EE 5323 - HW06

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HW06 – Lyapunov Stability Analysis, LaSalle's Extension, and UUB
EE 5323 – Nonlinear Systems
Dr. Frank Lewis

Exercise 1

LaSalle's Extension
Consider the system from HW05,

$$\begin{cases} \dot{x}_1 = x_2 + x_1(x_1^2 - 2) \\ \dot{x}_2 = -x_1 \end{cases}$$

We used a quadratic Lyapunov Function to show this system is locally SISL. And we found the region within which $\dot{V} \leq 0$. Use LaSalle's extension to verify that the system is AS. Find the equilibrium point.
Answer
1.a) Lyapunov function candidate: $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) > 0$

$$\dot{V} = \frac{\partial V^\top}{\partial x}\dot{x} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \Longrightarrow$$

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$$

Now we plug in system dynamics to check stability,

$$\dot{V} = x_1(x_2 + x_1(x_1^2 - 2)) + x_2(-x_1) \Longrightarrow$$

$$\dot{V} = \cancel{x_1x_2} - x_1^2(x_1^2 - 2) - \cancel{x_1x_2}$$
$$\dot{V} = -x_1^2(x_1^2 - 2) \leq 0$$

Thus, the system is *asymptotically stable* (AS) and it is bound by a region with radius of $\sqrt{2}$. Moreover, we know $\dot{x} \longrightarrow 0$; thus, per LaSalle's extension, $\ddot{x} \longrightarrow 0$ must hold true and that makes $x_1, x_2 \longrightarrow 0$ at $t = \infty$. We proceed with plugging in the resulting x_1 in the system dynamics equation.

$$\dot{V} = -x_1^2(x_1^2 - 2) \leq 0 \Longrightarrow \dot{V} \longrightarrow 0,\ x_1 \mid x_1^2 = 2; \Longrightarrow x_1 = \{0, \pm\sqrt{2}\},\ x_2 = 0$$

Where e.p.s would be $(-\sqrt{2},\ 0),\ (0,\ 0),\ (+\sqrt{2},\ 0)$.

1.b) Simulation:

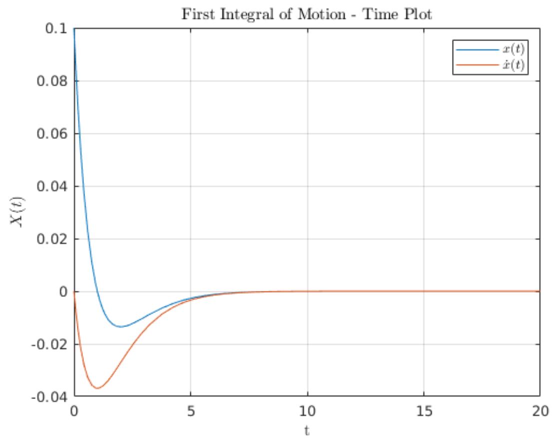

Figure 1:

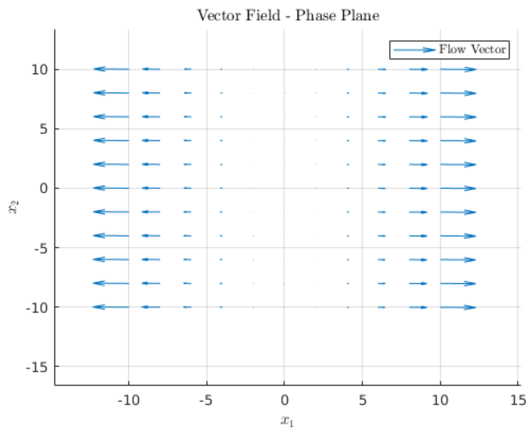

Figure 2:

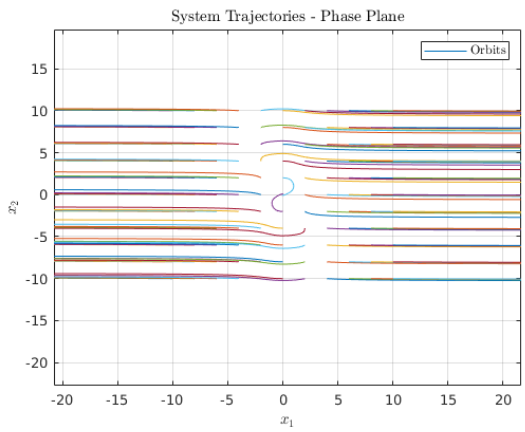

Figure 3:

