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Quadratic Formulas for Quaternions

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Abstract—In this paper, we derive explicit formulas for computing the roots of a quaternionic quadratic polynomial. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let $\mathbf{R}$ be the set of real numbers and $\mathbf{H}$ be the set of quaternions of the form $\alpha + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}$ where $\alpha, \beta, \gamma, \delta \in \mathbf{R}$, and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

For $a = \alpha + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}$, let $\bar{a} = \alpha - \beta\mathbf{i} - \gamma\mathbf{j} - \delta\mathbf{k}$ be the conjugate of a, $|a| = \sqrt{\bar{a}a} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$, $\operatorname{Re} a = (\bar{a} + a)/2 = \alpha$, and $\operatorname{Im} a = a - \operatorname{Re} a = \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}$. For $a, b \in \mathbf{H}$, we say that a is similar to b if there is a nonzero $q \in \mathbf{H}$ such that $a = q^{-1}bq$ or equivalently, $\operatorname{Re} a = \operatorname{Re} b$ and $|a|^2 = |b|^2$. For the basics of quaternions, see [1].

In this paper, we are interested in explicit formulas for computing the roots of a quadratic polynomial of the form

$$x^2 + bx + c,$$

where $b, c \in \mathbf{H}$. Let $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, $b = b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $c = c_0 + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

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Then $x^2 + bx + c = 0$ becomes the real system of nonlinear equations

$$
\begin{aligned}
x_0^2 - x_1^2 - x_2^2 - x_3^2 + b_0x_0 - b_1x_1 - b_2x_2 - b_3x_3 + c_0 &= 0,\\
2x_0x_1 + b_0x_1 + b_1x_0 + b_2x_3 - b_3x_2 + c_1 &= 0,\\
2x_0x_2 + b_0x_2 + b_2x_0 - b_1x_3 + b_3x_1 + c_2 &= 0,\\
2x_0x_3 + b_0x_3 + b_3x_0 + b_1x_2 - b_2x_1 + c_3 &= 0.
\end{aligned}
$$

It is not obvious at all that this nonlinear system will have an explicit solution. Nonetheless, there are several attempts in the literature. In [2], Zhang and Mu proposed to compute *some* roots of a quadratic polynomial by solving a real *linear* system. However, they did not discuss how to find *all* the roots. In [3], Porter reduced solving a quadratic polynomial to a linear polynomial of the form $px + xq + r$ provided a root of the given quadratic polynomial is already known. However, he did not discuss how to find such root. In [4], Niven determined how many roots a quadratic polynomial can have, but he did *not* give the explicit formulas for computing the roots. In Section 2, we adopt the idea of Niven to compute the roots of a quadratic polynomial using explicit formulas in terms of its coefficients (see Theorem 2.3). Then, we discuss some consequences and two applications of the quaternionic quadratic formulas.

2. QUATERNIONIC QUADRATIC FORMULAS

In this section, we solve the monic standard quadratic equation

$$x^2 + bx + c = 0,$$

where $b, c \in \mathbf{H}$. We begin with two lemmas on solutions of some special real polynomials, their proofs are left as routine exercises.

LEMMA 2.1. *Let B, E, and D be real numbers such that*

(i) $D \neq 0$, *and*
(ii) $B < 0$ *implies* $B^2 < 4E$.

Then the cubic equation

$$y^3 + 2By^2 + \left(B^2 - 4E\right) y - D^2 = 0$$

has exactly one positive solution y.

LEMMA 2.2. *Let B, E, and D be real numbers such that*

(i) $E \geq 0$, *and*
(ii) $B < 0$ *implies* $B^2 < 4E$.

Then the real system

$$
\begin{aligned}
N^2 - \left(B + T^2\right) N + E = 0,\\
T^3 + (B - 2N)T + D = 0,
\end{aligned}
$$

has at most two solutions (T, N) satisfying $T \in \mathbf{R}$ and $N \geq 0$ as follows.

