Error Dynamics

Define control error r(t) and suppose the error dynamics are

$$\dot{r} = f(x) - \tau$$

with *f(x)* some unknown nonlinearities and $\tau(t)$ a control input.

Adaptive Control

Assume: *f(x)* is known to be of the structure

$$f(x) = W^T \phi(x)$$

with *W* an unknown parameter vector and $\phi(x)$ a known basis set, called a regression vector.

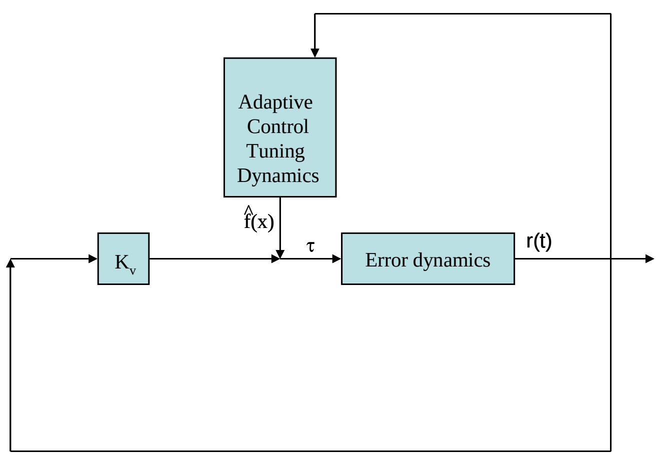
Note that this structure is LINEAR-IN-THE-PARAMETERS (LIP) *W*.

Note that the regression matrix $\phi(x)$ depends on the system, and CHANGES for different systems.

Then the error dynamics are

$$\dot{r} = W^T \phi(x) - \tau$$

Adaptive Controller: The adaptive controller is shown in the figure.



Adaptive Controller

Define the estimate for the nonlinear function as $\hat{f}(x) = \hat{W}^T \phi(x)$. Select the controller

$$\tau = \hat{f}(x) + K_v r = \hat{W}^T \phi(x) + K_v r$$

with $\hat{W}(t)$ a time-varying estimate of the unknown parameters W and control gain $K_v > 0$ any symmetric positive definite matrix.

Then the closed-loop system becomes

$$\dot{r} = W^T \phi(x) - \tau = W^T \phi(x) - \hat{W}^T \phi(x) - K_v r$$

$$\dot{r} = \tilde{W}^T \phi(x) - K_v r$$

with the parameter estimation error defined as

$$\tilde{W}(t) = W - \hat{W}(t)$$

Let the parameter estimate be updated (tuned) using

$$\frac{d\hat{W}}{dt} = \dot{\hat{W}} = F\phi(x) r^T$$

with parameter tuning gain $F > 0$ any symmetric positive definite gain matrix. This is the internal (tuning) dynamics of the adaptive controller. The state of the adaptive controller is given by the parameter estimates $\hat{W}(t)$.

Performance of Adaptive Controller: Using the adaptive controller, the closed-loop system is asymptotically stable, i.e. the control error *r(t)* goes to zero.

If an additional Persistence of Excitation (PE) condition holds, the parameter estimates converge to the actual unknown parameters.

Proof: Select the Lyapunov function

$$L = \tfrac{1}{2} r^T r + \tfrac{1}{2} tr\{\tilde{W}^T F^{-1} \tilde{W}\}$$

with *r(t)* the control error and $\tilde{W}(t) = W - \hat{W}(t)$ the parameter estimation error. Note that this trace *tr{}* of a quadratic form is a norm (the Frobenius norm).

Differentiate

$$\dot{L} = r^T \dot{r} + tr\{\tilde{W}^T F^{-1} \dot{\tilde{W}}\}$$

But $\dot{r} = \tilde{W}^T \phi(x) - K_v r$ so that

$$\dot{L} = r^T \left(\tilde{W}^T \phi(x) - K_v r \right) + tr\{\tilde{W}^T F^{-1} \dot{\tilde{W}}\} = -r^T K_v r + tr\{\tilde{W}^T \left(F^{-1} \dot{\tilde{W}} + \phi(x) r^T \right)\}$$

Now select tuning law

$$\dot{\hat{W}} = F\phi(x) r^T \quad \text{or} \quad \dot{\tilde{W}} = -F\phi(x) r^T$$

to give

$$\dot{L} = -r^T K_v r \, .$$

Therefore $L > 0$ and $\dot{L} \le 0$ and the system is Stable in the sense of Lyapunov (SISL), i.e. the states $r(t), \tilde{W}(t)$ are bounded.

