<u>EE 5323 Nonlinear Control Systems</u>

Homework Pledge of Honor

On all homeworks in this class - YOU MUST WORK ALONE.

Any cheating or collusion will be severely punished.

It is very easy to compare your software code and determine if you worked together
It does not matter if you change the variable names.

Please sign this form and include it as the first page of all of your submitted homeworks.

....

Typed Name: Shubham Gunjal

Pledge of honor:

"On my honor I have neither given nor received aid on this homework."

e-Signature: Shubham Raosaheb Gunjal

Digitally signed by Shubham Raosaheb Gunjal
Date: 2021.11.19 14:20:48 -06'00'

EE 5323 Homework 7

Lyapunov Controls Design, Feedback Linearization

1. **Controls Design.** A system is given by

$$\dot{x}_1 = x_2 \operatorname{sgn}(x_1)$$
$$\dot{x}_2 = x_1 x_2 + u$$

Select Lyapunov function candidate

$$V(x) = \tfrac{1}{2}(x_1^2 + x_2^2)$$

Use Lyapunov to design a controller *u(x)* to make system SISL.

2. **Multi-input Control.** Use Lyapunov to design controls u_1, u_2 to make this system

$$\dot{x}_1 = x_1 x_2^2 + u_1$$
$$\dot{x}_2 = x_1^3 x_2^7 + u_2$$

a. SISL, and then
b. AS

3. (Slotine and Li problem 6.3) A system is given by

$$\dot{x}_1 = \sin x_2$$
$$\dot{x}_2 = x_1^4 \cos x_2 + u$$

with output $y(t) = x_1(\mathrm{t})$

a. Design a FB linearization controller to make the output follow a desired trajectory $y_d(t)$ That is, find $u(t)$

b. Discuss the internal dynamics. Are they a problem?

<u>4. Effect of Output Choice in i/o FB Linearization</u>

It is desired to stabilize a system given by

$$\dot{x}_1 = x_2 \sin x_1 - x_1 + u$$
$$\dot{x}_2 = -x_1 + x_2^2$$

a. Select the output as $y = x_1$ and use FB lin. design to select the control *u(t)* to follow the desired trajectory $y_d(t)$. Check the internal dynamics. Set *y=0* to get the zero dynamics. Is the system minimum phase?

b. Select the new output $y = x_2$. Find the FB lin. controller u(t). Does this work? What about the internal dynamics?

EE 5323 Homework 7 Solution

Shubham Gunjal

November 19, 2021

Problem 1 - Lyapunov control design
A system is given by,

$$\dot{x}_1 = x_2 \sin x_1$$
$$\dot{x}_2 = x_1 x_2 + u$$

The states of the system are,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In state-space form,

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x, u)$$
$$\dot{X} = \begin{bmatrix} x_2 \sin x_1 \\ x_1 x_2 + u \end{bmatrix}$$

First, check the stability using Lyapunov function candidate. Consider a Lyapunov function candidate,

$$V(X) = \frac{1}{2}(x_1^2 + x_2^2) > 0$$

$$\begin{aligned}
\dot{V}(X) &= \left(\frac{\partial V}{\partial X}\right)^T \dot{X} \\
&= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\
&= x_1 \dot{x}_1 + x_2 \dot{x}_2 \\
&= x_1(x_2 \sin x_1) + x_2(x_1 x_2 + u) \\
\dot{V}(X) &= x_1 x_2 \sin x_1 + x_1 x_2^2 + x_2 u
\end{aligned}$$

To make this system SISL, $\dot{V}(X) \leq 0$. Select u as,

$$u = -x_1 x_2 - x_1 \sin x_1 - k x_2$$

Then $\dot{V}(X)$ reduces to,

$$\begin{aligned}
\dot{V}(X) &= x_1 x_2 \sin x_1 + x_1 x_2^2 + x_2 u \\
&= x_1 x_2 \sin x_1 + x_1 x_2^2 + x_2(-x_1 x_2 - x_1 \sin x_1 - k x_2) \\
&= x_1 x_2 \sin x_1 + x_1 x_2^2 - x_1 x_2^2 - x_1 x_2 \sin x_1 - k x_2^2 \\
\dot{V}(X) &= -k x_2^2 \leq 0
\end{aligned}$$

Which indicates that with this input, the system is SISL and $x_2 \to 0$ and $\dot{x}_2 \to 0$. Applying LaSalle extension to check what happens to x_1.

