STABILIZATION OF DOUBLE INVERTED PENDULUM ON CART: LQR APPROACH

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Abstract- Parameter identification and optimal control of a double inverted pendulum using LQR approach are presented. The mechanical model of the system was derived by the Lagrangian method, and the parameters were identified via the energetic description of the system. The model of the system was developed and experimental results verified stability and recovery of the system by applying impact disturbance.

Keywords: Optimal Control, LQR, Parameter Identification, Double Inverted Pendulum

I. INTRODUCTION

Inverted pendulum forms a highly non-linear class of problems in classical control theory. The interest in inverted pendulum problems is raised due to the fundamental similarities of these systems with a great variety of practical systems, e.g. balancing a stick on the hand palm, mimicking the human gait, balancing the rocket launcher, vertical motion of the human arm [1]. Since the number of actuators in the inverted pendulum systems are typically less than the degrees of freedom (DOF), such systems are categorized as under-actuated systems and are highly unstable yet controllable. As the DOFs of the system surpass the number of actuation, the complexity, and nonlinearity of the system increases and the controllability of the system worsens.

The controlling and stabilization problem of the double inverted pendulum on a cart was studied in this paper. The objective was to find the control parameters first and then using the mechanical model of the system; an LQR optimal controller was designed which should be relatively robust facing spontaneous environmental disturbances. Thus, these five steps of were taken:

1. Modeling the problem using energy approach.
2. Identify the value of different system parameters extracted from the modeling part.
3. Simulate the problem computationally in MATLAB® SIMULINK® and predict the behavior of the system under control.
4. Implement the controller on the system and perform data acquisition while controller in action.
5. Compare the performance of the system with the simulated predictions and optimize the parameters and controller design to minimize the error between the prediction and simulation.

Many researchers have already studied the single and double inverted pendulum on a cart. Different control approaches have already been proposed and implemented to such systems, e.g. PID[2-4], LQR[4-6], LQG[7, 8], neural network[9, 10], fuzzy logic[2, 10, 11].

In another effort, Jadlovska S. et al. [1], have designed and implemented a complete MATLAB® toolbox; i.e. "Inverted Pendulum Modeling and Control." Studies have shown the high sensitivity of the control performance on model parameters [12, 13]. Thus parameter identification process should be performed using a technique less prone to data acquisition noise and uncertainty. Through the literature concerning model-based parameter identification, two distinct methods are identifiable, i.e. parameter identification using the equation of motion of the system denoted as EOM from now on [14], and parameter identification energetic description of the system, ED.

II. MECHANICAL MODELING

The mechanical model of the system was derived based on the geometrical assumptions as depicted in fig. 1. The energetic description of the system, i.e. kinetic and potential energies of the components, were formulated and then used to derive the Lagrangian description of motion.
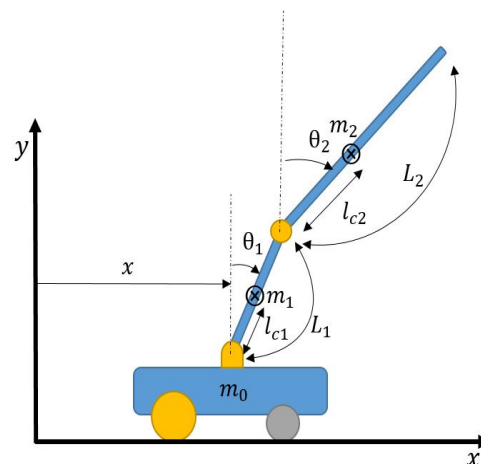

Fig 1: The geometrical model of the double inverted pendulum on a cart system.

