EE 5323 - HW03

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HW03 – Nonlinear System Simulations
EE 5323 – Nonlinear Systems
Dr. Lewis

Exercise 1

Voltera Predator-Prey System
Consider the Voltera predator-prey system,

$$\dot{x}_1 = -x_1 + x_1 x_2$$

$$\dot{x}_2 = x_2 - x_1 x_2$$

Find the equilibrium points and their nature.

Answer
State variable is given as:

$$\dot{x}_1 = -x_1 + x_1 x_2$$

$$\dot{x}_2 = x_2 - x_1 x_2$$

The Voltera predator-prey system has limit cycles therefore the system is at equilibrium when the population of both predator and prey remain constant; thus, the derivative should be zero. To find the equilibrium, I set $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. Solve the system for its roots.

$$\dot{x}_1 = 0 \Longrightarrow 0 = -x_1 + x_1 x_2$$

$$\dot{x}_2 = 0 \Longrightarrow 0 = x_2 - x_1 x_2$$

$$0 = x_1(\beta x_2 - \alpha) \implies x_1 = 0;\ x_2 = \alpha/\beta$$

$$0 = x_2(\gamma - \sigma x_1) \implies x_1 = \gamma/\sigma;\ x_2 = 0$$

There are two equilibrium points at $(x_1,\ x_2)$,

- At zero, $(0,\ 0)$,
- Any positive pair of integers $(\alpha/\beta,\ \gamma/\sigma)$ (1, 1). -3

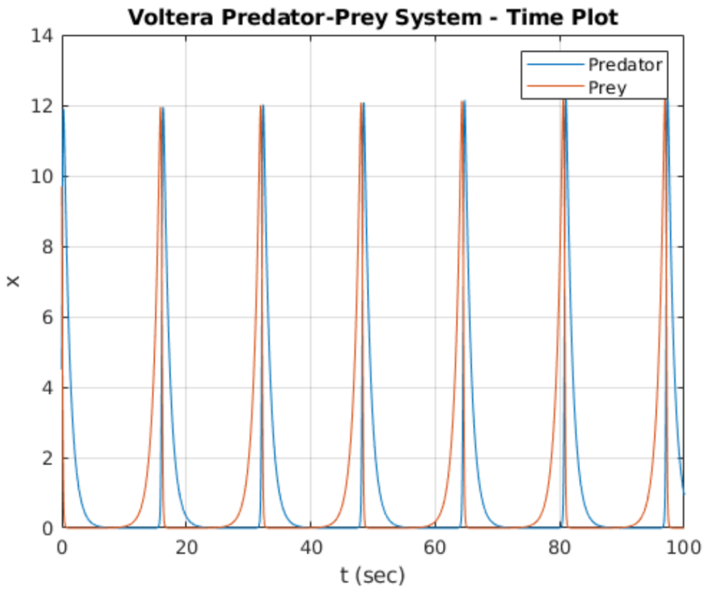
The equilibrium point nature of the zero is a stable center point that is a limit cycle. The other e.p. has a saddle point nature because it is stable in one dimension (goes to zero) and unstable in the other (goes to infinity).

