THE MATHEMATICAL THEORY OF CONTEXTUALITY
Lecture 5: Cohomological characterization

Samson Abramsky

Department of Computer Science, UCL

TACL 2024 Summer School

Based on...

- *Contextuality, Cohomology and Paradox (2015)*, in *Proceedings of CSL 2015*, S. Abramsky, R.S. Barbosa, K. Kishida, R. Lal and S. Mansfield
- S. Abramsky, S. Mansfield and R. Soares Barbosa, *The Cohomology of Non-Locality and Contextuality*, in *Proceedings of QPL 2011*, EPTCS 2011.

Based on...

- *Contextuality, Cohomology and Paradox (2015)*, in *Proceedings of CSL 2015*, S. Abramsky, R.S. Barbosa, K. Kishida, R. Lal and S. Mansfield
- S. Abramsky, S. Mansfield and R. Soares Barbosa, *The Cohomology of Non-Locality and Contextuality*, in *Proceedings of QPL 2011*, EPTCS 2011.

See also work by

- Giovanni Carù (1701.00656)
- Robert Raussendorf, Cihan Okay, Stephen Bartlett et al. (1701.01888)
- Adam Ó Conghaile (2206.15253)
- . . .

Contextuality

What *is* contextuality, as a problematic, non-classical phenomenon?

In a nutshell: where we have a family of data which is *locally consistent*, but *globally inconsistent*.

The Borders of Paradox

The phenomenon arises with *observable data*, reflecting physical reality, taking us to the borders of paradox!

The Borders of Paradox

The phenomenon arises with *observable data*, reflecting physical reality, taking us to the borders of paradox!

What saves us from a direct conflict between logic and experience is that the data *cannot* directly be observed globally.
We cannot observe all the variables *at the same time*.

The Borders of Paradox

The phenomenon arises with *observable data*, reflecting physical reality, taking us to the borders of paradox!

What saves us from a direct conflict between logic and experience is that the data *cannot* directly be observed globally.
We cannot observe all the variables *at the same time*.

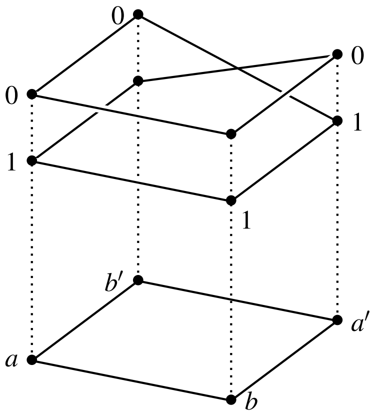
A “transcendental deduction” of the *incompatibility* (in general) of observables.

Bundle Pictures

Strong Contextuality

- E.g. the PR box:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	✓	×	×	✓
(a,b')	✓	×	×	✓
(a',b)	✓	×	×	✓
(a',b')	×	✓	✓	×



No event extends to a global valuation

Topological Characterization

Topological Characterization

- Local consistency — global inconsistency

Topological Characterization

- Local consistency — global inconsistency

- Contextuality is pervasive (e.g. physics, computation, logic, ...)

Topological Characterization

- Local consistency — global inconsistency
- Contextuality is pervasive (e.g. physics, computation, logic, ...)
- Goal: find the common mathematical structure in these diverse manifestations, and develop a widely applicable theory

Topological Characterization

- Local consistency — global inconsistency
- Contextuality is pervasive (e.g. physics, computation, logic, ...)
- Goal: find the common mathematical structure in these diverse manifestations, and develop a widely applicable theory
- Can be effectively visualised in topological terms

Topological Characterization

- Local consistency — global inconsistency
- Contextuality is pervasive (e.g. physics, computation, logic, ...)
- Goal: find the common mathematical structure in these diverse manifestations, and develop a widely applicable theory
- Can be effectively visualised in topological terms
- "Twisting" in bundle space gives rise to an obstruction to global consistency

Topological Characterization

- Local consistency — global inconsistency
- Contextuality is pervasive (e.g. physics, computation, logic, . . .)
- Goal: find the common mathematical structure in these diverse manifestations, and develop a widely applicable theory
- Can be effectively visualised in topological terms
- "Twisting" in bundle space gives rise to an obstruction to global consistency
- Idea: use *cohomology* to characterize contextuality

Why Cohomology?

Why Cohomology?

- A major theme of 20/21st century mathematics

Why Cohomology?

- A major theme of 20/21st century mathematics
- Constructive witnesses for non-existence, instead of proofs by contradiction

Why Cohomology?

- A major theme of 20/21st century mathematics
- Constructive witnesses for non-existence, instead of proofs by contradiction
- Often computable

Why Cohomology?

- A major theme of 20/21st century mathematics
- Constructive witnesses for non-existence, instead of proofs by contradiction
- Often computable
- Increasingly coming into applications (e.g. persistent homology, TDA)

Why Cohomology?

- A major theme of 20/21st century mathematics
- Constructive witnesses for non-existence, instead of proofs by contradiction
- Often computable
- Increasingly coming into applications (e.g. persistent homology, TDA)
- Part of the program of developing a widely applicable mathematical theory of contextuality

Cohomology from the ground up

Empirical models $(X, \mathscr{M}, O)$.

Cohomology from the ground up

Empirical models $(X, \mathscr{M}, O)$.

Joint outcomes of measuring variables in a context C are *sections* $s : C \to O$.

Cohomology from the ground up

Empirical models $(X, \mathscr{M}, O)$.

Joint outcomes of measuring variables in a context C are *sections* $s : C \to O$.

We consider "sections*": formal $\mathbb{Z}$-linear combinations of sections

$$\sum_{s:C\to O} z_s \cdot s$$

Cohomology from the ground up

Empirical models $(X, \mathscr{M}, O)$.

Joint outcomes of measuring variables in a context C are *sections* $s : C \to O$.

We consider "sections*": formal $\mathbb{Z}$-linear combinations of sections

$$\sum_{s:C \to O} z_s \cdot s$$

The "Original Sin" of cohomology: we need an abelian group structure to work with.

Cohomology from the ground up

Empirical models $(X, \mathscr{M}, O)$.

Joint outcomes of measuring variables in a context C are *sections* $s : C \to O$.