Barbalat's Lemma shows that $\dot{L} \to 0$ hence control error *r(t)* goes to zero. Viz.

$$L(t) = -\int_t^{\infty} \dot{L}(\sigma)\, d\sigma = \int_t^{\infty} r^T K_v r \, d\sigma$$

Is bounded so that $r^T K_v r \to 0$

Parameter Convergence and Persistence of Excitation:

We now want to know when $\tilde{W}(t) \to 0$, i.e. $\hat{W}(t) \to W$, so that the parameter estimates go to the actual parameters.

One has the dynamics of $\tilde{W}(t)$ as

$$\dot{\tilde{W}} = -F\phi(x) r^T$$

write the closed-loop error dynamics $\dot{r} = \tilde{W}^T \phi(x) - K_v r$ as

$$z(t) \equiv \tilde{W}^T \phi(x) = \dot{r} + K_v r$$

$$z^T(t) \equiv \phi^T(x)\tilde{W} = (\dot{r} + K_v r)^T$$

which is an output equation for $\tilde{W}(t)$ dynamics.

We have just shown that $r(t) \to 0, \dot{r}(t) \to 0$, so that the $\tilde{W}(t)$ dynamics has zero input and zero output. This means that the state $\tilde{W}(t)$ goes to zero if the system is uniformly completely observable, i.e.

$$\alpha_1 I \le \int_t^{t+T} \phi(x(\sigma))\phi^T(x(\sigma))\, d\sigma \le \alpha_2 I$$

for all, for some positive values of of α_1, α_2, T .

This is the same as a persistence of excitation condition on the regression vector $\phi(x)$.

Therefore, if $\phi(x)$ is PE, the parameters converge.

Example 1. Linear System.

Let $y = \dfrac{1}{s^2 + a_1 s + a_2} u$

Or

$$\ddot{y} + a_1\dot{y} + a_2 y = u$$

It is desired to track a reference input $y_d(t)$. Define the tracking error $e = y_d - y$. Then $\dot{e} = \dot{y}_d - \dot{y}$ and $\ddot{e} = \ddot{y}_d + a_1\dot{y} + a_2 y - u$.

Define the sliding error variable $r = \dot{e} + \Lambda e$ with time constant matrix $\Lambda > 0$ any symmetric positive definite matrix. Then

$$\dot{r} = \ddot{e} + \Lambda\dot{e} = \ddot{y}_d + \Lambda\dot{e} + a_1\dot{y} + a_2 y - u$$

Select now the auxiliary input *v(t)* so that $u = v + \ddot{y}_d + \Lambda\dot{e}$. Then

$$\dot{r} = a_1\dot{y} + a_2 y - v$$

is of the error dynamics form $\dot{r} = f(x) - \tau$ used above with $x = [y\ \dot{y}]^T$ and

$$f(x) = a_1\dot{y} + a_2 y = \begin{bmatrix} a_1 & a_2 \end{bmatrix}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = W^T\phi(x)$$

a function of the unknown system parameters and the regression matrix $\phi(x)$ given by *y(t)* and its derivative.

Note that the regression matrix $\phi(x)$ depends on the system, and CHANGES for different systems.

The overall adaptive control law is given by $u = \hat{W}^T\phi(x) + K_v r + \ddot{y}_d + \Lambda\dot{e}$, $\dot{\hat{W}} = F\phi(x)r^T$. This is a dynamical controller with internal state equal to $\hat{W}(t)$.

Example 2. Nonlinear Robot System.

Consider the robot-like system

$$\ddot{y} + d(y,\dot{y}) + k(y) = u$$

with $d(y,\dot{y})$ an unknown nonlinear damping term and *k(y)* an unknown nonlinear friction.

The same development as above yields $\dot{r} = f(x) - v$ with $x = [y\ \dot{y}]^T$ and $f(x) = d(y,\dot{y}) + k(y)$.

LIP Assumption Needed for Adaptive Control. Now we are stuck unless we assume there is a linear in the parameters (LIP) form for *f(x)*, e.g. in the form

$$f(x) = \begin{bmatrix} D & K \end{bmatrix}\begin{bmatrix} d_1(y,\dot{y}) \\ k_1(y,\dot{y}) \end{bmatrix} = W^T\phi(x)$$

With *D, K* the unknown damping and friction gains, and $d_1(y,\dot{y}), k_1(y,\dot{y})$ KNOWN damping and friction functions, possibly nonlinear (e.g. Stribeck friction). If this LIP assumption holds, we proceed as above to design the adaptive controller.

