2.1 Combinatorics Applied Structures

3) 4 shirts sizes, 6 shirt colors, 3 emblem choices

{XL, L, M, S} * {red, blue, green, white, orange, yellow} * {dragon, rone, dog}

(4)(6)(3) = 72 different combinations of shirts.

5) 6 officers, 2 rows, 3 people different arrangements

$6! = 6 * 5 * 4 * 3 * 2 * 1 = 720$

7) 2 blouses, $\overbrace{\text{2 pants, 1 skirt}}^{\text{3 bottoms}}$, 1 blazer

$2 \cdot 3 \cdot 1 = 6$ different outfits

Blazer optional $\overbrace{(\text{yes, no})}^{2}$

$2 \cdot 3 \cdot 2 = 12$ different outfits

1 sweater $2 \cdot 3 \cdot 2 \cdot 1 = 12$ different outfits

Sweater optional $\overbrace{(\text{yes, no})}^{2}$

$2 \cdot 3 \cdot 2 \cdot 2 = 24$ different outfits

9) a) 2 choices {0, 1} 8 bits/spots

$2^8 = 256$ different bit patterns

b) first 4 = last 4

$2^4 = 16$

c) half even → half odd

$\frac{2^8}{2} = 128$

11) a) A = {1, 2, 3, 4} $P(A) = 2^4 = 16$ diffrent subsetes

b) A = {1, 2, 3, 4, 5} proper subsets of A $2^5 - 1 = 31$

13) $\underset{1}{A}\ \underset{2}{B}\ \underset{3}{C}$ (1) $\underset{1}{C}\ \underset{2}{A}\ \underset{3}{B}$ (2) $\underset{1}{B}\ \underset{2}{C}\ \underset{3}{A}$ (3) 3 different way's

15) {fish, lamb, beef}, {peas, carrots}, {pie, icecream, cake}
$3 * 2 * 3 = 18$ diffent dinners

2.2

1) $1000 \cdot 999 \cdot 998 = 997{,}002{,}000$ different way's to distribute
or $P(1000,3) = \frac{1000!}{(1000-3)!} = \frac{1000 \cdot 999 \cdot 998 \cdot \cancel{997!}}{\cancel{997!}} = 997{,}002{,}000$

5) $15! = 15 \cdot 14 \cdot \ldots \cdot 1 = 1{,}307{,}674{,}000{,}000$
or $P(15,15) = \frac{15!}{(15-15)!} = \frac{15!}{1}$

7) a) $P(15,5) = \frac{15!}{(15-5)!} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot \cancel{10!}}{\cancel{10!}} = 360{,}360$

b) one spot, second spot ...
$15 \cdot 14 \cdot 13 \cdot 12 \cdot 11$
two players for one spot $\rightarrow 2 \cdot 14 \cdot 13 \cdot 12 \cdot 11 = 48{,}048$

2.3

1) $A = \{a, b, c\}$ Partition
$\{\{a\}, \{b\}, \{c\}\}$, $\{\{a,b\}, \{c\}\}$, $\{\{a\}, \{b,c\}\}$, $\{\{a,c\}, \{b\}\}$
$\{\{a,b,c\}\}$

2.4

1) 3 faculty 4 students 10 faculty 25 students

$\binom{10}{3} \cdot \binom{25}{4} = \frac{10!}{(10-3)!\,3!} \cdot \frac{25!}{(25-4)!\,4!}$

$= \frac{10!}{7! \cdot 3!} \cdot \frac{25!}{21! \cdot 4!} = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7}\,3!} \cdot \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot \cancel{21!}}{\cancel{21!} \cdot 4!}$

$= \frac{720}{6} \cdot \frac{303600}{24} = 1,518,000$

3) $\{1, 2, 3 .. 10\}$ How many subsets contain 7 elements

$\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} = \frac{10!}{(10-7)!\,7!} + \frac{10!}{(10-8)!\,8!} + \frac{10!}{(10-9)!\,9!} + \frac{10!}{(10-10)!\,10!}$

$= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{3!\,\cancel{7!}} + \frac{10 \cdot 9 \cdot \cancel{8!}}{2!\,\cancel{8!}} + \frac{10 \cdot \cancel{9!}}{1!\,\cancel{9!}} + \frac{10!}{10!} = \frac{720}{6} + \frac{90}{2} + 10 + 1 = 176$

7) 52 cards each player gets 5 cards

a) How many hands possible?

combination of 5 cards are chose from 52 cards

$\binom{52}{5} = \frac{52!}{(52-5!)\,5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{\cancel{47!} \cdot 5!} = 2,598,960$

b) 1st player 2nd 3rd 4th

$\binom{52}{5} \cdot \binom{47}{5} \cdot \binom{42}{5} \cdot \binom{37}{5} = 2,598,960 \cdot \frac{47!}{(47-5)!} \cdot \frac{42!}{(42-5)5!} \cdot \frac{37}{(37-5)!\,5!}$

$= 2,598,960 + 1,533,939 + 850,668 + 435,897 = 5,419,464$

9) 5 hand cards 52 cards 4 aces in 2 aces in 5 hand card

$\binom{4}{2} \cdot \binom{48}{3} = \frac{4!}{(4-2)!2!} \cdot \frac{48!}{(48-3)!3!} = 6 \cdot \frac{103776}{6} = 103,776$

11) $\binom{12}{5} \cdot \binom{7}{4} \cdot \binom{3}{3}$

group of 5 from 12 group of 4 from 7 group of 3 from 3

$\frac{12!}{(12-5)!5!} \cdot \frac{7!}{(7-4)!4!} \cdot \frac{3!}{(3-3)!3!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!} \cdot 5!} \cdot \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{3!\,\cancel{4!}} \cdot \frac{\cancel{3!}}{\cancel{3!}}$

$792 \cdot 35 \cdot 1 = 27,720$

Book of Proof

3.2

1) (T, H, E, O, R, Y) 6 Elements repitition allowed

a) length 4 list How many?

$6 \cdot 6 \cdot 6 \cdot 6 = 1296$

b) length 4 list begin with T

$\overset{T}{1} \cdot 6 \cdot 6 \cdot 6 = 216$

c) length 4 list dont begin with T

$5 \cdot 6 \cdot 6 \cdot 6 = 1080$

3) (A, B, C, D, E, F) length 3 list rep allowed from 6

$6 \cdot 6 \cdot 6 = 216$

b) rep not allowed $6 \cdot 5 \cdot 4 = 120$

c) rep not allowed contain A $(A \cdot 5 \cdot 4) + (5 \cdot A \cdot 4) + (5 \cdot 4 \cdot A) = 60$

d) rep allowed contain A $6 \cdot 6 \cdot 6 - 5 \cdot 5 \cdot 5 = 91$

8 bits 2 choices

5) a) $2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2 = 2^8 = 256$ reps allowed

b) $2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot1 = 2^7 = 128$ end in zero

c) $2\cdot1\cdot2\cdot1\cdot2\cdot2\cdot2\cdot2 = 2^6 = 64$ second and fourth = 1

d) $2^6 + 2^6 + 2^6 = 192$

$(n\cdot1\cdot n\cdot0\cdot n\cdot n\cdot n\cdot n)$ $(n\cdot0\cdot n\cdot1\cdot n\cdot n\cdot n\cdot n)$ $(n\cdot1\cdot n\cdot1\cdot n\cdot n\cdot n\cdot n)$

7) 4 letter codes from $\overbrace{ABCD\ldots Z}^{26}$

$26^4 = 456{,}976$

4 letter codes no 2 consecutive letters

$\underset{\text{any}}{26}\cdot\underset{\text{can't be}}{25}\cdot\underset{\text{can't be}}{25}\cdot\underset{\text{can't be}}{25} = 406{,}250$