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \sin x_1 \\ x_1 x_2 + u \end{bmatrix}$$

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ u \end{bmatrix}$$

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 \sin x_1 \end{bmatrix}$$

Thus, from above equation, $x_1 \to 0$, and $\dot{x}_1 \to 0$. So, the system is actually asymptotically stable.

Problem 2 - Lyapunov control design for multi-input system

$$\dot{x}_1 = x_1 x_2^2 + u_1$$
$$\dot{x}_2 = x_1^3 x_2^7 + u_2$$

The states of the system are,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In state-space form,

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x, u)$$

$$\dot{X} = \begin{bmatrix} x_1 x_2^2 + u_1 \\ x_1^3 x_2^7 + u_2 \end{bmatrix}$$

First, check the stability using Lyapunov function candidate. Consider a Lyapunov function candidate,

$$V(X) = \frac{1}{2}(x_1^2 + x_2^2) > 0$$

$$\begin{aligned} \dot{V}(X) &= \left(\frac{\partial V}{\partial X}\right)^T \dot{X} \\ &= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\ &= x_1 \dot{x}_1 + x_2 \dot{x}_2 \\ &= x_1(x_1 x_2^2 + u_1) + x_2(x_1^3 x_2^7 + u_2) \\ \dot{V}(X) &= x_1^2 x_2^2 + x_1 u_1 + x_1^3 x_2^8 + x_2 u_2 \end{aligned}$$

For this system to be SISL, $\dot{V}(X) \leq 0$. Select inputs as,

$$u_1 = -x_1 x_2^2$$
$$u_2 = -x_1^3 x_2^7 - k x_2$$

Then, $\dot{V}(X)$ reduces to,

$$\begin{aligned} \dot{V}(X) &= x_1^2 x_2^2 + x_1 u_1 + x_1^3 x_2^8 + x_2 u_2 \\ &= x_1^2 x_2^2 + x_1(-x_1 x_2^2) + x_1^3 x_2^8 + x_2(-x_1^3 x_2^7 - k x_2) \\ &= x_1^2 x_2^2 - x_1^2 x_2^2 + x_1^3 x_2^8 - x_1^3 x_2^8 - k x_2^2 \\ \dot{V}(X) &= -k x_2^2 \leq 0 \end{aligned}$$

Thus, with these inputs, the system is SISL. Above equation implies that $x_2 \to 0$, and $\dot{x}_2 \to 0$. x_1 is unknown. Apply LaSalle extension to check what happens to x_1.

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2^2 + u_1 \\ x_1^3 x_2^7 + u_2 \end{bmatrix}$$
$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -x_1 x_2^2 \\ -x_1^3 x_2^7 - kx_2 \end{bmatrix}$$
$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The above equation implies that $\dot{x}_1 \to 0$, but x_1 is still unknown and depends on system initial conditions. Which means the system is still SISL.

For this system to be asymptotically stable, $\dot{V}(X) < 0$.

$$\dot{V}(X) = x_1^2 x_2^2 + x_1 u_1 + x_1^3 x_2^8 + x_2 u_2$$

Select inputs as,

$$u_1 = -x_1 x_2^2 - k_1 x_1$$
$$u_2 = -x_1^3 x_2^7 - k_2 x_2$$

Then, $\dot{V}(X)$ reduces to,

$$\begin{aligned} \dot{V}(X) &= x_1^2 x_2^2 + x_1 u_1 + x_1^3 x_2^8 + x_2 u_2 \\ &= x_1^2 x_2^2 + x_1(-x_1 x_2^2 - k_1 x_1) + x_1^3 x_2^8 + x_2(-x_1^3 x_2^7 - k_2 x_2) \\ &= x_1^2 x_2^2 - x_1^2 x_2^2 - k_1 x_1^2 + x_1^3 x_2^8 - x_1^3 x_2^8 - k_2 x_2^2 \\ \dot{V}(X) &= -k_1 x_1^2 - k_2 x_2^2 < 0 \end{aligned}$$

Thus, with such inputs, $\dot{V}(X)$ is negative definite and the system is asymptotically stable.