(a) $T = 0,\ N = (B \pm \sqrt{B^2 - 4E})/2$ *provided that* $D = 0,\ B^2 \geq 4E$.
(b) $T = \pm\sqrt{2\sqrt{E} - B},\ N = \sqrt{E}$ *provided that* $D = 0,\ B^2 < 4E$.
(c) $T = \pm\sqrt{z},\ N = (T^3 + BT + D)/2T$ *provided that* $D \neq 0$ *and z is the unique positive root of the real polynomial* $z^3 + 2Bz^2 + (B^2 - 4E)z - D^2$.

THEOREM 2.3. *The solutions of the quadratic equation $x^2 + bx + c = 0$ can be obtained by formulas according to the following cases.*

CASE 1. If $b, c \in \mathbf{R}$ and $b^2 < 4c$, then

$$x = \frac{1}{2}\left(-b + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}\right),$$

where $\beta^2 + \gamma^2 + \delta^2 = 4c - b^2$ and $\beta, \gamma, \delta \in R$.

CASE 2. If $b, c \in \mathbf{R}$ and $b^2 \geq 4c$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

CASE 3. If $b \in \mathbf{R}$ and $c \notin \mathbf{R}$, then

$$x = \frac{-b}{2} \pm \frac{\rho}{2} \mp \frac{c_1}{\rho}\mathbf{i} \mp \frac{c_2}{\rho}\mathbf{j} \mp \frac{c_3}{\rho}\mathbf{k},$$

where $c = c_0 + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ and $\rho = \sqrt{(b^2 - 4c_0 + \sqrt{(b^2 - 4c_0)^2 + 16(c_1^2 + c_2^2 + c_3^2)})/2}$.

CASE 4. If $b \notin \mathbf{R}$, then

$$x = \frac{-\operatorname{Re} b}{2} - (b' + T)^{-1}(c' - N),$$

where $b' = b - \operatorname{Re} b = \operatorname{Im} b$, $c' = c - ((\operatorname{Re} b)/2)(b - (\operatorname{Re} b)/2)$, and (T, N) is chosen as follows.

1. $T = 0, N = (B \pm \sqrt{B^2 - 4E})/2$ provided that $D = 0$, $B^2 \geq 4E$.
2. $T = \pm\sqrt{2\sqrt{E} - B}$, $N = \sqrt{E}$ provided that $D = 0$, $B^2 < 4E$.
3. $T = \pm\sqrt{z}$, $N = (T^3 + BT + D)/2T$ provided that $D \neq 0$ and z is the unique positive root of the real polynomial $z^3 + 2Bz^2 + (B^2 - 4E)z - D^2$,

where $B = |b'|^2 + 2\operatorname{Re} c'$, $E = |c'|^2$, and $D = 2\operatorname{Re} \bar{b}'c'$.

PROOF.

CASE 1. $b, c \in \mathbf{R}$ AND $b^2 < 4c$. Note that x_0 is a solution if and only if $q^{-1}x_0q$ is also a solution for $q \neq 0$, and there are at least two complex solutions

$$\frac{-b \pm \sqrt{4c - b^2}\ \mathbf{i}}{2}.$$

Hence, the solution set is

$$\left\{q^{-1}\frac{-b + \sqrt{4c - b^2}\ \mathbf{i}}{2}q : q \neq 0\right\} = \left\{\frac{1}{2}\left(-b + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}\right) : \beta^2 + \gamma^2 + \delta^2 = 4c - b^2\right\}.$$

CASE 2. $b, c, \in \mathbf{R}$ AND $b^2 \geq 4c$. Note that x_0 is a solution if and only if $q^{-1}x_0q$ is also a solution for $q \neq 0$, and hence, there are at most two solutions, both are real

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

CASE 3. $b \in \mathbf{R}$ AND $c \notin \mathbf{R}$. Let $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and $c = c_0 + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Then $x^2 + bx + c = 0$ becomes the real system

$$\begin{aligned}\left(x_0 + \frac{b}{2}\right)^2 - x_1^2 - x_2^2 - x_3^2 &= \frac{b^2}{4} - c_0,\\ (2x_0 + b)x_1 &= -c_1,\\ (2x_0 + b)x_2 &= -c_2,\\ (2x_0 + b)x_3 &= -c_3.\end{aligned}$$