9) 5 colors • 3 engines • 2 transmissions = 30 different combinations

3.3

1) 52 cards Five hands 1 red card in five hands

$52\cdot51\cdot50\cdot49\cdot48 = 311{,}875{,}200$ Possible 5 hand combinations

26 red cards 26 black cards

$26\cdot25\cdot24\cdot23\cdot22 = 7{,}893{,}600$ black cards

$311{,}875{,}200 - 7{,}893{,}600 = \boxed{303{,}981{,}600}$

All black or All Hearts

$26\cdot25\cdot24\cdot23\cdot22 + 13\cdot12\cdot11\cdot10\cdot9 = \boxed{8{,}048{,}040}$

All black   All red

3) $(26\cdot25\cdot24\cdot23\cdot22) + (26\cdot25\cdot24\cdot23\cdot22) = 15{,}787{,}200$

5) 1 - 9999 no rep's

one digit #'s   2 digit #'s   3 digit #'s   4 digit #'s

$9 + 9\cdot9 + 9\cdot9\cdot8 + 9\cdot9\cdot8\cdot7$

$= 5274$

one repeated digit

$9999 - 5274 = 4725$

total # of integers   total # of no repeated

7) five long   26 letters   1 upper case

upper case and lower case

$52\cdot52\cdot52\cdot52\cdot52 = 380{,}204{,}032$

all lower case

$26\cdot26\cdot26\cdot26\cdot26 = 11{,}881{,}376$

$380{,}204{,}032 - 11{,}881{,}376 = \boxed{368{,}322{,}656}$

all combination's - all upper/lower case combinations

must mix $52^5 - 26^5 - 26^5 = \boxed{356{,}441{,}280}$

9) list of 6 no rep's A, B, C, D, E, F, G, H

AE**** + *AE*** + **AE** + ***AE*

+ ****AE $= (6\cdot5\cdot4\cdot3\cdot2\cdot1)\,5 = 3600$

11) integers divisible by 5 1-1000

1000/5 = 200 integers   1000-200 = 800 not divisible by 5

3.4

1) smallest n for n! has 10 digits

$13! = 13 \cdot 12 \cdots 1 = 6{,}227{,}020{,}800$

3) 5 digit positive, odd integers no rep

(1, 3, 5, 7, 9) $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$

5) $\frac{120!}{118!} = \frac{120 \cdot 119 \cdot \cancel{118!}}{\cancel{118!}} = 120 \cdot 119 = 14{,}280$

7) How many 9 digit #'s can be made 1-9

all odd start on the left no rep.

$\underbrace{1\,3\,5\,7\,9}_{odd}\,\underbrace{8\,2\,6\,4}_{even}$

$5!\,4! = (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)(4 \cdot 3 \cdot 2 \cdot 1) = 2880$

9) A B C D E F G permutations? ABC consecutively and order

$\boxed{ABC} * * * * = 5! = 120$

11) 7 hands 52 cards How many are not all red cards

$P(52,7) = \frac{52!}{(52-7)!} = 674{,}274{,}182{,}400$

$P(26,7) = \frac{26!}{(26-7)!} = 3{,}315{,}312{,}000$

$P(52,7) - P(26,7) = \boxed{670{,}958{,}870{,}400}$

13) 6 list no rep 26 letters

$P(26,6) = \frac{26!}{(26-6)!} = \frac{26\cdot25\cdot24\cdot23\cdot22\cdot21\cdot\cancel{20!}}{\cancel{20!}} = 165,765,600$

15) 15 people 4 list different ways

$P(15,4) = \frac{15!}{(15!-4)!} = \frac{15\cdot14\cdot13\cdot12\cdot\cancel{11!}}{\cancel{11!}} = 32,760$

17) 10 people 3 list different ways

$P(10,3) = \frac{10!}{(10-3)!} = \frac{10\cdot9\cdot8\cdot\cancel{7!}}{\cancel{7!}} = 720$

