Derivative Rules

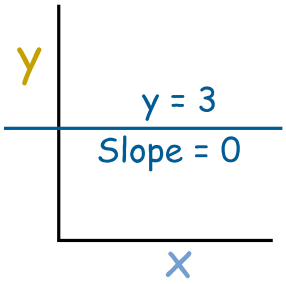
Advanced

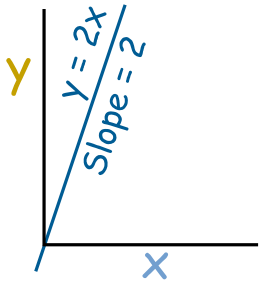
The Derivative tells us the slope of a function at any point.

There are **rules** we can follow to find many derivatives.

For example:

- The slope of a **constant** value (like 3) is always 0
- The slope of a **line** like 2x is 2, or 3x is 3 etc
- and so on.





Here are useful rules to help you work out the derivatives of many functions (with examples below). Note: the little mark ′ means "Derivative of".

Common Functions	Function	Derivative
Constant	c	0
Line	x	1
	ax	a
Square	x^2	$2x$
Square Root	$\sqrt{x}$	$(½)x^{-½}$
Exponential	e^x	e^x
	a^x	$\ln(a)\ a^x$
Logarithms	$\ln(x)$	$1/x$
	$\log_a(x)$	$1\ /\ (x\ \ln(a))$
Trigonometry (x is in radians)	$\sin(x)$	$\cos(x)$
	$\cos(x)$	$-\sin(x)$
	$\tan(x)$	$\sec^2(x)$

Inverse Trigonometry	$\sin^{-1}(x)$	$1/\sqrt{(1-x^2)}$
	$\cos^{-1}(x)$	$-1/\sqrt{(1-x^2)}$
	$\tan^{-1}(x)$	$1/(1+x^2)$

Rules	**Function**	**Derivative**
Multiplication by constant	cf	cf'
Power Rule	x^n	nx^{n-1}
Sum Rule	$f + g$	$f' + g'$
Difference Rule	$f - g$	$f' - g'$
Product Rule	fg	$f\,g' + f'\,g$
Quotient Rule	f/g	$(f'\,g - g'\,f)/g^2$
Reciprocal Rule	$1/f$	$-f'/f^2$
Chain Rule (as "Composition of Functions")	$f \circ g$	$(f' \circ g) \times g'$
Chain Rule (using ′)	$f(g(x))$	$f'(g(x))g'(x)$
Chain Rule (using $\frac{d}{dx}$)		$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

"The derivative of" is also written $\frac{d}{dx}$

So $\frac{d}{dx}\sin(x)$ and $\sin(x)'$ both mean "The derivative of sin(x)"

Examples

Example: what is the derivative of sin(x) ?

From the table above it is listed as being **cos(x)**

It can be written as:

$$\frac{d}{dx}\sin(x) = \cos(x)$$

Or:

$$\sin(x)' = \cos(x)$$

Power Rule

Example: What is $\frac{d}{dx}x^3$?

The question is asking "what is the derivative of x^3 ?"

We can use the Power Rule, where n=3:

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}x^3 = 3x^{3-1} = \mathbf{3x^2}$$

(In other words the derivative of x^3 is $3x^2$)

So it is simply this:

"multiply by power
then reduce power by 1"

It can also be used in cases like this:

Example: What is $\frac{d}{dx}(1/x)$?

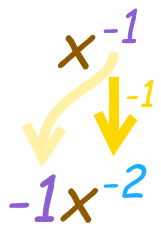
1/x is also $\mathbf{x^{-1}}$

We can use the Power Rule, where n = −1:

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}x^{-1} = -1x^{-1-1} = \mathbf{-x^{-2}}$$

So we just did this:



which simplifies to $\mathbf{-x^{-2}}$

Multiplication by constant

Example: What is $\frac{d}{dx}5x^3$?

the derivative of cf = cf′

the derivative of 5f = 5f′

We know (from the Power Rule):

$$\frac{d}{dx}x^3 = 3x^{3-1} = 3x^2$$

So:

$$\frac{d}{dx}5x^3 = 5\frac{d}{dx}x^3 = 5 \times 3x^2 = \mathbf{15x^2}$$

Sum Rule

Example: What is the derivative of x^2+x^3 ?

The Sum Rule says:

the derivative of f + g = f′ + g′

So we can work out each derivative separately and then add them.

Using the Power Rule:

- $\frac{d}{dx}x^2 = 2x$
- $\frac{d}{dx}x^3 = 3x^2$

And so:

$$\text{the derivative of } x^2 + x^3 = \mathbf{2x + 3x^2}$$

Difference Rule

It doesn't have to be **x**, we can differentiate with respect to, for example, **v**:

Example: What is $\frac{d}{dv}(v^3 - v^4)$?