The Lagrangian description of motion, implies that:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q \tag{1}$$

Where, q is the vector of degrees of freedom, i.e. generalized coordinates, L is the Lagrangian functional and Q is the external forces applied to the system. In this problem q is:

$$q = \begin{pmatrix} x \\ \theta_1 \\ \theta_2 \end{pmatrix} \tag{2}$$

And Q is:

$$Q = \begin{pmatrix} f_m - f_f \\ -\tau_{f1} \\ -\tau_{f2} \end{pmatrix} \tag{3}$$

Where, f_m is the horizontal driving force applied by the motor to the system, f_f is the viscous friction force against the motion of the cart, τ_{f1} is the viscous friction in the first pivot joint and τ_{f2} is the viscous friction in the second pivot joint. Jadlovska et al.[1] have shown that the force thrusted by a DC motor can be modeled as a linear function of input voltage. Moreover, the viscous friction of cart motion on the rack and pivot joints can be assumed as a linear function of the cart and joint velocities. Thus:

$$Q = \begin{pmatrix} K_m V - \delta_0 \dot{x} \\ -\delta_1 \dot{\theta}_1 \\ -\delta_2 \dot{\theta}_2 \end{pmatrix} \tag{4}$$

The Lagrangian functional is defines as the difference between the kinetic and the potential energy of the whole system. So:

$$L = \sum T_i - \sum V_i = T_0 + T_1 + T_2 - (V_0 + V_1 + V_2) \tag{5}$$

$$\sum V_i = V_0 + V_1 + V_2 \tag{6}$$

$$V_0 = 0 \tag{7}$$

$$V_1 = m_1 g y_1 = m_1 g l_{c1} \cos \theta_1 \tag{8}$$

$$V_2 = m_2 g y_2 = m_2 g (L_1 \cos \theta_1 + l_{c2} \cos \theta_2) \tag{9}$$

$$\sum T_i = T_0 + T_1 + T_2 \tag{10}$$

$$T_0 = \frac{1}{2} m_0 \dot{x}^2 \tag{11}$$

$$T_1 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) \tag{12}$$

$$= \frac{1}{2} m_1 \left[(\dot{x}_0^2 + l_{c1} \dot{\theta}_1^2 \cos \theta_1)^2 + (l_{c1} \dot{\theta}_1^2 \sin \theta_1)^2 \right] + \frac{1}{2} I_1 \dot{\theta}_1^2$$

$$= \frac{1}{2} m_1 \dot{x}_0^2 + \frac{1}{2} (m_1 l_{c1}^2 + I_1) \dot{\theta}_1^2 + m_1 l_{c1} \dot{x}_0 \dot{\theta}_1 \cos \theta_1$$

$$T_2 = \frac{1}{2} m_1 (\dot{x}_2^2 + \dot{y}_2^2) \tag{13}$$

$$= \frac{1}{2} m_2 \left[(\dot{x}_0 + L_1 \dot{\theta}_1 \cos \theta_1 + l_{c2} \dot{\theta}_2 \cos \theta_2)^2 + (l_{c2} \dot{\theta}_2 \sin \theta_2 + L_1 \dot{\theta}_1 \sin \theta_1)^2 \right] + \frac{1}{2} I_2 \dot{\theta}_2^2$$

$$= \frac{1}{2} m_2 \dot{x}_0^2 + \frac{1}{2} m_2 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} (m_2 l_{c2}^2 + I_2) \dot{\theta}_2^2 + m_2 L_1 \dot{x}_0 \dot{\theta}_1 \cos \theta_1 + m_2 l_{c2} \dot{x}_0 \dot{\theta}_2 \cos \theta_2 + m_2 L_1 l_{c2} \dot{\theta}_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

Thus, the equation of motion is derived as below:

$(m_0 + m_1 + m_2)\ddot{x} + (m_2 L_1 + m_1 l_{c1})\ddot{\theta}_1 \cos \theta_1 + m_2 l_{c2} \ddot{\theta}_2 \cos$	(14)
$(m_1 l_{c1}^2 + m_2 L_1^2 + I_1)\ddot{\theta}_1 + (m_2 L_1 + m_1 l_{c1})\ddot{x} \cos \theta_1$	
$+ m_2 L_1 l_{c2} \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 L_1 l_{c2} \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - (m_1 l_{c1} + $	(15)
$(m_2 l_{c2}^2 + I_2)\ddot{\theta}_2 + m_2 l_{c2} \ddot{x}_0 \cos \theta_2 + m_2 L_1 l_{c2} \ddot{\theta}_1 \cos(\theta_1 - \theta_2)$	(16)

Re-arranging the equations of motion in matrix notation, it can be identified as a second order mechanical system. Thus the system of differential equations can be expressed:

$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = Hu$	(17)
$u = K_m V_0 \,\& \, H = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	(18)

Where, u is the input vector and H is the input gain matrix.

