Value-at-Risk

Second Edition – by Glyn A. Holton

2.2.4 Functions

Notation to indicate that a function f maps elements of a set A to elements of a set B is:

$$f: A \to B \tag{2.4}$$

A is the function's domain; B contains its range. We are primarily interested in three types of functions:

- functions from $\mathbb{R}$ to $\mathbb{R}$,
- functions from $\mathbb{R}^n$ to $\mathbb{R}$,
- functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.

We call functions of the first form **real**—they map real numbers to real numbers. The natural logarithm function is a real function, which we denote *log*. We do not employ the logarithm base 10. If a function f has an inverse, we denote this f^{-1}. The derivative of a real function f may be indicated with differential notation or simply as f'.[2] We indicate the value of a function f at a particular point a as either $f(a)$ or $f|_a$. The former is read as "f of a". The latter is read as "f evaluated at a".

Consider $f : \mathbb{R}^p \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$. The **composition** of f and g is the function $f \circ g$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined as

$$f \circ g(x) = f(g(x)) \tag{2.5}$$

The **gradient** ∇f and **Hessian** $\nabla^2 f$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ are the vector of its first partial derivatives and matrix of its second partial derivatives:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \quad \text{and} \quad \nabla^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix} \qquad [2.6]$$

The Hessian is symmetric if the second partials are continuous.

The **Jacobian** Jf of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is the matrix of its first partial derivatives.

$$\boldsymbol{J}f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \qquad [2.7]$$

Note that the Hessian of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the Jacobian of its gradient.