Problem 3 - Feedback linearization
Consider a nonlinear system,

$$\dot{x}_1 = \sin x_2$$
$$\dot{x}_2 = x_1^4 \cos x_2 + u$$

The output of the system is selected to be $y(t) = x_1(t)$. Taking derivative of output y, until the input u shows up.

$$\dot{y} = \dot{x}_1 = \sin x_2$$
$$\ddot{y} = \ddot{x}_1 = \frac{\partial \sin x_2}{\partial t} \frac{\partial x_2}{\partial t}$$
$$\ddot{y} = \ddot{x}_1 = \dot{x}_2 \cos x_2$$
$$\ddot{y} = \ddot{x}_1 = (x_1^4 \cos x_2 + u) \cos x_2$$
$$\ddot{y} = \ddot{x}_1 = x_1^4 \cos^2 x_2 + u \cos x_2$$

Suppose it is desired for output to reach a desired state, $y_d = x_{1d}$. Define tracking error as,

$$\begin{aligned}
e &= y_d - y \\
\dot{e} &= \dot{y}_d - \dot{y} \\
\ddot{e} &= \ddot{y}_d - \ddot{y} \\
\ddot{e} &= \ddot{y}_d - \ddot{y} \\
\ddot{e} &= \ddot{y}_d - (x_1^4 \cos^2 x_2 + u \cos x_2)
\end{aligned}$$

For error to be asymptotically stable, it must be of the form as,

$$\ddot{e} = -k_v \dot{e} - k_p e$$

Select input u such as,

$$\begin{aligned}
\ddot{e} &= \ddot{y}_d - (x_1^4 \cos^2 x_2 + \{\frac{1}{\cos x_2}(-x_1^4 \cos^2 x_2 + \ddot{y}_d + k_v \dot{e} + k_p e)\} \cos x_2) \\
\ddot{e} &= \ddot{y}_d - (x_1^4 \cos^2 x_2 + \{(-x_1^4 \cos^2 x_2 + \ddot{y}_d + k_v \dot{e} + k_p e)\}) \\
\ddot{e} &= \ddot{y}_d - (x_1^4 \cos^2 x_2 - x_1^4 \cos^2 x_2 + \ddot{y}_d + k_v \dot{e} + k_p e) \\
\ddot{e} &= -k_v \dot{e} - k_p e
\end{aligned}$$

where the input is selected to be,

$$u = \frac{1}{\cos x_2}(-x_1^4 \cos^2 x_2 + \ddot{y}_d + k_v \dot{e} + k_p e)$$

The above equation for e is asymptotically stable, i.e. $e \to 0$, $\dot{e} \to 0$, and $\ddot{e} \to 0$ for any positive values for the gains, k_v and k_p. And thus, the system output reaches the desired trajectory, i.e. $y \to y_d$.

The system is second order and the error dynamics are second order. So, the internal dynamics are also stable. But they can still be checked as follows. Select state x_2 to check internal dynamics as it already has input u in it.

$$\begin{aligned}
\dot{x}_2 &= x_1^4 \cos x_2 + u \\
\dot{x}_2 &= x_1^4 \cos x_2 + \frac{1}{\cos x_2}(-x_1^4 \cos^2 x_2 + \ddot{y}_d + k_v \dot{e} + k_p e)
\end{aligned}$$

To check if the internal dynamics is stable, we check the zero dynamics, which is a simplified form of internal dynamics. To achieve zero dynamics set y_d, $\dot{y}_d$, $\ddot{y}_d$, e, and $\dot{e}$ equal to zero.

$$\begin{aligned}
\dot{x}_2 &= x_1^4 \cos x_2 + \frac{1}{\cos x_2}(-x_1^4 \cos^2 x_2 + \ddot{y}_d + k_v \dot{e} + k_p e) \\
\dot{x}_2 &= 0
\end{aligned}$$

The above equation of the zero dynamics is stable, i.e. $x_2 = x_2(0)$, $\forall\ t > 0$, and for any initial condition $x_2(0)$. Which in turn means the internal dynamics is stable. Thus, the input-output feedback linearization works.

Problem 4 - Feedback linearization, effect of output choice
Consider a nonlinear system,

$$\begin{aligned}
\dot{x}_1 &= x_2 \sin x_1 - x_1 + u \\
\dot{x}_2 &= -x_1 + x_2^2
\end{aligned}$$

The output of the system is selected to be $y(t) = x_1(t)$. Taking derivative of output y, until the input u shows up.

$$\dot{y} = \dot{x}_1 = x_2 \sin x_1 - x_1 + u$$

For this system, relative degree is 1. Suppose it is desired for output to reach a desired state, $y_d = x_{1d}$. Define tracking error as,

$$\begin{aligned}
e &= y_d - y \\
\dot{e} &= \dot{y}_d - \dot{y} \\
\dot{e} &= \dot{y}_d - (x_2 \sin x_1 - x_1 + u)
\end{aligned}$$

