1. MANIFOLDS (M)

- Hausdorff topological space with an atlas → covering by a collection of open sets U_i, with homeomorphisms $\varphi_i : U_i \to V_i \subset \mathbb{R}^n$, V_i is open, $\dim M = n$
- smooth (differentiable) manifold: $U_{jk} = U_j \cap U_k$ $\quad \varphi_{jk} : \varphi_j(U_{jk}) \mapsto \varphi_k(U_{jk})$ – two charts (coordinatizations) is a smooth diffeomorphism between open sets in $\mathbb{R}^n$.

- meaning – coordinatization (chart) gives a description in $\mathbb{R}^n$, this description is not unique but connected by diffeomorphisms.
- Diff : $M \to M$, diffeomorfically connected manifolds are the same manifold.
- vector fields defined on manifolds $\mathcal{X}(M)$ reside in a tangent bundle. Their action on scalar functions $F(M)$ is $L_x f$ $\quad x \in \mathcal{X}(M)$, $f \in F(M)$, coordinate independent. Under φ_i this becomes $a_i(x) \frac{\partial}{\partial x_i} f$, summation over i implied. $\frac{\partial}{\partial x_i}$ ← a coordinate basis, $a_i(x)$, $x \in \mathbb{R}^n$, $i = 1..n$ coordinate of a vector.

2. FLOWS OF VECTOR FIELDS

$\mathcal{U} \subset \mathbb{R}^n$ $\quad \mathcal{X}(\mathcal{U})$ – vector field on $\mathcal{U}$ is a smooth map

$$X : \mathcal{U} \mapsto \mathbb{R}^n$$

and the respective differential equation is:

$$\dot{y} = X(y) \quad y(0) = x \quad x \in \mathcal{U} \qquad (1)$$

$y(t)$ ← integral curve of the vector field → an orbit

$y(t) = \Phi_x^t(x)$ $\qquad \Phi^t$ ← one parameter family of mappings → flow

$\Phi_x^t : \mathcal{U} \to \mathcal{U}$ $\qquad t$ is the time

(1) is a dynamical system on $\mathcal{U}$, with an initial condition x. $\mathcal{U}$ could had also been one of $\mathcal{U}_i$, and then under the coordinatization φ_i one has

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n, \quad f(x) \in \mathbb{R}^n$$

However under a different coordinatisation Φ_x one finds

$$\dot{y} = g(y)$$

Now, if the flow is on a smooth manifold there is a diffeomorphism

$$x \to y \quad \varphi(x) = y$$

Then one finds:

$$\dot{y} = D\varphi_x \dot{x} = D\varphi_x f(x) = D\varphi_x(\varphi^{-1}(y)) f(\varphi^{-1}(y))$$
$$= g(y)$$

$D\varphi_x$ is a Jacobian of φ

conversely

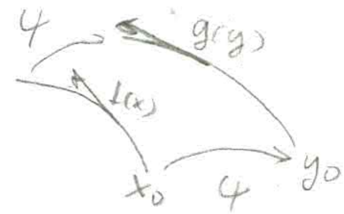
$$f(x) = D^{-1}\varphi_x(x)\, g(y(x))$$

Clearly φ takes flows $x(t) \to y(t)$

$$\varphi(\Phi^t(x)) = \psi^t(\varphi(x))$$

ϕ^t flow of $f(x)$

ψ^t flow of $g(y)$



if φ is C^k one sais that those two systems are C^k-diffeomorphic

- two diffeomorphically equivalent systems have similar Jacobians around an equilibrium point

$$A(x_0) = D\varphi^{-1}(x_0)\, B(y_0)\, D\varphi(y_0)$$

because

$$D f(x) = D^{-1}\varphi_x\, D g(y(x))$$

jacobian w.r.t. y

$$= D^{-1}\varphi_x\, D_y g \frac{dy}{dx} = D^{-1}\varphi_x \underbrace{D_y g}_{B(y_0)} D\varphi_x$$

Diffeomorphically equivalent systems $\to$ topologically equivalent systems, just two different coordinate representations of the same system.

