Game theory

Game theory deals with decisions, by involving two or more components.

Each one of the players aims to optimize its own decision in contrast to others or with others (coalitions).

- Initial development was by John Von Neumann (1928)

→ Relationship between game theory and linear programming

- Simplex Algorithm developed by George B. Dantzig, solved problems in game theory

- Nash → honoured Nobel Prize (Beautiful Mind) for discovering the Nash Equilibrium Point

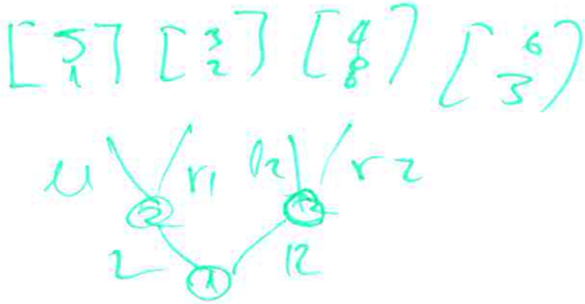
Kinds of games

- Number of players : $n=2$, $n>2$ . . .
  (competition, cooperation)
- Dynamic, Static Games.
- Number of strategies classify the games into finite and infinite.
- Classify them based on the characteristics of win, loss, draw.

Zero-Sum Games

Loss of the first player is the reward of the other. Their summation is zero.
Conflict between the interests of these two players.
Assumption: Every player selects a strategy which gives the best results by knowing that the other player knows the strategy that he follows.

e.g. Strategic form



players that communicate.

1. $\{L, R\}$
2. $\{l_1, r_1\}, \{l_2, r_2\}$

Combining the different strategies from different vertices we get

1. $\{L, R\}$
2. $\{l_1 l_2, l_1 r_2, l_2 r_1, r_1 r_2\}$

$$
\begin{array}{c|cccc}
 & l_1 l_2 & l_1 r_2 & l_2 r_1 & r_1 r_2 \\
\hline
L & 5 & 5 & 3 & 3 \\
R & 4 & 6 & 4 & 6
\end{array}
\; ; \;
\begin{pmatrix} 1 & 1 & 2 & 2 \\ 8 & 3 & 8 & 3 \end{pmatrix}
$$

Problem: How the players will choose their strategy in order to maximize his payoff?

Nash Equilibrium

The game is: $(N; X^0, Y^0; f_1, f_2)$

$N = \{1,2\}$ playerset

$\{X^0, Y^0\}$ sets of strategies for

$1 \longrightarrow X^0 = \{L, R\}$

$2 \longrightarrow Y^0 = \{l_1 l_2, l_1 r_2, r_1 l_2, r_1 r_2\}$

$f_1 : X^0 \times Y^0 \rightarrow \mathbb{R}$ $\quad f_1(i,j) = a_{ij}, \; i \in X^0$

$f_2 : X^0 \times Y^0 \rightarrow \mathbb{R}$ $\quad f_2(i,j) = b_{ij}, \; j \in Y^0$

where A and B are two matrices

How do we choose in X^0 and Y^0?

$(i^*, j^*) \in X^0 \times Y^0$ is an equilibrium point if

$$\begin{cases} f_1(i, j^*) \leq f_1(i^*, j^*) & \forall i \in X^0 \\ f_2(i^*, j) \leq f_1(i^*, j^*) & \forall j \in Y^0 \end{cases}$$

Any deviation (unilaterally) is not profitable

To find an equilibrium point we see:

$a_{ij^*} \leq a_{i^*j^*} \quad \forall i \in X^0$

$b_{i^*j} \leq b_{i^*j^*} \quad \forall j \in Y^0$

$a_{i^*j^*}$ is the largest entry in the column A

$b_{i^*j^*}$ is the largest entry in row B

In the previous example

in column 1, a_{11} is the largest but b_{11} is not the largest in row 1

in column 2, a_{22} is the largest but b_{22} is not the largest in row 2.

in column 3, a_{23} is the largest and b_{23} is the largest in row 2.

(2,3) is an E.P.

Unique E.P. $i^* = 2,\ j^* = 3 \longrightarrow a_{23} = 4$, $b_{23} = 8$

Sometimes there is no E.P. Sometimes there are more than one.

(i) Existence of E.P?

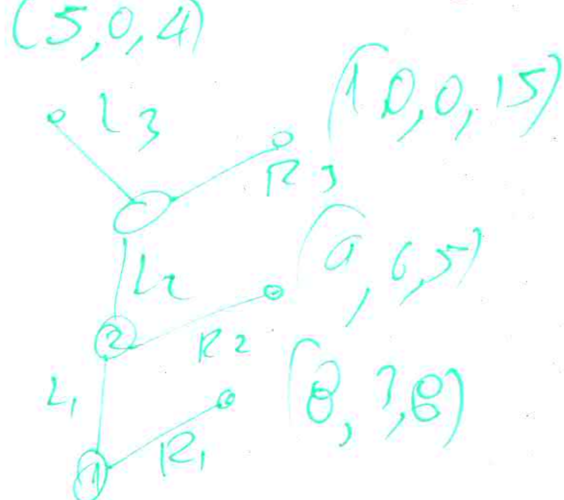
(ii) Refinement of E.P concept to decide what to choose among the E.P.'s.

