Non-Classical Temporal Logic in Topological Dynamics

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DAY 3

A Completeness Proof Sketch for LTL

Fix a formula φ and let Σ be the set of subformulas of φ (possibly with some extra stuff).

1. Build the canonical model, $\mathfrak{C}$.
2. Consider the filtration, $\mathfrak{C}/\Sigma$.
3. Assign a characteristic formula to each element $\mathfrak{m}$ of $\mathfrak{C}/\Sigma$.

   In this case, $\mathfrak{m}$ is just a Σ-type and $\chi(\mathfrak{m}) = \bigwedge \mathfrak{m}^+ \wedge \neg \bigvee \mathfrak{m}^-$.
4. Prove that $\mathfrak{C}/\Sigma$ is **ω-sensible.**
5. Conclude that any element of $\mathfrak{C}/\Sigma$ can be included in a **realizing path, and hence an** LTL **model.**

A Completeness Proof Strategy for ITL

Fix a formula φ and let Σ be the set of subformulas of φ.

1. Build the canonical model, $\mathfrak{C}$.
2. Replace ${}^{\mathfrak{C}}/_{\Sigma}$ by the domain of the **maximal simulation** from $\mathbb{I}_{\Sigma}$.
3. Assign a characteristic formula to each element $\mathfrak{m}$ of ${}^{\mathfrak{C}}/_{\Sigma}$.

   In this case, we replace $\chi(\mathfrak{m})$ by the **Jankov-De Jongh formula** $\mathrm{Sim}(\varphi)$.
4. Prove that ${}^{\mathfrak{C}}/_{\Sigma}$ is ω-sensible.
5. Conclude that ${}^{\mathfrak{C}}/_{\Sigma}$ is a **quasimodel** falsifying every **unprovable** formula.

THE CALCULUS $\mathrm{ITL}^0_\circ$

ITAUT Intuitionistic propositional axioms

TEMPORAL AXIOMS:

$\mathrm{NEXT}_\perp$	$\neg \circ \perp$
$\mathrm{NEXT}_\wedge$	$(\circ\varphi \wedge \circ\psi) \to \circ(\varphi \wedge \psi)$
$\mathrm{NEXT}_\vee$	$\circ(\varphi \vee \psi) \to (\circ\varphi \vee \circ\psi)$
$\mathrm{NEXT}_\to$	$\circ(\varphi \to \psi) \to (\circ\varphi \to \circ\psi)$

RULES:

MP $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$ NEC $\dfrac{\varphi}{\circ\varphi}$

THE CALCULUS $\mathrm{ITL}^0_\Diamond$

ITAUT Intuitionistic propositional axioms

TEMPORAL AXIOMS:

$\mathrm{NEXT}_\bot$	$\neg \circ \bot$
$\mathrm{NEXT}_\wedge$	$(\circ\varphi \wedge \circ\psi) \to \circ(\varphi \wedge \psi)$
$\mathrm{NEXT}_\vee$	$\circ(\varphi \vee \psi) \to (\circ\varphi \vee \circ\psi)$
$\mathrm{NEXT}_\to$	$\circ(\varphi \to \psi) \to (\circ\varphi \to \circ\psi)$
$\mathrm{FIX}_\Diamond$	$(\varphi \vee \circ\Diamond\varphi) \to \Diamond\varphi$

RULES:

MP $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$ NEC $\dfrac{\varphi}{\circ\varphi}$

MON $\dfrac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}$ $\mathrm{IND}_\Diamond$ $\dfrac{\circ\varphi \to \varphi}{\Diamond\varphi \to \varphi}$

THE CALCULI $\mathrm{ITL}^0_{\forall}/\mathrm{ITL}^0_{\Diamond\forall}$

Add the following to $\mathrm{ITL}^0_{\circ}/\mathrm{ITL}^0_{\Diamond}$

$\mathrm{K}_{\forall}$	$\forall(\varphi \to \psi) \to (\forall\varphi \to \forall\psi)$	$\mathrm{EM}_{\forall}$	$\forall\varphi \vee \neg\forall\varphi$
$\mathrm{DIST}_{\forall}$	$\forall(\varphi \vee \forall\psi) \to \forall\varphi \vee \forall\psi$	$\mathrm{T}_{\forall}$	$\forall\varphi \to \varphi$
$\mathrm{NEXT}_{\forall}$	$\forall\varphi \leftrightarrow \circ\forall\varphi$	$4_{\forall}$	$\forall\varphi \to \forall\forall\varphi$
$\mathrm{NEC}_{\forall}$	$\dfrac{\varphi}{\forall\varphi}$		