For error to be asymptotically stable, it must be of the form as,

$$\dot{e} = -ke$$

Pick u such as,

$$\begin{aligned}
\dot{e} &= \dot{y}_d - (x_2 \sin x_1 - x_1 + u) \\
\dot{e} &= \dot{y}_d - (x_2 \sin x_1 - x_1 + (-x_2 \sin x_1 + x_1 + \dot{y}_d + ke)) \\
\dot{e} &= -ke
\end{aligned}$$

The input u is selected as,

$$u = -x_2 \sin x_1 + x_1 + \dot{y}_d + ke$$

The above equation for error dynamics is stable for any positive value of gain, k. Thus, $e \to 0$, and $y \to y_d$. The system is second order while the error dynamics is first order. Thus, the internal dynamics of the system must be checked for stability. Select state x_2 to check internal dynamics.

$$\begin{aligned}
\dot{x}_2 &= -x_1 + x_2^2 \\
\ddot{x}_2 &= -\dot{x}_1 + 2x_2\dot{x}_2 \\
\ddot{x}_2 &= -(x_2 \sin x_1 - x_1 + u) + 2x_2(-x_1 + x_2^2) \\
\ddot{x}_2 &= -(x_2 \sin x_1 - x_1 + (-x_2 \sin x_1 + x_1 + \dot{y}_d + ke)) + 2x_2(-x_1 + x_2^2) \\
\ddot{x}_2 &= -(\dot{y}_d + ke) + 2x_2(-x_1 + x_2^2)
\end{aligned}$$

To check if the internal dynamics is stable, we check the zero dynamics, which is a simplified form of internal dynamics. To achieve zero dynamics set y_d, $\dot{y}_d$, and e equal to zero.

$$\begin{aligned}
\ddot{x}_2 &= 2x_2(x_2^2) \\
\ddot{x}_2 &= 2x_2^3
\end{aligned}$$

The zero dynamics are unstable as $x_2 \to \infty$, $\forall x_2(0) \neq 0$ and $\forall \dot{x}_2(0) \neq 0$. Thus, the internal dynamics are unstable. Thus, the input-output feedback linearization does not work for the output of choice $y = x_1$.

Select a new output $y = x_2$. Taking derivative of output y, until the input u shows up.

$$\begin{aligned}
\dot{y} &= \dot{x}_2 = -x_1 + x_2^2 \\
\ddot{y} &= \ddot{x}_2 = -\dot{x}_1 + 2x_2\dot{x}_2 \\
\ddot{y} &= \ddot{x}_2 = -(x_2 \sin x_1 - x_1 + u) + 2x_2(-x_1 + x_2^2) \\
\ddot{y} &= \ddot{x}_2 = -x_2 \sin x_1 + x_1 - u - 2x_1x_2 + 2x_2^3
\end{aligned}$$

For this system, relative degree is 2. Suppose it is desired for output to reach a desired state, $y_d = x_{1d}$. Define tracking error as,

$$\begin{aligned}
e &= y_d - y \\
\dot{e} &= \dot{y}_d - \dot{y} \\
\ddot{e} &= \ddot{y}_d - \ddot{y} \\
\ddot{e} &= \ddot{y}_d - (-x_2 \sin x_1 + x_1 - u - 2x_1x_2 + 2x_2^3)
\end{aligned}$$

For error to be asymptotically stable, it must be of the form as,

$$\ddot{e} = -k_v\dot{e} - k_p e$$

Select the input u such that,

$$\begin{aligned}
\ddot{e} &= \ddot{y}_d - (-x_2 \sin x_1 + x_1 - u - 2x_1x_2 + 2x_2^3) \\
\ddot{e} &= \ddot{y}_d + x_2 \sin x_1 - x_1 + 2x_1x_2 - 2x_2^3 + u \\
\ddot{e} &= \ddot{y}_d + x_2 \sin x_1 - x_1 + 2x_1x_2 - 2x_2^3 + (-\ddot{y}_d - x_2 \sin x_1 + x_1 - 2x_1x_2 + 2x_2^3 - k_v\dot{e} - k_p e) \\
\ddot{e} &= -k_v\dot{e} - k_p e
\end{aligned}$$

The input u is selected as,

$$u = -\ddot{y}_d - x_2 \sin x_1 + x_1 - 2x_1x_2 + 2x_2^3 - k_v\dot{e} - k_p e$$

The above equation of the error dynamics is stable for any positive values for the gains, k_v and k_p. Thus, $e \to 0$, and $y \to y_d$. Also, the system is second order and the error dynamics are second order. Thus, the internal dynamics are stable. Hence, the feedback linearization for output of choice $y = x_2$ works.