EE 5323- Exam 1

Fall 2021

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All questions require numerical calculations to arrive at the answers. To obtain full credit, show all your work. No partial credit will be given without the supporting work. This probably means you must do calculations by hand and type them up, not using MATLAB routines.

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Pledge of honor:

"On my honor I have neither given nor received aid on this examination."

Signature: Bardia Mojra

1. Equilibrium points and linearization

System is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + \frac{x_1^3}{9} - x_2$$

$$\dot{X} = A X \Rightarrow \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ (-1+\frac{x_1^2}{9}) & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

d. Find the Jacobian

$$J = \frac{d}{dx} A \Rightarrow J = \begin{bmatrix}\frac{df_1}{dx_1}(0) & \frac{df_1}{dx_2}(1)\\ \frac{d}{dx_1}(-1+\frac{x_1^2}{9}) & \frac{d}{dx_2}(-1)\end{bmatrix}$$

$$J = \begin{bmatrix}0 & 1\\ -1+\frac{2x_1}{9} & -1\end{bmatrix}$$

e. Find all equilibrium points

$$\dot{X} = 0 \Rightarrow \begin{bmatrix}\dot{x}_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} = \begin{bmatrix}x_2\\ -x_1 + \frac{x_1^3}{9} - x_2\end{bmatrix}$$

$$\Rightarrow \begin{cases} x_2 = 0 \\ -x_1 + \frac{x_1^3}{9} - \cancel{(0)} = 0 \end{cases} \Rightarrow x_1\left(\frac{x_1^2}{9} - 1\right) = 0 ; \Rightarrow \begin{cases} x_1 = 0 \rightarrow \text{Case 1} \\ \frac{x_1^2}{9} - 1 = 0 \Rightarrow \end{cases}$$

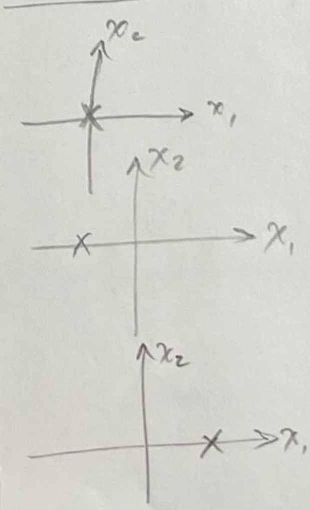
$$\frac{x_1^2}{9} = +9 \Rightarrow x_1 = \pm\sqrt{9} \Rightarrow \underline{x_1 = \pm 3} \quad \text{Cases 2 \& 3}$$

(x_1, x_2)

e.p. Case 1: $(0,0)$

e.p. Case 2: $(-3, 0)$

e.p. Case 3: $(+3, 0)$



phase plane

f. Find the nature of all e.p.s. Sketch the phase plane trajectories near each e.p.

$$\text{e.p.s.} = \begin{cases} (0,0) \\ (-3,0) \\ (+3,0) \end{cases} \qquad A_1 = \begin{bmatrix} 0 & 1 \\ -1 + \frac{2x_1}{9} & -1 \end{bmatrix}$$

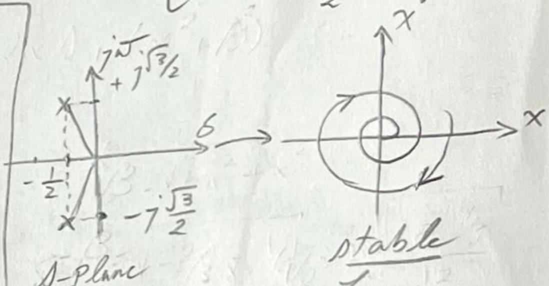
$$\Delta(s) = |sI - A| = 0 \Rightarrow \begin{vmatrix} s & -1 \\ 1 - \frac{2x_1}{9} & s+1 \end{vmatrix} \Rightarrow s^2 + s + \left(-1 \mp \frac{2x_1}{9}\right) = 0$$

$$\boxed{s^2 + s + 1 - \frac{2x_1}{9} = 0}$$

at (0,0):

$$s^2 + s + 1 = 0 \Rightarrow \underbrace{s^2 + \tfrac{1}{2}s + \tfrac{1}{2}s + \tfrac{1}{4}} - \tfrac{1}{4} + 1 = 0$$

$$\Rightarrow (s + \tfrac{1}{2})^2 - \tfrac{1}{4} + \tfrac{4}{4} = 0 \Rightarrow (s + \tfrac{1}{2})^2 = -\tfrac{3}{4} \Rightarrow s + \tfrac{1}{2} = \pm\frac{\sqrt{-3}}{2} \Rightarrow \left\{ s = \pm j\frac{\sqrt{3}}{2} - \tfrac{1}{2} \right. \text{ at } (0,0)$$

