanty for performing some of the simulations in this paper and for a discussion about the proof of Theorem 5 in [11].

REFERENCES

[1] D. J. Love, R. W. Heath Jr., V. K. N. Lau, D. Gesbert, B. D. Rao, and M. Andrews, "An overview of limited feedback in wireless communication systems," *IEEE J. Sel. Areas Commun.*, vol. 26, no. 8, pp. 1341–1365, Oct. 2008.

[2] K. K. Mukkavilli, A. Sabharwal, E. Erkip, and B. Aazhang, "On beamforming with finte-rate feedback in multiple-antenna systems," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2562–2579, Oct. 2003.

[3] D. J. Love, R. W. Heath Jr., and T. Strohmer, "Grassmannian beamforming for multiple-input multiple-output wireless systems," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2735–2747, Oct. 2003.

[4] R. W. Heath Jr. and D. J. Love, "Multimode antenna selection for spatial multiplexing systems with linear receivers," *IEEE Trans. Signal Process.*, vol. 53, no. 8, pp. 3042–3056, Aug. 2005.

[5] J. C. Rho and B. D. Rao, "Design and analysis of MIMO spatial multiplexing systems with quantized feedback," *IEEE Trans. Signal Process.*, vol. 54, no. 8, pp. 4730–4740, Dec. 2006.

[6] W. Dai, Y. Liu, and B. Rider, "Quantization bounds on Grassmann manifolds and applications to MIMO communications," *IEEE Trans. Inf. Theory*, vol. 54, no. 3, pp. 1108–1123, Mar. 2008.

[7] W. Dai, Y. Liu, V. K. N. Lau, and B. Rider, "On the information rate of MIMO systems with finite rate channel state feedback using power on/off strategy," *IEEE Trans. Inf. Theory*, vol. 55, no. 11, pp. 5032–5047, Nov. 2009.

[8] W. Santipach and M. L. Honig, "Capacity of a multiple-antenna fading channel with a quantized precoding matrix," *IEEE Trans. Inf. Theory*, vol. 55, no. 3, pp. 1218–1234, Mar. 2009.

[9] A. Grant, "Capacity of ergodic MIMO channels with complete transmitter channel knowledge," in *Proc. 6th Australian Commun. Theory Workshop*, Queensland, Australia, 2005.

[10] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *Eur. Trans. Telecommun.*, vol. 10, no. 6, pp. 585–595, 1999.

[11] R. T. Krishnamachari and M. K. Varanasi, "On the geometry and quantization of manifolds of positive semi-definite matrices," *IEEE Trans. Signal Process.*, Aug. 2013, to be published.

[12] A. D. Dabbagh and D. J. Love, "Feedback rate – capacity loss tradeoff for limited feedback MIMO systems," *IEEE Trans. Inf. Theory*, vol. 52, no. 5, pp. 2190–2202, May 2006.

[13] R. T. Krishnamachari and M. K. Varanasi, "Finite-rate feedback of input covariance matrices in MIMO systems," presented at the IEEE Int. Symp. Inf. Theory, Austin, Texas, USA, 2010.

[14] R. T. Krishnamachari and M. K. Varanasi, "MIMO performance under covariance matrix feedback," presented at the IEEE Int. Symp. Inf. Theory, St. Petersburg, Russia, 2011.

[15] V. Lau, Y. Liu, and T. A. Chen, "On the design of MIMO block-fading channels with feedback-link capacity constraint," *IEEE Trans. Commun.*, vol. 52, no. 1, pp. 62–70, Jan. 2004.

[16] J. C. Roh and B. D. Rao, "Multiple antenna channels with partial channel state information at the transmitter," *IEEE Trans. Wireless Commun.*, vol. 3, no. 2, pp. 677–688, Mar. 2004.

[17] B. D. Rao, S. Ghosh, and J. R. Zeidler, "Techniques for MIMO channel covariance matrix quantization," *IEEE Trans. Signal Process.*, vol. 60, no. 6, pp. 3340–3345, Jun. 2012.

[18] D. Sacristan-Murga and A. Pascual-Iserte, "Differential feedback of MIMO channel Gram matrices based on geodesic curves," *IEEE Trans. Wireless Commun.*, vol. 9, no. 12, pp. 3714–3727, Dec. 2010.

[19] Y. Jiang and M. K. Varanasi, "The RF-chain limited MIMO system-Part I: Optimum diversity-multiplexing tradeoff," *IEEE Trans. Wireless Commun.*, vol. 8, no. 10, pp. 5238–5247, Oct. 2009.

[20] A. M. Mathai, *Jacobians of Matrix Transformations and Functions of Matrix Argument*. Singapore: World Scientific, 1997.

[21] D. N. C. Tse and P. Viswanath, *Fundamentals of Wireless Communication*. Cambridge, U.K.: Cambridge University Press, 2005.

[22] S. Mukherjee, Q. Wu, and D. X. Zhou, "Learning gradients on manifolds," *Bernoulli*, vol. 16, no. 1, pp. 181–207, Feb. 2010.

Rajesh T. Krishnamachari received his B. Tech. degree in electrical engineering from the Indian Institute of Technology, Madras, India, in 2003 and the M.S. and Ph.D. degrees from the Electrical, Computer and Energy Engineering Department of the University of Colorado, Boulder, CO in 2008 and 2011, respectively. He was an R&D Engineer between 2003–2005 in the telecommunications industry. Since 2012, he has been employed in the financial services industry in New York, NY.

Mahesh K. Varanasi (S'87–M'89–SM'95–F'10) received the Ph.D. degree from the Electrical and Computer Engineering Department, Rice University, Houston, TX, in 1989.

He joined what is now the Electrical, Computer and Energy Engineering Department of the University of Colorado at Boulder as an Assistant Professor in 1989. In the same department, he was an Associate Professor between 1996–2001 and has been a full Professor since 2001. He is an affiliated faculty member of the Departments of Mathematics and Applied Mathematics at the University of Colorado and the Department of Computer Science at the King Abdulaziz University in Jeddah, KSA. His research and teaching interests are in the areas of network communication and information theory, wireless communications and coding, detection and estimation theory, and signal processing. He has published on a variety of topics in these fields and is a Highly Cited Researcher in the "Computer Science" category according to the ISI Web of Science. He also served as an Editor for the IEEE Transactions on Wireless Communications during 2007–2009.

Kaniska Mohanty, photograph and biography not available at the time of publication.