III. PARAMETER IDENTIFICATION

For Parameter Identification the energetic description of the system was employed. The concept is to form a set of equations stating the energetic mechanical balance of the system throughout the motion caused by excitation, τ. Since the system is starting from the resting position the kinetic energy is balanced at zero while, if no physical barrier is present, the potential energy is at its local minimum (static balance). Expediting of the excitation, motion of the cart would result in motion of the cart and pendulum, while the friction work, between cartwheels and rails and pivot joints of the pendulum, exhausts a portion of the external work done on the system. Equations 19 represents the energy balance of the cart-pendulum system.

$$\int_{t_1}^{t_2} \tau \dot{x} dt - \int_{t_1}^{t_2} f \dot{x} dt = \sum_i T_i^{t_2} + \sum_i V_i^{t_2} - \sum_i T_i^{t_1} - \sum_i T_i^{t_1} \tag{19}$$

where, τ is the external force excitation, f represents the viscous frictional forces vector acting on system, T_i is the kinetic energy of the i_{th} component,

Vi is the potential energy of the i_{th} component while t_1 and t_2 indicate the initial and final time of the excitation[14].

Energetic description of the double pendulum system for parameter identification introduced in equations 20 and 21.

$$E = \sum T_i + \sum V_i$$
$$= T_0 + T_1 + T_2 + V_0 + V_1 + V_2 \qquad (20)$$

$$E = \frac{1}{2}(m_0 + m_1 + m_2)\dot{x}_2^2$$
$$+ \frac{1}{2}(m_1 l_{c1}{}^2 + m_2 L_1{}^2 + I_1)\dot{\theta}_1^2$$
$$+ \frac{1}{2}(m_2 l_{c2}{}^2 + I_2)\dot{\theta}_2^2 \qquad (21)$$
$$+(m_2 L_1 + m_1 l_{c1})\dot{x}_0 \dot{\theta}_1 \cos\theta_1$$
$$+ m_2 l_{c2}\dot{x}_0 \dot{\theta}_2 \cos\theta_2$$
$$+ m_2 L_1 l_{c2}\dot{\theta}_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$
$$+(m_1 l_{c1} + m_2 L_1)g\cos\theta_1 + m_2 l_{c2}g\cos\theta_2$$

Thus, the parameters of the system, $d_1 - d_{11}$ were selected:

$d_1 = m_0 + m_1 + m_2$	(22)
$d_2 = m_2 L_1 + m_1 l_{c1}$	(23)
$d_3 = m_2 l_{c2}$	(24)
$d_4 = m_1 l_{c1}{}^2 + m_2 L_1{}^2 + I_1$	(25)
$d_5 = m_2 L_1 l_{c2}$	(26)
$d_6 = m_2 l_{c2}{}^2 + I_2$	(27)
$d_7 = \delta_0$	(28)
$d_8 = \delta_1$	(29)
$d_9 = \delta_2$	(30)
$d_{10} = m_1 g l_{c1} + m_2 g L_1$	(31)
$d_{11} = m_2 l_{c2} g$	(32)

By considering E_{t_2} and E_{t_1} as the state of energy in time t_2 and t_1, this relationship can be expressed:
$$E_{t_2} - E_{t_1} = w_m - w_f \qquad (33)$$

Where w_m, is the work of external forces which is applied by motor and the w_f is the work of frictional forces from time t_1 to t_2.

$$w_m = \int_{t_1}^{t_2} f_m \dot{x}\, dt = \int_{t_1}^{t_2} K_m V_0 \dot{x}\, dt \qquad (34)$$

$$w_f = w_{f_0} + w_{f_1} + w_{f_2} = \int_{t_1}^{t_2}\delta_0 \dot{x}_0^2\, dt + \int_{t_1}^{t_2}\delta_1 \dot{\theta}_1^2\, dt + \int_{t_1}^{t_2}\delta_2 \dot{\theta}_2^2\, dt \qquad (35)$$

