Toward Real-Time State Estimation and Tracking of Elastic Rods

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***Abstract*—This is abstract**

I. Introduction

This is the intro

A. Related Work

This is related

B. Objectives

The objective of this article is to present a concise, intuitive and transferable the solution to a classical optimal control problem. Our objective is to model the plant using Lagrange's equations of motion, derive state dynamics, obtain a generalized cost function, and to optimally control the double pendulum by reaching a stable configuration in the upright position while given minim-energy control input to the system.
Also — testing
Also —

C. Article Structure

II. Estimation Formulation

There are many benefits to using the *state space* framework, notably that is well generalized and conveniently implemented on modern computers.

A. System Setup

Consider a resting cart on a flat rigid surface along the x-axis, on top of which are stacked two inverted pendulums, figure 01. The pendulums resemble a robotic arm but rather without a motor or external torque input. At the end of each pendulum is assumed to be a point-mass, m_1, m_2. Each pendulum moves freely and independently and is connected by lengths l_1 and l_2 from their joints, q_1 and q_2 to their point-masses, m_1, m_2. The mass of the cart is considered point-mass as well and is denoted by m_c with its position q_c along the x-axis. θ_1 and θ_2 denote angular deviations from upright position for q_1 and q_2, respectively.

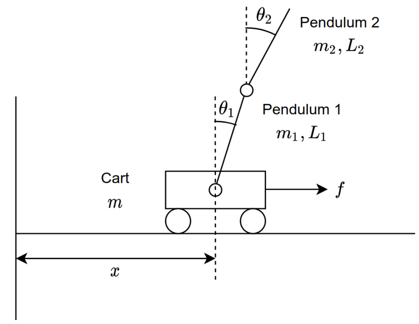

Fig. 1. Physical System

B. State Definition

Suppose we have a continuous-time, nonlinear system presented in state space form.

$$\dot{x} = f(x, u), \tag{1}$$

$$y = h(x). \tag{2}$$

whats f - prediction model whats h - obs model omit obs noise, it requires EKF??

Our goal is to stabilize the described highly nonlinear system in the upright configuration

C. Noise and Friction

D. Global Coordinate Frame

We need to form a homogenous configuration space to present state variables q, θ_1, θ_2 by forming a global coordinate frame with $x(t = 0)$ as the origin.

$$\begin{gathered} q_c = \begin{bmatrix} q \\ 0 \end{bmatrix}, \quad q_1 = \begin{bmatrix} q + l_1 \sin\theta_1 \\ l_1 \cos\theta_1 \end{bmatrix}, \\ q_2 = \begin{bmatrix} q + l_1 \sin\theta_1 + l_2 \sin\theta_2 \\ l_1 \cos\theta_1 + l_2 \cos\theta_2 \end{bmatrix}, \end{gathered} \tag{3}$$

We denote position of point-masses for m_c, m_1, and m_2 by $q_c, q_1, q_2 \in R^2$ on the x-y plane. We seek to define our system in terms of these state variables. Thus, we define state variable vector $q(t)$ as

$$q = [q_c, q_1, q_2]^T \tag{4}$$

In later section, we formally rewrite state variable definitions independent of time t via a linearization process. For simpler representation, we denote state variable vector as $q := q(t)$.

E. Other Considerations

Noise and Friction modeling, process noise versus observation noise.

III. Model Derivation

A. Equations of Motion

As it is laid out in *Lagrangian mechanics*; first we must obtain the Lagrangian of the physical model, $\mathcal{L} = K - P$. K and P represent the *kinetic* and *potential* energies of the system. K is the total kinetic energy from all three masses and in later sections, we will discuss transforming *Lagrangian mechanics* to *Hamiltonian mechanics* by replacing velocities with *momenta*. Here, we define K as

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m_c\|\dot{q_c}\|^2 + m_1\|\dot{q_1}\|^2 + m_2\|\dot{q_2}\|^2. \quad (5)$$

P denotes the overall potential energy of the system; but since the cart rolls flat along the x axis, it is inherently incapable of storing energy so we discard it. P is obtained by

$$P = g\left[m_1 l_1 \cos\theta_1 + m_2(l_1\cos\theta_1 + l_2 cos\theta_2)\right] \quad (6)$$