Since c is nonreal, $2x_0 + b$ is nonzero and so x_1, x_2, x_3 can be expressed in terms of x_0 and be substituted into the first equation to obtain

$$(2x_0 + b)^4 - \left(b^2 - 4c_0\right)(2x_0 + b)^2 - 4\left(c_1^2 + c_2^2 + c_3^2\right) = 0.$$

It follows that $2x_0 + b = \pm\sqrt{(b^2 - 4c_0 + \sqrt{(b^2 - 4c_0)^2 + 16(c_1^2 + c_2^2 + c_3^2)})/2}$ and so $x_0 = (1/2)$ $(-b \pm \rho)$, where $\rho = \sqrt{(b^2 - 4c_0 + \sqrt{(b^2 - 4c_0)^2 + 16(c_1^2 + c_2^2 + c_3^2)})/2} \neq 0$ since $c \notin \mathbf{R}$. Finally,

$$\begin{aligned} x &= x_0 - \frac{c_1}{2x_0 + b}\mathbf{i} - \frac{c_2}{2x_0 + b}\mathbf{j} - \frac{c_3}{2x_0 + b}\mathbf{k} \\ &= \frac{-b}{2} \pm \frac{\rho}{2} \mp \frac{c_1}{\rho}\mathbf{i} \mp \frac{c_2}{\rho}\mathbf{j} \mp \frac{c_3}{\rho}\mathbf{k}. \end{aligned}$$

CASE 4. $b \notin \mathbf{R}$. Rewrite the equation $x^2 + bx + c = 0$ as

$$y^2 + b'y + c' = 0,$$

where $y = x + (\operatorname{Re} b)/2$, $b' = b - \operatorname{Re} b \notin \mathbf{R}$, and $c' = c - ((\operatorname{Re} b)/2)(b - (\operatorname{Re} b)/2)$. Following the idea of Niven [4], we observe that the solution of the quadratic equation $y^2 + b'y + c' = 0$ also satisfies

$$y^2 - Ty + N = 0,$$

where $N = \bar{y}y \geq 0$ and $T = y + \bar{y} \in \mathbf{R}$. Hence, $(b' + T)y + (c' - N) = 0$, and so

$$y = -(b' + T)^{-1}(c' - N)$$

because $T \in \mathbf{R}$ and $b' \notin \mathbf{R}$ implies that $b' + T \neq 0$. To solve for T and N, we substitute y back into the definitions $T = y + \bar{y}$ and $N = \bar{y}y$ and simplify to obtain the real system

$$\begin{aligned} N^2 - \left(B + T^2\right)N + E &= 0, \\ T^3 + (B - 2N)T + D &= 0, \end{aligned}$$

where $B = b'\bar{b}' + c' + \bar{c}' = |b'|^2 + 2\operatorname{Re} c'$, $E = c'\bar{c}' = |c'|^2$, $D = \bar{b}'c' + \bar{c}'b' = 2\operatorname{Re}\bar{b}'c'$ are real numbers. Note that $E = |c'|^2 \geq 0$. If $B < 0$, then $c' + \bar{c}' < 0$ and $B^2 - 4E = |b'|^2B + |b'|^2(c' + \bar{c}') + (c' - \bar{c}')^2 \leq 0$ because of the fact that $(c' - \bar{c}')^2 \leq 0$. It follows that $B^2 - 4E < 0$, otherwise $B^2 - 4E = 0$ and so $|b'|^2B = |b'|^2(c' + \bar{c}') = (c' - \bar{c}')^2 = 0$, i.e., $b' = 0 \in \mathbf{R}$, a contradiction. Hence, by Lemma 2.2, such system can be solved explicitly as claimed. Consequently, $x = (-\operatorname{Re} b)/2 - (b' + T)^{-1}(c' - N)$. ∎