This example was a bimatrix game (2 players)

Another Example

1. $\{L_1, R_1\} = X$
2. $\{L_2, R_2\} = Y$
3. $\{L_3, R_3\} = Z$

$N = \{1, 2, 3\}$



payoff functions?

strategies payoffs

L_1	L_2	L_3	5	0	4
L_1	L_2	R_3	10	0	15
R_1	L_2	L_3	8	3	8
R_1	L_2	R_3	8	3	8
L_1	R_2	L_3	9	6	5
L_1	R_2	R_3	9	6	5
R_1	R_2	L_3	8	3	8
R_1	R_2	R_3	8	3	8

$(N; X^0, Y^0, Z^0; f_1, f_2, f_3)$

$(i^*, j^*, k^*) \in X^0 \times Y^0 \times Z^0$ is an equilibrium point if:

$f_1(i, j^*, k^*) \leq f_1(i^*, j^*, k^*) \quad \forall i \in X_0$

$f_2(i^*, j, k^*) \leq f_2(i^*, j^*, k^*) \quad \forall j \in Y_0$

$f_3(i^*, j^*, k) \leq f_3(i^*, j^*, k^*) \quad \forall k \in Z_0$

check:

(a) $(L_1 L_2 L_3)$

$f_1(L_1 L_2 L_3) = 5 \qquad f_1(R_1, L_2, L_3) = 8$ No

(b) (L_1, L_2, R_3)

$f_1(L_1, L_2, R_3) = 10 \qquad f_1(R_1, L_2, R_3) = 8$ No

$f_2(L_1, L_2, R_3) = 0 \qquad f_2(L_1, R_2, R_3) = 6$

(c) R_1, L_2, L_3

$f_1(R_1, L_2, L_3) = 8$ $f_1(L_1, L_2, L_3) = 0$

$f_2(R_1, L_2, L_3) = 3$ $f_2(R_1, R_2, L_3) = 3$

$f_3(R_1, L_2, L_3) = 8$ $f_3(R_1, L_2, R_3) = 8$

Yes E.P. (R_1, L_2, L_3)

~~Note~~

~~Going to coalitional form~~

Reduce the strategic form to a coalitional game $N = \{1, 2\}$. Any non empty subset of N is a coalition. The game should be given as a pair (N, v) N = set of players v is the characteristic function which gives the numbers $v(S)$ of all coalitions! If they decide together the strategy to be followed

$v(\emptyset) = 0$ is an assumption.

Problem: how do you divide fairly the sum of the grand coalition?

A solution is $x_i = v(\{i\}) + \frac{1}{n}[v(N) - \sum_{j \in N} v(j)]$ $\forall i \in N$.

Prisoner's dilemma (non-zero sum game)

The most famous non-zero game.
Two prisoners are held in separate cells. The district attorney knows that they jointly committed an armed robbery but only if at least one of them confesses will he have the evidence to guarantee a conviction. If neither of them confesses, they will be sentenced to 2 years in prison for illegal possession of firearms. The sentence for armed robbery is 20 years. However if they both plead guilty it will be reduced to 10 years. If one confesses and the other does not, the one who confesses will be set free and the other sentenced for 20 years. The DA visits each of the prisoners to invite them to confess. Should he?
The prisoners dilemma may be expressed by the following matrix where in each cell the number before the comma is the outcome for Row and the number after the comma is the outcome for column

The numbers represent years in prison, and are preceded by minus signs because more years in prison are worse than fewer.

	Column: Don't confess	Confess
Row: Don't confess	-2, -2	-20, 0
Confess	0, -20	-10, -10

Row does not know what Column will do. But he knows that if Column does not confess he will receive -2 if he does not confess and 0 if he confesses. If Column confesses he will receive -20 if he does not confess and -10 if he confesses irrespective of what Column does, it is therefore a "sure thing" that Row is better off if he confesses. The reasoning is symmetrical for Column. Therefore rational prisoners will confess, even though both of them know all along that it would be better for each if neither confessed.

