EE 5323- Exam 1

Fall 2021

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All questions require numerical calculations to arrive at the answers. To obtain full credit, show all your work. No partial credit will be given without the supporting work. This probably means you must do calculations by hand and type them up, not using MATLAB routines.

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Name: Binoy M. George

Pledge of honor:

"On my honor I have neither given nor received aid on this examination."

Signature: BGeo

Name: Binoy M. George
UTA ID: 1001649736

EE 5323 - Exam 1

1. System:

$$\dot{x}_1 = x_2 = f_1$$
$$\dot{x}_2 = -x_1 + \frac{x_1^3}{9} - x_2 = f_2$$

a) The Jacobian is given by:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + x_1^2/3 & -1 \end{bmatrix}$$

b)
$$f_1 = 0 \Rightarrow x_2 = 0$$
$$f_2 = 0 \Rightarrow -x_1 + \frac{x_1^3}{9} - 0 = 0$$
$$\Rightarrow -9x_1 + x_1^3 = 0$$
$$\Rightarrow x_1(x_1^2 - 9) = 0$$
$$\Rightarrow x_1 = 0, \quad x_1 = \pm 3$$

$\therefore$ Equilibrium points are $(0,0), (3,0), (-3,0)$

c) 1) At e.p. $(0,0)$

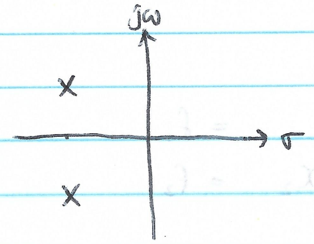
$$A = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$$

$$\therefore |sI - A| = \begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix} = s(s+1) + 1 = s^2 + s + 1$$

Solving for:

$s^2+s+1=0$

$\Rightarrow s = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{3}\,i}{2}$



stable focus

2) At e.p $(-3,0)$

$A = \begin{bmatrix} 0 & 1 \\ -1+3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$

$|sI-A| = \begin{vmatrix} s & -1 \\ -2 & s+1 \end{vmatrix} = s(s+1)-2 = s^2+s-2$

Solving for:

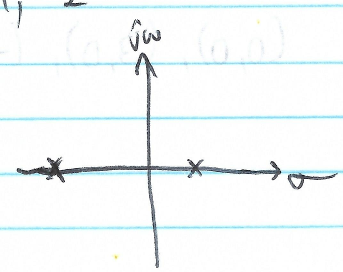
$s^2+s-2=0$

$\cancel{\Rightarrow s = -b \pm \sqrt{b^2-4ac}} \Rightarrow s^2+2s-s-2=0$

$\Rightarrow s(s+2)-(s+2)=0$

$\Rightarrow (s-1)(s+2)=0$

$\Rightarrow s=1,-2$



saddle point

3) At e.p $(3,0)$

$$A = \begin{bmatrix} 0 & 1 \\ -1+3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

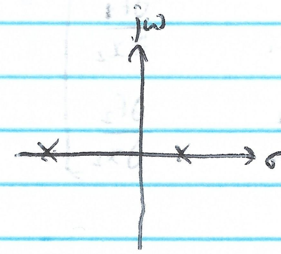
$$|sI - A| = \begin{vmatrix} s & -1 \\ -2 & s+1 \end{vmatrix} = s(s+1) - 2 = s^2 + s - 2$$