Typical Behavior of Adaptive Controllers.

The typical behavior of adaptive controllers is shown in the figure. Note that the errors go to zero and the parameter estimates converge. This assume that $f(x) = W^T\phi(x)$ holds exactly, and that there are no disturbances in the system.

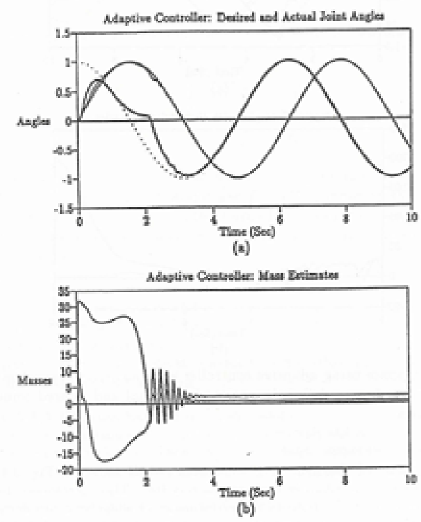


Figure 3.4.3: Response using adaptive controller. (a) Actual and desired joint angles. (b) Mass estimates.

Control errors go to zero

Parameter estimates converge

Robust Control

The error dynamics are

$$\dot{r} = f(x) - \tau$$

with *f(x)* some unknown nonlinearities and $\tau(t)$ a control input.

Assume: We know a fixed nominal value or estimate $\hat{f}(x)$ for unknown *f(x)*, and that the estimation error $\tilde{f} = f(x) - \hat{f}(x)$ is bounded like

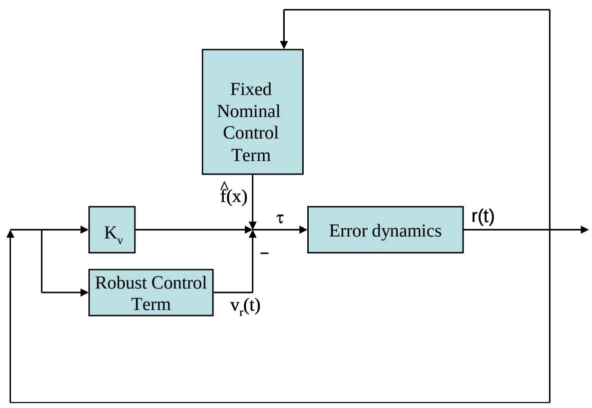
$$\left\| \tilde{f}(x) \right\| \le F(x)$$

with *F(x)* a known bound function, possibly nonlinear.

Note that robust control assumes less information than the adaptive controller, which needs the LIP structure

$$f(x) = W^T \phi(x)$$

with *W* an unknown parameter vector and $\phi(x)$ a known basis set or regression vector.



Robust Controller

Robust Saturation Controller: the robust controller is shown in the figure.

Select the controller

$$\tau = \hat{f}(x) + K_v r - v_r$$

with $v_r(t)$ a robust control term given by

$$v_r = \begin{cases} -r\dfrac{F(x)}{\|r\|}, & \|r\| \geq \varepsilon \\ -r\dfrac{F(x)}{\varepsilon}, & \|r\| < \varepsilon \end{cases}$$

with $\varepsilon > 0$ a small design parameter.

Then the closed-loop dynamics becomes

$$\dot{r} = f(x) - \tau = f(x) - \left(\hat{f}(x) + K_v r - v_r\right)$$

$$\dot{r} = \tilde{f}(x) - K_v r + v_r$$

This robust controller is easier to implement than the adaptive controller because the controller does not have any dynamics (i.e. $\dot{\hat{W}}$ for the adaptive controller).

Performance of Robust Controller: With this control, the closed-loop system is bounded stable with $\|r\|$ bounded with a magnitude near ε.

Proof: Select the Lyapunov function

$$L = \tfrac{1}{2} r^T r$$

Differentiate

$$\dot{L} = r^T \dot{r}$$

$$\dot{L} = r^T \left(\tilde{f}(x) - K_v r + v_r \right) = -r^T K_v r + r^T \left(\tilde{f}(x) + v_r \right)$$

Using norm bounds, one has

$$\dot{L} \le -\sigma_{\min}(K_v)\|r\|^2 + \|r\| F(x) + r^T v_r$$

with $\sigma_{\min}(K_v)$ the minimum singular value of K_v.

