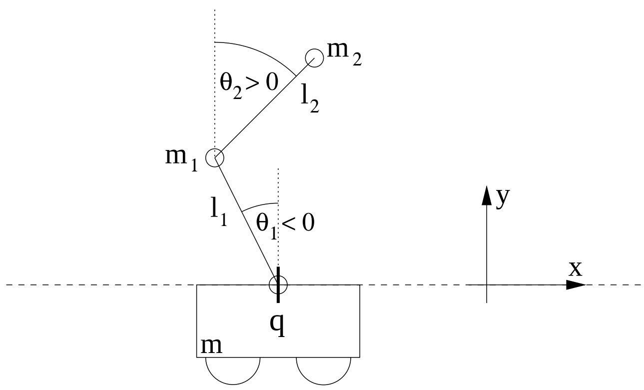
Equations of motion for an inverted double pendulum on a cart (in generalized coordinates)

Consider a double pendulum which is mounted to a cart, as in the following graphic:



The length of the first rod is denoted by l_1 and the length of the second rod by l_2. The mass of the cart is denoted by m. We assume that the rods have no mass, that on the top of the first rod (and thus at the bottom of the second rod) there is a weight of mass m_1, and that on the top of the second rod there is a weight of mass m_2. All masses are assumed to be concentrated into a point.

We denote by $\theta_1 = \theta_1(t)$ and $\theta_2 = \theta_2(t)$ the deviation of the rods from the upright position at time $t \in \mathbb{R}$ as depicted in the image above. By $q = q(t)$ we denote the horizontal position of the cart and we assume that the cart cannot move vertically. The derivatives with respect to time are denoted by

$$\frac{d}{dt}q(t) = \dot{q}, \quad \frac{d}{dt}\theta_1(t) = \dot{\theta}_1, \quad \frac{d}{dt}\theta_2(t) = \dot{\theta}_2.$$

The goal is to stabilize the pendulum in an upright position above the cart by only applying forces to the cart itself; think of only the cart having some kind of motor while the rods can dangle around freely. The control input $u = u(t)$ is thus the force that we can apply to the cart.

Furthermore, we assume that external distrubances w_1, w_2, w_3 act as forces on q, θ_1, θ_2; think of these external forces as wind or some human pushing the rods. The friction in the joints and the friction of the moving cart are modeled via a linear ansatz. We therefore introduce the damping coefficients d_1, d_2, d_3 and consider the friction/damping force of the cart to be $-d_1\dot{q}$ while the friction/damping forces in the joints are assumed to be $-d_2\dot{\theta}_1$ and $-d_3\dot{\theta}_2$.

The positions of the masses m, m_1, and m_2 are given by

$$q_0 := \begin{bmatrix} q \\ 0 \end{bmatrix}, \qquad q_1 := \begin{bmatrix} q + l_1 \sin\theta_1 \\ l_1 \cos\theta_1 \end{bmatrix}, \text{ and } \qquad q_2 := \begin{bmatrix} q + l_1 \sin\theta_1 + l_2 \sin\theta_2 \\ l_1 \cos\theta_1 + l_2 \cos\theta_2 \end{bmatrix},$$

respecitvely. Thus, the kinetic energy in the system is

$$
\begin{aligned}
K &= \frac{1}{2}\left\{m\|\dot{q}_0\|^2 + m_1\|\dot{q}_1\|^2 + m_2\|\dot{q}_2\|^2\right\} \\
&= \frac{1}{2}\left\{m\dot{q}^2 + m_1\left[\left(\dot{q} + l_1\dot{\theta}_1\cos\theta_1\right)^2 + \left(l_1\dot{\theta}_1\sin\theta_1\right)^2\right] + \right. \\
&\quad \left. m_2\left[\left(\dot{q} + l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2\right)^2 + \left(l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2\right)^2\right]\right\}
\end{aligned}
$$

and the potential energy can be given as

$$
P = g\left\{m_1 l_1\cos\theta_1 + m_2\left(l_1\cos\theta_1 + l_2\cos\theta_2\right)\right\}.
$$

