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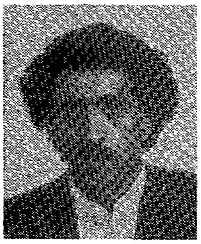

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Hassan Khalil (S'77–M'78), for a photograph and biography, see p. 651 of the July 1985 issue of this TRANSACTIONS.

On the Estimation of Asymptotic Stability Regions: State of the Art and New Proposals

ROBERTO GENESIO, MICHELE TARTAGLIA, MEMBER, IEEE, AND ANTONIO VICINO

Abstract—**This paper deals with the problem of the estimation of regions of asymptotic stability for continuous, autonomous, nonlinear systems. The first part of the work provides a comprehensive survey of the existing methods and of their applications in engineering fields. In the second part certain topological considerations are first developed and the "trajectory reversing method" is then presented together with a theorem on which it is based. In the final part, several examples of application are reported, showing the efficiency of the proposed technique for low-order (second and third) systems.**

INTRODUCTION

THE problem of the determination of regions of asymptotic stability for dynamical nonlinear systems is one of the most interesting aspects of classical stability studies both from a theoretical viewpoint and from that of applications. For this reason in the last 20 years several efforts have been made on the subject, generally arising from the Lyapunov theory of stability [1]–[4], and a number of applications have been attempted in the engineering field (electric power systems, chemical reactors) and in other areas such as ecology, biology, economics, etc.

This paper approaches the above problem and considers the important class of continuous, lumped-parameter nonlinear systems, excluding large-scale systems when studied by means of their peculiar methods. Discrete-time systems, to which much less attention is devoted in the literature, are also not taken into account. At the same time, the study of time-varying systems, and therefore of the domains of attraction of integral solutions apart from critical points are not considered here.

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The aim of the paper is twofold.

- To provide a state of the art on the methods for determining regions of asymptotic stability and on their applications. Indeed, to the authors' knowledge, a comprehensive survey on the subject has not yet appeared.
- To present an alternative method based on topological considerations which appears particularly powerful when dealing with low order systems.

The first point is developed in Section II, while Section III is concerned with some topological considerations which support the second point exposed in Section IV. Finally, Section V presents some examples of application of the method.

II. STATE OF THE ART

The numerous methods proposed in the literature for estimating the region of asymptotic stability (RAS) of an equilibrium point (asymptotically stable in the sense of Lyapunov [1]) for an nth order nonlinear system

$$\dot{x}=f(x) \tag{1}$$

may be divided in several classes according to Table I.

In the following the main features of the different classes will be described and the more relevant approaches will be analyzed in some detail while also outlining the corresponding applications.

A. Zubov Methods

The contributions included in this set refer to the Zubov theorem [5]–[7] which gives necessary and sufficient conditions

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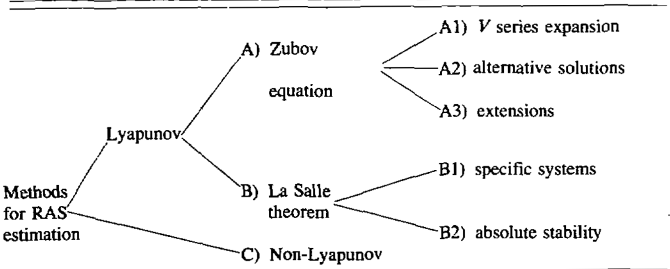
TABLE I
CLASSIFICATION OF THE METHODS FOR RAS ESTIMATION



for a certain region to be the RAS of an equilibrium point. Among these conditions the so-called Zubov equation must hold for the Lyapunov function (LF) V

$$\dot{V}(x)=[\nabla V(x)]^T f(x)=-\phi(x)[1-V(x)] \tag{2}$$

where $\phi(x)$ is an arbitrary positive definite function and the equation $V(x) = 1$ defines the boundary of the RAS. Unfortunately, (2) almost never admits a closed-form solution and therefore, different approximate methods have been proposed in order to estimate the RAS.

A1) A first approach to the problem is due to Zubov himself, who assumes the system (1) written as

$$\dot{x}=Ax+g(x) \tag{3}$$

with A eigenvalues having negative real parts and $g(x)$ admitting a Taylor expansion starting with second degree terms. In this case, the solution of (2) may be expressed (Lyapunov) in the form of the series

$$V(x)=V_2(x)+V_3(x)+\cdots+V_m(x)+\cdots \tag{4}$$

where $V_m(x)$ is a homogeneous function of mth degree. Zubov showed that for every approximate $\hat{V}$, derived from (4) truncated at the mth term, the boundary of the RAS lies between the surfaces

$$\check{V}(x)=\min_{\dot{\hat{V}}=0} \hat{V};\quad \hat{V}(x)=\max_{\dot{\hat{V}}=0} \hat{V}.$$

The computation of the successive forms $V_2, V_3, \cdots, V_m \cdots$ is made by solving sequentially linear equations in their coefficients derived by substitution in (2).

