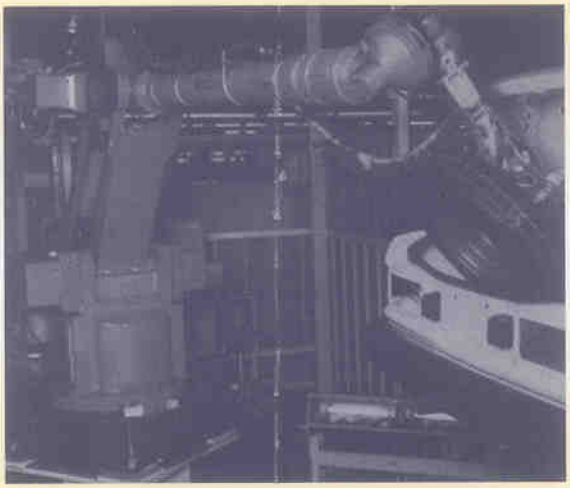
Control Engineering Series

Robot Manipulator Control
Theory and Practice

Second Edition, Revised and Expanded



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EXAMPLE 2.4-5: Equilibrium Points of Pendulum

The equilibrium points of the pendulum are isolated. On the other hand, a system described by $\dot{x} = 0$ has for equilibrium points any point in R and therefore, none of its equilibrium points is isolated.

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2.5 Vector Spaces, Norms, and Inner Products

In this section, we will discuss some properties of nonlinear differential equations and their solutions. We will need many concepts such as vector spaces and *norms* which we will introduce briefly. The reader is referred to [Boyd and Barratt], [Desoer and Vidyasagar 1975], [Khalil 2001] for proofs and details.

Linear Vector Spaces

In most of our applications, we need to deal with (linear) real and complex vector spaces which are defined subsequently.

DEFINITION 2.5-1 *A* real linear vector space *(resp.* complex linear vector space *is a set V, equipped with 2 binary operations: the addition (+) and the scalar multiplication (.) such that*

1. $x + y = y + x,\ \forall x, y \in V$
2. $x + (y + z) = (x + y) + z,\ \forall x, y, z \in V$
3. *There is an element 0_V in V such that $x + 0_V = 0_V + x = x,\ \forall x \in V$*
4. *For each $x \in V$, there exists an element $-x \in V$ such that $x + (-x) = (-x) + x = 0_V$*
5. *For all scalars $r_1, r_2 \in \Re$ (resp. $c_1, c_2 \in \mathbb{C}$), and each $x \in V$, we have $r_1.(r_2.x) = (r_1 r_2).x$ (resp. $c_1.(c_2.x) = (c_1 c_2).x$)*
6. *For each $r \in \Re$ (resp. $c \in \mathbb{C}$), and each $x_1, x_2 \in V$, $r.(x_1 + x_2) = r.x_1 + r.x_2$ (resp. $c.(x_1 + x_2) = c.x_1 + c.x_2$)*
7. *For all scalars $r_1, r_2 \in \Re$ (resp. $c_1, c_2 \in \mathbb{C}$), and each $x \in V$, we have $(r_1 + r_2).x = r_1.x + r_2.x$ (resp. $(c_1 + c_2).x = c_1.x + c_2.x$)*

8. *For each $x \in V$, we have $1.x = x$ where 1 is the unity in $\Re$ (resp. in $\mathbb{C}$).*

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EXAMPLE 2.5-1: Vector Spaces

The following are linear vector spaces with the associated scalar fields: $\Re^n$ with $\Re$, and $\mathbb{C}^n$ with $\mathbb{C}$.

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DEFINITION 2.5-2 *A subset M of a vector space V is a subspace if it is a linear vector space in its own right. One necessary condition for M to be a subspace is that it contains the zero vector.* ■

We can equip a vector space with many functions. One of which is the inner product which takes two vectors in V to a scalar either in $\Re$ or in $\mathbb{C}$, the other one is the norm of a vector which takes a vector in V to a positive value in $\Re$. The following section discusses the norms of vectors which is then followed by a section on inner products.

Norms of Signals and Systems

A *norm* is a generalization of the ideas of distance and length. As stability theory is usually concerned with the size of some vectors and matrices, we give here a brief description of some norms that will be used in this book. We will consider first the norms of vectors defined on a vector space X with the associated scalar field of real numbers $\Re$, then introduce the matrix induced norms, the function norms and finally the system-induced norms or operator gains.

Vector Norms

We start our discussion of norms by reviewing the most familiar normed spaces, that is the spaces of vectors with constant entries. In the following, $\| a \|$ denotes the absolute value of a for a real a or the magnitude of a if a is complex.

DEFINITION 2.5-3 *A norm $\| . \|$ of a vector x is a real-valued function defined on the vector space X such that*

1. $\| x \| > 0$ *for all $x \in X$ with* $\| x \| = 0$ *if and only if* $x = 0$.
2. $\| \alpha x \| = | \alpha | \| x \|$ *for all $x \in X$ and any scalar α.*

3. $\| x + y \| \leq \| x \| + \| y \|$ *for all* $x, y \in X$.