The principle of Lagrangian mechanics (as taught in "theoretical physics") states that to obtain the equations of motion for the cart, we have to define the Lagrangian $L := K - P$ and then the equations of motion are

$$
\begin{aligned}
u + w_1 - d_1\dot{q} &= \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}}\right\} - \left\{\frac{\partial L}{\partial q}\right\} \\
&= \frac{d}{dt}\left\{m\dot{q} + m_1\left(\dot{q} + l_1\dot{\theta}_1\cos\theta_1\right) + m_2\left(\dot{q} + l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2\right)\right\} - \{0\} \\
&= (m + m_1 + m_2)\,\ddot{q} + l_1(m_1 + m_2)\ddot{\theta}_1\cos\theta_1 - l_1(m_1 + m_2)(\dot{\theta}_1)^2\sin\theta_1 \\
&\quad + m_2 l_2\ddot{\theta}_2\cos\theta_2 - m_2 l_2(\dot{\theta}_2)^2\sin\theta_2 \\
&= (m + m_1 + m_2)\,\ddot{q} + l_1(m_1 + m_2)\ddot{\theta}_1\cos\theta_1 + m_2 l_2\ddot{\theta}_2\cos\theta_2 \\
&\quad - l_1(m_1 + m_2)(\dot{\theta}_1)^2\sin\theta_1 - m_2 l_2(\dot{\theta}_2)^2\sin\theta_2 \\
w_2 - d_2\dot{\theta}_1 &= \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\theta}_1}\right\} - \left\{\frac{\partial L}{\partial \theta_1}\right\} \\
&=.\star.= \left\{l_1(m_1 + m_2)\dot{q}\dot{\theta}_1\sin\theta_1 + l_1 l_2 m_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - g(m_1 + m_2)l_1\sin\theta_1\right\} \\
&\quad + \frac{d}{dt}\left\{l_1(m_1 + m_2)\dot{q}\cos\theta_1 + l_1^2(m_1 + m_2)\dot{\theta}_1 + l_1 l_2 m_2\dot{\theta}_2\cos(\theta_1 - \theta_2)\right\} \\
&= \left\{l_1(m_1 + m_2)\dot{q}\dot{\theta}_1\sin\theta_1 + l_1 l_2 m_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - g(m_1 + m_2)l_1\sin\theta_1\right\} \\
&\quad + \left\{l_1(m_1 + m_2)\ddot{q}\cos\theta_1 + l_1^2(m_1 + m_2)\ddot{\theta}_1 + l_1 l_2 m_2\ddot{\theta}_2\cos(\theta_1 - \theta_2)\right. \\
&\qquad \left. - l_1(m_1 + m_2)\dot{q}\dot{\theta}_1\sin\theta_1 - l_1 l_2 m_2\dot{\theta}_2(\dot{\theta}_1 - \dot{\theta}_2)\sin(\theta_1 - \theta_2)\right\} \\
&= l_1(m_1 + m_2)\ddot{q}\cos\theta_1 + l_1^2(m_1 + m_2)\ddot{\theta}_1 + l_1 l_2 m_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) \\
&\quad + l_1 l_2 m_2(\dot{\theta}_2)^2\sin(\theta_1 - \theta_2) - g(m_1 + m_2)l_1\sin\theta_1 \\
w_3 - d_3\dot{\theta}_2 &= \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\theta}_2}\right\} - \left\{\frac{\partial L}{\partial \theta_2}\right\} \\
&=.\star.= \left\{-l_2 m_2 g\sin\theta_2 + l_2 m_2\dot{q}\dot{\theta}_2\sin\theta_2 - l_1 l_2 m_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2)\right\} \\
&\quad + \frac{d}{dt}\left\{l_2^2 m_2\dot{\theta}_2 + l_2 m_2\dot{q}\cos\theta_2 + l_1 l_2 m_2\dot{\theta}_1\cos(\theta_1 - \theta_2)\right\} \\
&= \left\{-l_2 m_2 g\sin\theta_2 + l_2 m_2\dot{q}\dot{\theta}_2\sin\theta_2 - l_1 l_2 m_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2)\right\} \\
&\quad + \left\{l_2^2 m_2\ddot{\theta}_2 + l_2 m_2\ddot{q}\cos\theta_2 + l_1 l_2 m_2\ddot{\theta}_1\cos(\theta_1 - \theta_2)\right. \\
&\qquad \left. - l_2 m_2\dot{q}\dot{\theta}_2\sin\theta_2 - l_1 l_2 m_2\dot{\theta}_1(\dot{\theta}_1 - \dot{\theta}_2)\sin(\theta_1 - \theta_2)\right\} \\
&= l_2 m_2\ddot{q}\cos\theta_2 + l_1 l_2 m_2\ddot{\theta}_1\cos(\theta_1 - \theta_2) + l_2^2 m_2\ddot{\theta}_2 \\
&\quad - l_1 l_2 m_2(\dot{\theta}_1)^2\sin(\theta_1 - \theta_2) - l_2 m_2 g\sin\theta_2,
\end{aligned}
$$