This method was first applied by Margolis and Vogt [8] to second-order systems. The choice of the function ϕ (see [9], [10]) affects the shape of the surfaces $\hat{V}$ = constant. As m tends to infinity, these surfaces converge, not uniformly, to the true RAS boundary (see [11]). The essential point of the method consists in minimizing $\hat{V}$ subject to the constraint $\dot{\hat{V}} = 0$, as in other Lyapunov procedures. A possible solution was proposed by Rodden [12], who was the first to study the numerical aspects of the method by searching for the tangency point of two surfaces. Applications of the above method to power systems are in [13] and [14], while a similar approximate solution of the Zubov equation, assuming the V function as a ratio of polynomials, is proposed in [15]. At last, a recent work of Kirin, Nelepin, and Baidaev [16] provides a partial survey of Zubov's methods together with an interesting outline of the main results obtained by Russian authors.

Drawbacks of these methods appear to be the assumption of the analyticity of the system, the numerical burden, the arbitrariness of ϕ, and mostly the nonuniform convergence of the procedures. On the other hand, an analytical estimate of the RAS is obtained.

A2) A second class may include methods based on different approaches to the solution of Zubov equation (2).

Burnand and Sarlos [17] express the solution in the form of Lie series by means of differential geometry concepts; the same technique is also used by Kormanik and Li [18] who evaluate points close to the RAS boundary by backward integration of system (1) and the Zubov equation. The V function is used as an indicator of RAS boundary proximity. Reference [19] practically follows the same steps by means of hybrid integration of the Zubov equation according to the characteristics method. Applications to power systems may be found in [20], as well as in [21] and [22] where the Lagrange–Charpit method is used to solve a partial differential equation which is a modified form of (2). In [23] a numerical procedure has been recently presented to integrate the system equation in terms of the V function, so estimating RAS boundary points.

A3) Finally, a third approach to the RAS determination following the Zubov theorem is proposed by Szegö [24], [25] and completed by Szegö and Geiss [26]. The equation

$$\dot{V}(x)=[\nabla V(x)]^T f(x)=-\psi(x)/\beta[V(x)] \tag{5}$$

is studied as a Zubov generalized equation where the functions ψ and β must satisfy suitable conditions. The solution of (5) with respect to V may be simplified by means of a proposed variable transformation but it does not appear to be an easy task. On the other hand, to the authors' knowledge only one application of the method has been presented in [27] for power systems.

B. La Salle Methods

The largest class of methods for RAS estimation refers to an extension of Lyapunov theory due to La Salle [28] who gives the following conditions for a region Ω to be contained in the true RAS:

$$\begin{cases} V(x)>0 & \forall x \in \Omega,\ x\neq 0 \\ V(0)=0 & \\ \dot{V}(x)\leqslant 0 & \forall x \in \Omega,\ x\neq 0 \\ \dot{V}(0)=0 & \end{cases} \tag{6}$$

and no trajectory lies entirely in the regions $\dot{V} = 0$. Therefore, this approach has a more limited objective than the Zubov one and it is still based on the construction of a suitable LF.

B1) A first group includes methods which may be applied to nonlinear systems having an exactly defined structure. This class excludes approaches referring to whole classes of nonlinearities which will be included in the next group *B2)*.

From the analytical viewpoint, either using graphical or simple numerical implementations, all the classical construction procedures of LF's have been applied in order to satisfy (6), this generally for second-order systems. So quadratic or Lur'e LF's [29], Ingwerson [29], [30], Szego [29], [31], [32], variable gradient [29], [33], [34], and Krasowskii [35], [36] procedures have been used, generally providing rather conservative RAS estimates.

The need for more significant results, together with the availability of more powerful algorithms, have in a short time led to several optimization approaches. The first work in this direction is due to Weissenberger [37] who proposed an LF of the Lur'e type for a relay system, by choosing its coefficients in order to maximize the area of the estimated RAS. The above Rodden algorithm [12] was used to minimize V subject to $\dot{V} = 0$. Davison and Kurak [38] face the problem still using a quadratic LF and maximize the RAS volume with respect to its coefficients, while carefully considering the numerical aspects of the procedure. Shields and Storey [39] compare the already mentioned optimization methods by putting in evidence both the existence of multiple tangency points of V = constant to $\dot{V} = 0$ and the convergence properties of different algorithms. Finally, Michel, Sarabudla, and Miller [40] provide an improved algorithm for the method proposed in [38].

A quite original approach ensuring the nonexistence of limit cycles by means of a frequency criterion may be found in Noldus [41] and [42] where the RAS estimation is made at the same time for all stable equilibrium points of the system using a single LF. Moreover, improvements of the results by topological considerations are suggested in [42]. An application to the analysis of chemical reactors is in [43] while extensions in the study of power systems are contained in [44], [45]. A method to improve the choice of an LF by imbedding the studied system in an augmented one has been recently presented in [46]. Also recently Michel, Sarabudla, and Miller [40], [47] have developed a numerical algorithm which operates in two steps; in the first one an initial RAS estimate is obtained extending the results of Brayton and Tong on the stability set of matrices [48] and by defining a norm LF. The second step consists of enlarging by fast computations such an initial RAS. The applicability of the final result is extended to high dimensional systems by invoking the comparison principle.