We consider "sections*": formal $\mathbb{Z}$-linear combinations of sections

$$\sum_{s:C \to O} z_s \cdot s$$

The "Original Sin" of cohomology: we need an abelian group structure to work with.

Cochains:

- 0-cochains: $(r_i)_i$, where r_i is a section* over C_i.
- 1-cochains: $(r_{ij})_{i,j}$, where r_{ij} is a section* over $C_i \cap C_j$.
- 2-cochains: $(r_{ijk})_{i,j,k}$, where r_{ijk} is a section* over $C_i \cap C_j \cap C_k$.
- $\vdots$

Restriction and Coboundaries

We can restrict a 0-section* to a 1-section*, or a 1-section* to a 2-section*, by summing out:

Restriction and Coboundaries

We can restrict a 0-section* to a 1-section*, or a 1-section* to a 2-section*, by summing out:

E.g. $r_i|_{ij}$, where $r_i = \sum_{s \in O^{C_i}} z_s \cdot s$:

Restriction and Coboundaries

We can restrict a 0-section* to a 1-section*, or a 1-section* to a 2-section*, by summing out:

E.g. $r_i|_{ij}$, where $r_i = \sum_{s \in O^{C_i}} z_s \cdot s$:

for $t : C_i \cap C_j \to O$,

$$r_i|_{ij} \; :: \; t \mapsto \sum_{s|_{C_i \cap C_j} = t} z_s$$

Restriction and Coboundaries

We can restrict a 0-section* to a 1-section*, or a 1-section* to a 2-section*, by summing out:

E.g. $r_i|_{ij}$, where $r_i = \sum_{s \in O^{C_i}} z_s \cdot s$:

for $t : C_i \cap C_j \to O$,

$$r_i|_{ij} \ :: \ t \mapsto \sum_{s|_{C_i \cap C_j} = t} z_s$$

Coboundary maps

$$C^0 \to^{d^0} C^1 \to^{d^1} C^2$$

Restriction and Coboundaries

We can restrict a 0-section* to a 1-section*, or a 1-section* to a 2-section*, by summing out:

E.g. $r_i|_{ij}$, where $r_i = \sum_{s \in O^{C_i}} z_s \cdot s$:

for $t : C_i \cap C_j \to O$,

$$r_i|_{ij} \;::\; t \mapsto \sum_{s|_{C_i \cap C_j} = t} z_s$$

Coboundary maps

$$C^0 \to^{d^0} C^1 \to^{d^1} C^2$$

$$d^0(r_i)_i = (s_{ij})_{i,j}, \qquad s_{ij} := r_i|_{ij} - r_j|_{ij}$$

$$d^1(r_{ij})_i = (s_{ijk})_{i,j,k}, \qquad s_{ijk} := r_{ij}|_{ijk} - r_{ik}|_{ijk} + r_{jk}|_{ijk}$$

Cocycles and cohomology

Cocycles and cohomology

The elements of the kernel of d^i are called *i-cocycles*.

Cocycles and cohomology

The elements of the kernel of d^i are called *i-cocycles*.

The key fact is that $d^1 \circ d^0 = 0$, hence $\mathsf{im}\, d^0 \subseteq \mathsf{ker}\, d^1$, and we can mod out the cocycles Z^i by the coboundaries B^i to form the cohomology groups

$$H^i := Z^i / B^i.$$

Cocycles and cohomology

The elements of the kernel of d^i are called *i-cocycles*.

The key fact is that $d^1 \circ d^0 = 0$, hence $\mathsf{im}\, d^0 \subseteq \mathsf{ker}\, d^1$, and we can mod out the cocycles Z^i by the coboundaries B^i to form the cohomology groups

$$H^i := Z^i / B^i.$$

The 0-cocycles are those (r_i) with $d^0(r_i) = 0$, i.e. $r_i|_{ij} - r_j|_{ij} = 0$, i.e. $r_i|_{ij} = r_j|_{ij}$.

Cocycles and cohomology

The elements of the kernel of d^i are called *i-cocycles*.

The key fact is that $d^1 \circ d^0 = 0$, hence $\mathsf{im}\, d^0 \subseteq \mathsf{ker}\, d^1$, and we can mod out the cocycles Z^i by the coboundaries B^i to form the cohomology groups

$$H^i := Z^i / B^i.$$

The 0-cocycles are those (r_i) with $d^0(r_i) = 0$, i.e. $r_i|_{ij} - r_j|_{ij} = 0$, i.e. $r_i|_{ij} = r_j|_{ij}$.

These are exactly the *compatible families* of sections*.

Cocycles and cohomology

The elements of the kernel of d^i are called *i-cocycles*.

The key fact is that $d^1 \circ d^0 = 0$, hence $\mathsf{im}\, d^0 \subseteq \mathsf{ker}\, d^1$, and we can mod out the cocycles Z^i by the coboundaries B^i to form the cohomology groups

$$H^i := Z^i / B^i.$$

The 0-cocycles are those (r_i) with $d^0(r_i) = 0$, i.e. $r_i|_{ij} - r_j|_{ij} = 0$, i.e. $r_i|_{ij} = r_j|_{ij}$.

These are exactly the *compatible families* of sections*.

The 1-cocycles are those (r_{ij}) with $d^1(r_{ij}) = 0$, i.e. $r_{ij}|_{ijk} - r_{ik}|_{ijk} + r_{jk}|_{ijk} = 0$, i.e. $r_{ij} + r_{jk} = r_{ik}$.

Cocycles and cohomology

The elements of the kernel of d^i are called *i-cocycles*.

The key fact is that $d^1 \circ d^0 = 0$, hence $\mathsf{im}\, d^0 \subseteq \mathsf{ker}\, d^1$, and we can mod out the cocycles Z^i by the coboundaries B^i to form the cohomology groups

$$H^i := Z^i / B^i.$$

The 0-cocycles are those (r_i) with $d^0(r_i) = 0$, i.e. $r_i|_{ij} - r_j|_{ij} = 0$, i.e. $r_i|_{ij} = r_j|_{ij}$.

These are exactly the *compatible families* of sections*.