Solving for

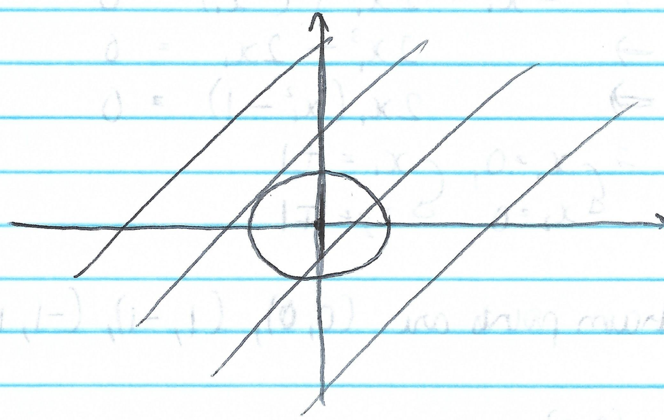
$s^2 + s - 2 = 0$

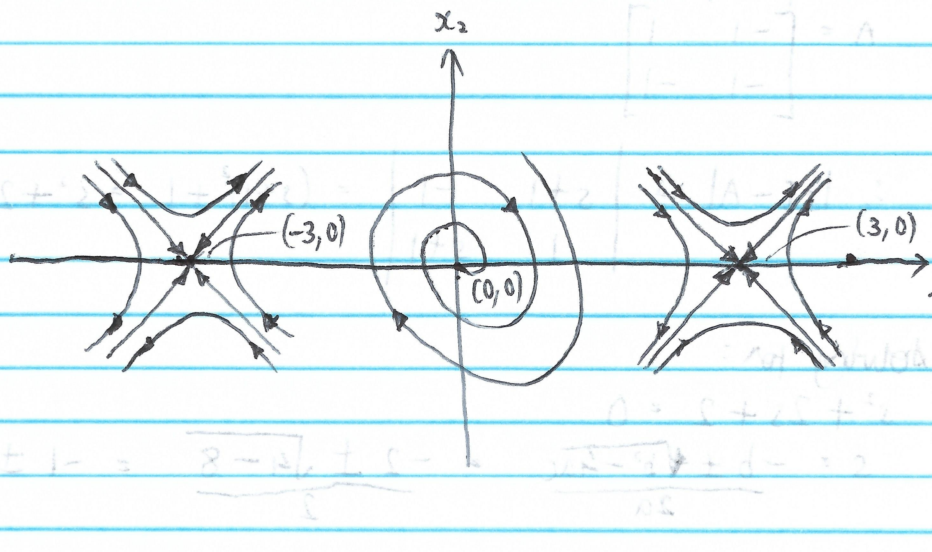
$\Rightarrow (s-1)(s+2) = 0$

$\Rightarrow s = 1, -2$



Saddle Point





2. System:

$$\dot{x}_1 = -x_1 + 2x_1^3 + x_2 = f_1$$
$$\dot{x}_2 = -x_1 - x_2 = f_2$$

a) The Jacobian is given by:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 + 6x_1^2 & 1 \\ -1 & -1 \end{bmatrix}$$

b) $f_2 = 0 \Rightarrow -x_1 - x_2 = 0 \Rightarrow x_1 = -x_2$

$f_1 = 0 \Rightarrow -x_1 + 2x_1^3 + (-x_1) = 0$

$\Rightarrow 2x_1^3 - 2x_1 = 0$

$\Rightarrow 2x_1(x_1^2 - 1) = 0$

$\therefore \begin{cases} x_1 = 0, \\ x_2 = 0 \end{cases} \begin{cases} x_1 = \pm 1 \\ x_2 = \mp 1 \end{cases}$

$\therefore$ Equilibrium points are $(0,0)$, $(1,-1)$, $(-1,1)$

c) 1) At e.p $(0,0)$

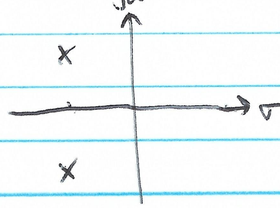
$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\therefore |sI - A| = \begin{vmatrix} s+1 & -1 \\ 1 & s+1 \end{vmatrix} = (s+1)^2 + 1 = s^2 + 2s + 2$$

Solving for:

$$s^2 + 2s + 2 = 0$$

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \frac{2i}{2} = -1 \pm i$$



stable focus

2) At e.p (1, −1)

$$A = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix}$$

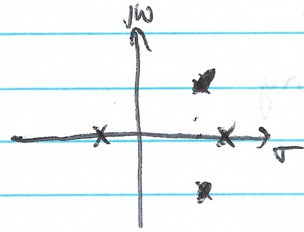
$$|sI - A| = \begin{vmatrix} s-5 & -1 \\ 1 & s+1 \end{vmatrix} = (s-5)(s+1)+1 = \cancel{s^2+6s+6}$$
$$= s^2 - 4s - 4$$