Several methods of this broad class have been largely applied to power system stability analysis (see the surveys of J.L. Willems [49], Ribbens-Pavella [50], and Fouad [51]) and an application to the ecology field is presented in [52].

As a general comment, it may be said that these methods do not provide a systematic approach to the problem but, as typical stability Lyapunov approaches, they lead to good results with a relatively small computational burden once a suitable LF has been chosen.

B2) A second class of methods collects approaches deriving from the application of the concepts of the absolute stability theory. They stem from the frequency domain Popov criterion and, generally according to the Kalman–Yakubovich lemma, they choose a Lur'e type V function holding for whole classes of nonlinear systems defined by a sector condition in the sense of Aizerman. As a consequence, the obtained results are expected to show appreciable properties of generality but lead to conservative RAS estimates when applied to a specific system.

The first papers in this direction are of Walker and McClamroch [53] and Weissenberger [54], [55] who consider the LF which proves the absolute stability of the system (Kalman–Yakubovich), i.e., a Lur'e form; they limit it to the finite domain where the sector condition holds and provide a generally conservative, but closed-form, solution to the RAS estimation problem. Extensions to multinonlinear systems based on the Anderson approach [56] are of J.L. Willems and J.C. Willems [57] and J.L. Willems [58] who apply the method to multimachine power systems (see also [59]). Pai and Mohan [60] and Pai [61] base the generalization to multinonlinear systems on the work of Narendra and Neumann [62]. The field where the methods of this class have most frequently been applied is that of power system transient stability analysis (see [49]–[51] and in particular, [63]–[67]).

C. Non-Lyapunov Methods

This class includes methods which do not explicitly employ Lyapunov functions. Early contributions of this kind are the obviously conservative linearization approach [35] and the "tracking function" method [68]–[71], which guarantees a practical stability region by using a consideration of La Salle [28] about the conditions for system trajectories so as not to cross a fixed surface. This last method essentially works for second-order systems. The procedures proposed in [72], [73] are mainly devoted to predicting the existence of limit cycles in second-order systems, but they could be helpful for practical stability region estimation. A possible use of the describing function method is outlined in [74]. Still oriented to second-order systems are the approaches proposed in [75] and more recently in [76]. Both methods derive from a geometrical interpretation of the system equation in light of the theory of flows with particular reference to the link between the divergence of $f(x)$ and the existence or not of closed trajectories in the plane. Davison and Cowan [77] develop a numerical procedure to determine a boundary point from which at least one segment of the RAS boundary is obtained: the method is only applicable to second-order systems and works particularly well for closed RAS. A similar approach is proposed in [78]; an application to power systems, consisting in an expansion of the RAS boundary in a power series around a given point of the boundary itself, may be found in [79]–[82]. Recently, an elegant approach based on Carleman linearization is due to Loparo and Blankenship [83] who express the nonlinear system solution in terms of Volterra series. Given an initial RAS estimate and an approximation level, the procedure iteratively evaluates enlarged RAS estimates according to the distance between the approximate and the true system solutions.

The method proposed in this paper and explained in the following and an analytical version of its [84] may also be included in this class.

III. Some Topological Considerations

The problem of determining the RAS is surely a difficult task, as was outlined in the previous section. The aim of this section is to provide some topological considerations which may result to be quite helpful in facing the problem with any method and especially with the one proposed in the next section. In particular, the knowledge of all the equilibrium points, which are generally of easy determination for low-order systems, may give indications about the topological configuration of the RAS.

Let the evolution of the autonomous nonlinear system be described by the equation

$$\dot{x} = f(x) \tag{1}$$

where $x(t) \in R^n$, $f: R^n \to R^n$ and satisfies the well-known sufficient conditions for the existence and the uniqueness of each solution $x(t,\ x_o)$ from initial condition $x(0) = x_o$ [85]. It is also assumed that the origin is an equilibrium point

$$f(o) = 0 \tag{7}$$

and that it is isolated and asymptotically stable, i.e., it is Lyapunov stable and attractive [1], [7]. System [1] may admit other equilibrium points satisfying

$$f(x) = 0. \tag{8}$$

The region of asymptotic stability of the origin is defined as the set of all points x_0 such that

$$\lim_{t \to \infty} x(t,\ x_0) = 0 \tag{9}$$

and will be denoted by Ω (simply connected, see Section IV) with boundary surface Γ.

The RAS is an open invariant set, and therefore the following theorem holds.

Theorem 1 [86], [87], [7]: The boundary Γ of the RAS is formed by whole trajectories.

As a consequence, excluding the trivial case $n = 1$, the following conclusions may be drawn.

- $n = 2$. If the RAS is bounded, its boundary is formed by either a limit cycle or a phase polygon (with unstable equilibrium points) or a closed curve of critical points.
- $n > 2$. If the RAS is bounded, there exist constraints on the number of equilibrium points and precisely the following holds.

