A Calculus for S^3-diagrams of Manifolds with Boundary*

Bojana Femić[1], Vladimir Grujić[2], Jovana Obradović[3], and Zoran Petrić[4]

[1] Mathematical Institute SANU, Belgrade, Serbia
femicenelsur@gmail.com
[2] Faculty of Mathematics, University of Belgrade, Belgrade, Serbia
vgrujic@matf.bg.ac.rs
[3] Mathematical Institute SANU, Belgrade, Serbia
jovana@mi.sanu.ac.rs
[4] Mathematical Institute SANU, Belgrade, Serbia
zpetric@mi.sanu.ac.rs

The aim of this paper is to introduce a calculus for a presentation of compact, orientable, connected 3-manifolds with boundary in terms of diagrams embedded in S^3 in a form akin to the standard surgery presentation of closed, orientable, connected 3-manifolds. Our motivation to introduce such a presentation of manifolds is to give a completely combinatorial description of the category 3Cob, whose arrows are 3-dimensional cobordisms, which is an ongoing project. We hope this could support further investigations of faithfulness of 3-dimensional Topological Quantum Field Theories. On the other hand, the calculus could be applied within some coherence results in Category theory (see [1, Section 9]).

That every closed, orientable, connected 3-manifold may be obtained by surgery on a link in S^3 was proved by Wallace, [9] and (independently) by Lickorish, [4]. This result provides a language for presentation of such manifolds. The rational surgery calculus for this language was introduced by Rolfsen, [7]. This calculus consists of two types of modifications and he proved that two surgery descriptions yield homeomorphic 3-manifolds if one can be transformed into the other by a finite sequence of these modifications. At about the same time, Kirby, [3] introduced another surgery calculus and he proved its completeness, i.e. that two surgery descriptions (with integral framing) yield homeomorphic 3-manifolds if and only if one can be transformed into the other by a finite sequence of operations from this calculus. By relying on Kirby's result, Rolfsen, [8], proved the completeness of his calculus. Fenn and Rourke, [2], merged two Kirby's operations into an infinite list of integral moves of one type, which is a special case of Rolfsen's second modification. Roberts, [6], developed a calculus for surgery data in arbitrary compact, connected 3-manifold (possibly with boundary, or non-orientable).

The first part of our work uses a generalization of Wallace-Lickorish result to compact, orientable, connected 3-manifolds with boundary in order to establish a diagrammatic language of these manifolds. This language is based on the well known language for closed manifolds that consists of surgery data in S^3 written in terms of framed links. We extend the "alphabet" by introducing some rigid "symbols" in the form of wedges of circles. The intuition behind a wedge of circles in a diagram is that its neighbourhood is removed from S^3 forming one component of the boundary of the resulting manifold S^3_-. Given a framed link in S^3_-, the result of performing the surgery along that link (which consists in removing the tubular neighbourhood of every link component and sewing it back according to the corresponding framing) is a manifold whose boundary is canonically identified with the original $\partial(S^3_-)$. Hence, a diagram in S^3 consisting of a collection of wedges of circles, together with a framed link, presents a manifold. We call

*Bojana Femić, Jovana Obradović and Zoran Petrić were supported by the Science Fund of the Republic of Serbia, Grant No. 7749891, Graphical Languages - GWORDS. Vladimir Grujić was supported by the Science Fund of the Republic of Serbia, Grant No. 7744592, Integrability and Extremal Problems in Mechanics, Geometry and Combinatorics - MEGIC.

such a diagram a $\mathbb{Z}$-*diagram* (see Figure 1). We prove that *every* manifold is presentable by a $\mathbb{Z}$-diagram.

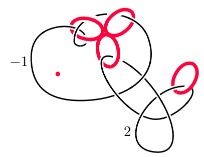

Figure 1: An example of a $\mathbb{Z}$-diagram. For the sake of better visualisation, we mark the wedges of circles in red.

The second part of our work adapts Roberts' calculus into a diagrammatic calculus adequate for our language. Our adaptation, shown in Figure 2, is akin to the adaptation of Kirby's calculus made by Fenn and Rourke.

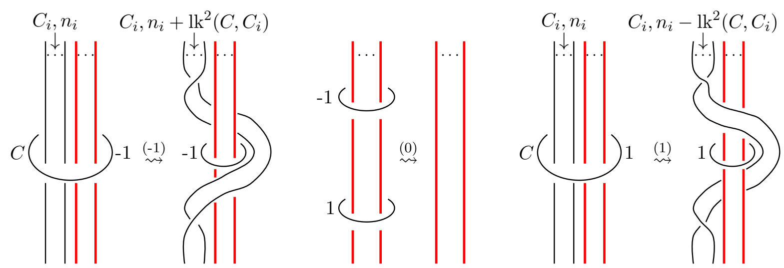

Figure 2: Moves (-1), (0) and (1)

As the Fenn and Rourke local moves can be reduced to a finite list, which is shown by Martelli, [5], our calculus is also presentable by a finite list of local moves. Finally, we develop a rational surgery calculus for our language. We prove the completeness for all the calculi.

References

[1] D. Đorđević, Z. Petrić and M. Zekić, *A diagrammatic calculus for categorical quantum protocols*, available at arXiv, 2022

[2] R.A. Fenn and C. Rourke, *On Kirby's calculus of links*, ***Topology***, vol. 75 (1979), pp. 1-15

[3] R. Kirby, *A calculus for framed links in* S^3, ***Inventiones Mathematicae***, vol. 45 (1978), pp. 35-56

[4] W.B.R. Lickorish, *A Representation of Orientable Combinatorial 3-Manifolds*, ***Annals of Mathematics***, vol. 76 (1962), pp. 531-540

[5] M. Martelli, *A finite set of local moves for Kirby calculus*, ***Journal of Knot Theory and its Ramifications***, vol. 21 1250126 (2012)

[6] J. Roberts, *Kirby calculus in manifolds with boundary*, ***Turkish Journal of Mathematics***, vol. 21 (1997), pp. 111-117

[7] D. Rolfsen, ***Knots and Links***, Publish or Perish, Berkeley, 1976

[8] ———, *Rational surgery calculus:extension of Kirby's theorem*, ***Pacific Journal of Mathematics***, vol. 110 (1984), pp. 377-386

[9] A.D. Wallace, *Modifications and cobounding manifolds*, ***Canadian Journal of Mathematics***, vol. 12 (1960), pp. 503-528