COROLLARY 2.4. *The quadratic equation $x^2 + bx + c = 0$ has infinitely many solutions if and only if $b, c \in \mathbf{R}$ and $b^2 < 4c$.*

EXAMPLE 2.5. Consider the quadratic equation $x^2 + 1 = 0$, i.e., $b = 0$ and $c = 1$. This belongs to Case 1 in Theorem 2.3. Then $x = (1/2)(\beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k})$ where $\beta^2 + \gamma^2 + \delta^2 = 4$. Consequently, the infinite solutions are $x = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, where $x_1^2 + x_2^2 + x_3^2 = 1$.

COROLLARY 2.6. *The quadratic equation $x^2 + bx + c = 0$ has a unique solution if and only if either*

(i) *$b, c \in \mathbf{R}$ and $b^2 - 4c = 0$, or*
(ii) *$b \notin \mathbf{R}$ and $D = 0 = B^2 - 4E$.*

EXAMPLE 2.7. Consider the quadratic equation $x^2 - 2x + 1 = 0$, i.e., $b = -2$ and $c = 1$. This belongs to Case 2 in Theorem 2.3. Then the unique solution is $x = 1$.

EXAMPLE 2.8. Consider the quadratic equation $x^2 + \mathbf{i}x + (1/2)\mathbf{j} = 0$, i.e., $b = \mathbf{i}$ and $c = (1/2)\mathbf{j}$. This belongs to Case 4 in Theorem 2.3. Then $b' = \mathbf{i}$ and $c' = (1/2)\mathbf{j}$. Moreover, $B = 1, E = 1/4$, and $D = 0$. It is Subcase 1 in Case 4. Hence, $T = 0$ and $N = 1/2$. Consequently, $x = (1/2)(\mathbf{k} - \mathbf{i})$ is the unique solution.

COROLLARY 2.9. *The quadratic equation $x^2 + bx + c = 0$ has exactly two distinct solutions if and only if*

(i) $b, c \in \mathbf{R}$ *and* $b^2 - 4c > 0$, *or*
(ii) $b \in \mathbf{R}$ *and* $c \notin \mathbf{R}$, *or*
(iii) $b \notin \mathbf{R}$, $D = 0$, $B^2 - 4E \neq 0$, *or*
(iv) $b \notin \mathbf{R}$ *and* $D \neq 0$.

EXAMPLE 2.10. Consider the quadratic equation $x^2 - 5x + 4 = 0$, i.e., $b = -5$ and $c = 4$. This belongs to Case 2 in Theorem 2.3. Then the two solutions are $x = 1$ and $x = 4$.

EXAMPLE 2.11. Consider the quadratic equation $x^2 - x + \mathbf{k} = 0$, i.e., $b = -1$ and $c = \mathbf{k}$. This belongs to Case 3 in Theorem 2.3. Then $c_0 = c_1 = c_2 = 0$, $c_3 = 1$, and $\rho = \sqrt{(1 + \sqrt{17})/2}$. Consequently, the two solutions are $x = (1 + \rho)/2 - \mathbf{k}/\rho$ and $x = (1 - \rho)/2 + \mathbf{k}/\rho$.

EXAMPLE 2.12. Consider the quadratic equation $x^2 + \mathbf{i}x + (1 + \mathbf{j}) = 0$, i.e., $b = \mathbf{i}$ and $c = 1 + \mathbf{j}$. This belongs to Case 4 in Theorem 2.3. Then $b' = \mathbf{i}$ and $c' = 1 + \mathbf{j}$. Moreover, $B = 3$, $E = 2$, and $D = 0$. It is Subcase 1 in Case 4. Hence, $T = 0$, $N = 2$ or 1. Consequently, the two solutions are $x = \mathbf{k} - \mathbf{i}$ and $x = \mathbf{k}$.