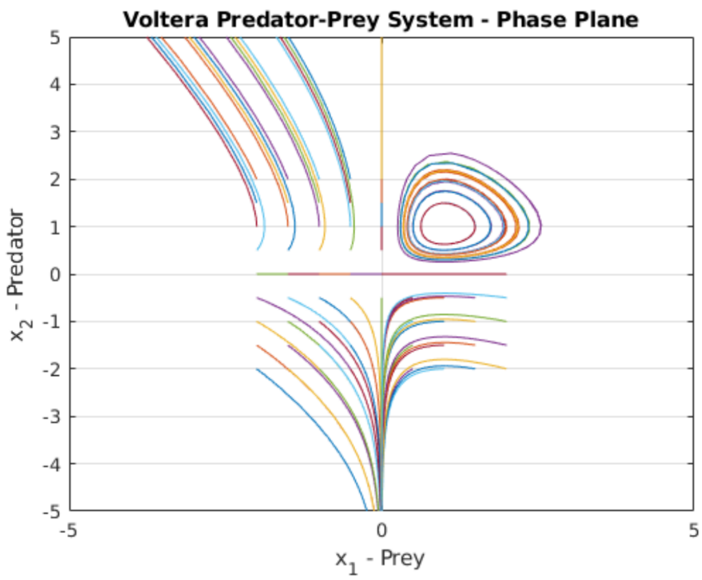
(0, 0) saddle & (1, 1) center -10

Matlab Code

```
%% HW03 - Q01 - Voltera Predator-Prey System
% @author: Bardia Mojra
% @date: 09/28/2021
% @title HW03 - Q01 - Voltera Predator-Prey System
% @class ee5323 - Nonlinear Systems
% @professor - Dr. Frank Lewis

clc
close all
warning('off','all')
warning

x0_set = -2:.5:2;
t_intv= [0 100];
x_0= [4.5, 9.7]'; % initial conditions for x(t)

figure
[t,x]= ode23('Voltera', t_intv, x_0);
plot(t,x)
hold on;
grid on;
title('Voltera Predator-Prey System - Time Plot');
ylabel('x');
xlabel('t (sec)');
legend('Predator', 'Prey');
t_intv= [0 10];

figure
for i=x0_set
  for j=x0_set
    x0 = [i; j];
    [t,x]= ode45('Voltera', t_intv, x0);
    plot(x(:,1),x(:,2))
    hold on;
  end
end
title('Voltera Predator-Prey System - Phase Plane');
ylabel('x_2 - Predator');
xlabel('x_1 - Prey');
axis([-5 5 -5 5]);
grid on;

function xdot = Voltera(t,x)
  xdot = [-x(1)+x(1)*x(2); x(2)-x(1)*x(2)];
end
```

Figures


Voltera Predator-Prey System - Time Plot
Predator
Prey
x
t (sec)
0
20
40
60
80
100
0
2
4
6
8
10
12
14


Voltera Predator-Prey System - Phase Plane
x_2 - Predator
x_1 - Prey
-5
0
5
-5
-4
-3
-2
-1
0
1
2
3
4
5

Exercise 2

Equilibrium points and linearization
Consider the following system,

$$\dot{x}_1 = x_2(-x_1 + x_2 - 1)$$
$$\dot{x}_2 = x_1(x_1 + x_2 + 1)$$

(a) Find all equilibrium points

(b) Find Jacobian

(c) Find the nature of all e.p.s

Answer
a. Find all e.p.s
At equilibrium points, all states reach their minimal energy state; therefore, the derivative of the state should equal zero. Then, we solve for the roots of the obtained characteristic equation.

$$f(x_1,\ x_2) = \dot{X} = 0 \Longrightarrow \begin{cases} \dot{x}_1 = 0 \Longrightarrow & x_2(-x_1 + x_2 - 1) = 0 \\ \dot{x}_2 = 0 \Longrightarrow & x_1(\ x_1 + x_2 + 1) = 0 \end{cases}$$

There are four possible cases for state derivative, $\dot{X}$, to equal zero, $\dot{X} = 0$; where the system is at an equilibrium point.

Case 1

$$\begin{cases} (x_2 = 0)\cancel{(-x_1 + x_2 - 1)} = 0 \\ (x_1 = 0)\cancel{(\ x_1 + x_2 + 1)} = 0 \end{cases}$$

e.p. at $(x_1, x_2) = (0, 0)$

Case 2

$$\begin{cases} (x_2 = 0)\cancel{(-x_1 + x_2 - 1)} = 0 \\ \cancel{(x_1)}(\ x_1 + x_2 + 1 = 0) = 0 \Longrightarrow x_1 = -1 \end{cases}$$

