Basic Results in Functional Analysis

Prepared by: F.L. Lewis
Updated: Sunday, August 17, 2014

$f(x): X \to Y$ is continuous on X if $\forall x \in X, \varepsilon > 0 \;\; \exists \delta(\varepsilon, x) > 0 \;\; \ni$

$$\|z - x\| < \delta \quad \Rightarrow \quad \|f(z) - f(x)\| < \varepsilon$$

$f(x): X \to Y$ is uniformly continuous on X if it is continuous and $\delta(\varepsilon)$ does not depend on x.

$f(x): X \to Y$ is Lipschitz if $\|f(z) - f(x)\| < \ell \, \|z - x\|$

Locally if for every *x, z* in a compact set, vs. Globally if for all *x, z.*

Derivative of *f(x)* at *x* is

$$f\,'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

f(x) is differentiable at *x* if derivative exists at *x*. Then $f(x) \in C^1$.

f(x) is continuously differentiable at *x* if derivative exists at *x* and is continuous. Then $f(x) \in C^2$.

f(x) is uniformly continuous if it is continuous and its derivative is bounded.

Lipschitz implies unif. cont.

Contin. diff. implies locally Lipschitz

Let *f(x)* be contin diff on *X*. Then globally Lipschitz implies

$|f\,'(x)| \le \ell$, for a const ℓ, $\forall x \in X$

Diff implies cont

Let $\dot{x} = f(x)$

f(x) contin implies exists solution *x(t)*

f(x) contin and Lipschitz implies exists a unique solution *x(t)*

Locally on a compact set, or globally.

f(x) contin diff implies exists a unique solution *x(t)*

Def. An inner product on a linear vector space X with field F is a function $<.,.>: X \times X \to F$ such that for $x, y, z \in X$, $\alpha \in F$

a. $< x, x > \geq 0$

b. $< x, x > = 0 \quad iff \quad x = 0$

c. $< x, \alpha y > = \alpha < x, y >$, homogeneous

d. $< x, y + z > = < x, y > + < x, z >$, linear

e. $< x, y > = \overline{< y, x >}$, complex conjugate

Usually $<.,.>: R^n \times R^n \to R$.

Def. A norm on R^n is a function $\|.\|: R^n \to R$ such that, for $x, y \in R^n$, $\alpha \in R$:

a. $\|x\| \geq 0$

b. $\|x\| = 0 \quad iff \quad x = 0$

c. $\|\alpha x\| = |\alpha| \, \|x\|$, homogeneity

d. $\|x + y\| \leq \|x\| + \|y\|$, triangle inequality

Fact. Every norm is a convex function.

Fact. Every inner product defines a norm

$$\|x\| = < x, x >^{1/2}$$

Def. A seminorm or pseudonorm does not have property b.

Def. A quasinorm has d. replaced by the milder property

$$\|x + y\| \leq K\left(\|x\| + \|y\|\right), \quad K > 1$$

Another Def. A quasinorm has c. replaced by the milder property

$$\|x\| = \|-x\|$$

Def. vector p-norm (Holder norms) for $x \in R^n$ with components $x_i \in R$

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$$

Fact. $\|x\|_1 \geq \|x\|_2 \geq \cdots \geq \|x\|_\infty$

$\|x\|_1 \leq k_2 \|x\|_2 \leq \cdots \leq k_\infty \|x\|_\infty$, for some k_i.

This means all vector norms on R^n are equivalent.

Minkowski inequality or triangle inequality

$$\|x+y\|_p \le \|x\|_p + \|y\|_p$$

Holder's inequality. Let $x, y \in R^n$ and $1 = \frac{1}{p} + \frac{1}{q}$. Then

$$|<x, y>| \le \|x\|_p \, \|y\|_q$$

Cauchy-Schwarz inequality is the special case *p=q=2*

$$|<x, y>| \le \|x\|_2 \, \|y\|_2$$

Sylvester's inequality

$$\sigma_{\min}(A)\|x\|^2 \le x^T A x \le \sigma_{\max}(A)\|x\|^2$$

with σ the singular values of *A*.