EXAMPLE 2.13. Consider the quadratic equation $x^2 + \mathbf{i}x + \mathbf{j} = 0$, i.e., $b = \mathbf{i}$ and $c = \mathbf{j}$. This belongs to Case 4 in Theorem 2.3. Then $b' = \mathbf{i}$ and $c' = \mathbf{j}$. Moreover, $B = 1$, $E = 1$, and $D = 0$. It is Subcase 2 in Case 4. Hence, $T = \pm 1$ and $N = 1$. Consequently, the two solutions are $x = (\mathbf{i} + 1)^{-1}(1 - \mathbf{j}) = (1/2)(1 - \mathbf{i} - \mathbf{j} + \mathbf{k})$ and $x = (\mathbf{i} - 1)^{-1}(1 - \mathbf{j}) = (1/2)(-1 - \mathbf{i} + \mathbf{j} + \mathbf{k})$.

EXAMPLE 2.14. Consider the quadratic equation $x^2 + \mathbf{i}x + (1 + \mathbf{i} + \mathbf{j}) = 0$, i.e., $b = \mathbf{i}$ and $c = 1 + \mathbf{i} + \mathbf{j}$. This belongs to Case 4 in Theorem 2.3. Then $b' = \mathbf{i}$ and $c' = 1 + \mathbf{i} + \mathbf{j}$. Moreover, $B = 3$, $E = 3$, and $D = 2$. It is Subcase 3 in Case 4. Now the unique positive roots of $z^3 + 6z^2 - 3z - 4$ is 1, and hence, $T = 1$ and $N = 3$, or $T = -1$ and $N = 1$. Consequently, the two solutions are $x = (1/2)(1 - 3\mathbf{i} - \mathbf{j} + \mathbf{k})$ and $x = (1/2)(-1 + \mathbf{i} + \mathbf{j} + \mathbf{k})$.

THEOREM 2.15. *If the quadratic equation $x^2 + bx + c = 0$ has exactly two distinct solutions x_1 and x_2, then $x_1 + b/2$ and $-(x_2 + b/2)$ are similar. Indeed, there exists nonzero $\alpha \in \mathbf{H}$ such that $b\alpha = \alpha b$ and $\alpha(x_1 + b/2)\alpha^{-1} = -(x_2 + b/2)$.*

PROOF. By Corollary 2.9, we have several cases to deal with.

(i) If $b, c \in \mathbf{R}$ and $b^2 - 4c > 0$, by Case 2 in Theorem 2.3, it is clear that $x_1 + b/2 = -(x_2 + b/2)$.
(ii) If $b \in \mathbf{R}$ and $c \notin \mathbf{R}$, by Case 3 in Theorem 2.3, it is clear that $x_1 + b/2 = -(x_2 + b/2)$.
(iii)
 (a) If $b \notin \mathbf{R}$, $D = 0$, and $B^2 - 4E > 0$, then by Subcase 1 in Case 4 of Theorem 2.3, we have

$$x_i = \frac{-\operatorname{Re} b}{2} - b'^{-1}\left(c' - \frac{B \pm \sqrt{B^2 - 4E}}{2}\right), \qquad \text{for } i = 1, 2.$$

Thus, it is easy to see that

$$x_1 + \frac{b}{2} = -b'^{-1}\left(\operatorname{Im} c' - \frac{\sqrt{B^2 - 4E}}{2}\right) = \frac{b'}{|b'|^2}\left(\operatorname{Im} c' - \frac{\sqrt{B^2 - 4E}}{2}\right)$$

and

$$x_2 + \frac{b}{2} = -b'^{-1}\left(\operatorname{Im} c' + \frac{\sqrt{B^2 - 4E}}{2}\right) = \frac{b'}{|b'|^2}\left(\operatorname{Im} c' + \frac{\sqrt{B^2 - 4E}}{2}\right).$$

