CSE 6363 - HW01

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Exercise 1

Poisson distribution (PMF):

$$P_\lambda(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Part a: MLE of Poisson distribution

$$P_{MLE}(\lambda \mid k_1, k_2, ..., k_n) = \prod_{i=1}^{n} f_K(k_i \mid \lambda)\ ; \quad where\ \ f_K = P_\lambda(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\Rightarrow P_{MLE}(\lambda \mid k_1, k_2, ..., k_n) = \prod_{i=1}^{n} \frac{(e^{-\lambda}\ \lambda^{k_i})}{k_i!}$$

For optimization, we use the log function and we end up with log-likelihood:

$$P_{MLE}(\lambda \mid k_1, k_2, ..., k_n) = log(\prod_{i=1}^{n} \frac{(e^{-\lambda}\ \lambda^{k_i})}{k_i!})$$

$$\Rightarrow \sum_{i=1}^{n} ln((e^{-\lambda})(\frac{1}{k_i!})(\lambda^{k_i})) \Rightarrow \sum_{i=1}^{n}[(-\lambda) - ln(k_i!) + k_i ln(\lambda)]$$

$$\Rightarrow ln\ \mathcal{L} = P_{MLE}(\lambda \mid k_1, k_2, ..., k_n) = -n\lambda - \sum_{i=1}^{n}[ln(k_i!)] + ln(\lambda)\sum_{i=1}^{n} k_i$$

Thus, one can generalize the following case:

$$\frac{\partial ln\mathcal{L}}{\partial\hat{\lambda}} = -n + \frac{\sum_{i=0}^{n} k_i}{\hat{\lambda}} \Rightarrow -n + \frac{\sum_{i=0}^{n} k_i}{\hat{\lambda}} = 0 \Rightarrow \hat{\lambda}_n = \frac{1}{n}\sum_{i=1}^{n} k_i$$

And since λ is equal to the expected value (mean) and variance we can estimate λ as:

$$\hat{\lambda}_n = \frac{1}{n}\sum_{i=1}^{n} k_i = E[K] = Var[K]$$

Part b: Calculate $P_{MLE}(\lambda \mid D)$, where $D = \{2, 5, 0, 3, 1, 3\}$:

$$P_{MLE}(\lambda \mid D) = \hat{\lambda}_n = \frac{1}{n}\sum_{i=1}^{n} k_i \implies P_{MLE}(\lambda \mid D) = \frac{2+5+0+3+1+3}{6} = \frac{14}{6}$$

$$\implies P_{MLE}(\lambda \mid D) = \hat{\lambda} = 2.33$$

Part c: Derive an optimization for a MAP using conjugate prior and the Gamma distribution. The Gamma distribution is given as:

$$P_{\alpha,\beta}(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}\ e^{-\beta\lambda}\ ;\ \ where\ \ \alpha = 2,\ \ \beta = 1$$

For this part, we use Bayesian theorem to calculate the posterior distribution considering prior density of λ, $P(\lambda)$, and likelihood of data being a match for a given expected value, $P(D \mid \lambda)$.

$$P(\lambda \mid D) \propto P(\lambda)P(D \mid \lambda)$$

Derive likelihood for Poisson distribution and prepare for Bayes' equation:

$$P(D \mid \lambda) = \prod_{i=1}^{n} \frac{e^\lambda \lambda^{k_i}}{k_i!} = \frac{\prod_{i=1}^{n} e^\lambda \prod_{i=1}^{n} \lambda^{k_i}}{\prod_{i=1}^{n} k_i!} = \frac{e^{n\lambda}\lambda^{\sum_{i=1}^{n} k_i}}{\prod_{i=1}^{n} k_i!}$$

And we are given:

$$P(\lambda \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}\ e^{-\beta\lambda}\ ;\ \ where\ \ \alpha = 2,\ \ \beta = 1,\ \ \lambda > 0,$$

So we can substitute and also make n negative for $e^{n\lambda}$:

$$P(\lambda \mid D) \propto (\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}\ e^{-\beta\lambda})(\frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} k_i}}{\prod_{i=1}^{n} k_i!})$$

$$\implies P(\lambda \mid D) \propto (\frac{\beta^\alpha}{\Gamma(\alpha)\ \prod_{i=1}^{n} k_i!})(\lambda^{\alpha-1+\sum_{i=1}^{n} k_i}\ e^{-\lambda(n+\beta)})\ ;\ \ where\ \ \alpha = 2,\ \ \beta = 1,\ \ k_i \in D$$

With the last move, now we have the first parenthesis remain constant for given data and distribution ($D\ and\ \lambda$) so proportional probability would be the second parenthesis.

So for posterior distribution we would have, which looks like a β distribution therefore we can also define estimations for its parameters:

$$\implies P(\lambda \mid D) \propto (\lambda^{\alpha-1+\sum_{i=1}^{n} k_i}\ e^{-\lambda(\beta+n)})$$

$$\implies P(\lambda \mid D) \propto (\lambda^{\hat{\alpha}-1} . e^{-\lambda\hat{\beta}})\ ;\ \ \hat{\alpha} = \alpha + \sum_{i=1}^{n} k_i\ ,\ \ \hat{\beta} = \beta + n$$

For $\hat{\lambda} = 2.33$ and $\alpha = 2,\ \beta = 1$, we would have:

$$P_{MAP}(\lambda \mid D) \propto (\lambda^{\hat{\alpha}-1} \ .\ e^{-\lambda\hat{\beta}}) = (\lambda \mid \alpha + \sum_{i=1}^{n} k_i \ , \beta + n)\ ;\ where \ \sum_{i=1}^{n} k_i = 14$$

$$P_{MAP}(\lambda \mid D) \propto (2.33^{(2-1+14)} \ .\ e^{-(2.33)(1+6)}) = 2.\bar{3}^{15} \ .\ e^{-7(2.\bar{3})}$$
$$\implies P_{MAP}(\lambda \mid D, \alpha, \beta) \approx 0.02\bar{6}$$

Exercise 2

K Nearest Neighbor The following data provides, height, weight, age and gender.

D = {
((170, 57, 32), W),
((190, 95, 28), M),
((150, 45, 35), W),
((168, 65, 29), M),
((175, 78, 26), M),
((185, 90, 32), M),
((171, 65, 28), W),
((155, 48, 31), W),
((165, 60, 27), W),
((182, 80, 30), M),
((175, 69, 28), W),
((178, 80, 27), M),
((160, 50, 31), W),
((170, 72, 30), M),
}

a. Using Cartesian distance as the similarity measure. Show predictions for the following test data. For values of K or 1, 3, and 5. Include distance, calculation, neighbor selection and predictions.

$$D_{test} = \{(162, 53, 28), (168, 75, 32), (175, 70, 30), (180, 85, 29)\}$$

b. Implement.

Exercise 3