EE 5323 - HW06

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HW06 – Lyapunov Stability Analysis, LaSalle's Extension, and UUB
EE 5323 – Nonlinear Systems
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Exercise 1

LaSalle's Extension
Consider the system from HW05,

$$\begin{cases} \dot{x}_1 = x_2 + x_1(x_1^2 - 2) \\ \dot{x}_2 = -x_1 \end{cases}$$

We used a quadratic Lyapunov Function to show this system is locally SISL. And we found the region within which $\dot{V} \leq 0$. Use LaSalle's extension to verify that the system is AS. Find the equilibrium point.
Answer
4.a) Lyapunov function candidate: $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) > 0$

$$\dot{V} = \frac{\partial V^\top}{\partial x}\dot{x} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \implies$$

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$$

Now we plug in system dynamics to check stability,

$$\dot{V} = x_1(x_2 + x_1(x_1^2 - 2)) + x_2(-x_1) \implies$$

$$\dot{V} = \cancel{x_1 x_2} - x_1^2(x_1^2 - 2) - \cancel{x_1 x_2}$$

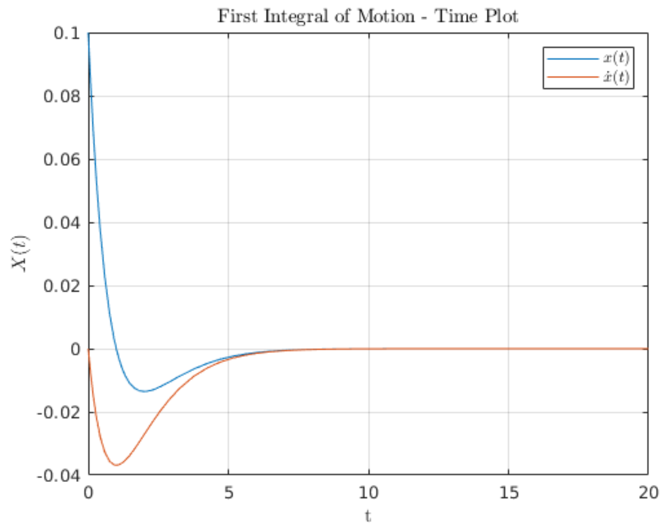
$$\dot{V} = -x_1^2(x_1^2 - 2) \leq 0$$

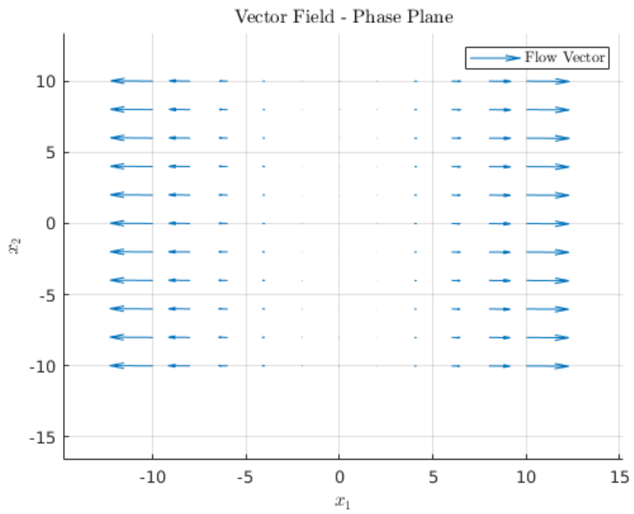
Thus, the system is *asymptotically stable* (AS) and it is bound by a region with radius of $\sqrt{2}$. Moreover, we know $\dot{x} \longrightarrow 0$; thus, per LaSalle's extension, $\ddot{x} \longrightarrow 0$ must hold true (not used in system dynamics). We proceed with plugging in the resulting x_1 in the system dynamics equation.

$$\dot{V} = -x_1^2(x_1^2 - 2) \leq 0 \implies \dot{V} \longrightarrow 0,\ x_1 \mid x_1^2 - 2 = 0; \implies x_1 = \pm sqrt2$$

$$\begin{cases} \dot{x}_1 = x_2 + x_1(x_1^2 - 2) \\ \dot{x}_2 = -x_1 \end{cases}$$

4.b) Simulation:



