Ill-Posed Linear Operator Equations

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Table of Content

Introduction

- Hadamard's definition of well-posedness
- Ill-posed problem in terms of linear operators

Moore-Penrose Generalized Inverse

- Definitions
- Propositions and Theorems

Compact Linear Operators

- Introduction
- Propositions and Theorems
- Picard Criterion and Stability

Spectral Theory and Functional Calculus

- Introduction
- Definitions, Propositions and Theorems
- More Propositions

Hadamard's Definition Of Well-Posedness

For all admissible data, a solution exists. (1)

For all admissible data, the solution is unique. (2)

The solution depends continuously on the data. (3)

1. (1) is ensured by relaxing the notion of a solution.
2. (2) is much more serious problem.In inverse problems, one is looking for the cause for and observed effect, not cause for a desired effect.
3. Violation of (3) creates numerical issues. Traditional numerical methods become unstable.

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Ill-posed problem in terms of linear operators

Definition

Let $T : X \longmapsto Y$ be a bounded linear operator between Hilbert spaces X and Y. We call y *Attainable* if

$$y \in R(T) \tag{4}$$

1. (1) is equivalent to the condition that
$$y \text{ is attainable for every } y \in Y.$$
2. (2) is equivalent to the condition that
$$T^{-1} \text{ exists} \iff N(T) = 0.$$
3. (3) is equivalent to
$$T^{-1} \text{ is continuous.}$$

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Table of Content

Introduction
Hadamard's definition of well-posedness
Ill-posed problem in terms of linear operators
Moore-Penrose Generalized Inverse
Definitions
Propositions and Theorems
Compact Linear Operators
Introduction
Propositions and Theorems
Picard Criterion and Stability
Spectral Theory and Functional Calculus
Introduction
Definitions, Propositions and Theorems
More Propositions

Definition (1.1)

Let $T : X \longmapsto Y$ be bounded linear operator.

1. $x \in X$ is called a least-squares solution of $Tx = y$, if

$$\|Tx - y\| = inf\{\|Tx - y\| \, |x \in X\} \tag{5}$$

Definition (1.1)

Let $T : X \longmapsto Y$ be bounded linear operator.

2. $x \in X$ is called best-approximate solution of $Tx = y$ if, x is a least-squares solution of $Tx = Y$ and

$$\|x\| = inf\{\|x\| \, |x \quad is \quad least - squares \quad solution\} \tag{6}$$

Definition (1.2)

The Moore-Penrose Generalized inverse $T^\dagger$ of $T \in L(X,Y)$ is defined as the unique linear extension of $\tilde{T}^{-1}$ to

$$D(T^\dagger) = R(T) + R(T)^\perp \tag{7}$$

$$N(T^\dagger) = R(T)^\perp \tag{8}$$

Where,

$$\tilde{T} := T|_{N(T)^\perp} : N(T)^\perp \to R(T) \tag{9}$$

Proposition (1.3)

Let P and Q be the orthogonal projectors onto NT and $\overline{R\{T\}}$, respectively. Then $R(T^\dagger) = N(T)^\perp$, and the four "Moore-Penrose Equations" hold:

$$TT^\dagger = Q|_{D(T^\dagger)}, \tag{10}$$

$$T^\dagger T = I - P, \tag{11}$$

$$TT^\dagger T = T, \tag{12}$$

$$T^\dagger TT^\dagger = T^\dagger. \tag{13}$$

Proposition (1.4)

The Moore-Penrose generalized inverse $T^\dagger$ has a closed graph $gr(T^\dagger)$. Furthermore, $T^\dagger$ is bounded(i.e. continuous) if and only if $R(T)$ is closed.

Theorem 1.5

Theorem

Let $y \in D(T^\dagger)$. Then, $Tx = y$ has a unique best-approximate solution, which is given by

$$x^\dagger := T^\dagger y. \tag{14}$$

The set of all least-square solutions is $x^\dagger + N(T)$.

Theorem 1.6

Theorem

Let $y \in D(T^\dagger)$. Then, $Tx = y$ is the least-squares solution of $Tx = y$ if and only if the normal equation

$$T^*Tx = T^*y \tag{15}$$

holds.