3.5

1) A has 37 elements How many subsets have 10 elements

$\binom{37}{10} = \frac{37!}{(37!-10)!\,10!} = \frac{37\cdot36\cdots\cancel{27!}}{\cancel{27!}\,10!} = 348,330,136$

subsets 30 elements

$\binom{37}{30} = \frac{37\cdot36\cdots\cancel{30!}}{(37-30)!\,\cancel{30!}} = \frac{37\cdot36\cdot35\cdot34\cdot33\cdot32\cdot31}{7!} = 10,295,472$

subsets 0 elements

$\binom{37}{0} = \frac{37!}{(37-0)!\,0!} = \frac{37!}{37!\cdot1} = 1$

3) 56 subsets with 3 elements $|X| = ?$

$\binom{z}{3} = 56$ $\cancel{\frac{6!}{(6-3)!\,3!} = \frac{6\cdot5\cdot4\cdot3!}{3!\,3!} = 20}$

$\frac{8!}{(8-3)!\,3!} = \frac{8\cdot7\cdot6\cdot\cancel{5!}}{\cancel{5!}\,3!} = 56$ $z=8$ $\binom{8}{3} = 56$ $|X| = 8$

5) 16 digit choose seven 1's

$\binom{16}{7} = \frac{16!}{(16-7)!7!} = \frac{16 \cdot 15 \ldots \cancel{9!}}{\cancel{9!}\, 7!} = 11{,}440$

7) $|\{X \in P(\{0,1,2,3,4 \ldots 9\}) : |X| < 4\}|$

$= \binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} = \frac{10!}{(10-0)!0!} + \frac{10!}{(10-1)!1!} + \frac{10!}{(10-2)!2!} + \frac{10!}{(10-3)!3!}$

$= 1 + 10 + 45 + 120 = 176$

9) 6 list ABCDEF no rep, D occurs before A

D A * * * * $\binom{6}{2} \cdot 4! = \frac{6!}{(6-2)!2!} \cdot (4 \cdot 3 \cdot 2 \cdot 1) = 360$

11) 10 digits integers no 0's three 6's

$\binom{10}{3} \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 = \frac{10!}{(10-3)!3!} \cdot 8^7 = 251{,}658{,}240$

13) $n, k \in \mathbb{Z}$ $0 \leq k \leq n$ $\binom{n}{k} = \binom{n}{n-k}$

$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ $\binom{n}{n-k} = \frac{n!}{(n-(n-k))!(n-k)!}$

$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-(n-k))!(n-k)!} = \binom{n}{n-k}$

15) 10 digits no # 1, fours

total 10 digits binaries

$2^{10} = 1024$

total of 10 digits with four 1's

$\binom{10}{4} = \frac{10!}{(10-4)!4!} = 210$

$1024 - 210 = 814$

17) 10 digit binary (four 1's or five 1's) or (not have)

have $\binom{10}{4} + \binom{10}{5} = \frac{10!}{(10-4)!4!} + \frac{10!}{(10-5)!5!} = 210 + 252 = 462$

not have $2^{10} - \binom{10}{4} - \binom{10}{5} = 1024 - 462 = 562$

19) 5 hand card 52 deck 4 suit each 13 cards

$\underset{\text{hearts}}{\binom{13}{5}} + \underset{\text{aces}}{\binom{13}{5}} + \binom{13}{5} + \binom{13}{5} = (1287)4 = 5148$

3.6

1) Row 11 Pascals triangle

$(x+y)^{11} = \binom{11}{0}\binom{11}{1}\binom{11}{2}\binom{11}{3}\binom{11}{4}\dots\binom{11}{11}$

$= 1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1$

3) coefficent x^8 in $(x+2)^{13}$

$(x+y)^{13} \to x^8y^5 = \binom{13}{5}x^8y^5 = 1287x^8y^5$

$y=2 \Rightarrow 1287x^8(2)^5 = 41{,}184x^8$

5) $\sum_{k=0}^{n}\binom{n}{k} = 2^n \qquad 2^n = (1+1)^n$

$= \binom{n}{0}1^n1^0 + \binom{n}{1}1^{n-1}1^1 + \dots + \binom{n}{n-1}1^1 1^{n-1} + \binom{n}{n}1^0 1^n$

