Linearization About a Nominal Trajectory

The equations of motion of a satellite in a planar orbit about a point mass M are:

$$\ddot{r} = r\dot{\theta}^2 - \frac{\mu}{r^2} + T\sin\phi$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \frac{T}{r}\cos\phi$$

where:

r= radial distance from mass M

θ= angle from a fixed reference point in the orbit

μ= GM, the gravitational constant of the attracting mass M

and the control inputs are:

T= thrust

ϕ= thrust angle.

Take the state as $x = \begin{bmatrix} r & \dot{r} & \theta & \dot{\theta} \end{bmatrix}^T$, and the control input vector as $u = \begin{bmatrix} \phi & T \end{bmatrix}^T$.

a. Write the linearized state equation $\dot{x} = Ax + Bu$. That is, find the Jacobians A(x) and B(x) as functions of the state.

b. Systems can be linearized about a nominal trajectory as well as a nominal equilibrium point. One solution of the satellite equations is a circular orbit, which has values of:

$r = r_0, \quad \dot{r} = 0, \quad \dot{\theta} = \omega, \quad T = 0, \quad and \quad \mu = r_0{}^3\omega^2$ (this is Kepler's third law).

The constant angular velocity is equal to ω, and the constant radius of the circular orbit is r_0. Assume that ϕ= 90° at nominal.

Linearize the system about this nominal trajectory. That is, evaluate the Jacobians from part a. about nominal.

Solution

The equations of motion of satellite in a planar orbit are:

$$\frac{d\dot{r}}{dt} = \ddot{r} = r\dot{\theta}^2 - \frac{\mu}{r^2} + 2T\sin\phi$$

$$\frac{d\dot{\theta}}{dt} = \ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r} + \frac{T}{2r}\cos\phi$$

State vector $\mathbf{x} = [r \ \ \dot{r} \ \ \theta \ \ \dot{\theta}]^T$

Control input vector $\mathbf{u} = [\phi \ \ T]^T$.

a. Linearized state equation : $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \frac{d\mathbf{x}}{dt} = \frac{d}{dt}\begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{r} \\ r\dot{\theta}^2 - \dfrac{\mu}{r^2} + 2T\sin\phi \\ \dot{\theta} \\ \dfrac{-2\dot{r}\dot{\theta}}{r} + \dfrac{T}{2r}\cos\phi \end{bmatrix} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} = f(\mathbf{x},\mathbf{u})$$

$$\mathbf{A} = \frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{\partial f}{\partial r} & \dfrac{\partial f}{\partial \dot{r}} & \dfrac{\partial f}{\partial \theta} & \dfrac{\partial f}{\partial \dot{\theta}} \end{bmatrix},$$

Therefore,

$$\mathbf{A} = \frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} \dfrac{\partial \dot{x}_1}{\partial r} & \dfrac{\partial \dot{x}_1}{\partial \dot{r}} & \dfrac{\partial \dot{x}_1}{\partial \theta} & \dfrac{\partial \dot{x}_1}{\partial \dot{\theta}} \\ \dfrac{\partial \dot{x}_2}{\partial r} & \dfrac{\partial \dot{x}_2}{\partial \dot{r}} & \dfrac{\partial \dot{x}_2}{\partial \theta} & \dfrac{\partial \dot{x}_2}{\partial \dot{\theta}} \\ \dfrac{\partial \dot{x}_3}{\partial r} & \dfrac{\partial \dot{x}_3}{\partial \dot{r}} & \dfrac{\partial \dot{x}_3}{\partial \theta} & \dfrac{\partial \dot{x}_3}{\partial \dot{\theta}} \\ \dfrac{\partial \dot{x}_4}{\partial r} & \dfrac{\partial \dot{x}_4}{\partial \dot{r}} & \dfrac{\partial \dot{x}_4}{\partial \theta} & \dfrac{\partial \dot{x}_4}{\partial \dot{\theta}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dot{\theta}^2 + \dfrac{2\mu}{r^3} & 0 & 0 & 2r\dot{\theta} \\ 0 & 0 & 0 & 1 \\ \dfrac{2\dot{r}\dot{\theta}}{r^2} - \dfrac{T\cos\phi}{2r^2} & \dfrac{-2\dot{\theta}}{r} & 0 & \dfrac{-2\dot{r}}{r} \end{bmatrix}$$

$$\mathbf{B} = \frac{\partial f}{\partial \mathbf{u}} = \begin{bmatrix} \dfrac{\partial f}{\partial \phi} & \dfrac{\partial f}{\partial T} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2T\cos\phi & 2\sin\phi \\ 0 & 0 \\ -\dfrac{T}{2r}\sin\phi & \dfrac{\cos\phi}{2r} \end{bmatrix}$$

b. At the nominal trajectory $r = r_0$, $\dot{r} = 0$, $\dot{\theta} = \omega$, $T = 0$, $\mu = r_0^3\omega^2$, and $\phi = 90^\circ$.
By substituting these values to above **A** and **B** matrices, we have

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega r_0 \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{2\omega}{r_0} & 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$