TOPOLOGY, ALGEBRA, AND CATEGORIES IN LOGIC

Research Abstract

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TOPOS OF EXISTENTIAL GRAPHS OVER RIEMANN SURFACES

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This research may be situated in a framework of mathematical geometrization, or, more precisely, of logical geometrization. Peirce's Existential Graphs (EG) offer a profound topological version of logic, through the transformation of Jordan curves in the complex plane [8]. This approach has been extended in the last decade with the introduction of Existential Graphs on Non Planar Surfaces [6], particularly on the sphere, the torus, and the cylinder. A geometrical broadening of logical tools is thus in action, something which can be extended to Riemann Surfaces in general. On another hand, through a careful study of sheaves, Grothendieck Toposes (GT) have offered strong geometrical tools [3], useful also in the more general context of elementary Toposes (ET) [4]. The purpose of our contribution –*Topos of Existential Graphs over Riemann Surfaces*– consists in blending all these tools together, in order to enlarge our understanding of geometric tools in logic. In particular, many crucial properties of locality and globality, both in Riemann Surfaces and in Toposes, acquire a new logical flavor through our perspective.

Peirce's (EG) offer a presentation, entirely diagrammatic, of many logical levels [10]. With exactly the *same rules* (writing/erasure, iteration/deiteration, double cut) [9], (EG) provide sound and complete axiomatizations of the classical propositional calculus (Peirce), first order purely relational logic (Peirce) [11], modal logics (Zeman) [16], and intuitionistic intermediate calculi (Oostra) [7]. Peirce's rules, stated originally on the continuous (complex) plane, are nevertheless entirely topological in nature, as intuitionistic logic entails [13], something which opens the way to extend them to more general surfaces. Part of a program of logical geometrization looks then for transfers (translations) of purely syntactic-deductive rules into geometrical actions, something that the (EG) fulfill fully and faithfully (a full and faithful functor may be considered in this setting). These connections pave the way to explore new logics on given surfaces, and, inversely, to offer new topological models for known logics.

Riemann Surfaces (RS) were introduced by Riemann in his Doctoral Thesis (1851) as tools to allow global inverses for many-valued functions, but from 20th century perspectives (following Weyl) they can be understood just as global patchworks of local fragments of the complex plane [15]. The surfaces contain local neighborhoods isomorphic to disks on the plane [1], and, in the case of simply connected surfaces, through Riemann's Uniformization Theorem [12], they are associated to the main archetypes of 19th century geometry: parabolic geometry (plane, Euclidean model), hyperbolic geometry (disk, Poincaré's model), and elliptic geometry (sphere, Riemann's model) [5]. Our proposal opens the way to study (EG) on (RS), and introduces new definitions of *local and global logics*, which capture in part many typological distinctions between classicism, intuitionism, and paraconsistency. Further, one can study the new logics in function of the number of the new negations which appear on the surfaces, and relate them to the genus of the surfaces.

Beyond analytic continuation on the complex plane, sheaves (French school, mid 20th century [14]) offer a generalization of glueing and extending procedures on very abstract settings. A sheaf consists of two topological spaces and a projection between them, which is a local homeomorphism (the upper space can then be seen as an unfolding of the base space). The main problem consists in looking for transfers (and obstructions) between local sections and global sections (inverses of the projection), something which resonates naturally with our new definitions of local and global (EG) negations on (RS). A further, deep, step in this direction is provided by considering *all possible* sheaves on the surface. This forces the emergence of a Grothendieck Topos (GT) –and, with simplifications, an Elementary Topos (ET)- associated *logically* to the surface. Here, we profit from combinatorial, linear, descriptions of the planar (EG) provided by Gangle and his school [2], a situation that we generalize to the (RS) setting, topological and nonlinear.

The exactness properties of the toposes, and their intrinsic algebraic geometry (following Grothendieck [14]), may help to characterize the external logics by their internal properties. In particular, the studies (still work in progress) of the classifier object in an appropriate (ET), and the fundamental group in an appropriate (GT), may offer interesting specific clues along the general program of logical geometrization. Thanks to locality/globality and linearity/nonlinearity distinctions, the transit from (EG) on the plane, to (EG) on Riemann Surfaces, may be understood as a complex-differential-homological-logical transit. This is similar to the transit between the local/linear and the global/nonlinear procedures which emerge naturally in differential equations, complex variables, and sheaf theory, opening the way to many unexpected mathematical and logical connections.

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