$\Rightarrow \sum_{k=0}^{n}\binom{n}{k} = 2^n$

7) $\sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n \qquad 4^n = (1+3)^n$

$$= \binom{n}{0} 1^n 3^0 + \binom{n}{1} 1^{n-1} 3^1 \ldots \binom{n}{n} 1^0 3^n$$

$$= \sum_{k=0}^{n} 3^k \binom{n}{k}$$

9) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots \pm \binom{n}{n} = 0$

$$0^n = (1+(-1))^n$$

$$= \binom{n}{0} 1^n (-1)^0 + \binom{n}{1} 1^{n-1} (-1)^1 \ldots + \binom{n}{n} 1^0 (-1)^n$$

$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \ldots \pm \binom{n}{n} = 0$$

11) $9^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (10^{n-k})$

$$9^n = (10 + (-1))^n$$

$$= \binom{n}{0} 10^n (-1)^0 + \binom{n}{1} 10^{n-1} (-1)^1 + \ldots \binom{n}{n} 10^0 (-1)^n$$

13) $\binom{n}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{n-1}{2}$

$$n \geq 3 \quad \binom{n}{3} = \binom{n-1}{3}\binom{n-1}{2} = \binom{n-2}{3}\binom{n-2}{2}\binom{n-1}{2} = \binom{n-3}{3}\binom{n-3}{2}\binom{n-2}{2}\binom{n-1}{2}$$

$$= \ldots = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} \ldots + \binom{n-1}{2}$$

3.7

1) $|A| = 100$ senior mm $\quad |B| =$ senior hm $\quad |A \cap B| = 33$ senior hm/mm

$|A \cup B| = 523$ seniors

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$523 = 100 + |B| - 33$$

$$|B| = 523 + 33 - 100 = 456$$

3) 4 digit positive integers even or zeros

$A =$ even $\qquad B =$ no zero's

0 000 $\qquad |A| = 9 \cdot 10 \cdot 10 \cdot 5 = 4500$

1 to 9 0-9 0-9 0,2,4,6,8

0 0 0 0

1 to 9 1 to 9 1-9 1,3,5,7,9 $\qquad |B| = 9 \cdot 9 \cdot 9 \cdot 5 = 3645$

$|A \cup B| = |A| + |B| = 8145$

5) 7 digits begin 1 or end 1 or four 1's

$2^6 + 2^6 + \binom{7}{4} - 2^5 - \binom{6}{3} - \binom{6}{3} + \binom{5}{2} = 101$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

7) 4 hands 52 card deck

all 4 same suit or all red.

$|A| = \binom{13}{4} 4 \qquad |B| = \binom{26}{4} \qquad |A \cap B| = 2\binom{13}{4}$

$|A \cup B| = |A| + |B| - |A \cap B| = 4\binom{13}{4} + \binom{26}{4} - 2\binom{13}{4}$

$= 16,380$

9) 4 list LISTED rep allowed all vowels or ends D

$|A| = 2 \cdot 2 \cdot 6 \cdot 6 = 144 \qquad |B| = 6 \cdot 6 \cdot 6 \cdot 1 = 216$

$|A \cap B| = 2 \cdot 2 \cdot 6 \cdot 1 = 24$

$|A \cup B| = 144 + 216 - 24 = 336$

11) 7 digits even or three 0's

$|A| = 9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 5 = 4{,}500{,}000$

$|B| = 9 \cdot \binom{6}{3} \cdot 9 \cdot 9 \cdot 9 = 131{,}220$

$|A \cap B| = 9 \cdot \binom{5}{3} \cdot 9 \cdot 9 \cdot 4 + 9 \cdot \binom{5}{2} \cdot 9 \cdot 9 \cdot 9$

$|A \cup B| = 4{,}500{,}000 + 131{,}220 - 29160 - 65610$

$= 4{,}536{,}450$

13) 8 digits end 1 or four 1's

$|A| = 2^7 = 128$ $\quad |B| = \binom{8}{4} = 70$

$|A \cap B| = \binom{7}{4} = 35$

$|A \cup B| = 128 + 70 - 35 = 163$

15) 10 digit begin in 1 or end 1

$|A| = 2^9$ $\quad |B| = 2^9$

$|A \cap B| = 2^8$

$|A \cup B| = 2^9 + 2^9 - 2^8 = 768$