Matlab Code

```matlab
%% HW05 - Q04 - AS
% @author: Bardia Mojra
% @date: 10/28/2021
% @title HW05 - Q04 - SISL Simulation
% @class ee5323 - Nonlinear Systems
% @professor - Dr. Frank Lewis

clc
clear
close all
warning('off','all')
warning

% part a
t_intv= [0 20];
x_0= [0.1, 0]'; % initial conditions for x(t)
figure
[t,x]= ode23('q04_sys', t_intv, x_0);
plot(t,x)
hold on;
grid on;
title('First Integral of Motion - Time Plot','Interpreter','latex');
ylabel('$X(t)$','Interpreter','latex');
xlabel('t','Interpreter','latex');
legend('$x(t)$', '$\dot{x}(t)$','Interpreter','latex');

% part b
figure();
hold on;
grid on;
mesh = -10:2:10;
[x1,x2] = meshgrid(mesh,mesh);
dx1=[];
dx2=[];
N=length(x1);
for i=1:N
  for j=1:N
    dx = q04_sys(0, [x1(i,j);x2(i,j)]);
    dx1(i,j) = dx(1);
    dx2(i,j) = dx(2);
  end
end
quiver(x1,x2,dx1,dx2);
ylabel('$x_2$','Interpreter','latex');
xlabel('$x_1$','Interpreter','latex');
legend('Flow Vector','Interpreter','latex');
```

```
title('Vector Field - Phase Plane','Interpreter','latex');
axis([-15 15 -15 15])

% part c
figure
for i=mesh
  for j=mesh
    init=[i, j];
    [t, x] = ode23(@q04_sys, [0 10], init);
    plot(x(:,1),x(:,2))
    hold on;
  end
end
ylabel('$x_2$','Interpreter','latex');
xlabel('$x_1$','Interpreter','latex');
legend('Orbits','Interpreter','latex');
title('System Trajectories - Phase Plane','Interpreter','latex
    ');
grid on;
axis([-50 50 -50 50])

%%
%
% function xdot = q04_sys(t,x)
%   xdot = [x(2) + x(1)*(x(1)^2-2); -x(1)];
% end
```

```
%% Part 1 Answer
%% Document Information:
% * Author: Bardia Mojra
% * Date: 10/28/2021
% * Title: HW 05 - Part 4 System
% * Term: Fall 2021
% * Class: EE 5323 - Nonlinear Systems
% * Dr. Lewis

function xdot = q04_sys(t,x)
    xdot = [x(2) + x(1)*(x(1)^2-2); -x(1)];
end
```

Exercise 2

Limit Cycles
Consider the following system,

$$\begin{cases} \dot{x} = 4x^2y - f_1(x)\left(x^2 + 2y^2 - 4\right) \\ \dot{y} = 2x^3 - f_2(y)\left(x^2 + 2y^2 - 4\right) \end{cases}$$

where the continuous functions $f_1(x), f_2(y)$ have the same sign as their argument. Show that the system tends towards a limit cycle independent of the explicit expressions of $f_1(x), f_2(y)$.
Answer
2.a) Weird Lyapunov function candidate: $V(x, y) = \frac{1}{2}(x^2 + 2y^2 - 4)^2 > 0$

$$\dot{V} = \frac{1}{2}2(x^2 + 2y^2 - 4)(2x\dot{x} + 4y\dot{y})$$

Now we plug in system dynamics to check stability,

$$\dot{V} = (x^2 + 2y^2 - 4)\left[2x\left(4x^2y - f_1(x)(x^2 + 2y^2 - 4)\right) + 4y\left(2x^3 - f_2(y)(x^2 + 2y^2 - 4)\right)\right] \Rightarrow$$

$$\dot{V} = (x^2 + 2y^2 - 4)^2\left[2x\left(4x^2y - f_1(x)\right) + 4y\left(2x^3 - f_2(y)\right)\right] \Rightarrow$$

$$\dot{V} = (x^2 + 2y^2 - 4)^2\left[16x^3y - 2xf_1(x) - 4yf_2(y)\right] \Rightarrow$$

Thus, $\dot{V} < 0$ unless $(x^2 + 2y^2 - 4) = 0$ which makes $\lim_{t\to\infty} \liminf \dot{V} \longrightarrow 0$. The invariant set forms a *Stable Limit Cycle* about $(x^2 + 2y^2 - 4) = 0$.

2.b) Simulation:

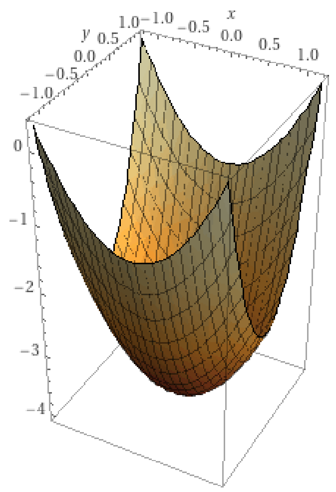

Figure 4: 3D Plot

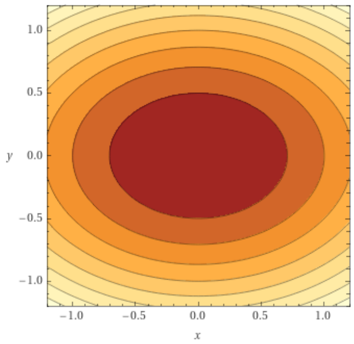

Figure 5: Contour Plot

Exercise 3

UUB of System with Disturbance
Consider the system on S&L p. 66 with a disturbance d,

