Linearization & e.p. example

$\dot{x} = y(1+x-y^2)$
$\dot{y} = x(1+y-x^2)$

1) Linearize

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \equiv \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y + yx - y^3 \\ x + xy - x^3 \end{bmatrix} \equiv f(x,y) = f(X)$$

$$\frac{\partial f}{\partial X} = \begin{bmatrix} \frac{\partial f}{\partial X_1} & \frac{\partial f}{\partial X_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} y & 1+x-3y^2 \\ 1+y-3x^2 & x \end{bmatrix} = \begin{bmatrix} X_2 & 1+X_1-3X_2^2 \\ 1+X_2-3X_1^2 & X_1 \end{bmatrix}$$

2) e.p. s

$\dot{x} = 0$
$\dot{y} = 0$

i) $x=0, y=0$ so $(0,0)$

ii) If $x=0$ then need $0 = 1+x-y^2 = 1-y^2$
or $y^2 = 1$ or $y = \pm 1$
so $(0,1)$
$(0,-1)$

iii) If $y=0$ then need $0 = 1+y-x^2 = 1-x^2$
or $x^2 = 1$ or $x = \pm 1$
so $(1,0)$
$(-1,0)$

iv) if $x \neq 0, y \neq 0$ then

① $1+x-y^2=0$ or $x = y^2-1$

② $0=1+y-x^2 = 1+y-(y^2-1)^2 = 1+y-(y^4-2y^2+1)$

$0 = y^4-2y^2-y = (y^3-2y-1)y$

$y = 0, -1, 1.618, -0.618$

if $\underline{y=0}$, $x = \pm 1$

If $\underline{y=-1}$ ① $\Rightarrow x=0$

② $\Rightarrow x^2=0$

If $\underline{y=1.618}$ ① $\Rightarrow x=1.618$

② $\Rightarrow x^2 = 2.618$, $x = \pm 1.618$

If $\underline{y=-0.618}$, ① $\Rightarrow x = 0.618$

② $\Rightarrow x^2 = 0.3820$, $x = \pm 0.618$

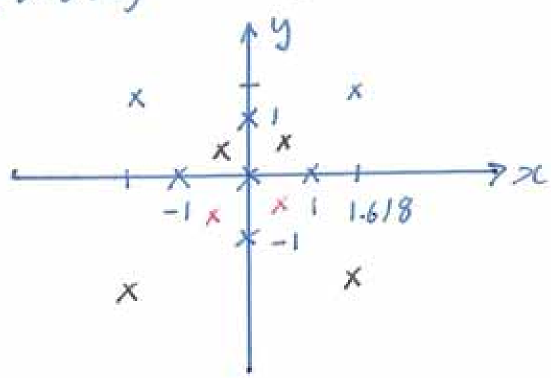
$\underline{so}$ $(1.618, 1.618)$

$(-1.618, 1.618)$

$(0.618, -0.618)$ x

$(-0.618, -0.618)$ x

also symmetric pts. ?



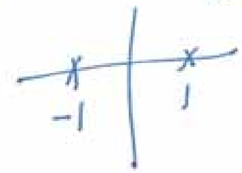
3) <u>nature of e.p.s</u> – Slotine & Li p. 33

<u>(0,0)</u> . lin. $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A$

$\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$ abt (0,0)

$\Delta(s) = |sI - A| = \begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix} = s^2 - 1 = 0$ (5)

$s^2 = 1, \quad s = \pm 1$

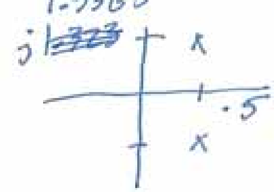


saddle point

<u>(0,1)</u> lin. $\left.\frac{\partial f}{\partial x}\right|_{(0,1)} = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix} = A$

$\Delta(s) = |sI - A| = \begin{vmatrix} s-1 & 2 \\ -2 & s \end{vmatrix} = s^2 - s + 4$ (5)

$s = \frac{1}{2} \pm j\,\cancel{1.323}\ 1.9365$



unstable focus

Fibonacci Numbers and the Golden Mean

The Fibonacci numbers are generated using the DT system

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k = Ax_k + Bu_k$$
$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \quad = Cx_k$$