The Difference Rule says

$$\text{the derivative of } f - g = f' - g'$$

So we can work out each derivative separately and then subtract them.

Using the Power Rule:

- $\frac{d}{dv}v^3 = 3v^2$
- $\frac{d}{dv}v^4 = 4v^3$

And so:

$$\text{the derivative of } v^3 - v^4 = \mathbf{3v^2 - 4v^3}$$

Sum, Difference, Constant Multiplication And Power Rules

Example: What is $\frac{d}{dz}(5z^2 + z^3 - 7z^4)$?

Using the Power Rule:

- $\frac{d}{dz}z^2 = 2z$

- $\frac{d}{dz} z^3 = 3z^2$
- $\frac{d}{dz} z^4 = 4z^3$

And so:

$$\frac{d}{dz}(5z^2 + z^3 - 7z^4) = 5 \times 2z + 3z^2 - 7 \times 4z^3 = \mathbf{10z + 3z^2 - 28z^3}$$

Product Rule

Example: What is the derivative of cos(x)sin(x) ?

The Product Rule says:

$$\text{the derivative of } fg = f\,g' + f'\,g$$

In our case:

- f = cos
- g = sin

We know (from the table above):

- $\frac{d}{dx} \cos(x) = -\sin(x)$
- $\frac{d}{dx} \sin(x) = \cos(x)$

So:

$$\text{the derivative of } \cos(x)\sin(x) = \cos(x)\cos(x) - \sin(x)\sin(x)$$

$$= \mathbf{\cos^2(x) - \sin^2(x)}$$

Reciprocal Rule

Example: What is $\frac{d}{dx}(1/x)$?

The Reciprocal Rule says:

$$\text{the derivative of } 1/f = -f'/f^2$$

With f(x)= x, we know that f'(x) = 1

So:

$$\text{the derivative of } 1/x = -1/x^2$$

Which is the same result we got above using the Power Rule.

Chain Rule

Example: What is $\frac{d}{dx}\sin(x^2)$?

$\sin(x^2)$ is made up of **sin()** and **x^2**:

- $f(g) = \sin(g)$
- $g(x) = x^2$

The Chain Rule says:

$$\text{the derivative of } f(g(x)) = f'(g(x))g'(x)$$

The individual derivatives are:

- $f'(g) = \cos(g)$
- $g'(x) = 2x$

So:

$$\frac{d}{dx}\sin(x^2) = \cos(g(x))\ (2x)$$

$$= 2x\ \cos(x^2)$$

Another way of writing the Chain Rule is: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Let's do the previous example again using that formula:

Example: What is $\frac{d}{dx}\sin(x^2)$?

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Have $u = x^2$, so $y = \sin(u)$:

$$\frac{d}{dx}\sin(x^2) = \frac{d}{du}\sin(u)\frac{d}{dx}x^2$$

Differentiate each:

$$\frac{d}{dx}\sin(x^2) = \cos(u)\ (2x)$$

Substitue back $u = x^2$ and simplify:

$$\frac{d}{dx}\sin(x^2) = 2x\cos(x^2)$$

Same result as before (thank goodness!)

Another couple of examples of the Chain Rule:

Example: What is $\frac{d}{dx}(1/\cos(x))$?

1/cos(x) is made up of **1/g** and **cos()**:

- $f(g) = 1/g$
- $g(x) = \cos(x)$

The Chain Rule says:

$$\text{the derivative of } f(g(x)) = f'(g(x))g'(x)$$

The individual derivatives are:

- $f'(g) = -1/(g^2)$
- $g'(x) = -\sin(x)$

So:

$$\mathbf{(1/\cos(x))' = -1/(g(x))^2 \times -\sin(x)}$$

$$\mathbf{= \sin(x)/\cos^2(x)}$$

Note: **$\sin(x)/\cos^2(x)$** is also **$\tan(x)/\cos(x)$**, or many other forms.

Example: What is $\frac{d}{dx}(5x-2)^3$?

The Chain Rule says:

$$\text{the derivative of } f(g(x)) = f'(g(x))g'(x)$$

$(5x-2)^3$ is made up of **g^3** and **$5x-2$**:

- $f(g) = g^3$
- $g(x) = 5x-2$

The individual derivatives are:

- $f'(g) = 3g^2$ (by the Power Rule)
- $g'(x) = 5$

So:

$$\mathbf{\frac{d}{dx}(5x-2)^3 = 3g(x)^2 \times 5 = 15(5x-2)^2}$$