DIGRESSION ON INVARIANT MANIFOLDS

Given an equilibrium point of $\dot{x} = f(x)$ as x_0 one has invariant spaces spanned by generalised eigenvectors whose eigenvalues have positive, negative or zero real part.

$$E^s = \text{span}\{v^1 \dots v^{n_s}\}$$
$$E^u = \text{span}\{u^1 \dots u^{n_u}\}$$
$$E^c = \text{span}\{w^1 \dots w^{n_c}\}$$

these linear vector spaces are tangent to corresponding stable, unstable and center manifold

- locally:

$$W^s_{loc}(x_0) = \{ x \in \mathcal{U}(x_0) \mid \phi^t(x) \to x^0 \text{ as } t \to \infty, \ \phi^t(x) \in U \ \forall t \geq 0 \}$$

$$W^u_{loc}(x_0) = \{ x \in \mathcal{U}(x_0) \mid \phi^t(x) \to x^0 \text{ as } t \to -\infty, \ \phi^t(x) \in \mathcal{U} \ \forall t \leq 0 \}$$

for a hyperbolic equilibrium $E^c = \phi$, $\underset{= \dim E^s}{\dim W^s = n_s}$, $\underset{= \dim E^u}{\dim W^u = n_u}$

- globally:

$$W^s(x_0) = \bigcup_{t \leq 0} \phi^t(W^s_{loc}(x_0)) \qquad W^u(x) = \bigcup_{t \geq 0} \phi^t(W^u_{loc}(x_0))$$

PARAMETER DEPENDENT DYNAMICAL SYSTEMS, TOPOLOGICAL EQUIVALENCE AND BIFURCATIONS OF EQUILIBRIA.

$\dot{x} = f(x, \mu)$ $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$ is a m parameter system

$f(x, \mu) = 0$ solution set, x_0 is a solution

$x_0(\mu)$ is a smooth function when Df_x is nonsingular (implicit fn. theorem

$$f(x, \mu) = 0 \to f(x + dx, \mu + d\mu) = 0$$
$$\Rightarrow D_x f \, dx + D_\mu f \, d\mu = 0$$
$$\frac{dx}{d\mu} = D_x f^{-1} D_\mu f$$

defines a flow (simplest case - when $m = 1$)

Equivalently there is a diffeomorphism connecting $f(x,0)$ and $f(x,\mu)$, i.e. they are topologically equivalent.

Def: A value of μ_0 where the system is not structurally stable is a bifurcation value, $x_0(\mu_0)$ is a bifurcation point

- structural stability: (1) $\dot{x} = f(x)$
(2) $\dot{x} = f(x) + \varepsilon g(x)$

} there must $\exists\, \varepsilon > 0$ s.t. (1) and (2) topologically equivalent.

imagine a change in μ resulting in a small perturbation

$$f(x,0) \rightarrow f(x,\delta\mu) = f(x,0) + \underbrace{Df_\mu(x,0)}_{g(x)}\delta\mu \qquad \delta\mu \ll 1$$

Now, under such a change, if $D_x f$ is nonsingular the change is smooth and $D_x f$ remains nonsingular ($x = x(\varepsilon)$).

$$A_\varepsilon = \left(D_x f + \varepsilon D_x g\right)\Big|_{x=x(\varepsilon)}$$

C^1 perturbations
$\|f-g\| + \|Df - Dg\| = d(f,g)$

therefore hyperbolic equilibria are structurally stable and remain topologically equivalent under small perturbations. Recall that in those cases $E^c = \emptyset$. Perturbation is small in C^1 sense.

If $E^c \neq 0$ equilibrium is non-hyperbolic, $D_x f$ is singular and a nonunique solution may exist for the equilibrium point.