The 1-cocycles are those (r_{ij}) with $d^1(r_{ij}) = 0$, i.e. $r_{ij}|_{ijk} - r_{ik}|_{ijk} + r_{jk}|_{ijk} = 0$, i.e. $r_{ij} + r_{jk} = r_{ik}$.

This *cocycle condition* occurs in many contexts in mathematics.

Relative Cohomology

The relative cohomology at i lets us focus on a section at context C_i.

Relative Cohomology

The relative cohomology at i lets us focus on a section at context C_i.

This lets us consider questions such as extending such a local section to a global one in an empirical model.

Relative Cohomology

The relative cohomology at i lets us focus on a section at context C_i.

This lets us consider questions such as extending such a local section to a global one in an empirical model.

We shall not give the abstract definition.

Relative Cohomology

The relative cohomology at i lets us focus on a section at context C_i.

This lets us consider questions such as extending such a local section to a global one in an empirical model.

We shall not give the abstract definition.

Proposition

For any $C_i \in \mathscr{M}$, the elements of the relative cohomology group H^0_i correspond bijectively to compatible families (r_j) such that $r_i = 0$.

Cohomology Obstruction

To each local section s at context C_i in an empirical model e, we associate an element $\gamma(s)$ of H^1_i, which can be regarded as an obstruction to s having an extension within the support of e to a global section.

Cohomology Obstruction

To each local section s at context C_i in an empirical model e, we associate an element $\gamma(s)$ of H_i^1, which can be regarded as an obstruction to s having an extension within the support of e to a global section.

In particular, the existence of such an extension implies that the obstruction vanishes, yielding *cohomological witnesses* for logical and strong contextuality in the *non-vanishing* of the obstruction.

Cohomology Obstruction

To each local section s at context C_i in an empirical model e, we associate an element $\gamma(s)$ of H^1_i, which can be regarded as an obstruction to s having an extension within the support of e to a global section.

In particular, the existence of such an extension implies that the obstruction vanishes, yielding *cohomological witnesses* for logical and strong contextuality in the *non-vanishing* of the obstruction.

Fix an element $s = s_1 \in S_e(C_1)$. Because of the compatibility of the empirical model, there is a family $\{s_i \in S_e(C_i)\}$ with $s_1|_{C_1 \cap C_i} = s_i|_{C_1 \cap C_i}$, $i = 2, \ldots, n$.

Cohomology Obstruction

To each local section s at context C_i in an empirical model e, we associate an element $\gamma(s)$ of H^1_i, which can be regarded as an obstruction to s having an extension within the support of e to a global section.

In particular, the existence of such an extension implies that the obstruction vanishes, yielding *cohomological witnesses* for logical and strong contextuality in the *non-vanishing* of the obstruction.

Fix an element $s = s_1 \in S_e(C_1)$. Because of the compatibility of the empirical model, there is a family $\{s_i \in S_e(C_i)\}$ with $s_1|_{C_1 \cap C_i} = s_i|_{C_1 \cap C_i}$, $i = 2, \ldots, n$.

We define the 0-cochain $c := (s_1, \ldots, s_n)$. The coboundary of this cochain is $z := d^0(c)$.

Cohomology Obstruction

To each local section s at context C_i in an empirical model e, we associate an element $\gamma(s)$ of H^1_i, which can be regarded as an obstruction to s having an extension within the support of e to a global section.

In particular, the existence of such an extension implies that the obstruction vanishes, yielding *cohomological witnesses* for logical and strong contextuality in the *non-vanishing* of the obstruction.

Fix an element $s = s_1 \in S_e(C_1)$. Because of the compatibility of the empirical model, there is a family $\{s_i \in S_e(C_i)\}$ with $s_1|_{C_1 \cap C_i} = s_i|_{C_1 \cap C_i}$, $i = 2, \ldots, n$.

We define the 0-cochain $c := (s_1, \ldots, s_n)$. The coboundary of this cochain is $z := d^0(c)$.

Proposition

The coboundary z of c vanishes under restriction to C_1, and hence is a cocycle in the relative cohomology with respect to C_1.

The Obstruction Defined

The Obstruction Defined

We define $\gamma(s_1)$ as the cohomology class $[z]$ in H^1_1.

The Obstruction Defined

We define $\gamma(s_1)$ as the cohomology class $[z]$ in H^1_1.

Remarks:
There is a more conceptual way of defining this obstruction, using the connecting homomorphism from the long exact sequence of cohomology (which in turn uses the "Snake Lemma").

The Obstruction Defined

We define $\gamma(s_1)$ as the cohomology class $[z]$ in H^1_1.

Remarks:
There is a more conceptual way of defining this obstruction, using the connecting homomorphism from the long exact sequence of cohomology (which in turn uses the “Snake Lemma”).

We have given a more concrete formulation, which may be easier to grasp, and is also convenient for computation.

The Obstruction Defined

We define $\gamma(s_1)$ as the cohomology class $[z]$ in H^1_1.

Remarks:
There is a more conceptual way of defining this obstruction, using the connecting homomorphism from the long exact sequence of cohomology (which in turn uses the "Snake Lemma").

We have given a more concrete formulation, which may be easier to grasp, and is also convenient for computation.

Note that, although $z = d^0(c)$, it is *not* necessarily a relative coboundary, since c is not a relative cochain, as $s_i|_{C_1 \cap C_i} \neq 0$.

The Obstruction Defined

We define $\gamma(s_1)$ as the cohomology class $[z]$ in H^1_1.

Remarks:
There is a more conceptual way of defining this obstruction, using the connecting homomorphism from the long exact sequence of cohomology (which in turn uses the "Snake Lemma").

We have given a more concrete formulation, which may be easier to grasp, and is also convenient for computation.

Note that, although $z = d^0(c)$, it is *not* necessarily a relative coboundary, since c is not a relative cochain, as $s_i|_{C_1 \cap C_i} \neq 0$.

Thus in general, we need not have $[z] = 0$.

Key Property of the Obstruction

Key Property of the Obstruction

Proposition

The following are equivalent:

1. *The cohomology obstruction vanishes:* $\gamma(s_1) = 0$.
2. *There is a* 0*-cochain* (r_i) *with* $s_1 = r_1$*, and for all* i, j*:*

$$r_i|_{C_i \cap C_j} = r_j|_{C_i \cap C_j}.$$

Application of contextuality

Application of contextuality

As an immediate application to contextuality, we have the following.