~~solving for:~~

$$\cancel{s^2 + 6s + 6 = 0}$$
$$\cancel{s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36 - 24}}{2}}$$
$$\cancel{= -3 \pm \frac{\sqrt{12}}{2} = -3 \pm \sqrt{3}}$$

solving for:

$$s^2 - 4s - 4 = 0$$
$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 + 16}}{2}$$
$$= 2 \pm \frac{4\sqrt{2}}{2} = 2 \pm 2\sqrt{2} \;\; \begin{cases} = -0.828 \\ = 4.828 \end{cases}$$



saddle point

3) At e.p $(-1, 1)$

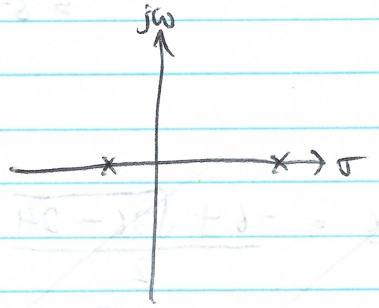
$$A = \begin{bmatrix} -5 & +1 \\ -1 & -1 \end{bmatrix}$$

$$|sI - A| = \begin{vmatrix} s+5 & -1 \\ 1 & s+1 \end{vmatrix} = (s+5)(s+1)+1 = s^2 - 4s - 4$$

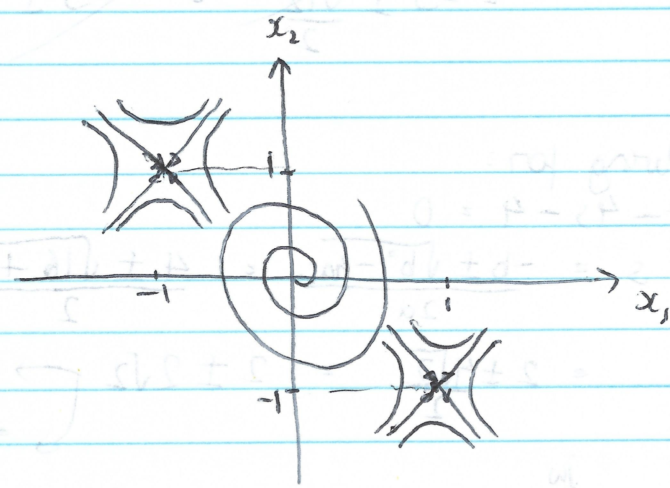
Solving for:

$s^2 - 4s - 4 = 0$

$s = -0.828, 4.828$



Saddle point



Phase Plane Trajectory

3. ~~Duffing's eqns~~

let $x_1 = x$

$x_2 = \dot{x}$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\alpha x_1 - x_1^3 \end{bmatrix} \begin{matrix} = f_1 \\ = f_2 \end{matrix}$$

a) Jacobian

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha - 3x_1^2 & 0 \end{bmatrix}$$

b) $f_1 = 0 \Rightarrow x_2 = 0$

$f_2 = 0 \Rightarrow -\alpha x_1 - x_1^3 = 0$

$x_1(x_1^2 + \alpha) = 0$

$x_1 = 0, \quad x_1 = \pm\sqrt{-\alpha}$

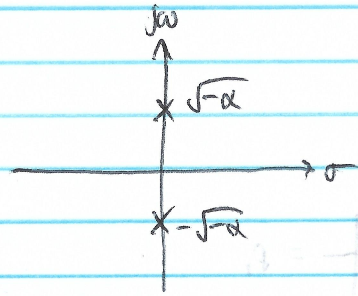
When $\alpha > 0$, the equilibrium point is $(0,0)$

Note: Points $(\sqrt{-\alpha}, 0)$ and $(-\sqrt{-\alpha}, 0)$ cannot exist since they will be imaginary

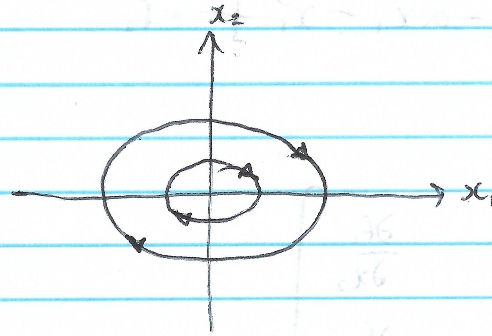
$$A = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix}$$

$$sI - A = \begin{vmatrix} s & -1 \\ \alpha & s \end{vmatrix} = s^2 + \alpha = 0$$