So, a bifurcation is the appearance of a topologically nonequivalent phase portrait under variation of parameters. If it involves equilibrium points, then those must be structurally unstable to bifurcate. Structural stability is somewhat more subtle if one is dealing with invariant sets other then equilibrium points

CENTRE MANIFOLD THEOREM AND NORMAL FORMS

- centre manifold unlike W^s and W^u may not be unique
- tangent to central eigenspace E^c

- TM. Let f be a C^r vector field on $\mathbb{R}^n$, having a singular point at the origin $f(0)=0$, and let $A = D_x f(0)$, then the spectrum of A $\sigma(A)$ can be devided into $\sigma_s, \sigma_u, \sigma_c$ with $Re\,\lambda < 0 \to \sigma_s$, $Re\,\lambda > 0 \to \sigma_u$, $Re\,\lambda = 0 \to \sigma_c$. Let E^s, E^u, E^c be the generalized eigenspaces of $\sigma_s, \sigma_u, \sigma_c$ then there exist C^r stable and unstable invariant manifolds W^u, W^s tangent to E^u, E^s at 0, and a C^{r-1} centre manifold W^c tangent to E^c at 0. All of them are invariant for the flow, and unique, except W^c.

Theorem implys local topological equivalence of a bifurcating system $\dot{x} = f(x)$ to

$$\begin{cases} \dot{\tilde{x}} = \tilde{f}(x) \\ \dot{\tilde{y}} = -\tilde{y} \\ \dot{\tilde{z}} = \tilde{z} \end{cases} \qquad (\tilde{x}, \tilde{y}, \tilde{z}) \in W^c \times W^s \times W^u$$

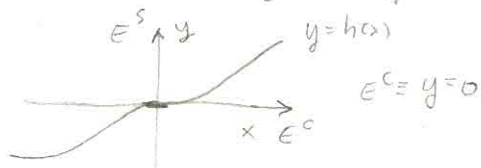
at the bifurcation point.

Forgetting W^u (applications) one can write linear and nonlinear parts of the above system as:

$$\begin{aligned} \dot{x} &= Bx + f(x,y) \\ \dot{y} &= Cy + g(x,y) \end{aligned} \qquad (x,y) \in \mathbb{R}^{n_c} \times \mathbb{R}^{n_s}$$

$$\begin{aligned} &Re\,\lambda_i(B) = 0 \quad \forall i = 1..n_c \\ &Re\,\lambda_i(C) < 0 \quad \forall i = 1..n_s \end{aligned}$$

Since W^c tangent to E^c ($E^c \equiv y=0$ space) then $W^c = \{(x,y) \mid y = h(x)\}$



$$\left.\begin{aligned} h(0) &= 0 \\ Dh(0) &= 0 \end{aligned}\right\} \text{tangency}$$

Then:
$$\dot{x} = Bx + g(x, h(x))$$
$$\dot{y} = Dh(x)\,\dot{x} = D_x h(x)[Bx + f(x, h(x))] = Ch(x) + g(x, h(x))$$
$$\Rightarrow N(h(x)) = D_x h(x)[Bx + f(x, h(x))] - Ch(x) - g(x, h(x)) = 0$$

possible to find $h(x) = h_0(x) + O(|x|^p)$ as $|x| \to 0$, a local approximation with a finite number of powers (Taylor series)

Obtaining this one seeks normal forms, only looking at the centre manifold since there is where the bifurcating solutions are.

Let us start with a system $\dot{x} = f(x)$ and apply a sequence of diffeomorphisms eliminating higher order terms.

NOTE: parameter dependent:
$\dot{x} = f(x, \alpha)$ $\quad \dot{y} = g(y, \beta)$
$x, y \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}^m$
$f \sim g \Leftarrow p : \mathbb{R}^n \to \mathbb{R}^m$ $\quad \beta = p(\alpha)$
$h_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ $\quad y = h_\alpha(x)$

$$\dot{x} = f(x)$$
$$x = h(y) \qquad D_y h(y)\,\dot{y} = f(h(y))$$
$$\dot{y} = D_y h^{-1}(y)\, f(h(y))$$

- one hopes to obtain a linear system

Assuming $Df(0)$ has distinct eigenvalues and that the system has been diagonalized using linear transformation, then:
$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + g_1(x_1 \ldots x_n) \\ \vdots \\ \dot{x}_n = \lambda_n x_n + g_n(x_1 \ldots x_n) \end{cases} \iff \dot{x} = \Lambda x + g(x)$$

$$\dot{x} = \Lambda x + g(x)$$

$$x = h(y) = y + P(y)$$

$\deg P$ = smallest degree of a nonvanishing derivative of some g_i.