Application of contextuality

As an immediate application to contextuality, we have the following.

Proposition

If the model e is possibilistically extendable, then the obstruction vanishes for every section in the support of the model. If e is not strongly contextual, then the obstruction vanishes for some section in the support.

Application of contextuality

As an immediate application to contextuality, we have the following.

Proposition

If the model e is possibilistically extendable, then the obstruction vanishes for every section in the support of the model. If e is not strongly contextual, then the obstruction vanishes for some section in the support.

Thus we have a *sufficient condition* for contextuality in the non-vanishing of the obstruction.

Application of contextuality

As an immediate application to contextuality, we have the following.

Proposition

If the model e is possibilistically extendable, then the obstruction vanishes for every section in the support of the model. If e is not strongly contextual, then the obstruction vanishes for some section in the support.

Thus we have a *sufficient condition* for contextuality in the non-vanishing of the obstruction.

The non-necessity of the condition arises from the possibility of "false positives": families of sections* (r_i)which do not determine a *bona fide* global section.

The Hardy Model

Support of the Hardy Model

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(A,B)	1	0	0	0
(A,B')	0	1	0	0
(A',B)	0	1	1	1
(A',B')	1	1	1	0

- Possibilistically non-local
- Not strongly contextual
- The section $(A,B) \to (0,0)$ witnesses non-locality
- All other sections belong to compatible families of sections

The Hardy Model

Support of the Hardy Model

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(A,B)	s_1	s_2	s_3	s_4
(A,B')	0	s_6	s_7	s_8
(A',B)	0	s_{10}	s_{11}	s_{12}
(A',B')	s_{13}	s_{14}	s_{15}	0

- Label non-zero sections
- Compatible family of $\mathbb{Z}$-linear combinations of sections:

$$r_1 = s_1, \quad r_2 = s_6 + s_7 - s_8, \quad r_3 = s_{11}, \quad r_4 = s_{15}$$

- One can check that

$$\begin{array}{rcl} r_2|A & = & 1\cdot(A \mapsto 0) + 1\cdot(A \mapsto 1) - 1\cdot(A \mapsto 1) & = & r_1|A, \\ r_2|B' & = & 1\cdot(B' \mapsto 1) + 1\cdot(B' \mapsto 0) - 1\cdot(B' \mapsto 1) & = & r_4|B' \end{array}$$

The Hardy Model

Support of the Hardy Model

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(A,B)	s_1	s_2	s_3	s_4
(A,B')	0	s_6	s_7	s_8
(A',B)	0	s_{10}	s_{11}	s_{12}
(A',B')	s_{13}	s_{14}	s_{15}	0

- Label non-zero sections
- Compatible family of $\mathbb{Z}$-linear combinations of sections:

$$r_1 = s_1, \quad r_2 = s_6 + s_7 - s_8, \quad r_3 = s_{11}, \quad r_4 = s_{15}$$

- One can check that

$$\begin{array}{lclcl} r_2|A & = & 1\cdot(A\mapsto 0) + 1\cdot(A\mapsto 1) - 1\cdot(A\mapsto 1) & = & r_1|A, \\ r_2|B' & = & 1\cdot(B'\mapsto 1) + 1\cdot(B'\mapsto 0) - 1\cdot(B'\mapsto 1) & = & r_4|B' \end{array}$$

The Hardy Model

Support of the Hardy Model

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(A,B)	s_1	s_2	s_3	s_4
(A,B')	0	s_6	s_7	s_8
(A',B)	0	s_{10}	s_{11}	s_{12}
(A',B')	s_{13}	s_{14}	s_{15}	0

- Label non-zero sections
- Compatible family of $\mathbb{Z}$-linear combinations of sections:

$$r_1 = s_1, \quad r_2 = s_6 + s_7 - s_8, \quad r_3 = s_{11}, \quad r_4 = s_{15}$$

- One can check that

$$\begin{array}{rcl} r_2|A & = & 1 \cdot (A \mapsto 0) + 1 \cdot (A \mapsto 1) - 1 \cdot (A \mapsto 1) & = & r_1|A, \\ r_2|B' & = & 1 \cdot (B' \mapsto 1) + 1 \cdot (B' \mapsto 0) - 1 \cdot (B' \mapsto 1) & = & r_4|B' \end{array}$$

The Hardy Model

- $\gamma(s_1)$ vanishes!
- This example illustrates that false positives do arise
- The cohomological obstruction does not show the non-locality of the Hardy model

The PR Box

Coefficients for Candidate Family $\{r_i\}$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$C_1 = (A,B)$	a	0	0	b
$C_2 = (A,B')$	c	0	0	d
$C_3 = (A',B)$	e	0	0	f
$C_4 = (A',B')$	0	g	h	0

The PR Box

Coefficients for Candidate Family $\{r_i\}$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$C_1 = (A,B)$	a	0	0	b
$C_2 = (A,B')$	c	0	0	d
$C_3 = (A',B)$	e	0	0	f
$C_4 = (A',B')$	0	g	h	0

Restrictions

$$
\begin{array}{lccc}
r_1|C_{1,2} = r_2|C_{1,2} & \longrightarrow & a = c & b = d \\
r_1|C_{1,3} = r_3|C_{1,2} & \longrightarrow & a = e & b = f \\
r_2|C_{2,4} = r_4|C_{2,4} & \longrightarrow & c = h & d = g \\
r_3|C_{3,4} = r_4|C_{3,4} & \longrightarrow & e = g & f = h
\end{array}
$$

The PR Box

Coefficients for Candidate Family $\{r_i\}$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$C_1 = (A,B)$	a	0	0	b
$C_2 = (A,B')$	c	0	0	d
$C_3 = (A',B)$	e	0	0	f
$C_4 = (A',B')$	0	g	h	0

Restrictions

$$
\begin{array}{lcll}
r_1|C_{1,2} = r_2|C_{1,2} & \longrightarrow & a = c & b = d \\
r_1|C_{1,3} = r_3|C_{1,2} & \longrightarrow & a = e & b = f \\
r_2|C_{2,4} = r_4|C_{2,4} & \longrightarrow & c = h & d = g \\
r_3|C_{3,4} = r_4|C_{3,4} & \longrightarrow & e = g & f = h
\end{array}
$$