Clearly, $\operatorname{Re}(x_1 + b/2) = \operatorname{Re}(-(x_2 + b/2)) = 0$ and $|x_1 + b/2|^2 = |-(x_2 + b/2)|^2$, thus, $x_1 + b/2$ and $-(x_2 + b/2)$ are similar, and $x_1 + b/2$ and $x_2 + b/2$ are also similar. Since $\overline{x_2 + b/2} = -(x_2 + b/2)$, it is easy to see that

$$b'\left(x_1 + \frac{b}{2}\right)b'^{-1} = -\left(x_2 + \frac{b}{2}\right).$$

(b) If $b \notin \mathbf{R}$, $D = 0$, and $B^2 - 4E < 0$, then by Subcase 2 in Case 4 of Theorem 2.3, we have

$$x_{1,2} = \frac{-\operatorname{Re} b}{2} - \left(b' \pm \sqrt{2\sqrt{E} - B}\right)^{-1} \left(c' - \sqrt{E}\right).$$

Thus,

$$\begin{aligned} x_1 + \frac{b}{2} = &= \frac{b'}{2} - \frac{-b' + \sqrt{2\sqrt{E} - B}}{2\left(\sqrt{E} - \operatorname{Re} c'\right)} \left(c' - \sqrt{E}\right) \\ &= \frac{b'}{2} + \frac{-b' + \sqrt{2\sqrt{E} - B}}{2} \left(1 - \frac{\operatorname{Im} c'}{\sqrt{E} - \operatorname{Re} c'}\right) \\ &= \frac{\sqrt{2\sqrt{E} - B}}{2} - \frac{\left(-b' + \sqrt{2\sqrt{E} - B}\right) \operatorname{Im} c'}{2\left(\sqrt{E} - \operatorname{Re} c'\right)}. \end{aligned}$$

Similarly, we have

$$-\left(x_2 + \frac{b}{2}\right) = \frac{\sqrt{2\sqrt{E} - B}}{2} + \frac{\left(-b' - \sqrt{2\sqrt{E} - B}\right) \operatorname{Im} c'}{2\left(\sqrt{E} - \operatorname{Re} c'\right)}.$$

Thus, it is clear that

$$\operatorname{Re}\left(x_1 + \frac{b}{2}\right) = \operatorname{Re}\left(-\left(x_2 + \frac{b}{2}\right)\right) = \frac{\sqrt{2\sqrt{E} - B}}{2}$$

and

$$\left|x_1 + \frac{b}{2}\right|^2 = \left|-\left(x_2 + \frac{b}{2}\right)\right|^2,$$

thus, $x_1 + b/2$ and $-(x_2 + b/2)$ are similar. Since

$$\operatorname{Im}\left(x_1 + \frac{b}{2}\right) = -\left(b' + \sqrt{2\sqrt{E} - B}\right)^{-1} \operatorname{Im} c'$$

and

$$\operatorname{Im}\left[-\left(x_2 + \frac{b}{2}\right)\right] = (\operatorname{Im} c')\left(b' + \sqrt{2\sqrt{E} - B}\right)^{-1},$$

note that $\operatorname{Im}[-(\bar{x}_2 + b/2)] = \operatorname{Im}(x_2 + b/2)$, it is easy to prove that

$$\left(b' + \sqrt{2\sqrt{E} - B}\right) \operatorname{Im}\left(x_1 + \frac{b}{2}\right) = \operatorname{Im}\left[-\left(x_2 + \frac{b}{2}\right)\right] \left(b' + \sqrt{2\sqrt{E} - B}\right).$$

Thus, we have

$$\left(b' + \sqrt{2\sqrt{E} - B}\right)\left(x_1 + \frac{b}{2}\right)\left(b' + \sqrt{2\sqrt{E} - B}\right)^{-1} = -\left(x_2 + \frac{b}{2}\right).$$