A JUNGLE OF LOGICS

Λ^1: Replace $\textsc{Next}_{\rightarrow}$ by

$$\textsc{Next}_{\leftrightarrow} := \circ(\varphi \rightarrow \psi) \leftrightarrow (\circ\varphi \rightarrow \circ\psi)$$

GDTL_*: ITL^1_* with the Gödel-Dummet axiom

$$\textsc{gd} := (\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$$

THEOREM

1. $\mathrm{ITL}^0_{\Diamond\forall}$ *is sound for the class of dynamical systems.*
2. $\mathrm{ITL}^1_{\Diamond\forall}$ *is sound for the class of dynamical systems with an interior map.*
3. $\mathrm{GDTL}_{\Diamond\forall}$ *is sound for the class of locally linear Alexandroff systems with an interior map*
   ...hence for its real-valued semantics.

Prime Types

Definition

Given sets of formulas Γ and Δ:

- $\Gamma \vdash \Delta$ if there exist finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\vdash \bigwedge \Gamma' \to \bigvee \Delta'$.
- $\Phi = (\Phi^+, \Phi^-)$ is **consistent** if $\Phi^+ \nvdash \Phi^-$.
- Φ is **saturated** if every formula belongs to either Φ^+ or Φ^-.

Definition

The **canonical model** is the structure $\mathfrak{C} = (W_\mathfrak{C}, \preccurlyeq_\mathfrak{C}, S_\mathfrak{C}, [\![\cdot]\!]_\mathfrak{C})$, where

$$\begin{aligned} W_\mathfrak{C}&: \text{ Set of prime theories} \\ \Gamma \preccurlyeq_\mathfrak{C} \Delta&: \ \Gamma^+ \subseteq \Delta^+ \\ S_\mathfrak{C}(\Gamma)^\pm&: \ \{\psi : \circ\psi \in \Gamma^\pm\} \\ [\![p]\!]_\mathfrak{C}&: \ \{\Gamma \in W_\mathfrak{C} : p \in \Gamma^+\} \end{aligned}$$

Properties of the Canonical Model

$\mathfrak{C}$ is a Σ-labelled quasimodel if $\ell(\Gamma) = (\Gamma^+ \cap \Sigma, \Gamma^- \cap \Sigma)$.

Proposition (Exercise)

$\mathfrak{C}$ is a **weak quasimodel:**

1. *$\preccurlyeq_{\mathfrak{C}}$ is a partial order respecting the semantics of $\to$.*
2. *$S_{\mathfrak{C}}$ is a sensible function (monotone and locally respects $\circ, \Diamond$).*
3. *$[\![\cdot]\!]_{\mathfrak{C}}$ is upward-closed.*

Proposition (Exercise)

1. *If* $\textsc{Next}_{\leftrightarrow}$ *is included then $S_{\mathfrak{C}}$ is an interior map.*
2. *If* GD *is included then $\preccurlyeq_{\mathfrak{C}}$ is locally linear.*

COMPLETENESS OF ○-LOGICS

THEOREM

1. $\mathrm{ITL}^0_\circ$ *is sound and strongly complete for*
   - (A) *The class of dynamical systems.*
   - (B) *The class of Alexandroff dynamical systems.*
2. $\mathrm{ITL}^1_\circ$ *is sound and strongly complete for*
   - (A) *The class of dynamical systems with an interior map.*
   - (B) *The class of Alexandroff dynamical systems with an interior map.*
3. $\mathrm{GDTL}_\circ$ *is sound and strongly complete for*
   - (A) *The class of locally linear Alexandroff systems with an interior map.*
   - (B) *The class of real-valued Gödel-Dummett models.*

Completeness of $\forall$-Logics

Recall: By convention, all logics include $\circ$.

Elements of the canonical model may disagree on $\forall$-formulas.

$\Gamma \sim_\forall \Delta$ if $\forall\psi \in \Gamma^+$ iff $\forall\psi \in \Delta^+$.

Universal slices: $\mathfrak{C}$ restricted to a single $\sim_\forall$-class.

Lemma

Every universal slice is honest.