where the MATLAB symbolic computations toolbox was used at the $=.\star.=$ symbols.

In matrix form and using the definition $y := \begin{bmatrix} q & \theta_1 & \theta_2 \end{bmatrix}^T$ this yields

$$\underbrace{\begin{bmatrix} m+m_1+m_2 & l_1(m_1+m_2)\cos\theta_1 & m_2 l_2 \cos\theta_2 \\ l_1(m_1+m_2)\cos\theta_1 & l_1^2(m_1+m_2) & l_1 l_2 m_2 \cos(\theta_1-\theta_2) \\ l_2 m_2 \cos\theta_2 & l_1 l_2 m_2 \cos(\theta_1-\theta_2) & l_2^2 m_2 \end{bmatrix}}_{=:M(y)} \underbrace{\begin{bmatrix} \ddot{q} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{=\ddot{y}}$$

$$= \underbrace{\begin{bmatrix} l_1(m_1+m_2)(\dot{\theta}_1)^2\sin\theta_1 + m_2 l_2(\dot{\theta}_2)^2\sin\theta_2 \\ -l_1 l_2 m_2(\dot{\theta}_2)^2\sin(\theta_1-\theta_2) + g(m_1+m_2)l_1\sin\theta_1 \\ l_1 l_2 m_2(\dot{\theta}_1)^2\sin(\theta_1-\theta_2) + g l_2 m_2 \sin\theta_2 \end{bmatrix} - \begin{bmatrix} d_1\dot{q} \\ d_2\dot{\theta}_1 \\ d_3\dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}_{=:w}}_{=:f(y,\dot{y},u,w)}$$

or

$$M(y)\ddot{y} = f(y,\dot{y},u,w). \tag{1}$$

Since the determinate of $M(y)$ is

$$\det M(y) \quad =.\star.= \quad l_1^2 l_2^2 m_2 \left(\underbrace{m m_1}_{>0} + \underbrace{m_1^2\sin^2\theta_1 + m_1 m_2 \sin^2\theta_1 + m m_2 \sin^2(\theta_1-\theta_2)}_{\geq 0} \right) > 0,$$

for all $y \in \mathbb{R}^3$, we conclude that $M(y)$ is invertible. Thus we can rewrite (1) into the form $\ddot{y} = M^{-1}(y)f(y,\dot{y},u,w)$ which with

$$x := \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

and via order reduction gives

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{y} \\ M^{-1}(y)f(y,\dot{y},u,w) \end{bmatrix}}_{=:F(x,u,w)}$$

or in short notation the ODE (control) system

$$\dot{x} = F(x,u,w).$$