Theorem 2: Given an odd order system ($n \neq 5$) without "degenerate"[1] equilibrium points, a necessary condition for the RAS to be bounded with smooth boundary Γ is that at least two

[1] "Degenerate" equilibrium points are considered those for which some of the eigenvalues of the linearized system are zero or pure complex in pairs.

other equilibrium points exist apart from the origin, an even number of which must lie on the boundary Γ.

Proof: On the assumptions of smoothness and boundedness Γ is a compact manifold [88]. Moreover, due to the way in which Γ can be generated (Theorem 3, see Section IV), it results to be an $(n-1)$-dimensional homotopy sphere and hence[2] is homeomorphic to an $(n-1)$ real sphere [88]. Then, its Euler characteristic is equal to 2 [88]. By the Poincarè-Hopf theorem [88] the sum of the indexes of the equilibrium points on Γ must be 2. By considering that the index of any non-"degenerate" equilibrium point can only be ± 1 [90], the conclusion of the theorem follows. ■

As a comment on this, observe that the presence of isolated "degenerate" critical points may be easily accounted for. Moreover, the proof of the above theorem can also be followed in the case of even-order system. Unfortunately, in this case the necessary condition for RAS boundedness is much weaker, since the sum of the indexes of the equilibrium points on Γ is equal to 0.

The above results may be easily applied through a simple analysis based on local properties of system equation (1) while the smoothness of Γ does not seem to be easily checkable.

IV. The Trajectory Reversing Method

Although, as outlined in Section II, several methods and procedures have been largely developed in the last 20 years, very little attention has been devoted to the possibility of estimating the RAS by reversing the system trajectory flow. Such a technique will be formalized in this section and it derives its effectiveness from the fact that the asymptotic behavior of the trajectories (as $t \to -\infty$) is related to the boundary of the RAS and always gives information about it. To the author's knowledge, reference to this idea may be found only in [18] and [19] where such a technique is viewed as a way of solving the Zubov equation and by the method of characteristics. Before going into detail, it may be recalled that time reversing in (1) [backward integration of (1)] is equivalent to considering the system

$$\dot{x} = -f(x) \tag{10}$$

which is characterized by the same trajectory configuration in state space as (1) but with reversed arrows on the trajectories. Beyond other modifications, this implies that the asymptotically stable equilibrium points of (1), and in particular the origin, become unstable for (10).

The general formulation of the trajectory reversing technique derives from the need, particularly felt in engineering applications, for enlarging an initial arbitrarily small estimate of the RAS. A theorem is now stated, which provides sufficient conditions for such an enlargement.

Theorem 3: Given the autonomous system (1) with continuous right member, if the origin is asymptotically stable, i.e., there exists a positive definite Lyapunov function $V(x)$ such that

a) $\Omega_0 = \{x : V(x) < K_0\}$

is simply connected with boundary Γ_0,

b) $\dot{V}(x) < 0 \quad \forall\, x \in \{x : V(x) \leqslant K_0\}, \qquad x \neq 0$

then the RAS may be approximated arbitrarily well by means of a convergent sequence of simply connected domains generated by the backward integration technique, starting from the initial RAS estimate Ω_0.

Proof: The continuity of $f(x)$ and the fact that for any $x \in \Gamma_0$ the backward mapping

$$\int_{t_0}^{t_1} -f(x)\; dt : x(t_0) \to x(t_1), \qquad t_1 > t_0 \tag{11}$$

is a homeomorphism ensure the one-to-one transformation of Γ_0 in Γ_1 and the simple connectedness of the domain Ω_1 bounded by Γ_1 [88]. Besides, hypothesis b) ensures that no point of Γ_0 may be taken by the backward mapping into the domain $V(x) \leqslant K_0$. As a consequence, the surface Γ_1 is such that

$$\Omega_1 \supset \Omega_0.$$

The definition of RAS and the way in which Γ_1 has been obtained ensure that Γ_1 bounds a domain wholly contained in the true RAS. Now, letting t_1 increase to $t_2, t_3, \cdots t_i, \cdots$, simply connected nested surfaces are obtained. Due to the uniqueness of the solutions of (1), these surfaces are such that

$$\Omega_{i+1} \supset \Omega_i$$

and they define domains wholly contained in the true RAS. Ω_i approximates arbitrarily well the RAS boundary. [3] In fact, assume that there exists a point x' out of Ω_∞, but interior to the RAS. In this case, forward integration of (1) with initial condition x' would lead to a point x'' belonging to an n-ball of arbitrarily small radius around the origin. In turn, this means, for the theorem of existence and uniqueness, that starting from x'' in the backward integration of (1) there would exist a time t_i such that

$$x(t_i, x'') = x'$$

which contradicts the hypothesis. ■

It is important to remark that, as a consequence of the above theorem, the RAS of an asymptotically stable equilibrium point must necessarily be simply connected. Indeed, this result agrees with a theorem [91], [92], stating that the RAS of (1) is homeomorphic to R^n. Furthermore, it may be observed that the theorem holds even if the asymptotically stable critical point is simply stable for the linearized system. This is because one of the converse theorems on stability [7] ensures the existence of a Lyapunov function for such a point, i.e., the existence of an arbitrarily small simply connected domain of attraction.