Mixed Strategies

Mixed strategy for a player is the probability distribution in the total of his pure strategies

Definition: If a player has m pure strategies then a mixed strategy is the m-vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

that satisfies the following conditions

$$\sum_{i=1}^{m} x_i = 1$$

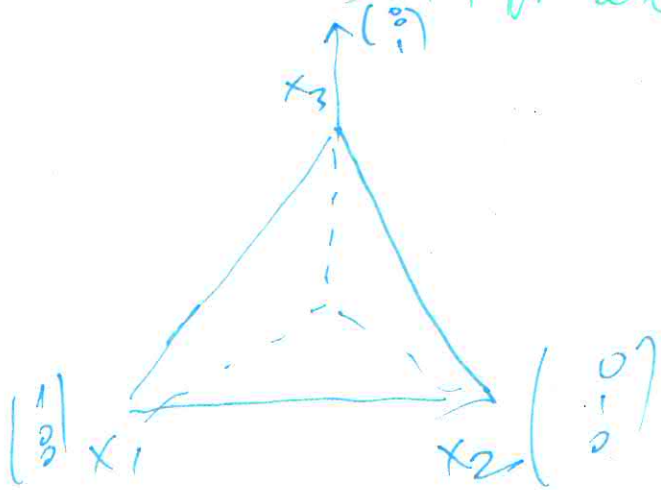
$$x_i \geq 0, \; i = 1 \ldots m.$$

Definition: The set of mixed strategies

$X = \{ x = (x_1, x_2 \ldots x_m)^T \in \mathbb{R}^m \,/\, \sum_{i=1}^{m} x_i = 1. \; x_i \geq 0, \; i = 1 \ldots m \}$

and $Y = \{ y\,(y_1, y_2 \ldots y_n)^T \in \mathbb{R}^n \,/\, \sum_{j=1}^{n} y_j = 1, \; y_j \geq 0, \; j = 1 \ldots n \}$

are called simplex probabilities in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively



So we can write:

$$x = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

So finally the mixed strategies are combination of pure ones

MinMax Theorem

Suppose that we have a zero sum player game with reward table A of dimension $m \times n$. The probabilities of the 2 players are independent such that the product $x_i y_j$ is the common probability that player I selects strategy i and player II selects strategy j. Sum of the probabilities is 1.

Definition: If player I selects mixed strategy x and player II mixed strategy y then the result is:

$$v(x, y) = x^T A y = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j$$

So suppose that player I selects the mixed strategy x and player II the pure strategy j then the expected reward of player I is:

$$v(x, e_j) = x^T A[:, j] = \sum_{i=1}^{m} x_i a_{ij}$$

the expected loss of player A is when he selects mixed strategy y and I pure strategy i:

$$v(e_i, y) = A[i,:]^T y = \sum_{j=1}^{n} a_{ij} y_j$$

e.g Game 2by2

Suppose game with payoff matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. If the game does not have E.P. in pure strategies then the optimal solutions of the 2 players must have all their parts positive. Suppose v is the value of the game. Then:

$$a_{11}x_1y_1 + a_{12}x_1y_2 + a_{21}x_2y_1 + a_{22}x_2y_2 = v$$

$$x_1(a_{11}y_1 + a_{12}y_2) + x_2(a_{21}y_1 + a_{22}y_2) = v$$

$a_{11}y_1 + a_{12}y_2 = v$ and $a_{11}x_1 + a_{21}x_2 = v$

$a_{21}y_1 + a_{22}y_2 = v$ $\quad\quad$ $a_{12}x_1 + a_{22}x_2 = v$.

linear complement deviation.

together with $x_1 + x_2 = 1$

$y_1 + y_2 = 1$

find the optimal by solving these equations.

Definition: Suppose we have 2 payoff matrices A, B with dimension m×n. Then we say that the couple of mixed strategies (x^*, y^*) has a Nash Eq. if for every other pair of strategies (x, y) the following inequalities hold:

$$x^T A y^* \geq x^{*T} A y^*$$

$$x^{*T} B y \geq x^{*T} B y^*$$

The Nash E.P is $a_0^* = a(x^*, y^*) = x^{*T} A y^*$, $b_0^* = b(x^*, y^*) = x^{*T} B y$

How to solve the inequalities?

The first inequality represents the minimization of $f(x) = x^T A y^*$ with respect to x and the second the minimization of the linear function $g(y) = x^T B y$ with respect to y.

e.g. Payoff.

I	Player II 1	2	3	4
1	4	0	6	-2
2	2	6	1	7

L.P model for player I is

Maximize v

subject to
$$4x_1 + 2x_2 \geq v$$
$$6x_2 \geq v$$
$$6x_1 + x_2 \geq v$$
$$-2x_1 + 7x_2 \geq v$$
$$x_1 + x_2 = 1$$
$$x_1, x_2 \geq 0$$

dual model for II

Maximize v

subject to
$$4y_1 + 6y_3 - 2y_4 \leq v$$
$$2y_1 + 6y_2 + y_3 + 7y_4 \leq v$$
$$y_1 + y_2 + y_3 + y_4 = 1$$
$$y_1, y_2, y_3, y_4 \geq 0$$

Solving we get $(x_1, x_2) = \left(\frac{5}{11}, \frac{6}{11}\right)$ with $v = 32/11$ for the primal and $(y_1, y_2, y_3, y_4) = \left(\frac{9}{11}, 0, 0, \frac{2}{11}\right)$ for the dual