Thus, the energy relationship between two time steps was determined as below:

$$E_{t_2} - E_{t_1} + \int_{t_1}^{t_2} d_7 \dot{x}^2\, dt + \int_{t_1}^{t_2} d_8 \dot{\theta}_1^2\, dt$$
$$+ \int_{t_1}^{t_2} d_9(\dot{\theta}_2 - \dot{\theta}_1)^2\, dt \qquad (36)$$
$$= \int_{t_1}^{t_2} K_m V \dot{x}_0\, dt$$

Using this energy equation, the parameters

identified by the non-negative least square minimization method in MATLAB®.

If one considers the K_m as a parameter to be determined, the numerical optimization procedure, used for finding the optimum parameters will lead to an all-zero solution. That is the trivial solution. The reason is as there is no constant number on the right-hand side of the set of equations, the procedure would find the optimum least square minimization of [A]{d}=0, which is trivially zero. To avoid numerical complexity, we tested different values of K_m=[0.6 0.7 0.8 0.9] to find the parameters. The parameter set with the minimum error was introduced as real parameters. The system was excited from resting position, fully extended downward configuration, with 4Volts of input and the position and angle of the pendulum were stored.

IV. STATE SPACE AND LQR CONTROLLER

The state space equations of a system of the cart and inverted pendulum were adopted from work done by Bogdanov [5]. The Lagrangian equations of motion were re-written as:

$$D(q) = \begin{bmatrix} d_1 & d_2 \cos\theta_1 & d_3 \cos\theta_2 \\ d_2 \cos\theta_1 & d_4 & d_5 \cos(\theta_1 - \theta_2) \\ d_3 \cos\theta_2 & d_5 \cos(\theta_1 - \theta_2) & d_6 \end{bmatrix} \qquad (38)$$

$$C(q,\dot{q}) = \begin{bmatrix} d_7 & -d_2 \sin(\theta_1)\dot{\theta}_1 & -d_3 \sin(\theta_2)\dot{\theta}_2 \\ 0 & d_8 + d_9 & d_5 \sin(\theta_1 - \theta_2)\dot{\theta}_2 + d_9 \\ 0 & -d_9 & d_9 \end{bmatrix} \qquad (39)$$

$$G(q) = \begin{pmatrix} 0 \\ -d_{10}\sin\theta_1 \\ -d_{11}\sin\theta_2 \end{pmatrix} \qquad (40)$$

In order to re-arrange equations, into a canonical state-space representation of the systems a state vector of X was defined.

$$x = \begin{pmatrix} x \\ \theta_1 \\ \theta_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\theta} \\ \dot{\boldsymbol{\theta}} \end{pmatrix} \qquad (41)$$

$$\dot{x} = Ax + Bu \qquad (42)$$

To alleviate the nonlinearity of the system and considering that the system is about to work at near zero-point state, the matrices were linearized using the Taylor's series expansion of sine and cosine functions and higher order was considered to be zero. Equations of linearized D, C and G matrices at zero neighborhood were formed and using the linearized matrices, A and B matrices were derived.

$$A = \begin{pmatrix} 0 & I \\ -D(0)^{-1}\dfrac{\partial G(0)}{\partial\theta} & -D^{-1}(0)C(0) \end{pmatrix} \qquad (43)$$

$$B = \begin{pmatrix} 0 \\ D(0)^{-1}\begin{pmatrix} K_m \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} \qquad (44)$$

Where I is the 3x3 identity matrix.

Linear Quadratic Regulator (LQR) method was implemented to stabilize double pendulum in the upright position. As a necessary condition of the stability in the LQR controller design, the input to the system, u, should be in the form of

u=-Kx, where K is the vector of LQR coefficients. To find K, the built-in function of lqr(A, B, Q, R) in MATLAB® was used. As per syntax of this function, A and B are the state space equation coefficient matrices while Q and R, are arbitrary error minimizing and controller effort minimizing weights.