The remainder of the section reports some considerations about the practical applicability of the method. The RAS estimation of the origin of system (1) involves five steps.

Step 1: Determine the equilibrium points of the system apart from the origin and perform the local stability analysis of such points.

Step 2: Determine an arbitrarily small stability region Ω_0 around each asymptotically stable equilibrium point in the state space domain of interest.[4] This means finding regions Ω_0 with boundaries Γ_0, as in Theorem 3, that may easily be obtained once Step 1 has been performed.

Step 3: Apply Theorem 3 by backward integration of the system (1) at the asymptotically stable equilibrium points of Step 2.

Step 4: Perform forward and backward integration of (1) starting in the neighborhood of the other critical points in the domain of interest.

Step 5: Derive an estimate of the RAS of the origin by means of topological considerations about the behavior of the obtained trajectories.

The performing of Steps 1 and 2 does not generally present relevant difficulties, while the backward integration of Step 3 may be accomplished for a limited number of (equally spaced) points on Γ_0 in order to derive the time evolution of the initial RAS estimate Ω_0. Step 4 must be carried out because the trajectories close to unstable equilibrium points may provide essential

[2] The inclusion of the case $n = 5$ in the theorem would be equivalent to assuming the validity of the well-known Poincaré conjecture.

[3] In the sense that along every direction from the origin, any point of the RAS boundary is the limit of successions made of points on the boundaries of Ω_i. If the RAS is unbounded, there is a succession of points on the boundaries of Ω_i tending to infinity.

[4] This means considering the domain where the state conserves a correct physical sense if (1) represents a real system, or where the flow concerning the studied RAS is known to lie (see Examples 5 and 7 of Section V).

information about the RAS boundary as outlined in Section III. Again, a limited number of starting points may be examined to this purpose. With reference to Step 5, the estimation of the RAS must be performed taking into account that trajectories obtained in Step 3, starting in the neighborhood of asymptotically stable points, usually behave as follows (see also the examples of Section V):

a) a set of trajectories runs to infinity in certain directions: the RAS extends to infinity in those directions;
b) a set of trajectories gathers along fixed surfaces tending to infinity. The presence of a piece of the RAS boundary can be recognized;
c) a trajectory tends to another critical point. This point lies on the RAS boundary and the analysis of the trajectories close to it (Step 4) may allow the extension of the RAS estimate in the surrounding region;
d) a trajectory tends to a closed path. In this case a limit cycle on the RAS boundary is recognized.

It must be emphasized that the whole of the above procedure appears to be particularly efficient for low-order systems because of its topological inference aspects. For second-order systems, the method provides an almost exact RAS estimate with few computations, as shown in the following section. For any example of Section V, the number of performed backward and forward integrations is indicated and it always results consistently lower than those used by simulation which usually obtains poorer results. With regard to third-order systems, when RAS sections are looked for, the method still works satisfactorily but with a greater amount of computations depending on the specific examined system, as illustrated in Example 7 of Section V. It is worth remarking that no trouble stems from deviations due to numerical approximation errors when in the RAS boundary proximity because the reversing of time generally makes such a surface asymptotically stable (apart from unusual cases, where the RAS is semistable).

For $n > 3$ the method may still be applied with the only aim the enlargement of an initial RAS estimate, in the sense of discretizing Γ_0 and carrying out an adequate number of backward integrations on an interval $t_0 - t_i$, obtaining an image Γ_i, which bounds a larger domain of attraction Ω_i.

In the next section, several examples show how the technique works when applied to second- and third-order systems.

V. Examples

Some applications of the trajectory reversing method are now presented where the equilibrium point whose RAS is looked for has been translated into the origin, and where the system equations have been integrated by the standard fourth-order Runge–Kutta method.

Example 1: Van der Pol oscillator

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 + \epsilon(x_1^2 - 1)x_2. \end{cases}$$

The origin is the only equilibrium point. With $\epsilon = 1$, Fig. 1 shows the obtained RAS performing only one backward integration (from inside or from outside the limit cycle). The behavior of trajectories is that of case d) of Section IV.

Example 2: From Hahn [4]

$$\begin{cases} \dot{x}_1 = -x_1 + 2x_1^2 x_2 \\ \dot{x}_2 = -x_2. \end{cases}$$

The origin is the only equilibrium point and Fig. 2 shows the RAS estimated from the reported trajectories. 14 backward integrations have been performed with behavior of trajectories a) and b).