Where $g \in R$ denotes gravitational acceleration. Next, we envoke the *Euler-Lagrange* equation to write the following ODE system whose solution is the position and velocity configuration vector of the cart and masses on the x-y plane, given the final time, previous state, system dynamics, control input and disturbances.

$$y(t) = f(t, q, \dot{q}, u, \omega) \quad (7)$$

The Lagrange Equations of Motions are defined as

$$Q = \frac{\partial\mathcal{L}}{\partial f_i} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial f_i}\right). \quad (8)$$

Where Q represents the vector of sum of external forces acting on the plan.

$$\mathcal{L} = \mathcal{L}(t, q, \dot{q}) \quad (9)$$

$$\left\{\begin{array}{rcl} u + \omega_1 - d_1\dot{q} & = & \frac{d}{dt}\{\frac{\partial\mathcal{L}}{\partial\dot{q}}\} - \{\frac{\partial\mathcal{L}}{\partial q}\} \\ \omega_2 - d_2\dot{\theta}_1 & = & \frac{d}{dt}\{\frac{\partial\mathcal{L}}{\partial\dot{\theta_1}}\} - \{\frac{\partial\mathcal{L}}{\partial\theta_1}\} \\ \omega_3 - d_3\dot{\theta}_2 & = & \frac{d}{dt}\{\frac{\partial\mathcal{L}}{\partial\dot{\theta_2}}\} - \{\frac{\partial\mathcal{L}}{\partial\theta_2}\} \end{array}\right. \quad (10)$$

So far our derivation has made our expressions more complicated but these steps are neccessary for capturing the full dynamical characteristics of the system. In the following section, we derive the continuous-time nonlinear model of described system. Moreover in *Optimal Control* section, we briefly explain how one could derive *the general optimal control solution* for a *positive definite and semidefinite energy system* with known characteristic parameters. Moreover, we intuitively explain and reference why we could consider the solution space to such a ODE as *smooth manifold* that maps current and future state through recursion and its energy state decays towards a local minima.

B. Continues-Time, Nonlinear Model

We used MATLAB Symbolic Math Toolbox to derive the resultant ODE. Obtaining symbolic or implicit expressions of our ODE will allow us to analytically derive to the optimal solution.

$$\begin{array}{rcl} u + \omega_1 - d_1\dot{q} & = & (m_c + m_1 + m_2)\ddot{q} + l_1(m_1+m_2)\ddot{\theta_1}\cos\theta_1 \\ & & +m_2 l_2\ddot{\theta_2}\cos\theta_2 - l_1(m_1+m_2)\dot{\theta_1}^2\sin\theta_1 \\ & & -m_2 l_2\dot{\theta_2}^2\sin\theta_2 \\ \omega_2 - d_2\dot{\theta_1} & = & l_1(m_1+m_2)\ddot{(q)}\cos\theta_1 + l_1^2(m_1+m_2)\ddot{\theta_1} \\ & & +l_1 l_2 m_2\ddot{\theta_2}\cos(\theta_1-\theta_2) + l_1 l_2 m_2\dot{\theta_2}^2\sin(\theta_1-\theta_2) \\ & & -g(m_1+m_2)l_1\sin\theta_1 \\ \omega_3 - d_3\dot{\theta_2} & = & l_2 m_2\ddot{q}\cos\theta_2 + l_1 l_2 m_2\ddot{\theta_1}\cos(\theta_1-\theta_2) + l_2^2 m_2\ddot{\theta_2} \\ & & -l_1 l_2 m_2\dot{\theta_1}^2\sin(\theta_1-\theta_2) - l_2 m_2 g\sin\theta_2 \end{array} \quad (11)$$

C. Optimal Control

D. Linearization

First, we rewrite the equations of motion into matrix form to batch acceleration, velocity, and position terms into separate matrices. A second order representation of the physical system is sufficient to provide an accurate enough estimate, per *Taylor Series Linearization* method.

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Hu \quad (12)$$

where matrices , $D(q) := D(q(t))$

E. Discrete, Linear Model

F. General Time-Invariant Cost Function

G. Linear Quadratic Regulator Solution

IV. Simulations and Results

V. Conclusion

The conclusion goes here.

Acknowledgment

The authors would like to thank...

References

[1] H. Kopka and P. W. Daly, *A Guide to LaTeX*, 3rd ed. Harlow, England: Addison-Wesley, 1999.