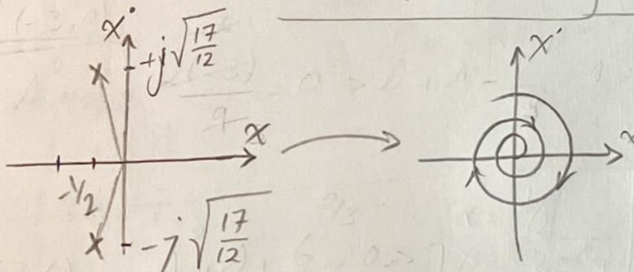


stable focus

at (-3,0) $\Rightarrow s^2 + s + 1 - \frac{2(-3)}{9} = 0 \Rightarrow s^2 + s + 1 + \frac{2}{3} = 0$

$$(s + \tfrac{1}{2})^2 - \tfrac{1}{4} + 1 + \tfrac{2}{3} = 0 \Rightarrow (s + \tfrac{1}{2})^2 = \frac{3 - 12 - 8}{12} = \frac{-17}{12}$$

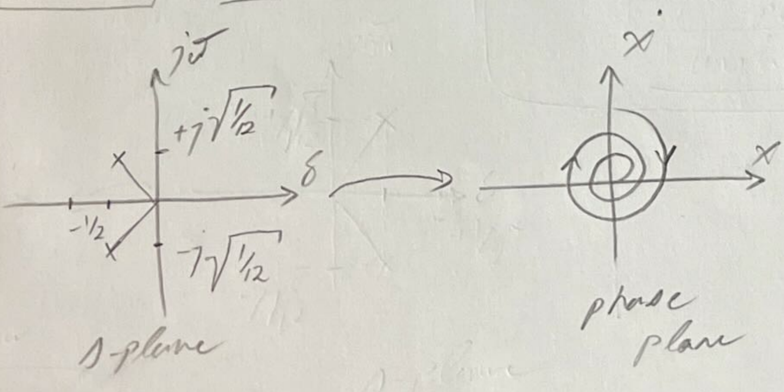
$$\Rightarrow s + \tfrac{1}{2} = \pm j\sqrt{\tfrac{17}{12}} \Rightarrow \boxed{s = \pm j\sqrt{\tfrac{17}{12}} - \tfrac{1}{2}} \text{ at } (-3,0)$$



stable focus about (-3,0)

at (+3,0) $\Rightarrow s^2 + s + 1 - \frac{2(3)}{9} = 0 \Rightarrow (s + \tfrac{1}{2})^2 = \tfrac{1}{4} - 1 + \tfrac{2}{3} = \frac{3 - 12 + 8}{12} = -\tfrac{1}{12} = (s + \tfrac{1}{2})^2$

$$\Rightarrow \boxed{s = \pm j\sqrt{\tfrac{1}{12}} - \tfrac{1}{2}} \text{ at } (3,0)$$



stable focus about (3,0)

$$\dot{X} = AX \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2x_1^2 - 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2. System is

$$\dot{x}_1 = -x_1 + 2x_1^3 + x_2$$

$$\dot{x}_2 = -x_1 - x_2$$

$$\dot{J} = \begin{bmatrix} \frac{df}{dx_1} & \frac{df}{dx_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow$$

a. Find the Jacobian

$$J_A = \begin{bmatrix} 4x_1 - 1 & 1 \\ -1 & -1 \end{bmatrix}$$

b. Find all equilibrium points $\Rightarrow$ Where the system is at its lowest point of energy $\Rightarrow$ thus $\dot{X} = 0$ & $\ddot{X} = 0$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{cases} 0 = -x_1 + 2x_1^3 + x_2 \\ 0 = -x_1 \quad\quad\quad - x_2 \end{cases} \text{ add}$$

$$0 = -2x_1 + 2x_1^3 + 0 \Rightarrow 2x_1(-1 + x_1^2) = 0 \Rightarrow \begin{cases} x_1 = 0 \\ x_1^2 = 1 \Rightarrow \\ x_1 = \pm\sqrt{1} \end{cases}$$

e.p.s. at $\begin{cases} (0,0) \\ (1,-1) \\ (-1,1) \end{cases}$

c. Find the nature of all e.p.s. Sketch the phase plane trajectories near each e.p.