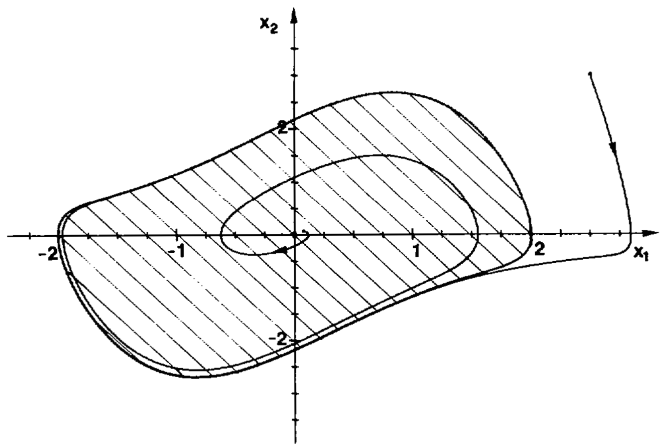


Fig. 1. Estimate of the RAS (dashed area) of Example 1 (Van der Pol oscillator).

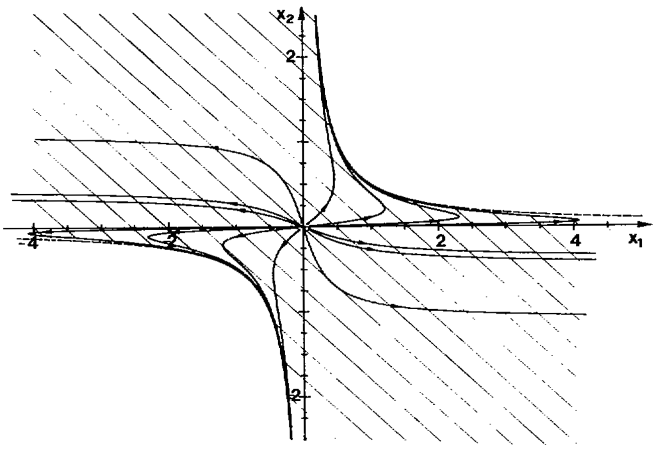


Fig. 2. Estimate of the RAS of Example 2.

Example 3: Synchronous generator

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -Dx_2 - \sin x_1 + \sin \delta_0 \end{cases}$$

where x_1 is the power angle and x_2 the corresponding speed deviation. Letting $D = 0.5$ and $\delta_0 = 0.412$, the equilibrium point of interest is $x_1 = 0.412$, $x_2 = 0$; the other critical points to be considered are $x_1 = 0.412 + 2\pi$, $x_2 = 0$ and $x_1 = -0.412 + \pi$, $x_2 = 0$ and both result unstable. Fig. 3 presents the obtained RAS. 12 backward and 8 forward integrations have been performed with behavior of trajectories a), b), and c).

Example 4: Continuous-flow stirred tank reactor [35].

$$\begin{cases} \dot{x}_1 = 700 - 2x_1 + 200x_2 \exp\left(\dfrac{25x_1 - 10^4}{x_1}\right) \\ \dot{x}_2 = 1 - x_2 - x_2 \exp\left(\dfrac{25x_1 - 10^4}{x_1}\right) \end{cases}$$

where x_1 is the temperature and x_2 the concentration. The critical points of the system are [35]:

$$x_1 = 354, \; x_2 = 0.964 \; \text{(point of interest)}$$

$$x_1 = 400, \; x_2 = 0.500 \; \text{(unstable)}$$

$$x_1 = 441, \; x_2 = 0.089 \; \text{(asymptotically stable).}$$

Fig. 4 shows the estimated RAS. 15 backward and 4 forward

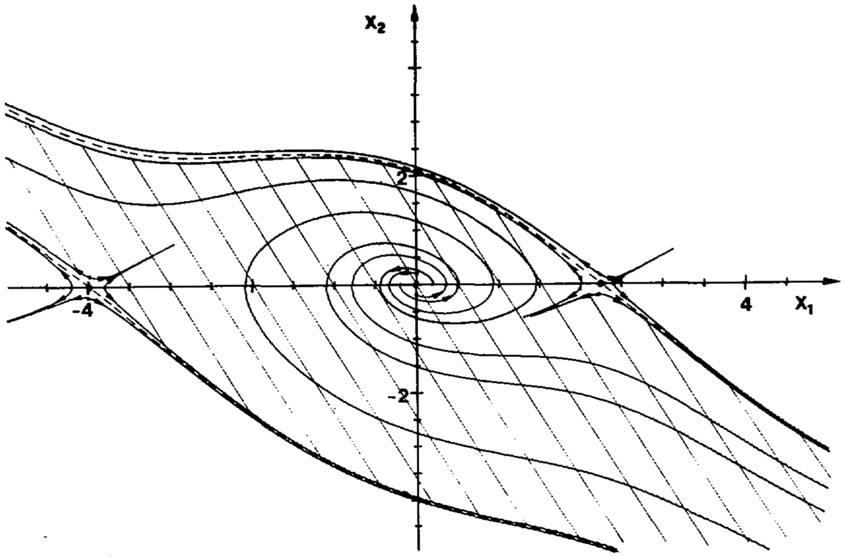


Fig. 3. Estimate of the RAS of Example 3 (synchronous generator).