There are two cases.

1. $\|r\| \ge \varepsilon$

Then

$$\dot{L} \le -\sigma_{\min}(K_v)\|r\|^2 + \|r\| F(x) - \|r\|^2 F(x)/\|r\|$$
$$= -\sigma_{\min}(K_v)\|r\|^2$$

So L is decreasing in this region, and $\|r\|$ decreases towards ε.

2. $\|r\| < \varepsilon$

$$\dot{L} \le -\sigma_{\min}(K_v)\|r\|^2 + \|r\| F(x) - \|r\|^2 F(x)/\varepsilon$$
$$= -\sigma_{\min}(K_v)\|r\|^2 + \|r\| F(x)\left(1 - \|r\|/\varepsilon\right)$$

With the last term generally positive. So nothing can be said about whether L is decreasing or increasing in this region.

The conclusion is that $\|r\|$ stays bounded with a magnitude near ε.

Example 3. Nonlinear Robot System.

Consider the robot-like system

$$\ddot{y} + d(y, \dot{y}) + k(y) = u$$

with $d(y, \dot{y})$ an unknown nonlinear damping term and *k(y)* an unknown nonlinear friction.

The development in Examples 1 and 2 results in $\dot{r} = f(x) - v$ with r(t) the sliding error, $x = [y \ \dot{y}]^T$, and $f(x) = d(y, \dot{y}) + k(y)$.

The robust control is given by $v = \hat{f}(x) + K_v r - v_r$ with the robust term $v_r(t)$ given above. The overall control input is then $u = v + \ddot{y}_d + \Lambda \dot{e}$.

Suppose in particular that the actual system nonlinearities are given by

$$\ddot{y} + 4\dot{y}^2 + 3y + 1.2g\cos(y) = u$$

with a coriolis term $d(y,\dot{y}) = 4\dot{y}^2$ and friction plus gravity given by $k(y) = 3y + 1.2g\cos(y)$.

Suppose the actual coefficients for the three nonlinear terms are unknown but one knows that the coefficients are respectively bounded by [2,8], [1,5], [1,2]. Select nominal values of these parameters at the midpoints of these ranges to get $\hat{f}(x) = 5\dot{y}^2 + 3y + 1.5g\cos(y)$. Then the control is $v = 5\dot{y}^2 + 3y + 1.5g\cos(y) + K_v r - v_r$ and the bound on the nonlinearities is

$$F(x) = 3\dot{y}^2 + 2y + 0.5g\cos(y)$$

Now the robust term can be defined and the controller implemented. The overall control input is then $u = v + \ddot{y}_d + \Lambda\dot{e}$.

Typical Behavior of Robust Controllers.

The typical behavior of robust controllers is shown in the figure. Note that the error does not go to zero but does indeed stay small.

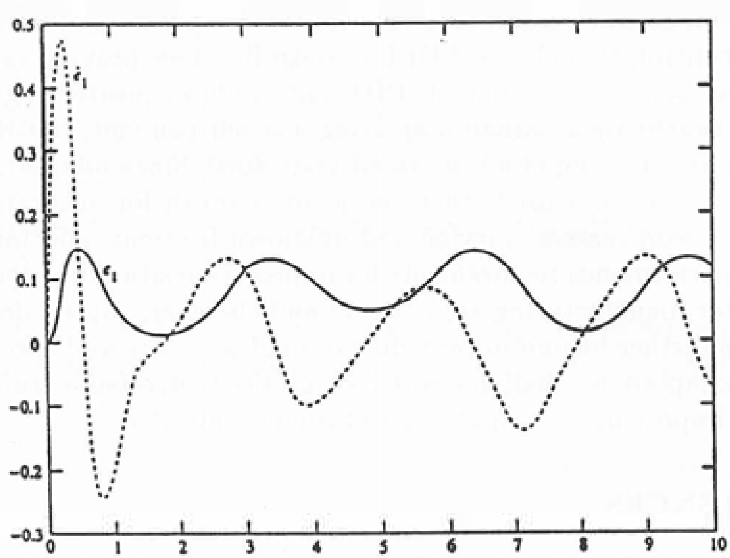


Figure 3.4.6: Typical behavior of robust controller.

Errors are bounded