- All coefficients are required to be equal
- Checking if a section is a member of a family amounts to setting its coefficient to 1 and all other coefficients in its context to 0
- The equations then require $1 = 0$

The PR Box

Coefficients for Candidate Family $\{r_i\}$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$C_1 = (A,B)$	a	0	0	b
$C_2 = (A,B')$	c	0	0	d
$C_3 = (A',B)$	e	0	0	f
$C_4 = (A',B')$	0	g	h	0

- All coefficients are required to be equal
- Checking if a section is a member of a family amounts to setting its coefficient to 1 and all other coefficients in its context to 0
- The equations then require $1 = 0$
- No family $\{r_i\}$ extending a section s ($\forall s.\ \gamma(s) \neq 0$)

The PR Box

Coefficients for Candidate Family $\{r_i\}$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$C_1 = (A,B)$	$a = 1$	0	0	$b = 0$
$C_2 = (A,B')$	c	0	0	d
$C_3 = (A',B)$	e	0	0	f
$C_4 = (A',B')$	0	g	h	0

- All coefficients are required to be equal
- Checking if a section is a member of a family amounts to setting its coefficient to 1 and all other coefficients in its context to 0
- The equations then require $1 = 0$
- No family $\{r_i\}$ extending a section s $(\forall s.\ \gamma(s) \neq 0)$

The PR Box

Coefficients for Candidate Family $\{r_i\}$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$C_1 = (A,B)$	$a = 1$	0	0	$b = 0$
$C_2 = (A,B')$	c	0	0	d
$C_3 = (A',B)$	e	0	0	f
$C_4 = (A',B')$	0	g	h	0

- All coefficients are required to be equal
- Checking if a section is a member of a family amounts to setting its coefficient to 1 and all other coefficients in its context to 0
- The equations then require $1 = 0$
- No family $\{r_i\}$ extending a section s ($\forall s.\ \gamma(s) \neq 0$)

Other Examples

The cohomology approach witnesses strong contextuality in a number of other well-known examples:

- GHZ model
- Peres-Mermin Square
- 18-vector Kochen-Specker model
- Other KS-type models

Other Examples

The cohomology approach witnesses strong contextuality in a number of other well-known examples:

- GHZ model
- Peres-Mermin Square
- 18-vector Kochen-Specker model
- Other KS-type models

It also witnesses contextuality in important *classes* of constructions, e.g. “All-versus-Nothing” arguments.

Further developments

Further developments

- In his thesis, Giovanni Carù gives a refined cohomological criterion which covers the vast majority of cases, kills all known counter-examples, and is conjectured to be complete.

Further developments

- In his thesis, Giovanni Carù gives a refined cohomological criterion which covers the vast majority of cases, kills all known counter-examples, and is conjectured to be complete.

- Following our work, Raussendorf and Okay have developed a related cohomological treatment of contextuality.

Further developments

- In his thesis, Giovanni Carù gives a refined cohomological criterion which covers the vast majority of cases, kills all known counter-examples, and is conjectured to be complete.
- Following our work, Raussendorf and Okay have developed a related cohomological treatment of contextuality.
- Sivert Aasnaess has shown that their work also falls under the scope of the sheaf cohomology invariants.

Further developments

- In his thesis, Giovanni Carù gives a refined cohomological criterion which covers the vast majority of cases, kills all known counter-examples, and is conjectured to be complete.
- Following our work, Raussendorf and Okay have developed a related cohomological treatment of contextuality.
- Sivert Aasnaess has shown that their work also falls under the scope of the sheaf cohomology invariants.
- Adam Ó Conghaile has applied the approach to obtain a novel polynomial-time approximation algorithm for CSP, which covers many known tractable cases.

CSP and the Feder-Vardi Conjecture

Given a finite relational structure B over a finite relational vocabulary σ, the *constraint satisfaction problem* $\mathsf{CSP}(B)$ is to decide, for an *instance* given by a finite σ-structure A, whether there is a homomorphism $A \to B$.

CSP and the Feder-Vardi Conjecture

Given a finite relational structure B over a finite relational vocabulary σ, the *constraint satisfaction problem* $\mathsf{CSP}(B)$ is to decide, for an *instance* given by a finite σ-structure A, whether there is a homomorphism $A \to B$.

The Feder-Vardi Conjecture (1993):

$$\boxed{\text{For every } B,\ \mathsf{CSP}(B) \text{ is either polynomial-time solvable, or NP-complete.}}$$

CSP and the Feder-Vardi Conjecture

Given a finite relational structure B over a finite relational vocabulary σ, the *constraint satisfaction problem* $\mathsf{CSP}(B)$ is to decide, for an *instance* given by a finite σ-structure A, whether there is a homomorphism $A \to B$.

The Feder-Vardi Conjecture (1993):

$$\boxed{\text{For every } B,\ \mathsf{CSP}(B) \text{ is either polynomial-time solvable, or NP-complete.}}$$

Evidence: known problems at the time were either NP-complete (e.g. 3-SAT), or fell into two "islands of tractability":

- bounded width – exactly solvable by local consistency, or
- "subgroup problems"

CSP and the Feder-Vardi Conjecture

Given a finite relational structure B over a finite relational vocabulary σ, the *constraint satisfaction problem* $\mathsf{CSP}(B)$ is to decide, for an *instance* given by a finite σ-structure A, whether there is a homomorphism $A \to B$.

The Feder-Vardi Conjecture (1993):

$$\boxed{\text{For every } B,\ \mathsf{CSP}(B) \text{ is either polynomial-time solvable, or NP-complete.}}$$

Evidence: known problems at the time were either NP-complete (e.g. 3-SAT), or fell into two "islands of tractability":

- bounded width – exactly solvable by local consistency, or
- "subgroup problems"

The two tractable classes identified by Feder and Vardi appeared to be quite different in character.

CSP and the Feder-Vardi Conjecture

Given a finite relational structure B over a finite relational vocabulary σ, the *constraint satisfaction problem* $\mathsf{CSP}(B)$ is to decide, for an *instance* given by a finite σ-structure A, whether there is a homomorphism $A \to B$.