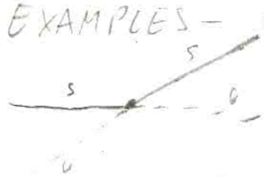
So $$\dot{y} = (I + DP(y))^{-1} f(y + P(y))$$

$$(I + DP(y))^{-1} = I - DP(y) \quad \text{up to first order}$$

P can be found if no eigenvalues of Λ have zero real part.
If this is not the case one uses the simplest polinomials topologically equivalent to the starting system

These low order polinomials are normal forms that describe qualitatively the nature of bifurcation. All bifurcation problems having the same normal form are equivalent.

- EXAMPLES - ONE PARAMETER BIFURCATIONS



$$\dot{x} = \mu x - x^2 \qquad \text{TRANSCRITICAL}$$

$$\dot{x} = \mu x \pm x^3 \qquad \text{PITCHFORK}$$

$$\dot{x} = \mu - x^2 \qquad \text{SADDLE NODE}$$

- bifurcation diagram

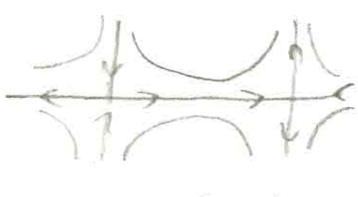
- def: stratum - maximally connected parameter set, for $\mu = \mu_0$, has all equivalent phase portraits.

- def: bifurcation diagram is a stratification of parameter space into strata (equivalence classes) induced by top. equivalence

HOPF → birth of a limit cycle → complex conjugate pairs passing through the imaginary axis

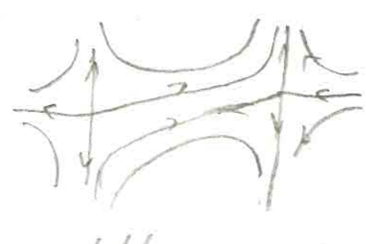
$$\begin{cases} \dot{x}_1 = \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2) \end{cases} \sim \begin{cases} \dot{\rho} = \rho(\alpha - \rho^2) \\ \dot{\theta} = 1 \end{cases}$$

NOTE: Global saddle connections must be transversal, otherwise a global bifurcation may occur:



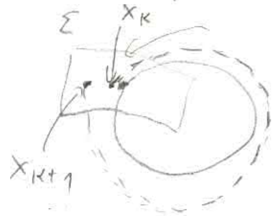
nontransversal

→



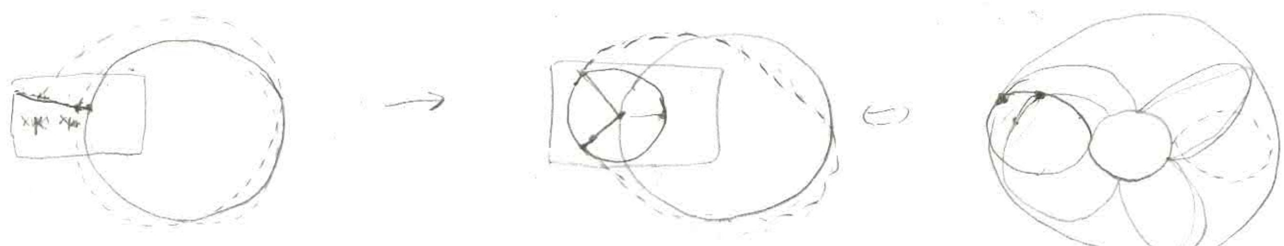
two saddle separatrices

- Global bifurcations more complicated, similar to several parameter bifurcations.
- bifurcations of limit tori, much more involved, bifurcations of chaotic attractors exhibiting continuous structural instability...
- generic bifurcations (Sotomayor) → which bifurcations are likely to occur in which types of systems - saddle node → the only generic one parameter bifurcation. (Generic = valid on a dense set)
- bifurcations of limit cycles can be analized as bifurcations of Poincaré map:



- dynamical system in $\mathbb{R}^n$ with a limit cycle defines a discrete system, a map on Σ, $\dim \Sigma = n-1$. Limit cycle corresponds to a fixed point of the Poincaré map.

