Derivation of HJI Constrained – Hong Kong Version

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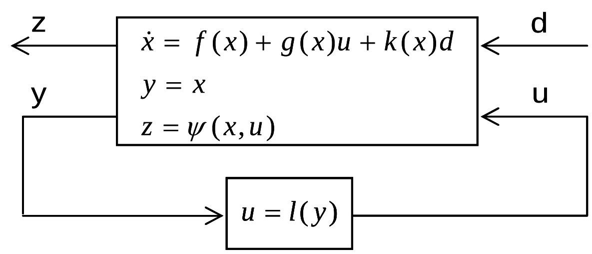
Bounded L_2 Gain Solution for Input-Constrained Systems

For the system

$$\dot{x} = f(x) + g(x)u + k(x)d$$
$$y = x$$
$$z = \psi(x,u)$$



where $\|z\|^2 = h^T h + \|u\|^2$, one desires to find a control *u(t)* such that, for a prescribed *γ*, when *x(0)= 0* and for all disturbances $d(t) \in L_2$ one has

$$\frac{\int_0^\infty \|z(t)\|^2 dt}{\int_0^\infty \|d(t)\|^2 dt} = \frac{\int_0^\infty (h^T h + \|u\|^2)dt}{\int_0^\infty \|d(t)\|^2 dt} \le \gamma^2,$$

i.e. L_2-gain less than or equal to *γ*.

Game Theory Saddle Point

Define the value functional

$$J(u,d) = \int_0^\infty \left(h^T h + \|u\|^2 - \gamma^2 \|d\|^2\right) dt$$

and consider the zero sum differential game where player *u(t)* (the optimal control) seeks to minimize *J* and player *d(t)* (the worst case disturbance) seeks to maximize *J*. If the resulting solution satisfies *J<0* for all *d(t)*, then the L_2-gain is less than or equal to γ. This optimal control problem has a unique solution if a game theoretic saddle point exists, i.e. if

$$\max_d \min_u J(x_0,u,d) = \min_u \max_d J(x_0,u,d) \qquad (1)$$

with $x(0)=x_0$ the initial condition.

To solve the optimal control problem, introduce a Lagrange multiplier *p(t)* and define the Hamiltonian

$$H(x,p,u,d) \equiv p^T\left(f + gu + kd\right) + h^T h + \|u\|^2 - \gamma^2 \|d\|^2 .$$

Since *H* is separable in *u, d*, necessary conditions for a stationary point are given by

$$0 = \frac{\partial H}{\partial u}, \quad 0 = \frac{\partial H}{\partial d}.$$

A unique saddle point $H(x,p,u^*,d^*)$ for *H(x,p,u,d)* exists at *(u*,d*)* if

$$\max_d \min_u H(x,p,u,d) = \min_u \max_d H(x,p,u,d) \tag{2}$$

which is equivalent to

$$H(x,p,u^*,d) \le H(x,p,u^*,d^*) \le H(x,p,u,d^*)$$

or

$$\frac{\partial^2 H}{\partial u^2} > 0, \quad \frac{\partial^2 H}{\partial d^2} < 0 .$$

Though (1) implies (2), the converse is not true. When (2) holds and (1) does not hold, one solves the minimax problem to obtain the practically useful solution

$$\min_u \max_d J(x_0,u,d) .$$

This is accomplished by first finding the worst case disturbance *d(t),* then holding it fixed while finding the optimal control *u(t).*

Constrained Controls

Define the norm $\|d\|^2 = d^T d$. To make sure the control *u(t)* is constrained with prescribed saturation function *ϕ(.),* define the object

$$\|u\|_q{}^2 = 2\int_0^u \phi^{-T}(\nu)d\nu$$

where $\phi^{-T}(u)u \ge 0$. This is a quasi-norm, i.e.

1. $\|x\|_q = 0 \Leftrightarrow x = 0$
2. $\|x+y\|_q \le \|x\|_q + \|y\|_q$
3. $\|x\|_q = \|-x\|_q$

Note that the third property of symmetry is weaker than the norm property of homogeneity.

Now one has the Hamiltonian

$$H(x,p,u,d) \equiv p^T\left(f + gu + kd\right) + h^T h + 2\int_0^u \phi^{-T}(\nu)d\nu - \gamma^2 d^T d$$

and the stationarity conditions

$$0 = \frac{\partial H}{\partial u} = g^T p + 2\phi^{-1}(u)$$

$$0 = \frac{\partial H}{\partial d} = k^T p - 2\gamma^2 d$$

so the optimal inputs are

$$u^* = -\tfrac{1}{2}\phi\left(g^T(x)p\right)$$

$$d^* = \frac{1}{2\gamma^2} k^T(x)p .$$

Note that the control is guaranteed to be saturated with saturation function *ϕ(.).*

One can show that

$$H(x,p,u,d) = H(x,p,u^*,d^*) - \gamma^2 \|d - d^*\|^2 + 2\left\{ \int_{u^*}^{u} \phi^{-T}(\nu)d\nu - \phi^{-T}(u^*)(u-u^*) \right\}^2$$

where the last term is positive definite if $\phi^{-1}(u)$ is monotonically increasing. This implies

$$H(x,p,u^*,d) \le H(x,p,u^*,d^*) \le H(x,p,u,d^*)$$

which guarantees a unique saddle point *(u*,d*)* for the Hamiltonian.