e.p. at $(x_1, x_2) = (-1, 0)$

Case 3

$$\begin{cases} \cancel{(x_2)}(-x_1 + x_2 - 1 = 0) = 0 \Longrightarrow x_2 = +1 \\ (x_1 = 0)\cancel{(\ x_1 + x_2 + 1 = 0)} = 0 \end{cases}$$

e.p. at $(x_1, x_2) = (0, +1)$

Case 4

$$\begin{cases} \cancel{(x_2)}(-x_1 + x_2 - 1 = 0) = 0 \\ \cancel{(x_1)}(\ \ x_1 + x_2 + 1 = 0) = 0 \end{cases}$$

$$\Longrightarrow x_2 = 0; \Longrightarrow x_1 = -1 \ \text{ as in } \textbf{Case 2} \text{ or } \textbf{Case 3}$$

b. Find the Jacobian

$$f(x_1,\ x_2) = \begin{cases} x_2(-x_1 + x_2 - 1) \\ x_1(\ \ x_1 + x_2 + 1) \end{cases} = \dot{X} = AX$$

$$\dot{X} = \begin{bmatrix} \frac{df}{dx_1} & \frac{df}{dx_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow \dot{X} = \begin{bmatrix} \frac{d}{dx_1}(-x_2) & \frac{d}{dx_2}(x_2 - 1) \\ \frac{d}{dx_1}(x_1 + 1) & \frac{d}{dx_2}(x_1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow$$

$$\dot{X} = AX \Longrightarrow \dot{X} = \begin{bmatrix} -x_2 & 0 \\ 0 & x1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow$$

$$A = \begin{bmatrix} -x_2 & 0 \\ 0 & x1 \end{bmatrix}$$

c. Find the nature of all e.p.s,
Compute the poles of the system using Eigen values,

$$\triangle(s) = |SI - A| = \begin{vmatrix} s + x_2 & 0 \\ 0 & s - x_1 \end{vmatrix} \Longrightarrow$$

$$(s + x_2)(s - x_1) = 0 \Longrightarrow \begin{cases} s = \ x_1 \\ s = \ -x_2 \end{cases}$$

$$(s + x_2)(s - x_1) = 0 \Longrightarrow s^2 - x_1 x_2 s - x_1 x_2 = 0$$

Standard form for characteristic equation,

$$s^2 + 2\alpha s + \alpha^2 + \beta^2 = s^2 + 2\alpha s + \omega_n^2 = 0; \quad \beta = j\omega$$

Case 1, at $(0, 0)$; $s^2 = 0 \implies s = 0$, center point with poles at the origin on the s-plane.
Case 2, at $(-1, 0)$; $s = 1$ and $s = 0$, center point with poles at the origin and +1 on the s-plane.
Case 3, at $(0, +1)$; $s = 0$ and $s = -1$, center point with poles at -1 and the origin on the s-plane.

(-1, 0) saddle -5

Exercise 3

System simulation
Consider the following system,

$$\dot{x}_1 = x_2(-x_1 + x_2 - 1)$$
$$\dot{x}_2 = x_1(x_1 + x_2 + 1)$$

Simulate the system using MATLAB for various initial conditions for the two cases:

(a) Take ICs spaced in a uniform mesh in the box x1=[-10,10], x2=[-10,10]. Make one phase plane plot with all the trajectories on it. Plot phase plane on square [-15,15]x[-15,15].

(b) Take ICs spaced in a uniform mesh in the box x1=[-3,3], x2=[-3,3]. Make one phase plane plot with all the trajectories on it. Plot phase plane on square [-5,5]x[-5,5].

