# Gödel-Mal'tsev type compactness theorems for second-order $(T_1, T_2)$-formulae

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**Abstract**

The existence of filtered products and ultraproducts in the corresponding new category of algebraic systems leads to a compactness type theorems for second-order $(T_1, T_2)$-formulae.

## Contents



## 1 The category of $(T_1, T_2)$-systems

By the arithmetic type of an algebraic system (for short, system) $\mathfrak{A} = (Q; \Sigma, \Omega)$ we understand the ordered pair $(T_1, T_2)$ of sets of natural numbers defined thus:

$$T_1 = \{|A| \,| A \in \Sigma\} \subseteq \mathbb{N},$$
$$T_2 = \{|P| \,| P \in \Omega\} \subseteq \mathbb{N},$$

where $|P|$ is the arity of the predicate $P$ and $|A|$ that of the operation $A$. An algebraic system of arithmetic type $(T_1, T_2)$ is called a $(T_1, T_2)$-system. If $\Sigma = \emptyset$, then $\mathfrak{A}$ is called a $T_2$-model. Two systems are said to be of the same arithmetic type if their arithmetic types are identical.

Let $\mathfrak{A} = (Q; \Sigma, \Omega)$ and $\mathfrak{A}' = (Q'; \Sigma', \Omega')$ be two systems of the same arithmetic type[1]. A triple $(\alpha, \tilde{\beta}, \tilde{\gamma})$ of maps $\alpha : Q \to Q'$, $\tilde{\beta} : \Sigma \to \Sigma'$ and $\tilde{\gamma} : \Omega \to \Omega'$ is called a homomorphism from $\mathfrak{A}$ to $\mathfrak{A}'$ (notation: $(\alpha, \tilde{\beta}, \tilde{\gamma}) : \mathfrak{A} \Rightarrow \mathfrak{A}'$) if $\tilde{\beta}$ and $\tilde{\gamma}$ preserve arity and

$$\alpha A(x_1, \ldots, x_n) = \left[\tilde{\beta}(A)\right](\alpha x_1, \ldots, \alpha x_n),$$
$$P(y_1, \ldots, y_m) = 1 \Longrightarrow [\tilde{\gamma}(P)]\,(\alpha y_1, \ldots, \alpha y_m) = 1\ ,$$

where $A \in \Sigma$, $P \in \Omega$, $|A| = n$, $|P| = m$, $x_1, \ldots, x_n, y_1, \ldots, y_m \in Q$.

Systems of the same arithmetic type form a category with the homomorphisms $(\alpha, \tilde{\beta}, \tilde{\gamma})$ as morphisms. In what follows the concepts of subsystem, kernel, congruence, factor system, direct product of systems, reduced product and ultraproduct are to be understood in the sense of this category.

Suppose we are given some set $\mathcal{K}$ of second order formulas [1, 2, 3] and any subsets $T_1$ and $T_2$ of $\mathbb{N}$. We say that a formula in this set has arithmetic type $(T_1, T_2)$, or is a $(T_1, T_2)$-formula,

[1] For simplicity, we consider only algebraic systems without nullary operations and predicates.

if the set of arities of all functional variables occurring in it is contained in $T_1$ while those of the predicate variables are contained in $T_2$. Values and validity (semantics) of a $(T_1, T_2)$-formula in the set $\mathcal{K}$ are defined here on an algebraic system $\mathfrak{A} = (Q; \Sigma, \Omega)$ of arithmetic type $(T_1, T_2)$ as the values and validity of formulae with bounded predicate (functional) quantifiers $(\forall X)$, $(\exists X)$. What that means is this: 'for every value $X = P \in \Omega$ ($X = A \in \Sigma$ respectively) of the corresponding arity', 'there exists a value $X = P \in \Omega$ ($X = A \in \Sigma$ respectively) of the corresponding arity'. The values (of corresponding arities) for free predicate and functional variables are also chosen from $\mathfrak{A}$, more precisely, from $\Omega$ and $\Sigma$ respectively.

An absolutely closed second-order formula is said to be a second-order proposition or property.

## 2 Main results

The expression $\mathfrak{A} \vdash \mathcal{F}$ means that the $(T_1, T_2)$-formula $\mathcal{F}$ or an appropriate value of it holds in the $(T_1, T_2)$-system $\mathfrak{A}$.

**Theorem 1.** *If a second-order proposition of arithmetic type $(T_1, T_2)$ holds in each member $\mathfrak{A}_i$ of a set $\{\mathfrak{A}_i : i \in I\}$ of $(T_1, T_2)$-systems, then it holds in every ultraproduct of the systems.*

An algebraic system $\mathfrak{A}$ of arithmetic type $(T_1, T_2)$ will be called a $(T_1, T_2)$-model of a set $\Gamma$ of second-order propositions of arithmetic type $(T_1, T_2)$ if $\mathfrak{A} \vdash \mathcal{F}$ for all $\mathcal{F}$ in $\Gamma$.

**Theorem 2.** *A set $\Gamma$ of second-order propositions of arithmetic type $(T_1, T_2)$ has a $(T_1, T_2)$-model if and only if every finite subset $\Gamma_0$ of $\Gamma$ has a $(T_1, T_2)$-model.*

Let $\Gamma_1$ and $\Gamma_2$ be sets of second-order propositions of arithmetic type $(T_1, T_2)$. We shall say that $\Gamma_2$ is a consequence of $\Gamma_1$ if every $(T_1, T_2)$-model for $\Gamma_1$ is a $(T_1, T_2)$-model for $\Gamma_2$. Sets $\Gamma_1$ and $\Gamma_2$ are equivalent if each is a consequence of the other.

**Theorem 3.** *If $\Gamma$ is a set of second-order propositions of arithmetic type $(T_1, T_2)$ that is equivalent to a set $\{\mathcal{F}\}$ consisting of a single proposition, then there exists a finite subset of $\Gamma$ equivalent to $\{\mathcal{F}\}$.*

A $(T_1, T_2)$-formula $F$ with free predicate, functional and/or object variables is said to be realizable in a $(T_1, T_2)$-system $\mathfrak{A}$ if there exist values in $\mathfrak{A}$ of all free variables occurring in $F$ for which $F$ is valid in $\mathfrak{A}$. A collection $\mathcal{F}$ of $(T_1, T_2)$- formulae is realizable in a $(T_1, T_2)$-system $\mathfrak{A}$ if there exist values in $\mathfrak{A}$ of all the free variables occurring in the members of $\mathcal{F}$ for which all the formulae in $\mathcal{F}$ are valid in $\mathfrak{A}$. A collection of $(T_1, T_2)$-formulae is said to be realizable if there is a $(T_1, T_2)$-system in which it is realizable.

For any $n \in \mathbb{N}$, a collection $\Gamma$ of $(T_1, T_2)$-formulae is said to be $n$-realizable if there exists a $(T_1, T_2)$-system in which every subset $\mathcal{F}$ of $\Gamma$ with at most $n$ members is realizable.

**Theorem 4.** *If every finite part of an infinite collection of $(T_1, T_2)$-formulae is realizable (n-realizable, respectively) then the whole collection is realizable (n-realizable, respectively) too.*

## References


[1] Mal'tsev A. I. Some questions of the theory of classes of models. *Proceedings of the IVth All-Union Mathematical Congress*, 1:169–198, 1963.

[2] Movsisyan Yu. M. *Introduction to the Theory of Algebras with Hyperidentities.* Yerevan State University Press, 1986.

[3] Movsisyan Yu. M. *Hyperidentities: Boolean and De Morgan Structures.* World Scientific, 2023.