

**EE 5323- Take Home Exam 2**

Fall 2021

This exam has 6 pages in all. There are 4 problems.

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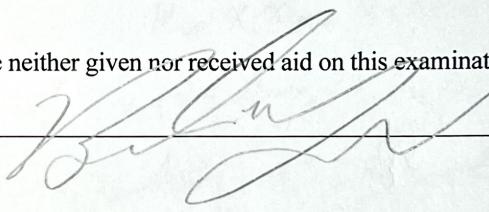
Almost all questions require numerical calculations to arrive at the answers. To obtain full credit, show all your work. No partial credit will be given without the supporting work.

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Pledge of honor:

"On my honor I have neither given nor received aid on this examination."

Signature: 

### 1. Lyapunov Function

Use Lyapunov function to examine the stability of the following systems. Be clear and show all steps.

a.

$$\begin{aligned} \dot{x}_1 &= x_2 \sin x_1 - x_1 \\ \dot{x}_2 &= -x_1 \sin x_1 - x_2 \end{aligned}$$

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) > 0$$

$$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2 = 0 \Rightarrow$$

$$x_1(x_2 \sin x_1 - x_1) + x_2(-x_1 \sin x_1 - x_2) = 0 \Rightarrow$$

$$x_1 x_2 \cancel{\sin x_1 - x_1^2} - x_1 x_2 \cancel{\sin x_1 - x_2^2} = 0 \Rightarrow -(x_1^2 + x_2^2) = 0$$

ep. (0,0) only

$$\dot{V} \leq 0 \quad \text{thus } \underline{\text{GDS}}$$

b.

$$\begin{aligned} \dot{x}_1 &= x_2 \sin x_1 - x_1 \\ \dot{x}_2 &= -x_1 \sin x_1 \end{aligned}$$

$$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2 = 0 \Rightarrow$$

$$x_1(x_2 \sin x_1 - x_1) + x_2(-x_1 \sin x_1) = 0$$

$$x_1 x_2 \cancel{\sin x_1 - x_1^2} - x_1 x_2 \cancel{\sin x_1} = 0$$

$$\dot{V} = -x_1^2 ; \quad \dot{V} = 0 ; \quad x_1 = 0$$

$$\dot{V} \leq 0 \quad \checkmark$$

SIDL  
marginally stable  
because it only depends  
on  $x_1$

## 2. LaSalle's Extension

$$\ddot{x} + k_1 \dot{x} + k_2 \dot{x}^3 + k_3 x^5 = 0 \Rightarrow \ddot{x} = -k_1 \dot{x} - k_2 \dot{x}^3 - k_3 x^5$$

- a. Use Lyapunov to check the stability. Hint: Use the energy as the Lyapunov function.  
Take the potential energy as

$$PE = \int_0^x (k_2 \dot{x}^3 + k_3 x^5) dx \quad V = \frac{1}{2} \dot{x}^2 + \int_0^x C(y) dy = R + U;$$

$$V = \frac{1}{2} \dot{x}^2 + \int_0^x C(y) dy \Rightarrow \begin{cases} \text{Per Leibniz's theorem: } \frac{d}{dt} \int_a^{\beta} F(x, t) dx = \\ = \beta \dot{F}(\beta, t) - \alpha \dot{F}(\alpha, t) + \int_{\alpha}^{\beta} \frac{d}{dt} F(x, t) dx \end{cases}$$

$$\Rightarrow \dot{x}(-k_1 \dot{x} - k_2 \dot{x}^3 - k_3 x^5) + \dot{x}(k_2 \dot{x}^3 + k_3 x^5) - (0) + (0+0) \Big|_{\dot{x}=0}$$

$$\Rightarrow V' = -k_1 \dot{x}^2 \leq 0 \text{ if } k_1 > 0;$$

$$V=0 \Rightarrow \dot{x} \rightarrow 0 \text{ or } k_1 = 0 \quad \text{SISL}$$

hints at use of LaSalle's extension  
we plug this in dynamics equation.

