S+L p. 33

First Integral of Motion

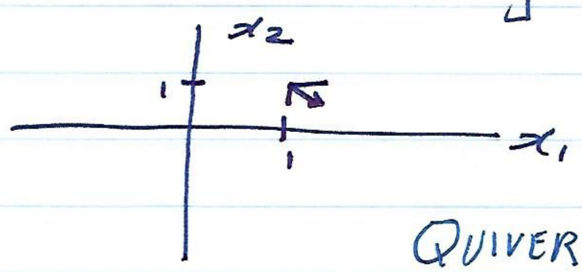
Verhulst p. 10, 16

$\dot{x} = f(x)$, Flow of Vector Field

ex! $\ddot{x} + x = 0$, $\ddot{x} = -x$

$$\frac{d}{dt} x = \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = f(x)$$

$$f(1,1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



QUIVER

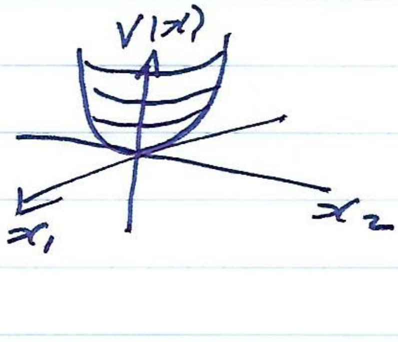
Orbital Derivative

Def FIM

1) Let $V(x) : R^n \Rightarrow R$
2) Let $\dot{V}(x) = 0$

$$\dot{V} = \frac{\partial V^T}{\partial x} \dot{x} = \frac{\partial V^T}{\partial x} f(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \end{bmatrix}$$

$$\dot{V} = \left\| \frac{\partial V}{\partial x} \right\| \, \| f(x) \| \cos \measuredangle$$



ex1 $\ddot{x} + x = 0$, $\ddot{x} = -x$

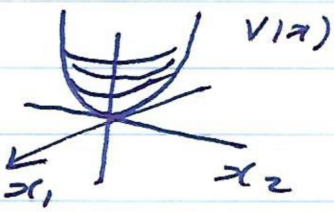
$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$$

$$\dot{x}\ddot{x} + \dot{x}x = 0$$

$$= \frac{d}{dt}\left(\frac{\dot{x}^2}{2} + \frac{x^2}{2} \right) = 0$$

so $V(x,\dot{x}) = \left(\frac{\dot{x}^2}{2} + \frac{x^2}{2} \right) = \left(\frac{x_1^2}{2} + \frac{x_2^2}{2} \right)$

is a FIM

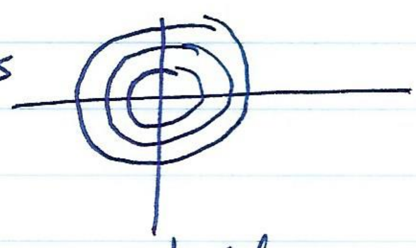


Double check

$$\dot{V} = \frac{\partial V^T}{\partial x} f(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = 0$$

Set of level sets



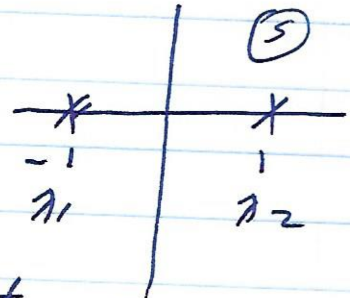
Family of curves

Integral Manifold

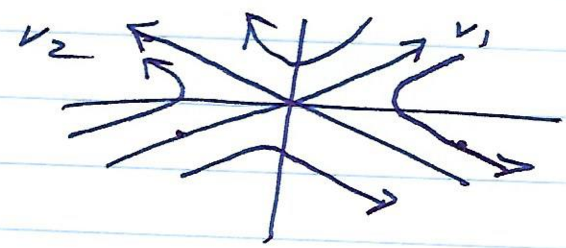
ex2 $\quad \ddot{x} - x = 0$

$s^2 - 1 = 0\ , \ s^2 = 1$

$s = \pm 1$



$$x(t) = v_1 e^{-t} w_1^T x(0) + v_2 e^{t} w_2^T x(0)$$

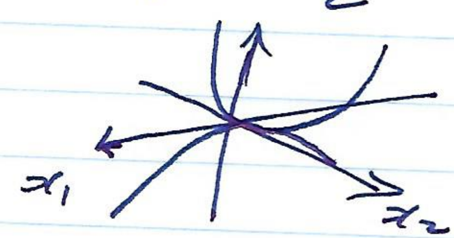


FIM

$$\dot{x}\ddot{x} - \dot{x}x = 0$$

$$= \frac{d}{dt}\left(\frac{\dot{x}^2}{2} - \frac{x^2}{2}\right) = 0$$

FIM $\quad V = \frac{\dot{x}^2}{2} - \frac{x^2}{2} = \frac{x_2^2}{2} - \frac{x_1^2}{2}$



saddle pt

ex 3 $\ddot{x} + f(x) = 0$ FIM

$$\dot{x}\ddot{x} + \dot{x} f(x) = 0$$

$$\frac{d}{dt}\left(\frac{\dot{x}^2}{2} + \int_0^x f(z)\,dz \right)$$

$$V = \frac{\dot{x}^2}{2} + \int_0^x f(z)\,dz$$

KE PE

ex 4 Hamilton's Eqs.

$$x = \begin{vmatrix} p \\ q \end{vmatrix} = \begin{bmatrix} \text{position} \\ \text{generalized momentum} \end{bmatrix}$$

$$\dot{x} = \left.\begin{bmatrix} \dot{p} = -\frac{\partial H}{\partial q} \\ \dot{q} = \frac{\partial H}{\partial p} \end{bmatrix}\right\} \text{Ham. eqs. of motion}$$

Hamiltonian $H(p, q)$, $x = \begin{bmatrix} p \\ q \end{bmatrix}$

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} = \begin{bmatrix} \frac{\partial H}{\partial p} & \frac{\partial H}{\partial q} \end{bmatrix} \begin{bmatrix} -\frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} = 0$$

Hamiltonian is First Integral of Motion