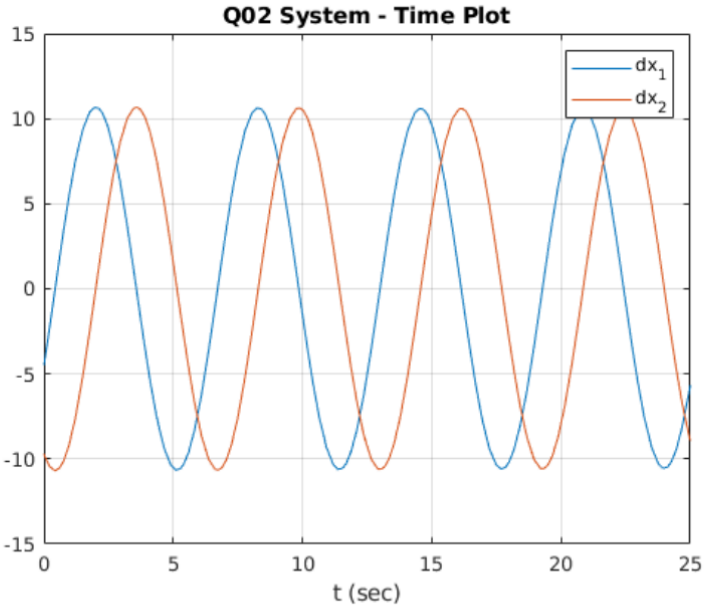
Answer
a. The phase plane rotation for all test cases are counter-clockwise.

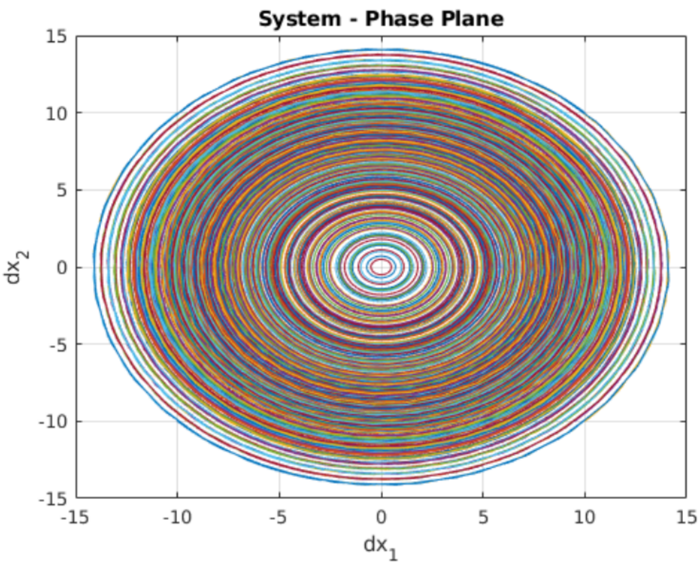
Matlab Code

```
%% HW03 - Q03a - System
% @author: Bardia Mojra
% @date: 09/28/2021
% @title HW03 - Q02 - System
% @class ee5323 - Nonlinear Systems
% @professor - Dr. Frank Lewis

clc
close all
%warning('off','all')
%warning

x0_set = -10:.5:10;
t_intv= [0 25];
x_0= [-4.5; -9.7]; % initial conditions for x(t)
%t = t_intv;

figure
[t, x] = ode23(@System, t_intv, x_0);
plot(t,x)
hold on;
grid on;
title('Q02 System - Time Plot');
xlabel('t (sec)');
legend('dx_1', 'dx_2');

t_intv= [0 25];
figure
for i=x0_set
```

```
  for j=x0_set
    x0 = [i; j];
    [t,x]= ode45(@System, t_intv, x0);
    plot(x(:,1),x(:,2))
    hold on;
  end
end
title('System - Phase Plane');
ylabel('dx_2');
xlabel('dx_1');
axis([-15 15 -15 15]);
grid on;

%%
%
% function xdot = System(t,x)
%   xdot = [-x(2); x(1)];
% end
```

Figures


Q02 System - Time Plot
dx_1
dx_2
15
10
5
0
-5
-10
-15
0
5
10
15
20
25
t (sec)


