PROFINITE QUASIVARIETIES OF ALGEBRAS

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A quasivariety $\mathfrak{N}$ is *profinite* if each profinite structure belonging to $\mathfrak{N}$ is isomorphic to an inverse limit of finite structures belonging to $\mathfrak{N}$.

The original motivation for this notion comes from the theory of natural dualities [1]. A natural duality establishes a dual equivalence between the quasivariety generated by a finite structure and a special dual category of Boolean topological structures (structure endowed by a compact, Hausdorff topology with a basis of clopen sets). The class of objects of this dual category coincides with the class $\mathbf{S}_c\mathbf{P}^+(A)$ of all homeomorphic copies of topologically closed substructures of nontrivial direct products of finite algebra A endowed with the product topology. It follows from general topological considerations that each member of $\mathbf{S}_c\mathbf{P}^+(A)$ is a Boolean topological structure. A quasivariety $\mathfrak{N}$ is *standard* if each Boolean topological structure with its algebraic reduct in $\mathfrak{N}$ belongs to $\mathbf{S}_c\mathbf{P}^+(\mathfrak{N}_{fin})$ where $\mathfrak{N}_{fin}$ denotes the class of finite structures from $\mathfrak{N}$. A finite structure A is standard if the quasivariety $\mathbf{SP}(A)$ is standard. It follows that each standard quasivariety is profinite.

The following problems were suggested in Clark et al. [2]; they are one of the key problems in the area of standard topological quasivarieties: 1) If a finite structure generates a standard quasivariety, must it be finitely based? 2) Which finite lattices generate a standard topological quasivariety? Nowadays there are many results concerning these problems. Actually, almost all of these results are about the standardness of varieties or the non-standardness (non-profiniteness) of proper quasivarieties (see for example [2, 3]). In particular, in [3] was proved that if a quasivariety $\mathfrak{N}$ is profinite, particularly standard, then $\mathfrak{N}$ has an independent basis of quasi-identities, that is, there is a set of quasi-identiteis Σ such that $\mathfrak{N} = \mathrm{Mod}(\Sigma)$ and $\mathfrak{N} \neq \mathrm{Mod}(\Sigma \backslash \{\varphi\})$ for every $\varphi \in \Sigma$. This allowed us to find wide classes of finite lattices and other algebras that generate non-standard quasivarieties.

The main goal of the talk is to discuss and present some results concerning the above mentioned problems. We provide the sufficient conditions for a quasivariety for being a profinite quasivariety in terms of the finite height of congruences. The definition of quasivariety with a finite height of congruences is quite technical and lengthy. Hence, we present one stronger sufficient condition here.

Let $\mathfrak{N}$ be a quasivariety. A congruence α on an algebra A is called an $\mathfrak{N}$-*congruence* or *relative congruence* provided $A/\alpha \in \mathfrak{N}$. The set $\mathrm{Con}_{\mathfrak{N}} A$ of all $\mathfrak{N}$-congruences of A forms an algebraic lattice with respect to inclusion $\subseteq$ which is called a *relative congruence lattice*. For a non-empty subset X of $A \times A$, $\theta_{\mathfrak{N}}(X)$ is the least $\mathfrak{N}$-congruence containing X. The least $\mathfrak{N}$-congruence $\theta_{\mathfrak{N}}(a, b)$ on algebra $A \in \mathfrak{N}$ containing the pair $(a, b) \in A \times A$ is called a *principal* $\mathfrak{N}$-*congruence* or a *relative principal congruence*. An existential primitive positive formula $\Sigma(x, y, u, v)$ is an $\mathfrak{N}$-congruence formula if $\mathfrak{N} \models (\forall xyz)\,[\Sigma(x, y, z, z) \to x \approx y]$. A quasivariety $\mathfrak{N}$ has *definable relative principal congruences* if there exists an $\mathfrak{N}$-congruence formula $\Sigma(x, y, u, v)$ such that $\theta_{\mathfrak{N}}(a, b) = \{(c, d) \in A \times A \mid A \models \Sigma(c, d, a, b)\}$ for all $A \in \mathfrak{N}$ and $a, b \in A$. A quasivariety $\mathfrak{N}$ is *relative congruence principal* if every compact $\mathfrak{N}$-congruence on an algebra A is a principal $\mathfrak{N}$-congruence for every $A \in \mathfrak{N}$. The quasivarieties with these properties are well represented in logics. Examples of this kind include the finitely generated quasivarieties of pointed Boolean algebras (Boolean algebras with additional constants in the signature), cylindric algebras of dimension n, Lukasiewicz algebras (MV-algebras), and many others.

Theorem 1. *Suppose a locally finite quasivariety* $\mathfrak{N}$ *of algebras has definable relative principal congruences and is relative congruence principal. Then* $\mathfrak{N}$ *is profinite.*

Thus, the finitely generated quasivarieties of the pointed Boolean algebras, cylindric algebras of dimension n, Lukasiewicz algebras (MV-algebras) are profinite. We summarise these results as follows. A ternary term $t(x, y, z)$ is called a discriminator term for an algebra A if the following universal sentences $(\forall xy)[t(x, x, y) = y]$ and $(\forall xyz)[x \neq y \to t(x, y, z) = x]$ are valid in A. A variety generated by a class of algebras with a common discriminator term is called a discriminator variety. We note that a quasivariety generated by a class of algebras with a common discriminator term is a variety.

Theorem 2. *A finitely generated subquasivariety of discriminator variety is profinite.*

At the end we note that a finitely generated subquasivariety of discriminator variety is finitely based and can be axiomatised by the identities and the quasi-identities of the form $(\forall\bar{x})\,[p_0(\bar{x}) \approx q_0(\bar{x}) \to p(\bar{x}) \approx q(\bar{x})]$ for some terms p_0, q_0, p, q.

References

[1] Clark, D.M., and B.A. Davey, Natural Dualities for the Working Algebraist, Cambridge University Press, Cambridge, 1998.

[2] Clark, D.M., B.A. Davey, M.G. Jackson, and J.G. Pitkethly, The axiomatizability of topological prevarieties, Advances in Mathematics 218(5):1604–1653, 2008.

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