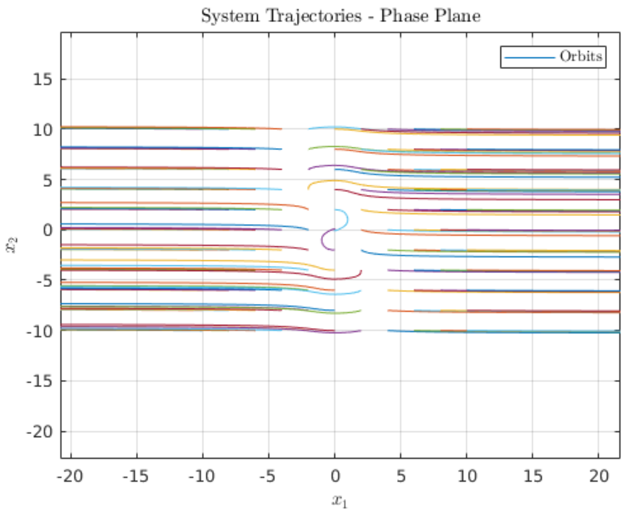


Matlab Code

```
%% HW05 - Q04 - AS
% @author: Bardia Mojra
% @date: 10/28/2021
% @title HW05 - Q04 - SISL Simulation
% @class ee5323 - Nonlinear Systems
% @professor - Dr. Frank Lewis

clc
clear
close all
warning('off','all')
warning

% part a
t_intv= [0 20];
x_0= [0.1, 0]'; % initial conditions for x(t)
figure
```




```
[t,x]= ode23('q04_sys', t_intv, x_0);
plot(t,x)
hold on;
grid on;
title('First Integral of Motion - Time Plot','Interpreter','
    latex');
ylabel('$X(t)$','Interpreter','latex');
xlabel('t','Interpreter','latex');
legend('$x(t)$', '$\dot{x}(t)$','Interpreter','latex');

% part b
figure();
hold on;
grid on;
mesh = -10:2:10;
[x1,x2] = meshgrid(mesh,mesh);
dx1=[];
dx2=[];
N=length(x1);
for i=1:N
  for j=1:N
    dx = q04_sys(0, [x1(i,j);x2(i,j)]);
    dx1(i,j) = dx(1);
    dx2(i,j) = dx(2);
  end
end
quiver(x1,x2,dx1,dx2);
ylabel('$x_2$','Interpreter','latex');
xlabel('$x_1$','Interpreter','latex');
legend('Flow Vector','Interpreter','latex');
title('Vector Field - Phase Plane','Interpreter','latex');
axis([-15 15 -15 15])

```

```
% part c
figure
for i=mesh
  for j=mesh
    init=[i, j];
    [t, x] = ode23(@q04_sys, [0 10], init);
    plot(x(:,1),x(:,2))
    hold on;
  end
end
ylabel('$x_2$','Interpreter','latex');
xlabel('$x_1$','Interpreter','latex');
legend('Orbits','Interpreter','latex');
title('System Trajectories - Phase Plane','Interpreter','latex');
grid on;
axis([-50 50 -50 50])

%%
%
% function xdot = q04_sys(t,x)
%   xdot = [x(2) + x(1)*(x(1)^2-2); -x(1)];
% end
```

```
%% Part 1 Answer
%% Document Information:
% * Author: Bardia Mojra
% * Date: 10/28/2021
% * Title: HW 05 - Part 4 System
% * Term: Fall 2021
% * Class: EE 5323 - Nonlinear Systems
% * Dr. Lewis

function xdot = q04_sys(t,x)
    xdot = [x(2) + x(1)*(x(1)^2-2); -x(1)];
end
```

Exercise 2

Limit Cycles
Consider the following system,

$$\begin{cases} \dot{x} = 4x^2 y - f_1(x)\left(x^2 + 2y^2 - 4\right) \\ \dot{y} = 2x^3 - f_1(y)\left(x^2 + 2y^2 - 4\right) \end{cases}$$

where the continuous functions $f_1(x)$, $f_2(y)$ have the same sign as their argument. Show that the system tends towards a limit cycle independent of the explicit expressions of $f_1(x)$, $f_2(y)$.

Answer

Exercise 3

UUB of a System with Disturbance
Consider the system on S&L p. 66 with a disturbance d,

$$\dot{x} + c(x) + d = 0$$

Assume that $xc(x)ax^2$ with $a > 0$ a known positive constant.
a. Assume that d is unknown but is bounded by $||d|| < D$ with D a known positive constant. Prove that the system is UUB and find the bound on $x(t)$.
b. Assume that d is unknown but is bounded by $||d|| < D||x||$ with D a known positive constant. Prove that the system is UUB and find the bound on $x(t)$.

Answer

Exercise 4

Lyapunov Analysis
Use Lyapunov Equation to check the stability of the linear systems.

a. $\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} x$

b. $\dot{x} = Ax = \begin{bmatrix} -7 & 4 \\ -7 & 3 \end{bmatrix} x$

c. $\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} x$

Answer