Def. Convergence of sequences. A sequence of vectors $\{x_k\} \equiv x_0, x_1, \cdots \in X$ is said to converge to a limit vector *x* if

$$\|x_k - x\| \to 0 \quad as \quad k \to \infty$$

or equivalently, $\forall \in > 0 \; \exists \; N \ni$

$$\|x_k - x\| < \varepsilon \quad \forall \; k \ge N$$

Def. *x* is an accumulation point of sequence $\{x_k\}$ if there is a subsequence of $\{x_k\}$ that converges to *x*. That is, there is an infinite subset *K* of nonnegative integers such that $\{x_k\}|_{k \in K}$ converges to *x*.

Def. A set $S \subset X$ is closed if and only if every convergent sequence with elements in *S* has a limit in *S*.

Def. A sequence $\{x_k\} \in X$ is said to be a Cauchy sequence if

$$\|x_k - x_m\| \to 0 \quad as \quad k, m \to \infty$$

Fact. Every convergent sequence is Cauchy, but not vice versa.

Def. Banach Space. A normed linear space *X* is complete if every Cauchy sequence converges to a vector in *X*. A complete normed linear space is a Banach Space.

Def. A pre-Hilbert Space is a linear space X with an inner product.

Def. Hilbert Space. A linear space *X* with inner product is complete if every Cauchy sequence converges to a vector in *X*. A complete linear space with inner product is a Hilbert Space.

Fact. A Hilbert space has an inner product, hence a norm, hence is a Banach Space.

Fact. $X = R^n$ is a Hilbert Space with inner product $< x.y >= x^T y, \quad x, y \in R^n$. The associated norm is $\|x\| = (x^T x)^{1/2}$, the L_2 vector norm.

Fact. A bounded sequence $\{x_k\}$ in R^n has at least one accumulation point in R^n.

Fact. A sequence of real numbers $\{r_k\}$ which is monotonically nondecreasing and bounded from above, i.e.

$$r_k \le r_{k+1} < R, \quad R = const$$

converges to a real number

Fact. A sequence of real numbers $\{r_k\}$ which is monotonically nonincreasing and bounded from below, i.e.

$$r_k \ge r_{k+1} > R, \quad R = const$$

converges to a real number

Contraction Mapping. Let *S* be a closed subset of a Banach space *X* and let *T* be a mapping that maps *S* into *S*. Suppose that

$$\|T(x) - T(y)\| \le \rho \|x - y\|, \quad \forall x, y \in S, \quad 0 < \rho < 1$$

Then

a. There exists a unique vector $x^* \in S$ such that $T(x^*) = x^*$. (fixed point)

b. x^* can be obtained by the method of successive approximation starting from any initial point in *S*.

Fact. Let $f(t): R^+ \to R^n$. Then

$$\left\| \int f(t)\, dt \right\| \le \int \|f(t)\|\, dt$$

This is actually the triangle inequality, for note that the Lebesgue integral is

$$\int_0^b f(t)\, dt = \lim_{T \to 0} \sum_{k=0}^{b/T} f(kT)T$$

So that

$$\left\| \int_0^b f(t)\, dt \right\| = \lim_{T \to 0} \left\| \sum_{k=0}^{b/T} f(kT)T \right\| \le \lim_{T \to 0} \sum_{k=0}^{b/T} \|f(kT)\| T$$

L_2 inner product.

$$< f, g >_p = \int_D f^T(\mu) g(\mu)\, d\mu$$

μ can denote time t and D a time interval, or can take $\mu = x$ and integrate over a region $D \subset R^n$.

Def. L_p norm (also denoted L_p^n norm) of function $f(x): R^n \to Y$

$$\| f(.) \|_p = \left(\int_D \| f(\mu) \|_p^{\ p} \, d\mu \right)^{1/p}, \quad \text{also denoted as } |||\, f \,|||_p .$$

μ can denote time t and D a time interval, or can take $\mu = x$ and integrate over a region $D \subset R^n$.