Theorem

The logics $\mathrm{ITL}^0_\forall$, $\mathrm{ITL}^1_\forall$, $\mathrm{GDTL}_\forall$ *are sound and strongly complete for their respective class of relational models.*

In the sequel, $\mathfrak{C}$ will denote a universal slice without explicit mention.

Simulations Revisited

A **simulation** between labelled spaces (X, ℓ_X) and (Y, ℓ_Y) is a continuous relation $E \subseteq X \times Y$ which preserves labels:

$$x \; E \; y \Rightarrow \ell_X(x) = \ell_Y(y)$$

A simulation between **pointed/rooted** labelled spaces preserves the designated point:

$$(X, \ell_X, x) \rightharpoonup (Y, \ell_Y, y)$$

if there is a simulation $E \subseteq X \times Y$ with $x \; E \; y$.

Simulations between labelled posets are forward-confluent.

Dynamic Simulations Revisited

A **dynamic simulation** between **labelled systems** (X, R_X, ℓ_X) and (Y, R_Y, ℓ_Y) is a simulation $E \subseteq X \times Y$ which is backwards-confluent with respect to R.

Example: Simulation between our quasimodel and model falsifying

$$\forall(\neg p \vee \Diamond p) \to (\Diamond p \vee \neg\Diamond p)$$

Proposition

Let E be a surjective, dynamic simulation between weak quasimodels $\mathfrak{X} = (X, R_X, \ell_X)$ and $\mathfrak{Y} = (Y, R_Y, \ell_Y)$ and $\mathfrak{Z} = (Z, R_Z, \ell_Z)$ the restriction of $\mathfrak{X}$ to $\mathrm{dom}(E)$.

1. *$\mathfrak{Z}$ is a weak quasimodel.*
2. *If R_Y is ω-sensible, so is R_Z.*
3. *If ℓ_Y is honest, so is ℓ_Z.*

Reminder: $\mathbb{I}_\Sigma$

$$\mathbb{I}_\Sigma = (I_\Sigma, \preccurlyeq_\Sigma, R_\Sigma, \ell_\Sigma)$$

Elements of I_Σ are called **(irreducible) moments**.

Given any dynamic topological model $(X, S, [\![\cdot]\!])$, the maximal simulation $E^* \subseteq I_\Sigma \times X$ is a surjective dynamic simulation.

Theorem

If $\mathfrak{X}$ is any dynamic topological model **or weak quasimodel** *then there is a surjective, dynamic simulation $E^* \subseteq I_\Sigma \times X$.*

Canonical Pseudo-Quotients

In particular, there is a surjective simulation E^* between $\mathbb{I}_\Sigma$ and $\mathfrak{C}$.

The restriction of $\mathbb{I}_\Sigma$ to $\mathrm{dom}(E^*)$ is denoted $\mathfrak{C}/\Sigma$.

Lemma
$\mathfrak{C}/\Sigma$ is an honest, weak quasimodel.

Theorem
$\mathrm{ITL}^0_\forall$ and $\mathrm{GDTL}_\forall$ are decidable and have the finite quasimodel property.

Possible Extensions

Exercise: $\mathrm{ITL}^0_\forall$, $\mathrm{GDTL}_\forall$, and $\mathrm{ITL}^1_\circ$ have the finite **model** property.

Open Question: Is $\mathrm{ITL}^1_\forall$ decidable?

Question: What about logics with $\diamond$?

We need ω-sensibility!

Jankov-De Jongh Formulas

Theorem

Given a locally finite labelled preorder $(W, \preccurlyeq, \ell)$*, there exist formulas*

$$\big(\mathsf{Sim}(w)\big)_{w \in W} \in \mathcal{L}_*$$

such that for any dynamic topological model $\mathcal{M} = (X, S, [\![\cdot]\!])$*, tfae:*

1. $(\mathcal{M}, x) \not\models \mathsf{Sim}(w)$
2. *there are* $y \succcurlyeq x$ *and* $E \subseteq W \times X$ *such that* $w \; E \; y$

Theorem

Given $\mathfrak{m} \in I_\Sigma$ *and* $\Gamma \in W_{\mathfrak{c}}$*, tfae:*

1. *there are* $\Delta \succcurlyeq \Gamma$ *and* $E \subseteq I_\Sigma \times W_{\mathfrak{c}}$ *such that* $\mathfrak{m} \; E \; \Delta$
2. $\mathsf{Sim}(\mathfrak{m}) \in \Gamma^-$

Corollary

A moment $\mathfrak{m}$ *belongs to* $\mathfrak{c}/\Sigma$ *iff* $\not\vdash \mathsf{Sim}(\mathfrak{m})$.