e.p.s $\begin{cases} (0,0) \\ (1,-1) \\ (-1,1) \end{cases}$; $J = \begin{bmatrix} 4x_1 - 1 & 1 \\ -1 & -1 \end{bmatrix}$

characteristic equation

$$\Delta(s) = |sI - A| = 0 \Rightarrow \begin{vmatrix} s - 4x_1 + 1 & -1 \\ +1 & s+1 \end{vmatrix} = 0 \Rightarrow (s+1)(s-4x_1+1)+1$$

$$s^2 - 4x_1 s + s + s - 4x_1 + 1 + 1$$

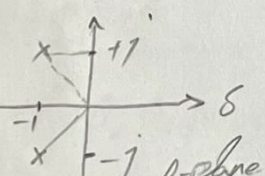
$$\boxed{0 = s^2 + (-4x_1 + 2)s + (-4x_1 + 2)}$$

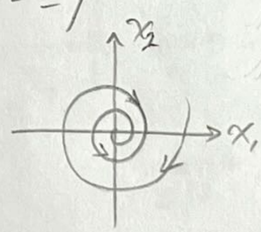
at (0,0)

$$s^2 + (2)s + 2 = 0 \Rightarrow \underbrace{s^2 + 2s + 1}_{(s+1)^2} + 1 = 0$$

$$\Rightarrow (s+1)^2 + 1 = 0 \Rightarrow (s+1)^2 = -1 \Rightarrow s+1 = \pm j$$

$$\Rightarrow \boxed{s = -1 \pm j}$$





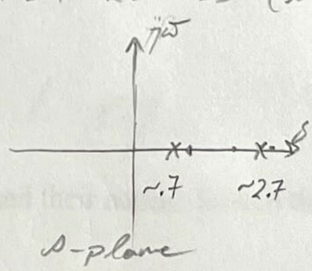
phase Plane
stable Focus

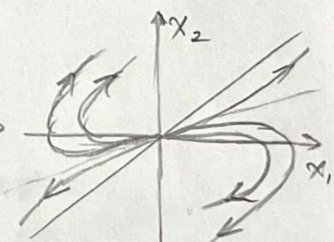
at (1,-1) $\Rightarrow s^2 \overset{-2s}{-4s + 2s} - 2 = 0 \Rightarrow \underbrace{s^2 - 2s + 1} - 1 - 2 = 0 \Rightarrow (s-1)^2 - 3 = 0 \Rightarrow$

$$(s-1)^2 = 3 \Rightarrow s - 1 = \pm\sqrt{3} \Rightarrow \boxed{s = 1 \pm \sqrt{3}}$$

$\boxed{\sqrt{3} \approx 1.7}$

$$\Rightarrow \begin{cases} s_1 \approx 2.7 \\ s_2 \approx 0.7 \end{cases}$$





phase plane
unstable node

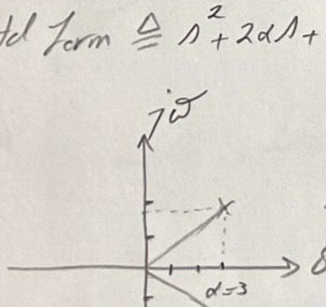
at (-1,1) $\Rightarrow s^2 + 6s + 6 = 0$; std form $\triangleq s^2 + 2\alpha s + \omega_n^2 = 0 \Rightarrow$

$$\Rightarrow \begin{cases} 2\alpha = 6 \Rightarrow \alpha = 3 \\ \omega_n^2 = 6 \Rightarrow \beta = \pm\sqrt{\omega_n^2 - \alpha^2} \end{cases}$$

