Persistent Homology on a lattice-ordered of multigraphs

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Abstract

A multicomplex structure is defined in an ordered lattice. This will help to observe the features of Persistent Homology in that context, its interaction with order and the repercussions of mixing multigraphs in the calculation of Betti numbers.

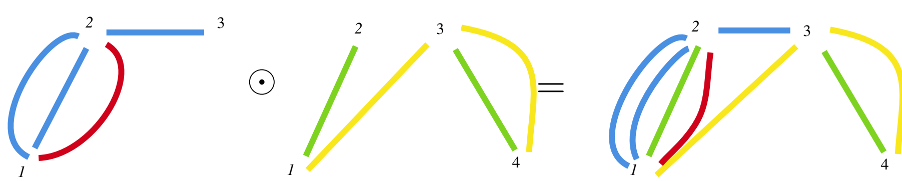
Our original interest on such structures is due to the fact that they provide an algebraic framework for an abstract notion of *embodiment* in Neuroscience by means of multigraphs with a partial structure developed in [1]. This structure opens the possibility to a dynamical behaviour, which needs a suitable setting for being studied in the line of TDA (Topological Data Analysis). We obviate the classical interaction of an static network and focus on the algebraic structure that we define and how it can change the network structure. The ideas we develop are mainly oriented to the original example described in [1].

Keywords: Multicomplex, Persistent Homology, Topological Data Analysis.

1 Introduction

A *multigraph* G on a set of *nodes* $V(G)$ is a multisubset of *edges* $E(G)$ that corresponds to pairs of elements of $V(G)$, together with the multiplicity function $m_G\colon E(G) \to \mathbb{Z}^+$. Similarly, the edges could have different colors. Let C be a finite set of colors, and $col_G\colon E(G) \to \mathscr{P}(C)$ a mapping that assigns to each edge a subset of colors. We will consider a multigraph as the pair (G, col_G) and, for $s \in \mathbb{N}$, we say that a multigraph is *s-colored* if col_G is onto and $s = |C|$, i.e. s denotes the number of colors included into the multigraph. Let the set of nodes indexed by the set $\{1,\ldots,n\}$ and denote by $MG(n)$ the set of multigraphs with such n nodes. Let c be a single color, then we denote by $MG^c(n)$ the set of 1-colored multigraphs with n nodes. Let us fix $C = \{c_1,\ldots,c_k\}$ as a set of colors, then we define the set of $|C|$-*colored multigraphs* with n nodes as $MG^C(n)$.

Example 1. *For $n = 3, m = 4, s = q = 2$ and $p = 3$:*



Let us consider a different non-commutative operation $\otimes$ such that

$$G \otimes H \odot K = G \otimes (H \odot K)$$

Now, having defined two different ways of composing multigraphs: $\otimes$ and $\odot$ we are considering sets MG^C of concatenations in the form $G_1 \oslash^1 \cdots \oslash^{k-1} G_k$ with $\oslash^i \in \{\otimes, \odot\}$ for $i = 1,\ldots,k-1$. The operation $\odot$ can be seen as an *accumulation of vertices and edges* of two given multigraphs that becomes a new multigraph with more colors than the original ones. For example, given $G \otimes H \otimes K, G \odot H \otimes K \in MG^C$, we understand that $G \odot H \otimes K$ is *over* $G \otimes H \otimes K$. By convention we say that $G \otimes H \otimes K \leq G \odot H \otimes K$,

TACL (eds.): easychair 1.0, 2024, volume 1, issue: 1, pp. 1-2

since we consider that $G \odot H$ is more complex, in some sense, than $G \otimes H$. We see that this partial order endows a set of multigraphs with a lattice structure.

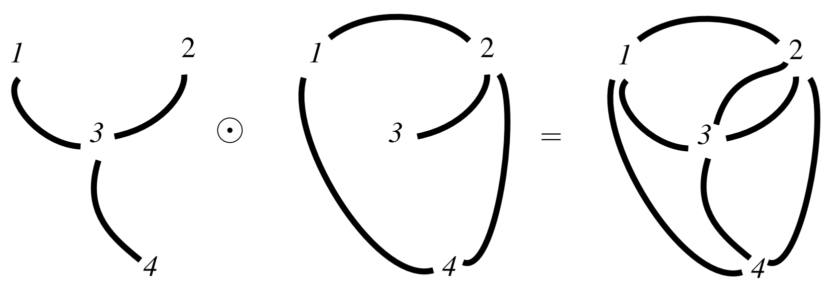
We now show how to define multicomplexes out of multigraphs. Our main references for this will be [2,3]. This idea is developed in [3], where the complexes are ordered and coloured in vertices. We will make use of the concepts introduced so far, the operation $\odot$ in particular to merge graphs, and will colour the edges rather than the vertices.

Definition 1. *Given a multigraph* (G, col_G) *its* clique multicomplex*, denoted by* $Cml(G)$*, has:*

- *the vertices of G as its vertices and*
- *the* $k-cliques$ *as its* $(k-1)-multicells.$

Now one can *close a multigraph* by means of the operation $\odot$ for the monochrome case.

Example 2. *In the following picture, four 2-cells and one 3-cell have been created:*



Finally, we colour the vertices in the usual manner: given a finite dimensional complex S and a set of colours $\mathscr{C}$ we colour a finite dimensional multicomplex by colouring the underlying complex. Once we have all the structure of coloured multicomplexes out of multigraphs, we define the operation $\odot$ for the simplices rather than the graphs. This entails as will be seen the definition of a tensor product for the *category of multicomplexes* and in fact a symmetric monoidal structure for it.

The idea behind this is to investigate how much the complex structure changes because of the merging process. In particular, try to know in which extent the dimension of the complex grows up through $\odot$. This is done by extending the $\odot$ operator for multisimplex, endow the category of multicomplexes with symmetric monoidal structure and calculate the Betti numbers for this setting.

2 References

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