Strict Leibniz Hierarchy and Categories of Logics

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Abstract

The Leibniz hierarchy is a classification system for propositional logics in terms of their behavior with respect to the Leibniz congruence/operator Ω. In [AP], we propose the introduction of two new classes of logics into the Leibniz hierarchy: the class of Ω-natural logics and its intersection with the class of truth-equational logics, which we called truth-natural. In short, we say that a logic has the property of Ω-naturality when its Leibniz operator commutes with inverse substitutions (endomorphisms on the algebra of formulas), or equivalently, inverse arbitrary homomorphisms. It is then clear that the well-known classes of equivalential logics and algebraizable logics, for instance, represent the intersection between the class of Ω-natural logics and the classes of protoalgebraic and weakly algebraizable logics, respectively.

In order to substantiate the relevance of these new classes, we looked to the attempted formalization of the Leibniz hierarchy conducted in [JM1, JM2, JM3], which also discusses the collection of all logics viewed as a poset whose partial order is given by existence of interpretations (in a specific sense) between logics, drawing inspiration from the Maltsev hierarchy of Universal Algebra. Furthermore, the question of whether there is a precise relation between the so-called Leibniz classes and the behavior of the Leibniz operator is raised in [JM2, Prob. 2].

We show in [AP, Prop. 3.3, Thm. 4.14] not only a sufficient condition for Ω-naturality determined solely by the underlying language of a given logic, but also that the class of Ω-natural logics do not, in fact, comprise a Leibniz class [JM2, Thm. 2.2 (ii)]. In this case, we have a negative answer to the aforementioned open problem associating Leibniz classes and the Leibniz operator.

With this issue in mind, we will opt to distinguish the definition of Leibniz-reduced interpretation (LR-interpretation, for short) from that of Suszko-reduced interpretation (SR-interpretation, for short):

Definition. *(translation, SR-interpretation, LR-interpretation)*

- *Given two languages Σ and Σ', a* **translation** *[JM1, Def. 3.1] from Σ to Σ' is any arity-preserving map $\tau : \Sigma \to Fm_{\Sigma'}$. We can see that any translation τ induces a contravariant functor $\tau^* : \Sigma'{-}Str \to \Sigma{-}Str$.*
- *Given a Σ-logic S and a Σ'-logic S', an* **SR-interpretation** *[JM1, Def. 3.2] of S into S' is a translation τ from Σ to Σ' such that $\tau^*[Mod^{\Xi}(S')] \subseteq Mod^{\Xi}(S)$.*
- *Given a Σ-logic S and a Σ'-logic S', an* **LR-interpretation** *of S into S' is a translation τ from Σ to Σ' such that $\tau^*[Mod^*(S')] \subseteq Mod^*(S)$,*

where Mod^{Ξ} (resp. Mod^*) denotes the class of all matrix models for a given logic whose filters have Suszko (resp. Leibniz) congruences coinciding with the identity relation, which are called ***Suszko-reduced*** *(resp.* ***Leibniz-reduced****, or simply* **reduced***) matrices.*

It is therefore easy to see that any LR-interpretation is also an SR-interpretation, since the Suszko congruence is always contained in the Leibniz congruence. This observation, together with the fact [JM1, Prop. 3.3] that SR-interpretations are also flexible morphisms of logics in the sense of [AFLM], shows that the following inclusion of (wide) subcategories

holds: $Log_{LR} \hookrightarrow Log_{SR} \hookrightarrow Log_f$, which denote the categories of all logics with LR-interpretations, SR-interpretations and flexible morphisms, respectively. In fact, Log_{SR} is simply the categorical reframing of the poset of all logics Log as defined in [JM1, Def. 3.5] before passing through the quotient of equi-interpretability. In this sense, we can then investigate the relationships between those three categories, e.g. finding possible equivalences, adjoints, their associated monads and algebras.

Moreover, we would also like to point out that the distinction between the Suszko and Leibniz congruences (and therefore between SR-interpretations and LR-interpretations) is non-existent among protoalgebraic logics [Cze, Thm. 1.5.4]. In Czelakowsi's words, "the list of plausible properties of the Suszko operator, parallel to those of the Leibniz one, may thus serve as a basis for distinguishing a hierarchy of all logics which, when restricted to protoalgebraic logics, agrees with the [Leibniz] hierarchy" [Cze, p. 9].

Now, in the context of Leibniz conditions and classes (see [JM2, Def. 2.1], if we replace (i.e. strengthen) SR-interpretability with LR-interpretability in the corresponding definitions, we can then define what we shall call **strict Leibniz conditions**, **strict Leibniz classes** and the **strict Leibniz hierarchy**. Therefore, once we make the appropriate tweaks to the surrounding concepts, this adaptation also preserves most, if not all properties analogous to those of Leibniz classes (such as the appealing [JM2, Thm. 2.2]). Even so, it remains to be verified whether Ω-natural (resp. truth-natural) logics do indeed form a strict Leibniz class via this characterization.

In conclusion, we also leave as a possibility for further investigation that of determining which categorical properties Log_{SR} and Log_{LR} possess. It is already known that Log_{SR} admits weak products, but not necessarily even finite weak coproducts, given that the poset Log admits arbitrary infima but not necessarily even finite suprema (see [JM1, Thm. 4.6, 5.1]). In addition, both Log_{SR} and Log_{LR} seem to be very good candidates for being factorization systems, considering the decomposition [JM1, Prop. 3.8] of any SR-interpretation into a compatible expansion and a term-equivalence.

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References

[AP] U. C. Almeida and D. C. Pinto. **Expanding the Leibniz hierarchy**. To appear in: Journal of Logic and Computation. Oxford University Press, 2024. `https://doi.org/10.1093/logcom/exae025`.

[AFLM] P. Arndt, R. A. Freire, O. O. Luciano and H. L. Mariano. **A global glance on categories in Logic**, Logica Universalis, 1, 3–39, 2007.

[AMP] P. Arndt, H. L. Mariano and D. C. Pinto. **Congruence Filter Pairs, Adjoints and Leibniz Hierarchy**. To appear in: Journal of Logic and Computation. Oxford University Press, 2024. `https://doi.org/10.1093/logcom/exae002`.

[Cze] J. Czelakowski. **Protoalgebraic logics**. Trends in Logic - Studia Logica Library, v. 10. Kluwer Academic Publishers, Dordrecht, 2001.

[Fon] J. M. Font. **Abstract Algebraic Logic - An Introductory Textbook**. Imperial College Press, London, 2016. ISBN 978-1-84890-207-7.

[JM1] R. Jansana and T. Moraschini. **The poset of all logics I: interpretations and lattice structure**. Journal of Symbolic Logic, 86(3):935–964, 2021a. `https://doi.org/10.1017/jsl.2021.48`.

[JM2] R. Jansana and T. Moraschini. **The poset of all logics II: Leibniz classes and hierarchy**. Published online in the Journal of Symbolic Logic, 88(1):324–362, 2021b. `https://doi.org/10.1017/jsl.2021.49`.

[JM3] R. Jansana and T. Moraschini. **The poset of all logics III: Finitely presentable logics**. Studia Logica, 109(3):539—580, 2021c. `https://doi.org/10.1007/s11225-020-09916-z`.