Bounded L_2 Gain

Note that for any C^1 function $V(x): R^n \to R$ one has, along the system trajectories,

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}^T \dot{x} = \frac{\partial V}{\partial x}^T (f + gu + kd),$$

so that

$$\frac{dV}{dt} + h^T h + 2\int_0^u \phi^{-T}(\nu)d\nu - \gamma^2 d^T d = H(x,V_x,u,d).$$

Suppose now there exists a C^1 function $V(x): R^n \to R$, with *V(0)=0*, whose gradient satisfies

$$H(x,V_x,u^*,d^*) = \frac{\partial V}{\partial x}^T \left(f + gu^* + kd^*\right) + h^T h + 2\int_0^{u^*} \phi^{-T}(\nu)d\nu - \gamma^2 d^{*T} d^* = 0 \qquad (3)$$

for all *x*. Then one has

$$H(x,V_x,u,d) = H(x,V_x,u^*,d^*) - \gamma^2 \|d - d^*\|^2 + 2\left\{ \int_{u^*}^{u} \phi^{-T}(\nu)d\nu - \phi^{-T}(u^*)(u-u^*) \right\}^2$$

$$H(x,V_x,u,d) = -\gamma^2 \|d - d^*\|^2 + 2\left\{ \int_{u^*}^{u} \phi^{-T}(\nu)d\nu - \phi^{-T}(u^*)(u-u^*) \right\}^2$$

Therefore,

$$\frac{dV}{dt} + h^T h + 2\int_0^u \phi^{-T}(\nu)d\nu - \gamma^2 d^T d = -\gamma^2 \|d - d^*\|^2 + 2\left\{ \int_{u^*}^{u} \phi^{-T}(\nu)d\nu - \phi^{-T}(u^*)(u-u^*) \right\}^2$$

Selecting *u(t)=u*(t)* yields

$$\frac{dV}{dt} + h^T h + 2\int_0^{u^*} \phi^{-T}(\nu)d\nu - \gamma^2 d^T d \le 0. \qquad (4)$$

Integrating this equation yields

$$V(x(T)) - V(x(0)) + \int_0^T \left(h^T h + 2\int_0^{u^*} \phi^{-T}(\nu)d\nu - \gamma^2 d^T d \right) dt \le 0 \qquad (5)$$

Select x(0)=0 and assume the system is asymptotically stable so that $\lim_{T\to\infty} x(T) = 0$. Noting that *V(0)=0* one has

$$\int_0^\infty \left(h^T h + 2\int_0^{u*} \phi^{-T}(\nu)d\nu \right) dt \le \gamma^2 \int_0^\infty (d^T d)\, dt$$

so the L_2 gain is less than γ.

Value Functions

Note that, for any prescribed *u(t), d(t)* one may set the Hamilton to zero,

$$0 = H(x, V_x, u, d) \equiv \frac{\partial V}{\partial x}^T \left(f + gu + kd \right) + h^T h + 2\int_0^u \phi^{-T}(\nu)d\nu - \gamma^2 d^T d$$

$$= \frac{dV}{dt} + h^T h + 2\int_0^u \phi^{-T}(\nu)d\nu - \gamma^2 d^T d \qquad (6)$$

This provides an infinitesimal equivalent to *J(u,d).* In fact, the value function or cost to go for the selected *u(t), d(t)* is defined as

$$V(x(t)) = \int_t^\infty \left(h^T h + \|u\|^2 - \gamma^2 \|d\|^2 \right) dt$$

whence Leibniz's formula reveals the infinitesimal equivalent

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}^T \dot{x} = -\left(h^T h + \|u\|^2 - \gamma^2 \|d\|^2 \right).$$

Therefore, solving (6) gives the cost to go for any given *u(t), d(t).* The *optimal* cost to go is given by the value function which solves (3).

Stability of Closed-Loop System

So far *V(x)* only needs to satisfy the HJI equation. If in fact there is a solution *V(x)* to the HJI with *V(x)>0,* and the system is zero-state observable through *z(t),* then the closed-loop system using *u(t)= u*(t)* is locally asymptotically stable. In fact, letting $d(t) = 0$ in (4) one has

$$\frac{dV}{dt} \le -h^T h - 2\int_0^{u*} \phi^{-T}(\nu)d\nu = -\|z(t)\|^2$$

However, ZS observable means $z(t) \equiv 0 \Rightarrow x(t) \equiv 0$. The converse $x(t) \ne 0 \Rightarrow z(t) \ne 0$ shows that $\dot{V} < 0$, so that the system is locally AS with Lyapunov function *V(x).*

Note further that if V(x)>0 then, for any *d(t),* the closed-loop system has bounded output *z(t).* In fact if *V(x)> 0,* then according to (5) one has

$$\int_0^T \left(h^T h + 2\int_0^{u*} \phi^{-T}(\nu)d\nu - \gamma^2 d^T d \right) dt \le -V(x(T)) < 0$$

and the system has bounded L_2 gain.

Determining the Optimal Controls

To determine the optimal controls one must solve the equation

$$H(x, V_x, u^*, d^*) = \frac{\partial V}{\partial x}^T \left(f + gu^* + kd^* \right) + h^T h + 2\int_0^{u^*} \phi^{-T}(v)dv - \gamma^2 d^{*T} d^* = 0$$

with the optimal control and worst case disturbance

$$u^* = -\tfrac{1}{2}\phi\left(g^T(x)V_x\right)$$

$$d^* = \frac{1}{2\gamma^2} k^T(x)V_x .$$

Substituting these into the optimal Hamiltonian yields the Hamilton-Jacobi-Isaacs equation

$$\frac{dV}{dx}^T \left(f - g \cdot \phi\left(\frac{1}{2} g^T \frac{dV}{dx} \right) \right) + h^T h + 2 \int_0^{-\phi\left(\frac{1}{2} g^T \frac{dV}{dx}\right)} \phi^{-1}(v)dv + \frac{1}{4\gamma^2} \frac{dV}{dx}^T kk^T \frac{dV}{dx} = 0.$$

This equation is generally impossible to solve analytically for most nonlinear systems.