(iv) If $b \notin \mathbf{R}$ and $D \neq 0$, from Theorem 2.3, Case 4, Subcase 3, we have

$$x_1 = -\frac{\operatorname{Re} b}{2} - (b' + T)^{-1} \left[c' - \frac{T^3 + BT + D}{2T}\right]$$

and

$$x_2 = -\frac{\operatorname{Re} b}{2} - (b' - T)^{-1}\left[c' - \frac{T^3 + BT - D}{2T}\right],$$

where $T = \sqrt{z}$ and z is the unique positive solution of the cubic equation $z^3 + 2Bz^2 + (B^2 - 4E)z - D^2 = 0$. Using the fact that $b' = \operatorname{Im} b$ and $B = |b'|^2 + 2\operatorname{Re} c'$, we have

$$x_1 + \frac{b}{2} = \frac{T}{2} - \frac{(T - b')}{T^2 + |b'|^2}\left(\operatorname{Im} c' - \frac{D}{2T}\right),$$

and also the fact that $D = 2\operatorname{Re}\bar{b}'c'$, we have

$$\operatorname{Re}\left\{(T - b')\left(\operatorname{Im} c' - \frac{D}{2T}\right)\right\} = 0.$$

Hence, $\operatorname{Re}(x_1 + b/2) = T/2$ and

$$\operatorname{Im}\left(x_1 + \frac{b}{2}\right) = \frac{-1}{T^2 + |b'|^2}\left\{(T - b')\left(\operatorname{Im} c' - \frac{D}{2T}\right)\right\}.$$

Similarly, we have

$$-\left(x_2 + \frac{b}{2}\right) = \frac{T}{2} - \frac{(T - b')}{T^2 + |b'|^2}\left(\operatorname{Im} c' + \frac{D}{2T}\right),$$

$\operatorname{Re}\{-(x_2 + b/2)\} = T/2$ and

$$\operatorname{Im}\left\{-\left(x_2 + \frac{b}{2}\right)\right\} = \frac{-1}{T^2 + |b'|^2}\left\{(T + b')\left(\operatorname{Im} c' + \frac{D}{2T}\right)\right\}.$$

Clearly, we have $|x_1 + b/2|^2 = |-(x_2 + b/2)|^2$, thus $x_1 + b/2$ and $-(x_2 + b/2)$ are similar. Consequently, $(T + b')(x_1 + b/2) = -(x_2 + b/2)(T + b')$. ∎

REMARK 2.16. If the quadratic equation $x^2 + bx + c = 0$ has only two distinct solutions x_1 and x_2, then by Theorem 2.15, there exists $0 \neq \alpha \in \mathbf{H}$ such that $b\alpha = \alpha b$ and $\alpha(x_1 + b/2)\alpha^{-1} = -(x_2 + b/2)$. Thus, we have $\alpha x_1 \alpha^{-1} + x_2 = -b$ and $\alpha x_1 \alpha^{-1} x_2 = c$, which resemble the sum and product formulas for solutions of complex quadratic equations.

Finally, we mention two applications of these results. While computing left eigenvalues of 2×2 quaternionic matrices [5], it is required to solve quadratic equations of the form $x^2 + xb + c = 0$, with $b, c \in \mathbf{H}$, whose solutions can be obtained as follows.

- The equation $x^2 + xb + c = 0$ is equivalent to $y^2 + \bar{b}y + \bar{c} = 0$ where $y = \bar{x}$. Now Theorem 2.3 can be used to solve for y and then $x = \bar{y}$.

While characterizing fixed points of the quaternionic linear fractional transformations $T(x) = -(bx + d)(ax + c)^{-1}$ [3,6], it is required to consider quadratic equations of the form $xax + bx + xc + d = 0$ with $a \neq 0$. It can be solved as follows.

- Since $a \neq 0$, the equation $xax + bx + xc + d = 0$ is equivalent to $y^2 + (aba^{-1} - c)y + (ad - aba^{-1}c) = 0$ with $y = ax + c$. Now Theorem 2.3 can be used to solve for y and then $x = a^{-1}(y - c)$.

From Corollaries 2.4, 2.6, and 2.9, we observe that the set

$$\{(b, c) : x^2 + bx + c = 0 \text{ has exactly two solutions}\}$$

is the complement of a union of two closed sets. Hence, it is an open set. Consequently, the set

$$\{(a, b, c, d) : xax + bx + xc + d = 0 \text{ has exactly two solutions}\}$$

is also open. This provides another proof for the conjecture posed at the end of [3].

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