In order to perform the simulation of the controlled system, a SimMechanics® mechanical model of the system was constructed for stabilization. The physical values of the system parameters were calculated from the identified parameters of the system. For experimental setup, the Quanser® double pendulum system on a cart was operated and it is shown in fig. 2 [15].

Fig 2: Quanser® Double inverted pendulum system

V. RESULTS AND DISCUSSION

Fig3 depicts the velocity of cart and angles of first and second pendula on the cart along with the 30 points moving average filtered data just as a sample of data which was used for parameter identification.
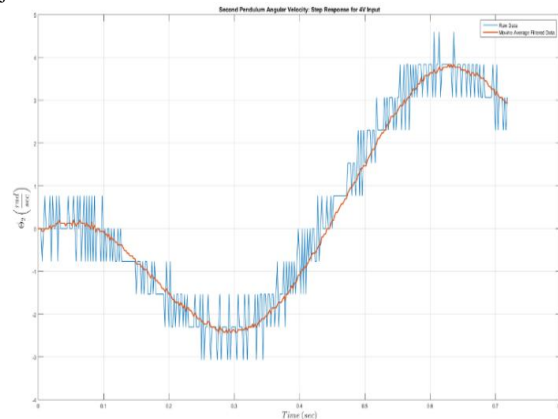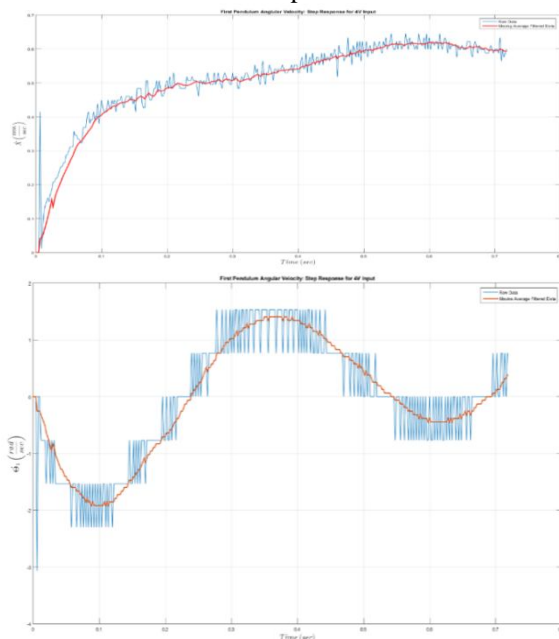
Fig3: Temporal variation in velocity of cart, angle of first and second pendulum subjected to step 4V excitation.

The motor constant of 0.7 could result in parameter-set with the minimum error which was implemented for controller design. Table I summarizes the results of parameters. It is obvious that since the effect of the rotational friction on the performance of the system was small; the least square algorithm was unable to identify its effects and returned $d_8 = d_9 = 0$

Table I: Parameters obtained for double pendulum system

K_m	d_1	d_2	d_3	d_4	d_5
0.7	0.83	0.026	0.013	0.0112	0.0009
d_6	d_7	d_8	d_9	d_{10}	d_{11}
0.0017	4.4793	0	0	0.4156	0.1886