$\therefore s = \pm\sqrt{-\alpha}$



Center Point



Phase Plane Trajectory

c) $f_1 = 0 \Rightarrow x_2 = 0$

$f_2 = 0 \Rightarrow x_1 = 0, \pm\sqrt{-\alpha}$

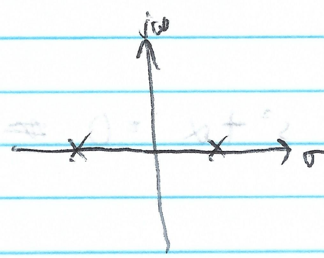
When $\alpha < 0$, the equilibrium points are:
$(0,0), (\sqrt{-\alpha}, 0), (-\sqrt{-\alpha}, 0)$

At $(0,0)$

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix}$$

$$sI - A = \begin{vmatrix} s & -1 \\ \alpha & s \end{vmatrix} = s^2 + \alpha = 0$$

$\Rightarrow s = \pm\sqrt{-\alpha} \Rightarrow$ along x-axis since $-\alpha = +ve$



Saddle Point

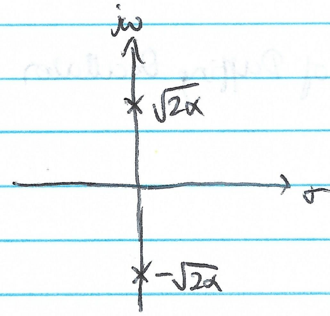
At e.p $(\sqrt{-\alpha}, 0)$

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha - 3(-\alpha) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2\alpha & 0 \end{bmatrix}$$

$$|sI - A| = \begin{vmatrix} s & -1 \\ -2\alpha & s \end{vmatrix} = s^2 - 2\alpha = 0$$

$\therefore s = \pm\sqrt{2\alpha}$

imaginary roots



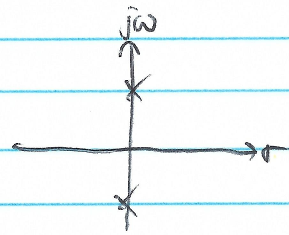
Center point

At e.p $(-\sqrt{-\alpha}, 0)$

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha - 3(-\sqrt{-\alpha})^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha + 3\alpha & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2\alpha & 0 \end{bmatrix}$$

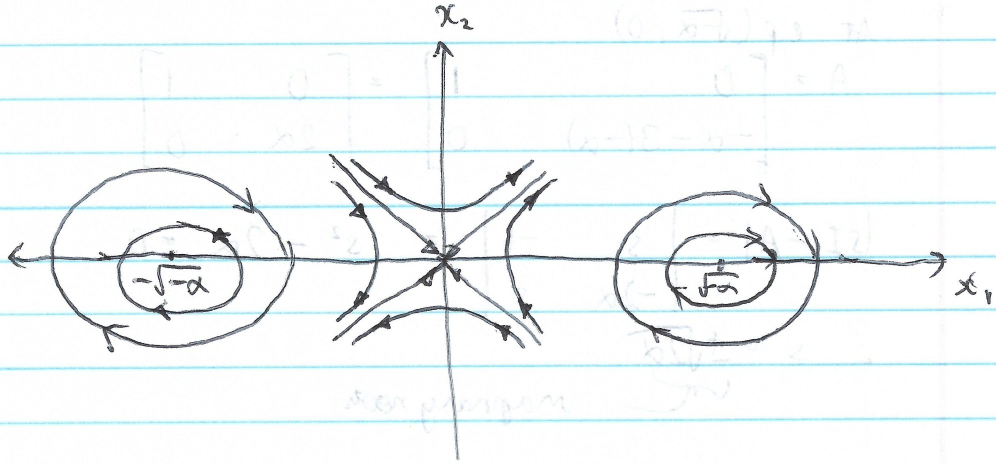
$$|sI - A| = s^2 - 2\alpha = 0$$

$\therefore s = \pm\sqrt{2\alpha}$



Center point

Phase Plane trajectory on next page



Phase Plane Trajectory of Duffing Oscillator