It follows from Theorem 1.6 that $T^\dagger y$ is the solution of $T^*Tx = T^*y$ of minimal norm,i.e.,

$$T^\dagger = (T^*T)^\dagger T^* \tag{16}$$

Table of Content

Introduction
- Hadamard's definition of well-posedness
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Moore-Penrose Generalized Inverse
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Compact Linear Operators
- Introduction
- Propositions and Theorems
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- Introduction
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Compact Linear Operators

- We study Compact Linear operators since integral operators are compact under suitable assumptions.
- For selfadjoint linear operator *Eigensystem* $(\lambda_n; v_n)$ consists of :-
  - All non-zero eigenvalues λ_n
  - A corresponding complete set of eigenvectors v_n
- The operator K can be diagonalized as follows:

$$Kx = \sum_{n=1}^{\infty} \lambda_n \langle x, v_n \rangle v_n$$

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$$Kx = \sum_{n=1}^{\infty} \lambda_n \langle x, v_n \rangle v_n$$

- If K is not sefladjoint, no eigenvalues need to exist.
- By the (15), we construct a *singular system* $(\sigma_n; v_n, u_n)$
- If $K^* : Y \longrightarrow X$ denotes the adjoint of K,
    - $\{\sigma_n^2\}_{n\in N}$ are the non zero eigenvalues of the self adjoint operator K^*K(and also KK^*),
    - $\{\sigma_n^2\}_{n\in N}$ are the corresponding complete orthonormal system of eigenvectors of K^*K.
    - $\{u_n^2\}_{n\in N}$ are defined as vectors -

$$u_n = \frac{Kv_n}{\|Kv_n\|}$$

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- $\{v_n^2\}_{n \in N}$ span $\overline{R(K^*)} = \overline{R(K^*K)}$
- $\{u_n^2\}_{n \in N}$ span $\overline{R(K)} = \overline{R(KK^*)}$
- The following formulas hold:

$$Kv_n = \sigma_n u_n, \tag{17}$$

$$K^* u_n = \sigma_n v_n, \tag{18}$$

$$Kx = \sum_{n=1}^{\infty} \sigma_n < x, v_n > u_n, \tag{19}$$

$$K^* y = \sum_{n=1}^{\infty} \sigma_n < y, u_n > v_n \tag{20}$$

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Proposition (1.7)

Let $K : X \longrightarrow Y$ be compact, $dimR(K) = \infty$. Then $K^{\dagger}$ is a densely defined unbounded linear operator with closed graph.

Theorem (1.8)

Let $(\sigma_n; v_n, u_n)$ be a singular system for the compact linear operator K , $y \in Y$. Then we have:

- $$y \in D(K^\dagger) \iff \sum_{n=1}^{\infty} \frac{|\langle y, u_n\rangle|^2}{\sigma_n^2} < \infty \tag{21}$$
- *For $y \in D(k^\dagger)$,*
$$K^\dagger y = \sum_{n=1}^{\infty} \frac{\langle y, u_n\rangle}{\sigma_n} v_n. \tag{22}$$

Picard Criterion

- A best-approximate solution of $Kx = y$ exists only if the generalized Fourier coefficients $(\langle y, u_n \rangle)$ decay fast enough relative to σ_n

- **Stability**
  - Error components which correspond to large singular value are harmless.
  - Error which correspond to small σ_n are dangerous.
  - If $dim R(K) = \infty$, data errors of fixed size can be amplified arbitrarily without bound.