The LQR controller constant was obtained using parameters identified. By testing different values of Q and R, the system response was improved regarding controller effort and error. The stabilization of the double pendulum with LQR controller constants of K [45, -305, 450, 15, -5, 40] for different initial conditions was captured from experiments. The experimental results showed a fair agreement with the simulation results and the comparison is plotted in fig. 5. Fig 6 shows the recovery of the system from multiple environmental (impact) disturbance.
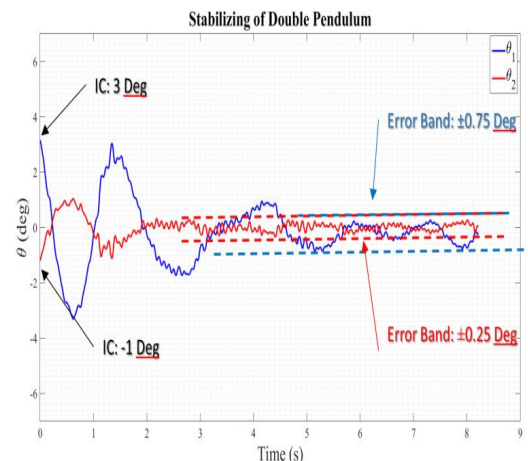
Fig4: (a) Stabilization of θ_1 and θ_2 and settling down to the error band of ±2 and ±0.75 degree
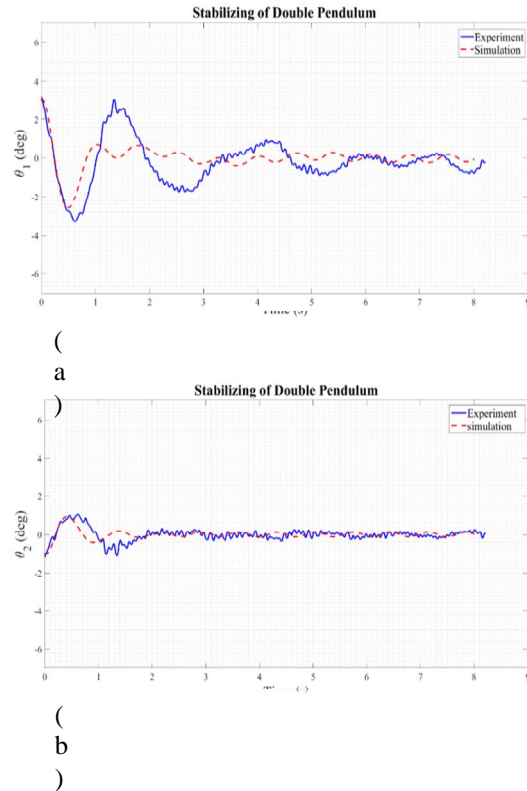
(a)

(b)

Fig5: (a) Comparison of changes in θ_1 from experiment (solid) and simulation (dashed) and (b) comparison of θ_2 from experiment (solid) and simulation (dashed).
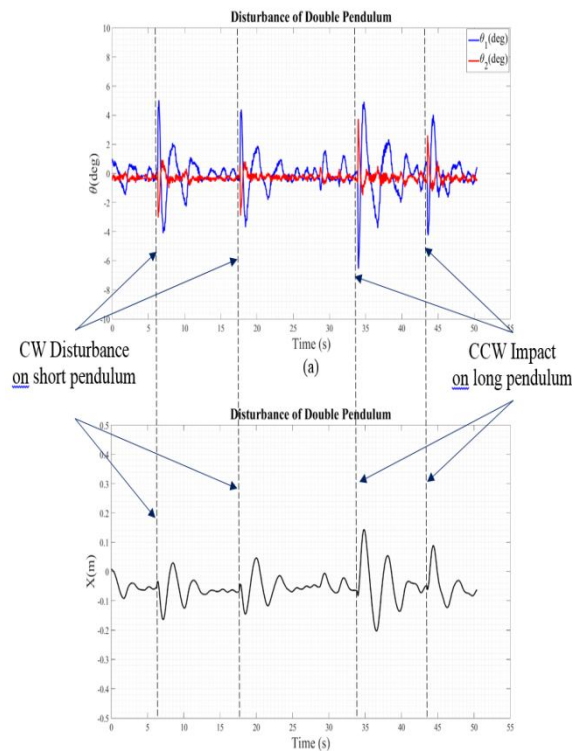

Fig6: (a) Disturbance response of short (blue line) and long pendulum (red line) to environmental disturbance (impact) (b) disturbance response of cart position due to environmental disturbance (impact).

CONCLUSION

In this study, the parameter identification a double inverted pendulum system followed, and its stabilization by optimal control was performed. Using the system response subjected to 4 Volts excitation, the parameter of the system were identified including the viscous friction of cart and rack. The LQR controller designed by using the identified parameters could properly stabilize pendulum without any extra fluctuating. Experimental results could successfully illustrate the ability of the controller to recover stability after applying impact disturbance on both first and second pendulum.

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