Neimark-Sacker bifurcation - birth of a limit torus



EXAMPLE OF GLOBAL BIFURCATIONS

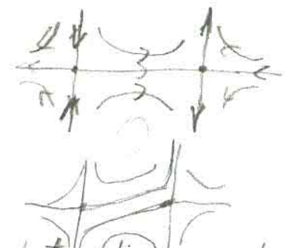
- prior example of nontransversal saddle connections. Now a homoclinic orbit.
- homoclinic and heteroclinic orbits occur when an equilibriums invariant manifolds intersect. $W^u \cap W^s \neq \phi$, unstable and stable manifolds belonging to the same (homo) or different (hetero) equilibrium.

Note that $W_1^u \cap W_2^u = \phi$, $W_1^s \cap W_2^s \neq \phi$

<u>In plane</u> – example of heteroclinic (saddle connection) and homoclinic bifurcations:

HETEROCLINIC:

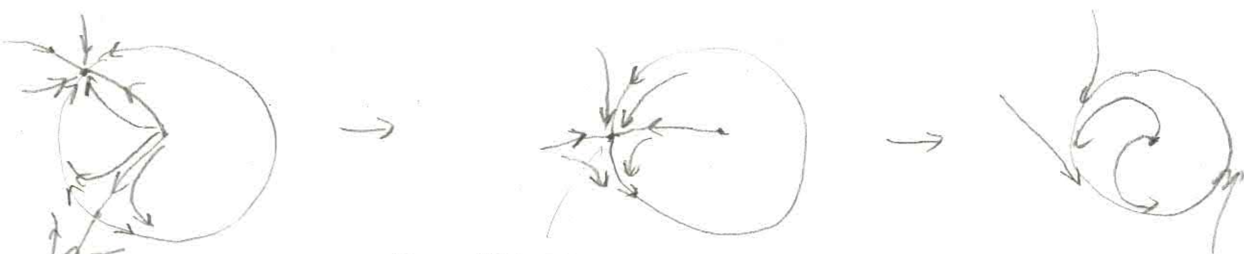
$$\dot{x}_1 = 1 - x_1^2 - \alpha x_1 x_2 \qquad \forall \alpha \quad x_{10} = (-1,0)$$
$$\dot{x}_2 = x_1 x_2 + \alpha(1 - x_1^2) \qquad x_{20} = (1,0)$$



$\alpha = 0$, x_1 axis is invariant, and it has a heteroclinic orbit, however for $\alpha \neq 0$ it is no longer invariant, and global saddle connection disappears.

HOMOCLINIC

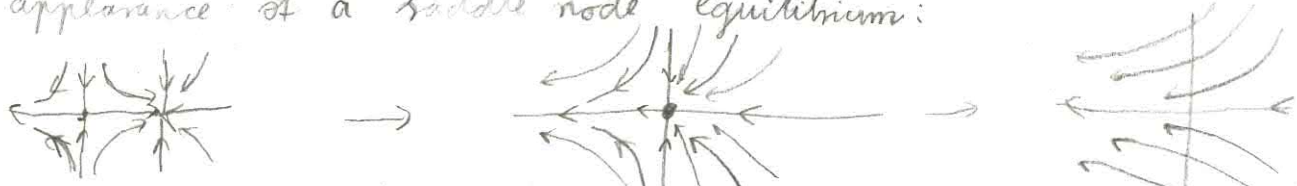
$$\begin{cases} \dot{x}_1 = x_1(1 - x_1^2 - x_2^2) - x_2(1 + \alpha + x_1) \\ \dot{x}_2 = x_1(1 + \alpha + x_1) + x_2(1 - x_1^2 - x_2^2) \end{cases} \sim \quad \begin{aligned} \dot{\rho} &= \rho(1 - \rho^2) \\ \dot{\theta} &= 1 + \alpha + \rho\cos\theta \end{aligned}$$



LOCALLY SADDLE NODE

$\alpha < 0$ $\qquad$ $\alpha = 0$ $\qquad$ $\alpha > 0$ LIMIT CYCLE

– the appearance of a saddle node equilibrium:



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distinct saddle and the node $\qquad$ degenerate singular point $\qquad$ no equilibrium