$$\dot{x} + C(x) + d = 0$$

Assume that $xC(x) > ax^2$ with $a > 0$ a known positive constant.
a. Assume that d is unknown but is bounded by $\|d\| < D$ with D a known positive constant. Prove that the system is UUB and find the bound on $x(t)$.
b. Assume that d is unknown but is bounded by $\|d\| < D\|x\|$ with D a known positive constant. Prove that the system is UUB and find the bound on $x(t)$.
Answer
1.a) We proceed with selecting a Lyapunov function candidate and normalizing its first derivative.

$$\dot{x} + C(x) + d = 0 \implies \dot{x} = -C(x) - d$$
$$V = \frac{1}{2}x^2 \implies \dot{V} = x\dot{x} \implies \dot{V} = -xC(x) - xd$$

per Cauchy-Schwarz $\mathbb{R}^2$:

$$\langle x, y\rangle^{1/2} = \|x^T y\| \le \|x\| \cdot \|y\| \;;\; \textit{thus, } \|Ax\| \le \|A\| \cdot \|x\|$$
$$\dot{V} = -\|x\|C(x) - \|xd\| \le -\|x\|C(x) - \|x\|\|d\|$$
$$\dot{V} \le -\|x\|C(x) - \|x\|\|d\| = -\|x\|(C(x) + \|d\|);$$

Since xC(x) » 0, x and C(x) must have the same sign, thus $\dot{V} \le 0$ when,

$$C(x) + \|d\| > 0 \implies C(x) > -\|d\|; \|d\| < D \implies -D < C(x) < D$$

Thus, C(x) term is bounded by $\pm D$.

1.b) We continue from the last step.

$$\dot{V} \le -\|x\|(C(x) + \|d\|),$$

So $\dot{V} \le 0$ when, $C(x) + \|d\| > 0$ where $\|d\| < D\|x\|$

$$C(x) > -\|d\| \implies -D\|x\| < C(x)\|x\| < +D\|x\| \implies$$
$$-D < \frac{C(x)}{\|x\|} < +D$$

Exercise 4

Lyapunov Equation
Use Lyapunov Equation to check the stability of the linear systems.

a. $\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x$

b. $\dot{x} = Ax = \begin{bmatrix} -7 & 4 \\ -7 & 3 \end{bmatrix} x$

c. $\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} x$

Answer
a.) $A^T P + PA = -Q$

$$\begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = -Q$$

$$\begin{bmatrix} a_1p_1 + a_3p_3 & a_1p_2 + a_3p_3 \\ a_2p_1 + a_4p_2 & a_2p_2 + a_4p_3 \end{bmatrix} \begin{bmatrix} a_1p_1 + a_3p_2 & a_2p_1 + a_4p_2 \\ a_1p_2 + a_3p_3 & a_2p_2 + a_4p_3 \end{bmatrix} = -Q$$

$$\begin{bmatrix} a_1p_1 + a_3p_3 + a_1p_1 + a_3p_2 & a_1p_2 + a_3p_3 + a_2p_1 + a_4p_2 \\ a_2p_1 + a_4p_2 + a_1p_2 + a_3p_3 & a_2p_2 + a_4p_3 + a_2p_2 + a_4p_3 \end{bmatrix} = -Q$$

$$\begin{bmatrix} -6p_3 + -6p_2 & -6p_3 + p_1 + -5p_2 \\ p_1 + -5p_2 + -6p_3 & p_2 + -5p_3 + p_2 + -5p_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} .0167 & 0 \\ 0 & .1 \end{bmatrix}$$

where $m_{11} = .0167$, $m_{22} = \ .00167$ so it is positive definite.

b.) $A^T P + PA = -Q$

$$\begin{bmatrix} -7 & -7 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -7 & 4 \\ -7 & 3 \end{bmatrix} = -Q$$

$$\begin{bmatrix} -7p_1 + -7p_3 + -7p_1 + -7p_2 & -7p_2 + -7p_3 + 4p_1 + 3p_2 \\ 4p_1 + 3p_2 + -7p_2 + -7p_3 & 4p_2 + 3p_3 + 4p_2 + 3p_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} .2857 & .5 \\ .5 & 1 \end{bmatrix}$$

where $m_{11} = .2857$, $m_{22} = \ .0357$ so it is positive definite.

c.) $A^T P + PA = -Q$

$$\begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} = -Q$$

$$\begin{bmatrix} -4p_3 + -4p_2 & -4p_3 + 1p_1 \\ 1p_1 + -4p_3 & 1p_2 + 1p_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow P = \text{undetermined}$$

The system is unstable and does not have a unique solution.