Eigenvectors + Directions in Phase Plane

$\dot{x} = Ax$

$J = M^{-1} A M$ Jordan Form

$A = M J M^{-1}$

$$[v_1\ v_2 \cdots v_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix}$$

right eigenvectors

left eigenvectors

a) right e-vectors

$AM = MJ$

$$A[v_1\ v_2 \cdots v_n] = [v_1\ v_2 \cdots v_n] \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$$

$Av_i = v_i \lambda_i$

$(A - \lambda_i I) v_i = 0$

b) left e-vectors

$M^{-1} A = J M^{-1}$

$$\begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \end{bmatrix} A = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \end{bmatrix}$$

$w_i^T A = \lambda_i w_i^T$

$w_i^T (A - \lambda_i I) = 0$

c) <u>Modal Decomposition</u>

$$e^{At} = M e^{Jt} M^{-1}$$

$$= \begin{bmatrix} v_1 & v_2 & \cdots \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \end{bmatrix}$$

$$= \sum_{i=1}^{n} v_i e^{\lambda_i t} w_i^T$$

$$\dot{x} = Ax \quad , \; x(0)$$

$$x(t) = e^{At} x(0)$$

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} v_i \left(w_i^T x(0) \right)$$

d) <u>2-D phase plane</u>

$$x(t) = e^{\lambda_1 t} v_1 \left(w_1^T x(0) \right) + e^{\lambda_2 t} v_2 \left(w_2^T x(0) \right)$$