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Lyapunov Design for Feedback Controls

$$\dot{x} = f(x,u) = f(x) + g(x)u \qquad (1)$$

affine in control input

open-loop (feedforward) control $u(t)$

closed-loop (feedback) control

$$u = h(x) \equiv \mu(x)$$

D $V(x)$ is a Control Lyapunov Function if:

i) $V(x)$ is real, continuous, + $V(x) > 0$

ii) $\min_u \frac{\partial V}{\partial x}^T (f(x) + g(x)u) < 0$

or

$$\exists u \ni \frac{\partial V}{\partial x}^T (f + gu) < 0$$

Thm If $V(x)$ is a CLF, then $\exists u \ni$ system (1) is AS.

proof note that $\dot{V} = \frac{\partial V}{\partial x}^T \dot{x} = \underbrace{\frac{\partial V}{\partial x}^T (f + gu)}_{< 0}$

<u>optimal Design</u> yields CLF

Define performance index

$$J(x;u) = \int_0^\infty \left(l(x) + u^T R u \right) d\tau$$

$l(x) > 0$, e.g. $l(x) = x^T Q x$

$R = R^T > 0$

select FB policy $u = \mu(x)$ + define <u>Value function</u> (cost to go)

$$V(x(t)) = \int_t^\infty \left(l(x) + \mu^T R \mu \right) d\tau > 0$$

<u>then</u> $\dot{V} = \frac{\partial V}{\partial x}^T \dot{x} = \frac{\partial V}{\partial x}^T (f + g\mu)$

Leibniz $\dot{V} = -\left(l(x(t)) + \mu^T R \mu \right)$

$\therefore \dot{V} = \frac{\partial V}{\partial x}^T (f + g\mu) = -\left(l + \mu^T R \mu \right)$

<u>so</u> if $V(x(t))$ is finite, it is a c.l.f. + $\mu(x)$ is a stabilizing policy.

optimal design

$$V^*(x(t)) = \min_u \int_t^\infty (l(x) + u^T R u)\, d\tau$$

if $\exists$ optimal control $u^*(x)$ then

V^* is a c.l.f. and $u^*(x)$ is stabilizing

Bellman Equation

$$\dot V = \frac{\partial V^T}{\partial x}(f+gu) = -l(x) + u^T R u$$

Bellman eq. is

$$H(x,u) = \frac{\partial V^T}{\partial x}(f+gu) + l(x) + u^T R u = 0$$

Bellman Optimality eq.

$H(x,u)$ = Hamiltonian

Bellman Opt. Eq. is

$$0 = \min_u H(x,u) = \min_u \left(\frac{\partial V^{*T}}{\partial x}(f+gu) + l(x) + u^T R u\right)$$

so $g^T \frac{\partial V^*}{\partial x} + 2Ru = 0$, $\quad u^* = -\frac{1}{2} R^{-1} g^T \frac{\partial V^*}{\partial x}$

& Bellman opt eq is

$$0 = H(x,u^*) = \frac{\partial V^T}{\partial x}(f+gu^*) + l(x) + u^{*T} R u^*$$

= Hamilton-Jacobi-Bellman eq.