The Feder-Vardi Conjecture (1993):

$$\boxed{\text{For every } B,\ \mathsf{CSP}(B) \text{ is either polynomial-time solvable, or NP-complete.}}$$

Evidence: known problems at the time were either NP-complete (e.g. 3-SAT), or fell into two “islands of tractability”:

- bounded width – exactly solvable by local consistency, or
- “subgroup problems”

The two tractable classes identified by Feder and Vardi appeared to be quite different in character.

This conjecture was recently proved (independently) by Bulatov and Zhuk (*c.* 2016).

Contextuality

In a nutshell: contextually arises where we have a family of overlapping pieces of data which is *locally consistent, but globally inconsistent*.

Contextuality

In a nutshell: contextually arises where we have a family of overlapping pieces of data which is *locally consistent, but globally inconsistent*.

Illustration: local consistency

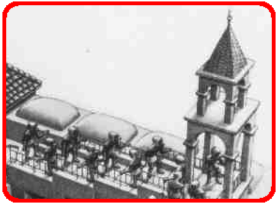
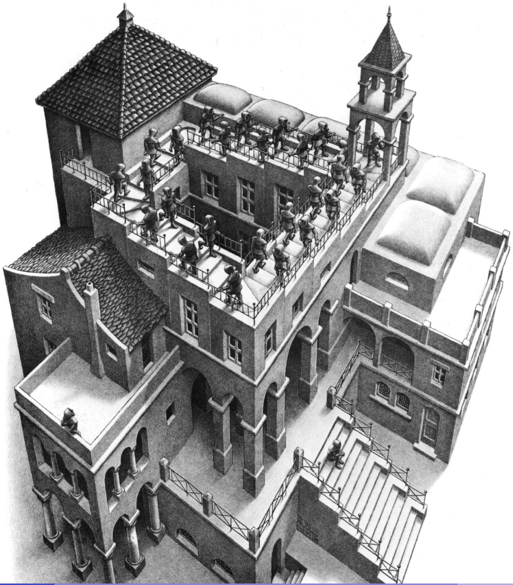


Illustration: global inconsistency



Topology of Paradox

- Clearly, the staircase *as a whole* cannot exist in the real world. Nonetheless, the constituent parts make sense *locally*.
- Quantum contextuality shows that the logical structure of quantum mechanics exhibits exactly these features of *local consistency*, but *global inconsistency*.
- This can happen because *not all variables can be measured at the same time* (non-commuting observables).
- We note that Escher's work was inspired by the *Penrose stairs*.
- Indeed, these figures provide more than a mere analogy. Penrose has studied the topological "twisting" in these figures using cohomology. This is quite analogous to our use of sheaf cohomology to capture the logical twisting in contextuality.
- Recent cross-over of these ideas into Constraint Satisfaction and structure isomorphism (refinements of Weisfeiler-Leman).

Local consistency as strong k-consistency: the Kolaitis-Vardi representation

Local consistency as strong k-consistency: the Kolaitis-Vardi representation

Given an approximation level k, we test if there are solutions for all $\leq k$-element subsets of the instance.

Local consistency as strong k-consistency: the Kolaitis-Vardi representation

Given an approximation level k, we test if there are solutions for all $\leq k$-element subsets of the instance.

More precisely, this amounts to the existence of a non-empty family S of homomorphisms $f : C \to B$, where C is an induced substructure of A with $|C| \leq k$.

Local consistency as strong k-consistency: the Kolaitis-Vardi representation

Given an approximation level k, we test if there are solutions for all $\leq k$-element subsets of the instance.

More precisely, this amounts to the existence of a non-empty family S of homomorphisms $f : C \to B$, where C is an induced substructure of A with $|C| \leq k$.

This is subject to the following conditions:

- **down-closure**: If $f : C \to B \in S$ and $C' \subseteq C$, then $f|_{C'} : C' \to B \in S$.
- **forth condition**: If $f : C \to B \in S$, $|C| < k$, and $a \in A$, then for some $f' : C \cup \{a\} \to B \in S$, $f'|_C = f$.

Local consistency as strong k-consistency: the Kolaitis-Vardi representation

Given an approximation level k, we test if there are solutions for all $\leq k$-element subsets of the instance.

More precisely, this amounts to the existence of a non-empty family S of homomorphisms $f : C \to B$, where C is an induced substructure of A with $|C| \leq k$.

This is subject to the following conditions:

- **down-closure**: If $f : C \to B \in S$ and $C' \subseteq C$, then $f|_{C'} : C' \to B \in S$.
- **forth condition**: If $f : C \to B \in S$, $|C| < k$, and $a \in A$, then for some $f' : C \cup \{a\} \to B \in S$, $f'|_C = f$.

This is equivalent to the existence of a winning strategy for Duplicator in the existential k-pebble game from A to B.

Local consistency as strong k-consistency: the Kolaitis-Vardi representation

Given an approximation level k, we test if there are solutions for all $\leq k$-element subsets of the instance.

More precisely, this amounts to the existence of a non-empty family S of homomorphisms $f : C \to B$, where C is an induced substructure of A with $|C| \leq k$.

This is subject to the following conditions:

- **down-closure**: If $f : C \to B \in S$ and $C' \subseteq C$, then $f|_{C'} : C' \to B \in S$.
- **forth condition**: If $f : C \to B \in S$, $|C| < k$, and $a \in A$, then for some $f' : C \cup \{a\} \to B \in S$, $f'|_C = f$.

This is equivalent to the existence of a winning strategy for Duplicator in the existential k-pebble game from A to B.

Notation: $A \to_k B$.

Local consistency as strong k-consistency: the Kolaitis-Vardi representation

Given an approximation level k, we test if there are solutions for all $\leq k$-element subsets of the instance.

More precisely, this amounts to the existence of a non-empty family S of homomorphisms $f : C \to B$, where C is an induced substructure of A with $|C| \leq k$.

This is subject to the following conditions:

- **down-closure**: If $f : C \to B \in S$ and $C' \subseteq C$, then $f|_{C'} : C' \to B \in S$.
- **forth condition**: If $f : C \to B \in S$, $|C| < k$, and $a \in A$, then for some $f' : C \cup \{a\} \to B \in S, f'|_C = f$.