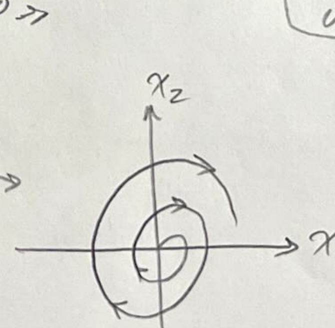
$$\Rightarrow \beta = \pm\sqrt{6 - 9}$$

$$\Rightarrow \beta = \pm\sqrt{-3} = \pm j\sqrt{3}$$

$$\boxed{s = 3 \pm 1.7j}$$



s-plane



phase plane
unstable Focus

3. Duffing's equation is interesting in that it exhibits bifurcation, or dependence of stability properties and number of equilibrium points on a parameter. The undamped Duffing equation is

$$\ddot{x} + \alpha x + x^3 = 0$$

$\dot{x} = \dot{x}$

$\ddot{x} = -\alpha x - x^3 \Rightarrow \dot{X} = AX \Rightarrow \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha - x^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

c. Find the Jacobian

$$J_A = \begin{bmatrix} 0 & 1 \\ -\alpha - x^2 & 0 \end{bmatrix}$$

e.p.s. $\dot{X} = 0 \Rightarrow \alpha x + x^3 = 0 \Rightarrow$

$\begin{cases} \boxed{x = 0} \\ \alpha + x^2 = 0 \Rightarrow \boxed{x^2 = \sqrt{\alpha}} \Rightarrow \boxed{x = \pm\sqrt{\alpha}} \end{cases}$

$\Delta(s) = 0 \Rightarrow |sI - A| = 0 \Rightarrow \begin{vmatrix} s & -1 \\ \alpha + x^2 & s \end{vmatrix} = 0 \Rightarrow s^2 + (+\alpha + x^2) = 0$

$\Rightarrow \Delta(s) = \boxed{s^2 + \alpha + x^2 = 0}$

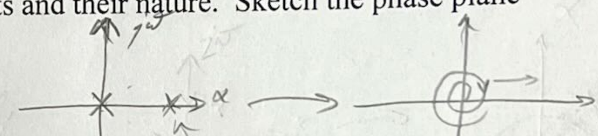
d. Let $\alpha > 0$. Find the equilibrium points and their nature. Sketch the phase plane trajectories near each e.p.

$s^2 + \alpha + x^2 = 0$

$\alpha > 0 \quad x^2 > 0$

$\Rightarrow s_1 = 0$ / if $x^2 > 0$

if $x^2 < 0$; $s_{2,3} \in Im\{\}$



stable e.p.

stable focus

& are complex conjugate

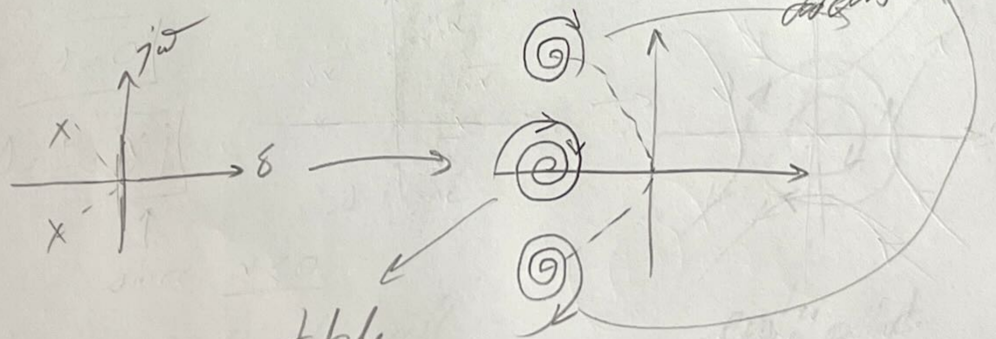
& add up to zero

So in this case, there only 1 eq. (0,0)

e. Let $\alpha < 0$. Find the equilibrium points and their nature. Sketch the phase plane trajectories near each e.p.

$s^2 + \alpha + x^2 = 0 \rightarrow$

$\begin{cases} s = \pm\sqrt{\alpha} \\ s = 0 \end{cases}$



unstable focus

3 e.p.s

stable focus