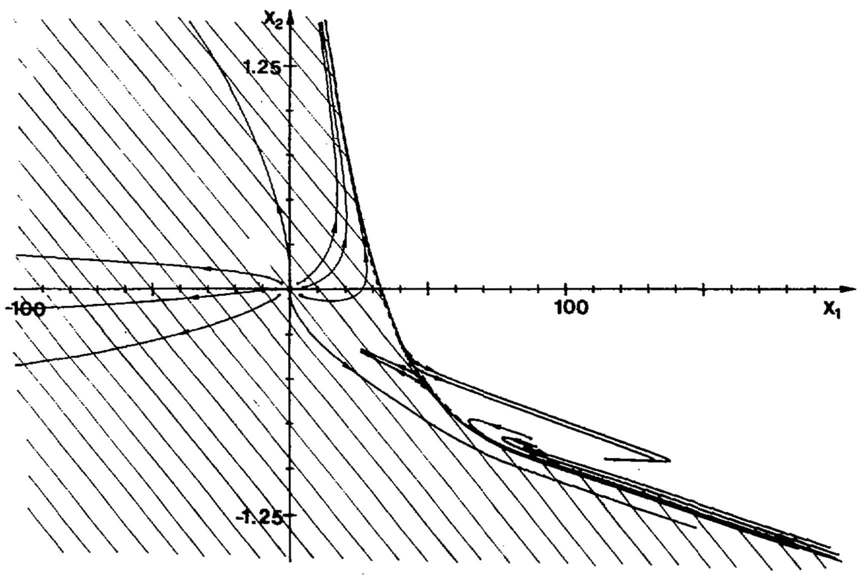


Fig. 4. Estimate of the RAS of Example 4.

integrations have been performed with behavior of trajectories a), b), and c).

Example 5: Predator–prey from [52]

$$\begin{cases} \dot{x}_1 = -3x_1 + 4x_1^2 - 0.5x_1x_2 - x_1^3 \\ \dot{x}_2 = -2.1x_2 + x_1x_2 \end{cases}$$

where x_1 and x_2 are the prey and predator populations. The equilibrium points are

$x_1 = 0,\ x_2 = 0$	(asymptotically stable)
$x_1 = 2.1,\ x_2 = 1.98$	(point of interest)
$x_1 = 1.0,\ x_2 = 0$	(unstable)
$x_1 = 3.0,\ x_2 = 0$	(unstable)

and the estimated RAS is shown in Fig. 5. Taking into account that the domain of physical interest is $x_1 \geqslant -2.1$, $x_2 \geqslant -1.98$ (corresponding to positive number of prey and predators), 7 backward and 6 forward integrations have been performed with behavior of trajectories a), b), and c).

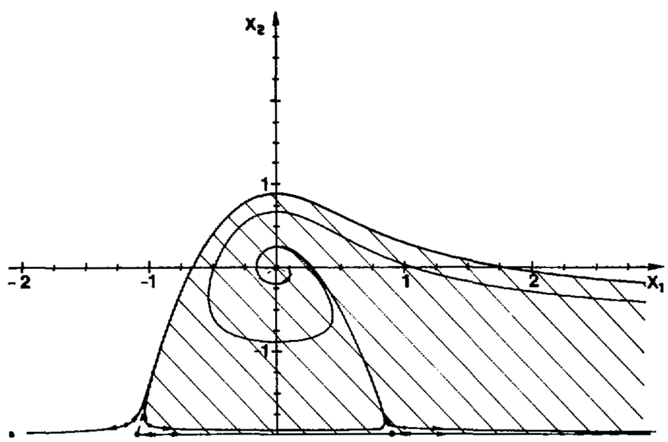


Fig. 5. Estimate of the RAS of Example 5 (predator–prey).

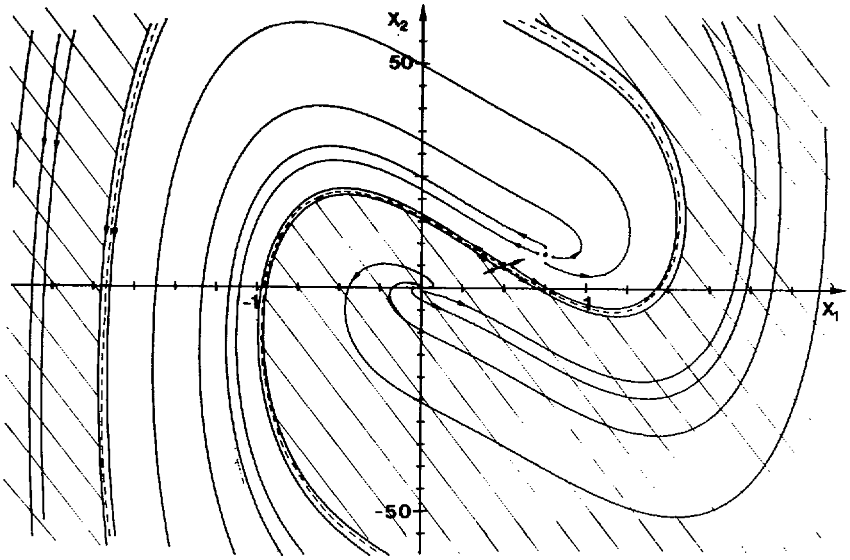


Fig. 6. Estimate of the RAS of Example 6.