This is equivalent to the existence of a winning strategy for Duplicator in the existential k-pebble game from A to B.

Notation: $A \to_k B$.

This fits perfectly into the sheaf-theoretic language used to capture contextuality by Abramsky-Brandenburger et al!

Strategies as Presheaves

Strategies as Presheaves

- We write $\Sigma_k(A)$ for the poset of subsets of A of cardinality $\leq k$. Each such subset gives rise to an induced substructure of A.

Strategies as Presheaves

- We write $\Sigma_k(A)$ for the poset of subsets of A of cardinality $\leq k$. Each such subset gives rise to an induced substructure of A.

- We define a presheaf $\mathscr{H}_k : \Sigma_k(A)^{\mathsf{op}} \to \mathbf{Set}$ by $\mathscr{H}_k(C) = \mathrm{hom}(C, B)$.
If $C' \subseteq C$, then the restriction maps are defined by $\rho^C_{C'}(h) = h|_{C'}$.

Strategies as Presheaves

- We write $\Sigma_k(A)$ for the poset of subsets of A of cardinality $\leq k$. Each such subset gives rise to an induced substructure of A.

- We define a presheaf $\mathscr{H}_k : \Sigma_k(A)^{\mathsf{op}} \to \mathbf{Set}$ by $\mathscr{H}_k(C) = \mathrm{hom}(C, B)$.
If $C' \subseteq C$, then the restriction maps are defined by $\rho^C_{C'}(h) = h|_{C'}$.

- This is the *presheaf of partial homomorphisms*.

Strategies as Presheaves

- We write $\Sigma_k(A)$ for the poset of subsets of A of cardinality $\leq k$. Each such subset gives rise to an induced substructure of A.

- We define a presheaf $\mathscr{H}_k : \Sigma_k(A)^{\mathsf{op}} \to \mathbf{Set}$ by $\mathscr{H}_k(C) = \mathrm{hom}(C, B)$.
If $C' \subseteq C$, then the restriction maps are defined by $\rho^C_{C'}(h) = h|_{C'}$.

- This is the *presheaf of partial homomorphisms*.

- A subpresheaf of $\mathscr{H}_k$ is a presheaf $\mathscr{S}$ such that $\mathscr{S}(C) \subseteq \mathscr{H}_k(C)$ for all $C \in \Sigma_k(A)$, and moreover if $C' \subseteq C$ and $h \in \mathscr{S}(C)$, then $\rho^C_{C'}(h) \in \mathscr{S}(C')$.

Strategies as Presheaves

- We write $\Sigma_k(A)$ for the poset of subsets of A of cardinality $\leq k$. Each such subset gives rise to an induced substructure of A.

- We define a presheaf $\mathscr{H}_k : \Sigma_k(A)^{\mathsf{op}} \to \mathbf{Set}$ by $\mathscr{H}_k(C) = \hom(C, B)$.
  If $C' \subseteq C$, then the restriction maps are defined by $\rho^C_{C'}(h) = h|_{C'}$.

- This is the *presheaf of partial homomorphisms*.

- A subpresheaf of $\mathscr{H}_k$ is a presheaf $\mathscr{S}$ such that $\mathscr{S}(C) \subseteq \mathscr{H}_k(C)$ for all $C \in \Sigma_k(A)$, and moreover if $C' \subseteq C$ and $h \in \mathscr{S}(C)$, then $\rho^C_{C'}(h) \in \mathscr{S}(C')$.

- A presheaf is *flasque* (or "flabby") if the restriction maps are surjective. This means that if $C \subseteq C'$, each $h \in \mathscr{S}(C)$ has an extension $h' \in \mathscr{S}(C')$ with $h'|_C = h$.

Proposition

There is a bijective correspondence between

1. *positional strategies from A to B*
2. *flasque sub-presheaves of $\mathscr{H}_k$.*

Proof.

The property of being a subpresheaf of $\mathscr{H}_k$ is equivalent to the down-closure property, while being flasque is equivalent to the forth condition. □

Global sections and cohomology

A *global section* is a family of partial homomorphisms $\{s_C : C \to B\}_{C \subseteq A, |C| \leq k}$ which agrees on overlaps:

$$\forall C, C' : \; s_C|_{C \cap C'} = s_{C'}|_{C \cap C'}$$

Global sections and cohomology

A *global section* is a family of partial homomorphisms $\{s_C : C \to B\}_{C \subseteq A, |C| \leq k}$ which agrees on overlaps:

$$\forall C, C' : \;\; s_C|_{C \cap C'} = s_{C'}|_{C \cap C'}$$

Global sections are in $1-1$ correspondence with homomorphisms $A \to B$.

Global sections and cohomology

A *global section* is a family of partial homomorphisms $\{s_C : C \to B\}_{C \subseteq A, |C| \leq k}$ which agrees on overlaps:

$$\forall C, C' : \ \ s_C|_{C \cap C'} = s_{C'}|_{C \cap C'}$$

Global sections are in $1-1$ correspondence with homomorphisms $A \to B$.

In previous work with Barbosa, Mansfield, Kishida and Lal (ABKLM) I used *cohomology* to characterise contextuality, *i.e.* the *non-existence* of such a homomorphism.

Global sections and cohomology

A *global section* is a family of partial homomorphisms $\{s_C : C \to B\}_{C \subseteq A, |C| \leq k}$ which agrees on overlaps:

$$\forall C, C' : \; s_C|_{C \cap C'} = s_{C'}|_{C \cap C'}$$

Global sections are in $1-1$ correspondence with homomorphisms $A \to B$.

In previous work with Barbosa, Mansfield, Kishida and Lal (ABKLM) I used *cohomology* to characterise contextuality, *i.e.* the *non-existence* of such a homomorphism.

This gives linear-algebraic approximations to global sections: a family $\{r_C\}_C$ where r_C is a $\mathbb{Z}$*-linear combination* of local sections, satisfying the compatibility (*i.e.* agree-on-overlap) conditions, suitably extended to linear combinations.

Global sections and cohomology

A *global section* is a family of partial homomorphisms $\{s_C : C \to B\}_{C \subseteq A, |C| \leq k}$ which agrees on overlaps:

$$\forall C, C' : \; s_C|_{C \cap C'} = s_{C'}|_{C \cap C'}$$

Global sections are in $1-1$ correspondence with homomorphisms $A \to B$.