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Degree of Ill-Posedness

- A problem is *mildly ill-posed* if $\sigma_n = O(n^{-\alpha})$
- A problem is *severely ill-posed* if $\sigma_n = O(e^{-n})$

Example 1.9

One dimesional backwards heat equation

$$\frac{du}{dt}(x,t) = \frac{d^2u}{dx^2}(x,t), \quad x \in [0,\pi], t \geqslant 0, \tag{23}$$

with homogeneous Dirichlet boundary conditions

$$u(0,t) = u(\pi,t) = 0, \quad x \in [0,\pi], \tag{24}$$

and assume the final temperature

$$f(x) := u(x,1), \quad x \in [0,\pi] \tag{25}$$

is given with $f(0) = f(\pi) = 0$; Find the initial temperature

$$v_0(x) := u(x,0), \quad x \in [0,\pi]. \tag{26}$$

Table of Content

Introduction
Hadamard's definition of well-posedness
Ill-posed problem in terms of linear operators
Moore-Penrose Generalized Inverse
Definitions
Propositions and Theorems
Compact Linear Operators
Introduction
Propositions and Theorems
Picard Criterion and Stability
Spectral Theory and Functional Calculus
Introduction
Definitions, Propositions and Theorems
More Propositions

Spectral Theory and Functional Calculus

- For $\lambda \in R$ and $x \in X$, we define

$$E_\lambda x := \sum_{\substack{n=1 \\ \sigma_n^2 < \lambda}}^{\infty} \langle x, v_n \rangle v_n \quad (+P) \tag{27}$$

where P is the orthogonal projector onto $N(K^*K)$ and only when $\lambda > 0$.

- For all λ, E_λ is an orthogonal projector and projects onto

$$\chi_\lambda := span\{v_n | n \in N, \sigma_n^2 < \lambda\} \quad (+N(K^*K), if \quad \lambda > 0) \tag{28}$$

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Definition

- E_λ is known as the spectral family.
- E_λ has jumps at $\lambda = \sigma_n^2$ of "height"
$$\sum_{\substack{n=1 \\ \sigma_n^2 < \lambda}}^{\infty} \langle \cdot, v_n \rangle v_n$$
- Hence we write

$$K^* K x = \sum_{n=1}^{\infty} \sigma_n^2 \langle x, v_n \rangle v_n = \int \lambda E_\lambda x. \tag{29}$$

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Definition

- This motivates the following notation:-
  - $$\int_{-\infty}^{+\infty} f(\lambda)dE_\lambda x = \sum_{n=1}^{\infty} f(\sigma_n^2) \langle x, v_n \rangle v_n \tag{30}$$
  - $$\int_{-\infty}^{+\infty} f(\lambda)d\langle E_\lambda x, y\rangle = \sum_{n=1}^{\infty} f(\sigma_n^2) \langle x, v_n \rangle \langle y, u_n \rangle \tag{31}$$
  - $$\int_{-\infty}^{+\infty} f(\lambda)d\|E_\lambda x\|^2 = \sum_{n=1}^{\infty} f(\sigma_n^2) |\langle x, v_n \rangle|^2 \tag{32}$$

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This in turn , motivates the definitions -

- $$f(K^*Kx) := \int f(\lambda)E_\lambda x := \sum_{n=1}^{\infty} f(\sigma_n^2)\,\langle x, v_n\rangle\, v_n \tag{33}$$
- $$f(KK^*x) := \int f(\lambda)F_\lambda x := \sum_{n=1}^{\infty} f(\sigma_n^2)\,\langle x, u_n\rangle\, u_n \tag{34}$$
- We can derive this result

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Definition (1.10)

A family $\{E_\lambda\}_{\lambda \in R}$ of orthogonal projectors in X i called a spectral family or a resolution of the identity if it satisfies the following conditions:

- $E_\lambda E_\mu = E_{min\{\lambda,\mu\}}, \quad \lambda, \mu \in R,$
- $E_{-\infty} = 0, E_{+\infty} = I$, where $E_{\pm\infty} x = \lim_{\lambda \to \pm\infty} E_\lambda x$ for all $x \in X$.
- $E_{\lambda-0} = E_\lambda$, where $E_{\lambda-0} x = \lim_{\varepsilon \to 0^+} E_{\lambda-\varepsilon} x$ for all $x \in X$.