Example 6: A pathological case from [93]

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3. \end{cases}$$

The equilibrium points are

$x_1 = 0,\ x_2 = 0$	(asymptotically stable)
$x_1 = -2.55,\ x_2 = -2.55$	(unstable)
$x_1 = -7.45,\ x_2 = -7.45$	(point of interest)

and the estimated RAS is shown in Fig. 6. 12 backward and 4 forward integrations have been performed with behavior of trajectories a), b), and c).

Example 7: Electric arc [94].

In this application, the classical Mayr model [95] of the electric arc is considered in order to study its behavior to derive the region of initial conditions which lead to the extinction of the arc. Assuming an RLC circuit connected to the arc, the system equations result to be

$$\begin{cases} \dot{x}_1 = -\dfrac{R}{L}x_1 - \dfrac{1}{L}x_2 + \dfrac{V}{L} \\ \dot{x}_2 = \dfrac{1}{C}x_1 - \dfrac{1}{C}x_2x_3 \\ \dot{x}_3 = x_3\left(\dfrac{1}{Q_0}x_2^2x_3 - \dfrac{1}{\tau}\right) \end{cases}$$

where x_1 is the curcuit current, x_2 the arc voltage, and x_3 the arc conductance. With reasonable values of parameters, there exist three equilibrium points [94], the first of which ($x_1 = 0$, $x_2 = V$, $x_3 = 0$) is the point of interest and corresponds to the condition of extinguished arc, while the second and the third correspond to the states of unstable and stable arc. Of course, the physically meaningful state space domain is defined by $x_3 \geqslant 0$. Letting

$$R = 21 \cdot 10^{-3},\ L = 68.2 \cdot 10^{-6},\ V = 424,\ Q_0 = 50 \cdot 10^{-3},$$
$$\tau = 5 \cdot 10^{-6}$$

and moreover

case a) $C = 0.15 \cdot 10^{-6}$

case b) $C = 3.7 \cdot 10^{-6}$

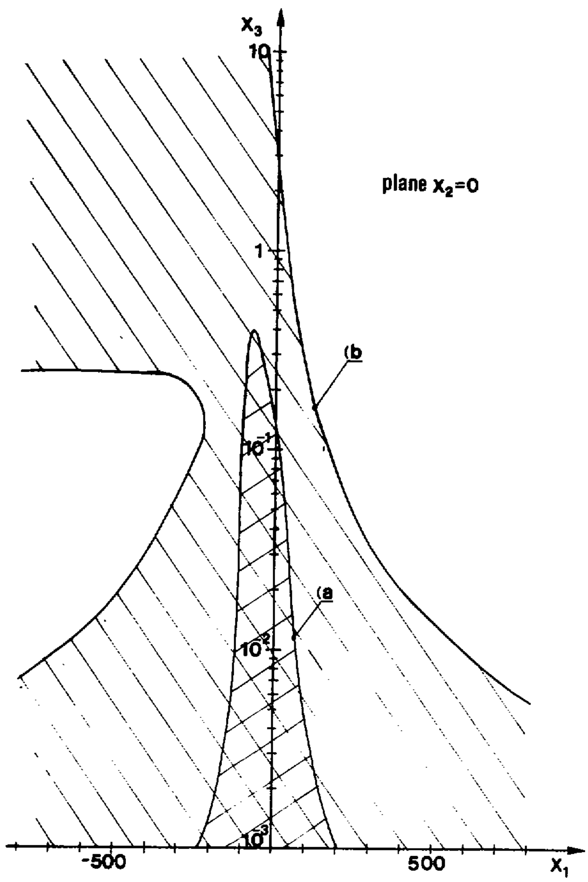


Fig. 7. Estimates of RAS sections of Example 7 (electric arc).

the two sections of the estimated RAS with the plane $x_2 = 0$ reported in Fig. 7 are obtained by the reversing trajectory method. It must be remarked that case a) does not cause any problem, while case b) requires some more computations to determine the upper part of the RAS.

VI. Conclusions

The problem of estimating the regions of asymptotic stability (RAS) of autonomous nonlinear systems has been dealt with in this paper. In the first part, with reference to continuous lumped-parameter systems, a comprehensive survey of the methods proposed in the literature has been presented together with a view of main applications in engineering fields. The second part of the paper is devoted to the description of the "trajectory reversing method" which allows the RAS estimation with the support of suitable topological considerations. Finally, several examples of applications are provided which show the efficiency of the proposed procedure especially for low-order systems.

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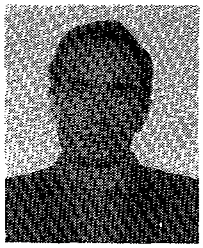

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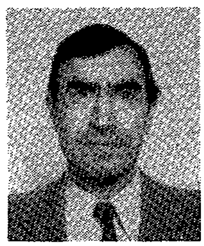

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