Hybrid logic for strict betweenness*

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The purpose of this talk is to present a hybrid logic for the ternary geometric betweenness relation in the sense of (Borsuk and Szmielew, 1960).

Let $\mathfrak{F} \coloneqq \langle U, B\rangle$ be a 3-frame, i.e., a frame with a ternary relation on the set of points. We will read $B(x,y,z)$ as *y is between x and z*. $\#(x,y,z)$ means that the elements of the set x, y and z are pairwise different. The betweenness axioms we take into account are:

$$B(x,y,z) \rightarrow \#(x,y,z)\,, \tag{B1}$$
$$B(x,y,z) \rightarrow B(z,y,x)\,, \tag{B2}$$
$$B(x,y,z) \rightarrow \neg B(x,z,y)\,, \tag{B3}$$
$$B(x,y,z) \wedge B(y,z,u) \rightarrow B(x,y,u)\,, \tag{B4}$$
$$B(x,y,z) \wedge B(y,u,z) \rightarrow B(x,y,u)\,, \tag{B5}$$
$$\#(x,y,z) \rightarrow B(x,y,z) \vee B(x,z,y) \vee B(y,x,z)\,, \tag{B6}$$
$$x \neq z \rightarrow \exists y\, B(x,y,z)\,, \tag{B7}$$
$$\forall y \exists x \exists z\, B(x,y,z)\,. \tag{B8}$$

Any 3-frame $\mathfrak{F}$ that satisfies (B1)–(B5) will be called *a betweenness frame* or simply a *b-frame*. The class of all b-frames will be denoted by '**B**'. We also distinguish the following classes:

(1) **LB** $\coloneqq$ **B** + (B6) of *linear* b-frames,

(2) **DLB** $\coloneqq$ **B** + (B6) + (B7) of *dense* linear b-frames,

(3) **LBWE** $\coloneqq$ **B** + (B6) + (B8) of *linear* b-frames *without endpoints*,

(4) **DLBWE** $\coloneqq$ **B** + (B6) + (B7) + (B8) of dense linear b-frames without endpoints.

As the betweenness relation is ternary for its modal analysis we are going to need binary modal operators. The basic idea for such an operator $\langle B\rangle$ comes from van Benthem and Bezhanishvili (2007). Given a model $\mathfrak{M} \coloneqq \langle \mathfrak{F}, V\rangle$ based on a b-frame $\mathfrak{F}$ we characterize the semantics for $\langle B\rangle$ in the following way

$$\mathfrak{M}, w \Vdash \langle B\rangle(\varphi,\psi) \;:\longleftrightarrow\; (\exists x, y \in W)\,(\mathfrak{M}, x \Vdash \varphi \text{ and } \mathfrak{M}, y \Vdash \psi \text{ and } B(x,w,y))\,. \tag{df $\langle B\rangle$}$$

$\langle B\rangle$ gives rise to a natural unary *convexity* operator

$$\mathrm{C}\,\varphi \;:\longleftrightarrow\; \langle B\rangle(\varphi,\varphi)\,. \tag{df C}$$

As has already been said, we are going to study the properties of $\langle B\rangle$ in the hybrid language with two sorts of variables: *propositional letters* p, q, r and so on, and *nominals* i, j, k, l, indexed if necessary. The set of all propositional letters will be denoted by 'Prop', and the set

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of nominals by 'Nom'. We assume that $\mathrm{Prop} \cap \mathrm{Nom} = \varnothing$. The *valuation* function is any function $V : \mathrm{Prop} \cup \mathrm{Nom} \to \mathcal{P}(U)$ such that for every nominal i, $V(i)$ is a singleton subset of the universe.

Recall that the semantics of the *at* operator—for which we standardly use @—is given by the following:

$$\mathfrak{M}, w \Vdash @_i \varphi :\longleftrightarrow \mathfrak{M}, V(i) \Vdash \varphi\,. \tag{df $@_i$}$$

We can see that:

$$\mathfrak{M}, w \Vdash \langle B \rangle (i, j) \longleftrightarrow (\exists x \in U)(\exists y \in U)\,(V(i) = \{x\} \text{ and } V(j) = \{y\} \text{ and } B(x, w, y))\,.$$

Using the standard techniques from (Blackburn et al., 2001) and generalized tools from (ten Cate, 2005), we are going to prove that

Theorem 0.1. *For every $i \in \{1, 3, 4, \ldots, 7\}$ the class of frames that satisfies* (Bi) *is not modally definable. Moreover, the class of* (B7)*-frames is not @-definable.*

Theorem 0.2. **DLBWE** *is @-definable. Indeed,* **DLBWE** *is @-definable by pure formulas.*

Making use of some results from (Bezhanishvili et al., 2023) we also show that

Theorem 0.3. *For any 3-frame $\mathfrak{F} \in$* **LBWE**: *$\mathfrak{F} \models$ (B7) iff $\mathfrak{F} \Vdash \mathrm{C}p \to \mathrm{C}\mathrm{C}p$, i.e., density is modally definable with respect to the class* **LBWE**.

We also discuss a system of hybrid logic L_B of strict betweenness built in the language with @-operator (following the style of ten Cate, 2005, Definition 5.1.2).

Finally, we analyze the case of the real line treated as the model of betweenness with the second-order completeness axiom, and we discuss methods to grasp the axiom in different languages.

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