Proposition (1.11)

Let $f : R \to R$ be a continuous function. Then the limit of the Riemann sum

$$\sum_{i=1}^{n} f(\xi_i)(E_{\lambda_i} - E_{\lambda_{i-1}})x,$$

exists in X for $\max\limits_{1 \leqslant i \leqslant n} |\lambda_i - \lambda_{i-1}| \to 0$*, where,*

$$-\infty < a = \lambda_0 < \ldots < \lambda_n = b < \infty, \; \xi_i \in (\lambda_{i-1}, \lambda_i],$$

and is denoted by

$$\int_a^b f(\lambda) dE_\lambda x$$

Definition (1.12)

For any given $x \in X$ and any continuous function F on R the integral $\int_{-\infty}^{+\infty} f(\lambda) dE_\lambda x$ is defined as the limit in X if it exists, of $\int_a^b f(\lambda) dE_\lambda x$ when $a \to -\infty$ and $b \to +\infty$

Proposition (1.13)

For $x \in X$ and $f : R \to R$ a continuous function, the following conditions are equivalent:

$$\int_{-\infty}^{+\infty} f(\lambda) dE_\lambda x \quad \text{exists}, \tag{36}$$

$$\int_{-\infty}^{+\infty} f^2(\lambda) d\|E_\lambda x\|^2 < \infty \tag{37}$$

Proposition (1.14)

Let A be a selfsufficient operator in X. Then there exists a unique spectral family $\{E_\lambda\}_{\lambda \in R}$, *such that*

$$D(A) = \{x \in X \quad | \quad \int_{-\infty}^{+\infty} \lambda^2 d\|E_\lambda x\|^2 < \infty\} \tag{38}$$

and

$$Ax = \int_{-\infty}^{+\infty} \lambda dE_\lambda x, \quad x \in D(A). \tag{39}$$

We use the symbolic notation

$$A = \int_{-\infty}^{+\infty} \lambda dE_\lambda. \tag{40}$$

Definition (1.15)

Let A be a selfadjoint operator in X with spectral family $\{E_\lambda\}_{\lambda \in R}$. Moreover let M_0 denote the set of all functions measurable with respect to the measure $d\|E_\lambda x\|^2$ for all $x \in X$. Then $f(A)$ is the operator defined by the formula

$$f(A)x = \int_{-\infty}^{+\infty} f(\lambda) dE_\lambda x, \quad x \in D(f(A)), \tag{41}$$

where,

$$D(f(A)) = \{x \in X \quad | \quad \int_{-\infty}^{+\infty} f^2(\lambda) d\|E_\lambda x\|^2 < \infty\}. \tag{42}$$

Proposition (1.16)

Let A be a self adjoint operator in X with spectral family $\{E_\lambda\}_{\lambda \in R}$. and let $f, g \in M_0$.

- *If $x \in D(f(A))$ and $y \in D(f(A))$, then*
$$\langle f(A)x, g(A)y \rangle = \int_{-\infty}^{+\infty} f(\lambda)g(\lambda)d\,\langle E_\lambda x, y\rangle$$
- *If $x \in D(f(A))$,*
then $f(A)x \in D(g(A))$ if and only if $x \in D((gf)(A))$; furthermore
$$g(A)f(A)x = (gf)(A)x.$$
- *If $D(f(A))$ is dense in X, then $f(A)$ is selfadjoint.*
- *$f(A)$ commutes with E_λ for all $\lambda \in R$.*

Proposition (1.17)

Let A be a selfadjoint operator in X with spectral family $\{E_\lambda\}_{\lambda \in R}$.

- $\lambda_0 \in \sigma(A)$ *if and only if* $E_{\lambda_0} \neq E_{\lambda_0+\varepsilon}$ *for all* $\varepsilon > 0$.
- λ_0 *is an eigenvalue of* A *if and only if* $E_{\lambda_0} \neq E_{\lambda_0+0^+} = \lim_{\varepsilon \to 0} E_{\lambda_0+\varepsilon}$; *the corresponding eigenspace is given by* $(E_{\lambda_0+0^+} - E_{\lambda_0})(X)$

Proposition (1.18)

Let $T : X \to Y$ be a linear bounded operator. Then

$$R(T^*) = R((T^*T^{1/2})) \tag{43}$$