A Kalman construction on Tense ICRDL-algebras

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Abstract

In this paper we introduce and study an alternative definition of tense operators on residuated lattices. We give a categorical equivalence for the class of tense residuated lattices, which is motivated by an old construction due to J. Kalman. The paper concludes with some applications regarding the description of congruences and a 2-contextual translation.

Classical tense logic is a logical system obtained from introducing within the classical propositional logic the notion of time, i.e., an expansion of propositional logic by new unary operators which are called tense operators. It is customary to denote these operators by G, H, F and P, and usually we define F and P via G and H as $F(x) = \neg G(\neg x)$ and $P(x) = \neg H(\neg x)$, where $\neg x$ denote the Boolean negation of x. It is well known that the class of tense Boolean algebras provides an algebraic semantics for classical tense logic. Afterwards, tense operators in intuitionistic logic were introduced by Ewald and the corresponding intuitionistic tense logical system, called IKt was established. In [1], Figallo and Pelaitay gave an algebraic axiomatization of the IKt system and showed that the algebraic axiomatization given by Chajda in [2] of the tense operators P and F in intuitionistic logic is not in accordance with the Halmos approach to existential operators. Thereafter, the study of tense operators has been extended to different algebraic structures associated with non-classical logics [3]. This can be evidenced by the approach applied in [1, 4], and also in [6], in where a Kalman's construction for the class of tense distributive lattices with implication was studied.

Definition 1. Let L be a integral commutative residuated distributive lattice (ICRDL-algebra, for short). Let G, H, F and P be unary operations on L satisfying:

(T0) $G(0) = 0$ and $H(0) = 0$,

(T1) $P(x) \leq y$ if and only if $x \leq G(y)$,

(T2) $F(x) \leq y$ if and only if $x \leq H(y)$,

(T3) $G(x) \wedge F(y) \leq F(x \wedge y)$ and $H(x) \wedge P(y) \leq P(x \wedge y)$,

(T4) $G(x) \cdot F(y) \leq F(x \cdot y)$ and $H(x) \cdot P(y) \leq P(x \cdot y)$,

(T5) $G(x \vee y) \leq G(x) \vee F(y)$ and $H(x \vee y) \leq H(x) \vee P(y)$,

(T6) $G(x \to y) \leq G(x) \to G(y)$ and $H(x \to y) \leq H(x) \to H(y)$.

An algebra $\mathbf{L} = \langle L, G, H, F, P\rangle$ will be called *tense ICRDL-algebra* and G,H, F and P will be called *tense operators*.

Definition 2. A structure $\langle A, \vee, \wedge, *, \sim, c, 0, 1\rangle$ is a *centered integral involutive residuated lattice* iff it satisfies the following conditions:

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1. $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,
2. $\langle A, *, 1\rangle$ is a commutative monoid,
3. $\sim$ is an involution of the lattice that is a dual automorphism,
4. c is a fixed point of the involution,
5. $x * y \leq z$ iff $x \leq \sim (y * (\sim z))$, for all $x, y, z \in A$.

Moreover, $\langle A, \vee, \wedge, *, \sim, c, 0, 1\rangle$ is said to be a *c-differential residuated lattice*, or *DRL-algebra* for short, if the following equation, called *Leibniz condition*, holds:

(LC) $(x * y) \wedge c = ((x \wedge c) * y) \vee (x * (y \wedge c))$.

Definition 3. An algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \sim, G, H, c, 0, 1\rangle$ is said to be a *tense DRL-algebra* if $\langle A, \vee, \wedge, *, \sim, c, 0, 1\rangle$ is a DRL-algebra and G and H are unary operations on A satisfying:

(t0) $G(1) = 1$ and $H(1) = 1$,

(t1) $G(c) = c$ and $H(c) = c$,

(t2) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$,

(t3) $x \leq GP(x)$ and $x \leq HF(x)$,

(t4) $G(x \vee y) \leq G(x) \vee F(y)$ and $H(x \vee y) \leq H(x) \vee P(y)$,

(t5) $G(x \Rightarrow y) \leq G(x) \Rightarrow G(y)$ and $H(x \Rightarrow y) \leq H(x) \Rightarrow H(y)$.

where F and P are defined by $F(x) :=\sim G(\sim x)$ and $P(x) :=\sim H(\sim x)$, for any $x \in A$; and $x \Rightarrow y := (x * (\sim y))$.

Let us write tICRDL for the category whose objects are tense ICRDL-algebras, and $\mathsf{tDRL_c}$ for the category whose objects are tense DRL-algebras such that for every $x, y \geq c$ such that $x \cdot y \leq c$, there exists $z \in A$ such that $z \vee c = x$ and $\sim z \vee c = y$. In both cases, the morphisms are the corresponding algebra homomorphisms. In this talk we extend the Kalman construction presented in [4] in order to establish the following result:

Theorem. The categories tICRDL and $\mathsf{tDRL_c}$ are equivalent.

The applications are twofold: algebraically, we note a correlation between tense filters in tense ICRDL-algebras and tense DRL-algebras. Logically, leveraging categorical equivalence and [5], we establish a finite 2-contextual translation between equational consequence relations in tense DRL-algebras and tense ICRDL-algebras.

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