BASIC PROPERTIES OF Sim

Fix finite Σ closed under subformulas and let $\mathbb{I}_\Sigma = (I_\Sigma, \succcurlyeq, R, \ell)$.

- If $\psi \in \ell^-(\mathfrak{m})$, then $\vdash \psi \to \mathsf{Sim}(\mathfrak{m})$
- If $\psi \in \ell^+(\mathfrak{m})$, then $\vdash (\psi \to \mathsf{Sim}(\mathfrak{m})) \to \mathsf{Sim}(\mathfrak{m})$.
- $\vdash \bigwedge_{\psi \in \ell^-(\mathfrak{m})} \mathsf{Sim}(\mathfrak{m}) \to \psi$
- $\vdash \circ \bigwedge_{\mathfrak{m} R \mathfrak{n}} \mathsf{Sim}(\mathfrak{n}) \to \mathsf{Sim}(\mathfrak{m})$

LEMMA (ANCESTRAL LEMMA)

Let $\mathfrak{m}$ be a moment of $\mathfrak{C}/\Sigma$ and $\Diamond\psi \in \ell^+(\mathfrak{m})$.
Let $R^*(w)$ be the set of moments reachable from $\mathfrak{m}$ in $\mathfrak{C}/\Sigma$.
Then, $\psi \in \ell^+(\mathfrak{n})$ for some $\mathfrak{n} \in R^*(\mathfrak{m})$.

PROOF.

By contradiction.

1. $\vdash (\Diamond\psi \to \mathsf{Sim}(\mathfrak{m})) \to \mathsf{Sim}(\mathfrak{m})$
2. $\vdash \psi \to \bigwedge_{\mathfrak{n}\in R^*(\mathfrak{m})} \mathsf{Sim}(\mathfrak{n})$
3. $\vdash \Diamond\psi \to \Diamond \bigwedge_{\mathfrak{n}\in R^*(\mathfrak{m})} \mathsf{Sim}(\mathfrak{n})$
4. $\vdash \circ \bigwedge_{\mathfrak{n}\in R^*(\mathfrak{m})} \mathsf{Sim}(\mathfrak{n}) \to \bigwedge_{\mathfrak{n}\in R^*(\mathfrak{m})} \mathsf{Sim}(\mathfrak{n})$
5. $\vdash \Diamond \bigwedge_{\mathfrak{n}\in R^*(\mathfrak{m})} \mathsf{Sim}(\mathfrak{n}) \to \bigwedge_{\mathfrak{n}\in R^*(\mathfrak{m})} \mathsf{Sim}(\mathfrak{n})$
6. $\vdash \Diamond\psi \to \mathsf{Sim}(\mathfrak{m})$
7. $\vdash \mathsf{Sim}(\mathfrak{m})$ $\square$

Completeness and Decidability of ITL

Let $* \subseteq \{\Diamond, \forall\}$.

Theorem

For any $\varphi \in \mathcal{L}_*$, tfae.

1. $\mathrm{ITL}^0_* \vdash \varphi$
2. φ *is valid on the class of finite quasimodels*
3. φ *is valid on the class of dynamical systems*

Corollary

$\mathrm{ITL}^0_{\Diamond\forall}$ *is decidable.*

COMPLETENESS AND DECIDABILITY OF GDTL

Let $* \subseteq \{\Diamond, \forall\}$.

THEOREM

For any $\varphi \in \mathcal{L}_*$, tfae.

1. $\vdash \varphi$
2. φ *is valid on the class of finite, locally linear quasimodels*
3. φ *is valid on the class of locally linear Alexandroff systems with an interior map*
4. φ *is valid with respect to the real-valued semantics.*

COROLLARY

$\mathrm{GDTL}_{\Diamond\forall}$ *is decidable.*

Open Questions

1. What is the complexity of ITL^0_*, for $* \subseteq \{\Diamond, \forall\}$?

2. Is $\mathrm{ITL}^0_{\Diamond\Box}$ decidable? Does it have a natural axiomatisation?†

3. Is ITL^1_* decidable, for $* \subseteq \{\Diamond, \Box\}$?

4. Is $\mathrm{GDTL}_{\Diamond\Box}$ complete?†

$\dagger$ The answer is **yes** if we include co-implication!

Thank you!

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