System - Phase Plane
15
10
5
0
-5
-10
-15
-15
-10
-5
0
5
10
15
dx_2
dx_1

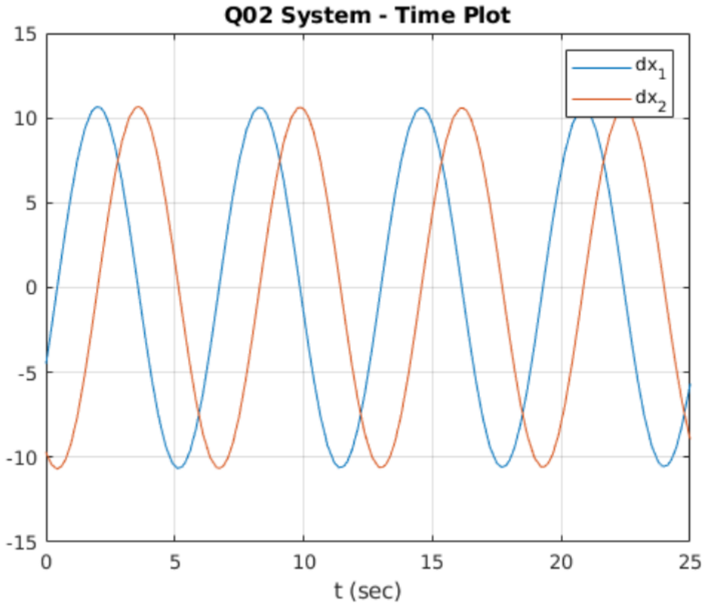
b. The phase plane rotation for all test cases are counter-clockwise.

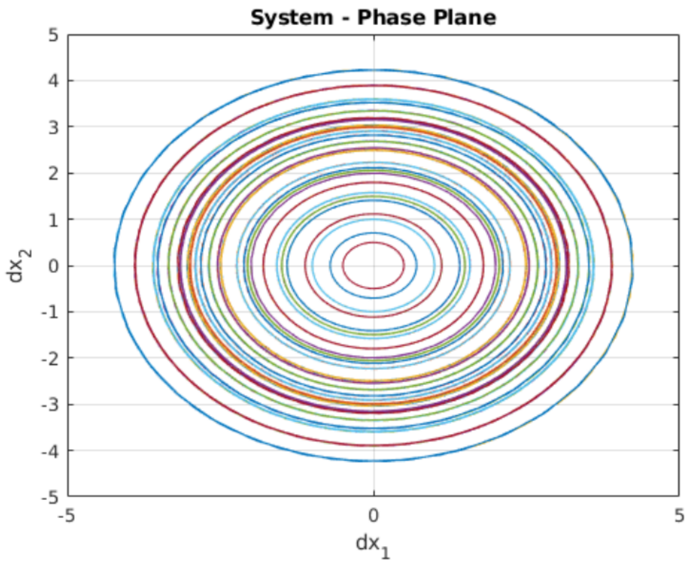
Matlab Code

```
%% HW03 - Q03b - System
% @author: Bardia Mojra
% @date: 09/28/2021
% @title HW03 - Q02 - System
% @class ee5323 - Nonlinear Systems
% @professor - Dr. Frank Lewis

clc
close all
%warning('off','all')
%warning

x0_set = -3:.5:3;
t_intv= [0 25];
x_0= [-4.5; -9.7]; % initial conditions for x(t)
%t = t_intv;

figure
[t, x] = ode23(@System, t_intv, x_0);
plot(t,x)
hold on;
grid on;
title('Q02 System - Time Plot');
xlabel('t (sec)');
legend('dx_1', 'dx_2');

t_intv= [0 25];
figure
for i=x0_set
  for j=x0_set
    x0 = [i; j];
    [t,x]= ode45(@System, t_intv, x0);
    plot(x(:,1),x(:,2))
    hold on;
  end
end
title('System - Phase Plane');
ylabel('dx_2');
xlabel('dx_1');
axis([-5 5 -5 5]);
grid on;

%%
%
% function xdot = System(t,x)
```

```
%     xdot = [-x(2); x(1)];
% end
```

Figures


Q02 System - Time Plot
15
10
5
0
-5
-10
-15
0
5
10
15
20
25
t (sec)
dx1
dx2


System - Phase Plane
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
0
5
dx2
dx1