Iterative Solution

Given an initial condition $x_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and an input sequence u_k, it is very easy to compute the state and output sequences of a DT system using simple iteration. Set x_0= 1 and take zero input u_k= 0. Then one has

$$x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad y_0 = Cx_0 = 0$$
$$x_1 = Ax_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad y_1 = Cx_1 = 1$$
$$x_2 = Ax_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad y_2 = Cx_2 = 1$$
$$x_3 = Ax_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad y_3 = Cx_3 = 2$$
$$x_4 = Ax_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad y_4 = Cx_4 = 3$$
$$x_5 = Ax_4 = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \qquad y_5 = Cx_5 = 5$$

Thus the system generates the sequence y_k equal to 1,1,2,3,5,8,13,… where each number is the sum of the previous two numbers.

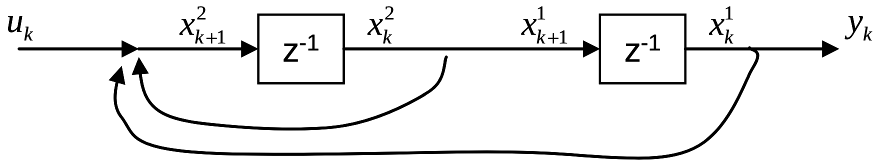
The Fibonacci numbers often appear in nature: The leaf patterns of many types of plants occur in bunches of only 1,2,3,5,8,… The spirals of certain types of sea shells appear only in groups reflecting the Fibonacci numbers.

System Structure and Block Diagram

Note that the first row of the A matrix is $\begin{bmatrix} 0 & 1 \end{bmatrix}$, so that the first state is just the previous value of the second state. The second row of the A matrix is $\begin{bmatrix} 1 & 1 \end{bmatrix}$, so the second state is the sum of the previous values of the first and second states. In fact, writing the state in terms of its components as $x_k = \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix}$, the state equations become

$$x_{k+1}^1 = x_k^2$$
$$x_{k+1}^2 = x_k^1 + x_k^2 + u_k$$
$$y_k = x_k^1$$

A block diagram of this system is shown. Note that the delay elements store previous values of the states and act as a memory device. This is in fact a shift register of length *n=2*, with *n* the number of states.



Block Diagram of Fibonacci System

Poles and The Golden Mean

The poles of the Fibonacci system are intriguing. The characteristic equation is

$$\Delta(z) = |zI - A| = \begin{vmatrix} z & -1 \\ -1 & z-1 \end{vmatrix} = z^2 - z - 1$$

which has roots $z = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = 1.618, 0.618$. The number $z = \frac{1+\sqrt{5}}{2} = \frac{3.236}{2} = 1.618$ is known as the GOLDEN MEAN. This ratio was used by the Greeks in their architecture, and was viewed as the perfect ratio between the width and height of Greek temple fronts.

Eigenvectors + Directions in Phase Plane

$\dot{x} = Ax$

$J = M^{-1}AM$ Jordan Form

$A = MJM^{-1}$

$$[v_1\ v_2 \cdots v_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix}$$

right eigenvectors

left eigenvectors

a) right e-vectors

$AM = MJ$

$$A[v_1\ v_2 \cdots v_n] = [v_1\ v_2 \cdots v_n] \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$$

$Av_i = v_i \lambda_i$

$(A - \lambda_i I)v_i = 0$

b) left e-vectors

$M^{-1}A = JM^{-1}$

$$\begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \end{bmatrix} A = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \end{bmatrix}$$

$w_i^T A = \lambda_i w_i^T$

$w_i^T (A - \lambda_i I) = 0$

c) Modal Decomposition

$$e^{At} = M e^{Jt} M^{-1}$$
$$= [v_1 \; v_2 \; \cdots] \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \end{bmatrix}$$
$$= \sum_{i=1}^{n} v_i e^{\lambda_i t} w_i^T$$

$$\dot{x} = Ax \;, \quad x(0)$$
$$x(t) = e^{At} x(0)$$
$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} v_i (w_i^T x(0))$$

d) 2-D phase plane

$$x(t) = e^{\lambda_1 t} v_1 (w_1^T x(0)) + e^{\lambda_2 t} v_2 (w_2^T x(0))$$