- b. Use LaSalle's extension to find a stronger type of stability for the system.

if  $\dot{x} \rightarrow 0 \Rightarrow \ddot{x} \rightarrow 0$ ; plug in sys. dynamic

$$\ddot{x} + k_1 \dot{x} + k_2 \dot{x}^3 + k_3 x^5 = 0 \Rightarrow k_3 x^5 = 0 \Rightarrow x = 0$$

$\Rightarrow$  thus the system is dissipative  
it reaches equilibrium at  $x=0$ ; thus it is  
considered Global Asymptotic Stable (GAS)

### 3. Lyapunov Equation for Linear Systems

Use Lyapunov Equation to check the stability of the linear systems

a.  $\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ 0 & -6 \end{bmatrix}x$

not stable  
underdamped

$$A^T P + PA = -Q \Rightarrow \begin{bmatrix} a_1 & a_3 \\ a_2 & a_3 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} = -Q$$

$$\begin{bmatrix} a_1 P_1 + a_3 P_3 + a_1 P_1 + a_3 P_2 & a_1 P_2 + a_3 P_3 + a_2 P_1 + a_4 P_2 \\ a_2 P_1 + a_4 P_2 + a_1 P_2 + a_3 P_3 & a_2 P_2 + a_4 P_3 + a_2 P_2 + a_4 P_3 \end{bmatrix} = -Q;$$

where I test the following  
Q matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \dots$$

I a unique solution  
DNE

b.  $\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}x$

stable w/

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underline{\underline{m_{11} = .83}} \\ \underline{\underline{m_{22} = .1667}}$$

Some  
A<sub>x,y</sub> matrix  
from 3.a.

$$\left[ \begin{array}{c} \\ \\ \end{array} \right] = -Q \quad \left[ \begin{array}{cc} .83 & -.5 \\ -.5 & .5 \end{array} \right]$$

$$m_{22} = (.83 \cdot .5) - (.25)$$

#### 4. UUB

Use Lyapunov to show that the system

$$\dot{x}_1 = x_1 x_2^2 - x_1(x_1^2 + x_2^2 - 3)$$

$$\dot{x}_2 = -x_1^2 x_2 - x_2(x_1^2 + x_2^2 - 3)$$

is uniformly ultimately bounded UUB. That is, show that the Lyapunov derivative is NEGATIVE OUTSIDE A BOUNDED REGION. Find the radius of the bounded region outside which  $\dot{V} < 0$ . Any states outside this region are attracted towards the origin.

$$V(x_1, x_2) = \frac{1}{2} (x_1^2 + x_2^2 - 3)^2 > 0 \Rightarrow V = \frac{1}{2} (x_1^2 + x_2^2 - 3)(2x_1 \dot{x}_1 + 2x_2 \dot{x}_2)$$

$$\begin{aligned} \Rightarrow \dot{V} &= (x_1^2 + x_2^2 - 3) \left( 2x_1 \left( x_1 \dot{x}_2 - x_2 (x_1^2 + x_2^2 - 3) \right) + 2x_2 \left( -x_1 \dot{x}_2 - x_2 (x_1^2 + x_2^2 - 3) \right) \right) \\ &= (x_1^2 + x_2^2 - 3) \underbrace{2x_1(-x_1^3 - x_1 x_2^2 + 3x_1)}_{-2x_1^2(x_1^2 + x_2^2 - 3)} + 2x_2(-x_1^2 x_2 - x_2^3 + 3x_2) \\ &\quad \underbrace{-2x_2^2(x_1^2 + x_2^2 - 3)} \end{aligned}$$

$$\Rightarrow \dot{V} = (x_1^2 + x_2^2 - 3)^2 (-2x_1^2 - 2x_2^2) = 0$$

$> 0$        $\leq 0$        $\rightarrow$  Per Cauchy-Schwarz

$$\Rightarrow \| \dot{V} \| \leq \| x_1^2 + x_2^2 - 3 \| \cdot \| (-2) \| \| x_1^2 + x_2^2 \| \quad \boxed{\| x^T y \| \leq \| x \| \cdot \| y \|}$$

given on an orthonormal space