In previous work with Barbosa, Mansfield, Kishida and Lal (ABKLM) I used *cohomology* to characterise contextuality, *i.e.* the *non-existence* of such a homomorphism.

This gives linear-algebraic approximations to global sections: a family $\{r_C\}_C$ where r_C is a $\mathbb{Z}$*-linear combination* of local sections, satisfying the compatibility (*i.e.* agree-on-overlap) conditions, suitably extended to linear combinations.

Given $s : C_0 \to B$, we can ask if it has an extension to such a $\mathbb{Z}$-linear family $\{r_C\}$, with $r_{C_0} = 1 \cdot s$.

Global sections and cohomology

A *global section* is a family of partial homomorphisms $\{s_C : C \to B\}_{C \subseteq A, |C| \leq k}$ which agrees on overlaps:

$$\forall C, C' : \; s_C|_{C \cap C'} = s_{C'}|_{C \cap C'}$$

Global sections are in $1-1$ correspondence with homomorphisms $A \to B$.

In previous work with Barbosa, Mansfield, Kishida and Lal (ABKLM) I used *cohomology* to characterise contextuality, *i.e.* the *non-existence* of such a homomorphism.

This gives linear-algebraic approximations to global sections: a family $\{r_C\}_C$ where r_C is a $\mathbb{Z}$*-linear combination* of local sections, satisfying the compatibility (*i.e.* agree-on-overlap) conditions, suitably extended to linear combinations.

Given $s : C_0 \to B$, we can ask if it has an extension to such a $\mathbb{Z}$-linear family $\{r_C\}$, with $r_{C_0} = 1 \cdot s$.

We can use this test to filter out those local sections from the k-consistency approximation which *do not have* such extensions, getting a sharper approximation.

Cohomological k-consistency

Key insight by Adam O' Conghaile: this cohomological refinement of k-consistency is *efficiently computable*!
(Since the predicate "s has a $\mathbb{Z}$-linear extension" translates into solvability of a polynomial size system of $\mathbb{Z}$-linear equations).

Cohomological k-consistency

Key insight by Adam O' Conghaile: this cohomological refinement of k-consistency is *efficiently computable*!
(Since the predicate "s has a $\mathbb{Z}$-linear extension" translates into solvability of a polynomial size system of $\mathbb{Z}$-linear equations).

So we get a new refined approximation algorithm for $\mathsf{CSP}(B)$.

Cohomological k-consistency

Key insight by Adam O' Conghaile: this cohomological refinement of k-consistency is *efficiently computable*!
(Since the predicate "s has a $\mathbb{Z}$-linear extension" translates into solvability of a polynomial size system of $\mathbb{Z}$-linear equations).

So we get a new refined approximation algorithm for $\mathsf{CSP}(B)$.

A key property which follows from the ABKLM results on contextuality:

Proposition

Cohomological k-consistency is exact for affine templates B.

Cohomological k-consistency

Key insight by Adam O' Conghaile: this cohomological refinement of k-consistency is *efficiently computable*!
(Since the predicate "s has a $\mathbb{Z}$-linear extension" translates into solvability of a polynomial size system of $\mathbb{Z}$-linear equations).

So we get a new refined approximation algorithm for $\mathsf{CSP}(B)$.

A key property which follows from the ABKLM results on contextuality:

Proposition

Cohomological k-consistency is exact for affine templates B.

So cohomological k-consistency captures the two main tractable classes identified by Feder-Vardi in a unified fashion.

Cohomological k-consistency

Key insight by Adam O' Conghaile: this cohomological refinement of k-consistency is *efficiently computable*!
(Since the predicate "s has a $\mathbb{Z}$-linear extension" translates into solvability of a polynomial size system of $\mathbb{Z}$-linear equations).

So we get a new refined approximation algorithm for $\mathsf{CSP}(B)$.

A key property which follows from the ABKLM results on contextuality:

Proposition

Cohomological k-consistency is exact for affine templates B.

So cohomological k-consistency captures the two main tractable classes identified by Feder-Vardi in a unified fashion.

Question

Is cohomological k-consistency exact for all tractable cases?

Cohomological k-equivalence

The whole story adapts quite straightforwardly to yield cohomological refinements of the Weisfeiler-Leman approximations to graph and structure isomorphism.

Cohomological k-equivalence

The whole story adapts quite straightforwardly to yield cohomological refinements of the Weisfeiler-Leman approximations to graph and structure isomorphism.

Recall that these correspond to equivalence in k-variable logic with counting $\mathbf{C}_k$.

Cohomological k-equivalence

The whole story adapts quite straightforwardly to yield cohomological refinements of the Weisfeiler-Leman approximations to graph and structure isomorphism.

Recall that these correspond to equivalence in k-variable logic with counting $\mathbf{C}_k$.

Moreover, the result on completeness of cohomological k-consistency for affine templates is leveraged to show that $\equiv_k^{\mathbb{Z}}$ is discriminating enough to defeat two important families of counter-examples:

- the CFI (Cai-Furer-Immerman) construction used to show that $\mathbf{C}_k$ is not strong enough to characterise polynomial time, and
- the constructions due to Lichter and Dawar et al. which are used to show similar results for linear algebraic extensions of $\mathbf{C}_k$.

Cohomological k-equivalence

The whole story adapts quite straightforwardly to yield cohomological refinements of the Weisfeiler-Leman approximations to graph and structure isomorphism.

Recall that these correspond to equivalence in k-variable logic with counting $\mathbf{C}_k$.

Moreover, the result on completeness of cohomological k-consistency for affine templates is leveraged to show that $\equiv_k^{\mathbb{Z}}$ is discriminating enough to defeat two important families of counter-examples:

- the CFI (Cai-Furer-Immerman) construction used to show that $\mathbf{C}_k$ is not strong enough to characterise polynomial time, and
- the constructions due to Lichter and Dawar et al. which are used to show similar results for linear algebraic extensions of $\mathbf{C}_k$.

References:

- `https://arxiv.org/abs/2206.15253` (AOC paper appeared in MFCS 2022)
- `https://arxiv.org/abs/2206.12156` (SA notes)