Def. Uniform function norm, or supremum norm, or L_∞ norm, or Chebyshev norm

$$\| f \|_\infty = \sup \left\{ \| f(x) \| : x \in R^n \right\}$$

Def. f is said to belong to L_p if $\| f(.) \|_p$ is bounded.

Def. The Lebesque normed space L_p (also denoted L_p^n) is

$$L_p = \{ f(.) \in Y : \| f(.) \|_p < \infty \}$$

If time integral, then L_2 inner product of $f(t), g(t) : R^+ \to R^n$ is

$$< f, g >_p = \int_0^\infty f^T(t) g(t) \, dt$$

and L_p norm of function $f(t) : R \to R^n$ is

$$\| f(.) \|_p = \left(\int_0^\infty \| f(t) \|_p^{\ p} \, dt \right)^{1/p},$$

If over time

$$\| f \|_\infty = \sup \left\{ \| f(t) \| : 0 \le t \right\}$$

Define inner product

$$< f, g >_{p,T} = \int_0^T f^T(t) g(t) \, dt$$

and norm

$$\| f(.) \|_{p,T} = \left(\int_0^T \| f(t) \|_p^{\ p} \, dt \right)^{1/p}$$

Def. The extended Lebesque normed space L_{pe} (also denoted L_{pe}^n) is

$$L_{pe} = \{ f(.) \in Y : \| f(.) \|_p < \infty, \forall T > 0 \}$$

Mean Value Theorem. Let $f(x): R^n \to R^m$ be differentiable at each point $x \in \Omega \subset R^n$, Ω an open set. Let $x, y \in \Omega$ with the line segment $L(x, y) \subset \Omega$. Then there exists a point z of $L(x, y)$ such that

$$\left.\frac{\partial f}{\partial x}\right|_{x=z} = \frac{f(y) - f(x)}{y - x}$$

Implicit Function Theorem. Let $f(x, y): R^n \times R^m \to R^n$ be continuously differentiable at each point $(x, y) \in \Omega \subset R^n \times R^m$, Ω an open set. Let $(x_0, y_0) \in \Omega$ such that $f(x_0, y_0) = 0$ and Jacobian matrix $\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}$ is nonsingular.

Then there exist neighborhoods $U \subset R^n$ of x_0 and $V \subset R^m$ of y_0 such that $\forall y \in V$ the equation $f(x, y) = 0$ has a unique solution $x \in U$. Moreover the solution can be written as

$$x = g(y)$$

where $g(.)$ is continuously differentiable at $y=y_0$.

Leibniz Formula.

If $\phi(t) = \int_{\alpha(t)}^{\beta(t)} f(x,t)\, dt$

Then $\frac{d}{dt}\phi(t) = \phi'(t) = f(\beta(t), t,)\beta'(t) - f(\alpha(t), t,)\alpha'(t) + \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t} f(x,t)\, dt$

Bellman-Gronwall Lemma. Let there be continuous functions $\alpha, x: R \to R$, and a continuous nonnegative function $\beta: R \to R$. If

$$x(t) \le \alpha(t) + \int_{t_0}^{t} \beta(s)x(s)\, ds, \quad t \ge t_0$$

Then

$$x(t) \le \alpha(t) + \int_{t_0}^{t} \alpha(s)\beta(s)e^{\int_s^t \beta(\tau)d\tau}\, ds, \quad t \ge t_0$$

Also, if α is a const then

$$x(t) \le \alpha e^{\int_{t_0}^t \beta(\tau)d\tau}, \quad t \ge t_0$$

If $\beta > 0$ is a const then

$$x(t) \le \alpha e^{\beta(t-t_0)}, \quad t \ge t_0 \qquad \text{(Gronwall Lemma)}$$

Jensen's Inequality. Given matrix $P > 0$, scalars $b > a \geq 0$, and $x \in R^n$

$$\int_{t-b}^{t-a} \dot{x}^T(\tau) P \dot{x}(\tau)\, d\tau \geq \frac{1}{b-a}\left(x(t-a) - x(t-b)\right)^T P \left(x(t-a) - x(t-b)\right)$$