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EXAMPLE 2.5-2: Vector Norms (1)

The following are common norms in $X = \Re^n$ where $\Re^n$ is the set of $n \times 1$ vectors with real components.

1. 1-norm: $\| x \|_1 = \sum_{i=1}^{n} | x_i |$.
2. 2-norm: $\| x \|_2 = \sqrt{\sum_{i=1}^{n} | x_i^2 |}$, also known as the Euclidean norm
3. p-norm: $\| x \|_p = \left(\sum_{i=1}^{n} | x_i^p |\right)^{\frac{1}{p}}$.
4. ∞ - norm : $\| x \|_\infty = \max | x_i | \;\; \forall i = 1, \cdots, n$.

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EXAMPLE 2.5-3: Vector Norms (2)

Consider the vector

$$x = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

Then, $\| x \|_1 = 5$, $\| x \|_2 = 2$ and $\| x \|_\infty = 2$.

■

We now present an important property of norms of vectors in $\Re^n$ which will be useful in the sequel.

LEMMA 2.5-1: *Let* $\| x \|_a$ *and* $\| x \|_b$ *be any two norms of a vector* $x \in \Re^n$. *Then there exists finite positive constants* k_1 *and* k_2 *such that*

$$k_1 \| x \|_a \leq \| x \|_b \leq k_2 \| x \|_a \quad \forall x \in \Re^n$$

■

The two norms in the lemma are said to be equivalent and this particular property will hold for any two norms on $\Re^n$.

EXAMPLE 2.5-4: Equivalent Vector Norms

1. It can be shown that for $x \in \Re^n$

$$||x||_1 \;\leq\; \sqrt{n}||x||_2$$

$$||x||_\infty \;\leq\; ||x||_1 \;\leq\; n||x||_\infty$$

$$||x||_2 \;\leq\; \sqrt{n}||x||_\infty$$

2. Consider again the vector of Example 2.5.3. Then we can check that

$$||x||_1 \;\leq\; \sqrt{3}||x||_2$$

$$||x||_\infty \;\leq\; ||x||_1 \;\leq\; 3||x||_\infty$$

$$||x||_2 \;\leq\; \sqrt{3}||x||_\infty$$

■

Matrix Norms

In systems applications, a particular vector x may be operated on by a matrix A to obtain another vector $y = Ax$. In order to relate the sizes of x and Ax we define the *induced matrix norm* as follows.

DEFINITION 2.5-4 *Let $\| x \|$ be a given norm of $x \in \Re^n$. Then each $m \times n$ matrix A, has an* induced norm *defined by*

$$||A||_i \;=\; \sup_{||x||=1} \;||Ax||$$

where sup stands for the supremum. ■

It is always imperative to check that the proposed norms verify the conditions of Definition 2.5.3. The newly defined matrix norm may also be shown to satisfy

$$||AB||_i \;\leq\; ||A||_i \;||B||_i$$

for all $n \times m$ matrices A and all $m \times p$ matrices B.

EXAMPLE 2.5-5: Induced Matrix Norms

Consider the ∞ induced matrix norm, the 1 induced matrix norm and the 2 induced matrix norm,

$$||A||_{i\infty} = \max_i \sum_j |a_{ij}|$$

$$||A||_{i1} = \max_j \sum_i |a_{ij}|$$

$$||A||_{i2} = \sqrt{\lambda_{max}(A^T A)}$$

where λ_{max} is the maximum eigenvalue. As an illustration, consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

Then, $||A||_{i1} = max(4, 4, 5) = 5$, $||A||_{i2} = 4.4576$, and $||A||_{i\infty} = max(4, 7, 2) = 7$.

■

Function Norms

Next, we review the norms of time-dependent functions and vectors of functions. These constitute an important class of signals which will be encountered in controlling robots.

DEFINITION 2.5-5 *Let $f(.) : [0, \infty) \longrightarrow R$ be a* uniformly continuous function. *A function f is uniformly continuous if for any $\epsilon > 0$, there is a $\delta(\epsilon)$ such that*

$$|t - t_0| < \delta(\epsilon) \implies |f(t) - f(t_0)| < \epsilon$$

Then, f is said to belong to L_p if for $p \in [1, \infty)$,

$$\int_0^\infty |f(t)|^p dt < \infty$$

f is said to belong to L_∞ if it is bounded i.e. if $\sup_{t \in [0,\infty)} | f(t) | \leq B < \infty$, *where sup $f(t)$ denotes the supremum of $f9t)$ i.e. the smallest number that*

is larger than or equal to the maximum value of $f(t)$. L_1 denotes the set of signals with finite absolute area, while L_2 denotes the set of signals with finite total energy. ■

The following definition of the norms of vector functions is not unique. A discussion of these norms is found in [Boyd and Barratt].

DEFINITION 2.5-6 *Let L_p^n denote the set of $n \times 1$ vectors of functions f_i, each of which belonging to L_p. The norm of $f \in L_p^n$ is*

$$||f(.)||_p \;=\; [\int_0^\infty \sum_{i=1}^n |f_i(t)|^p dt\;]^{1/p}$$

for $p \in [1, \infty)$ and

$$||f(.)||_\infty \;=\; \max_{1\,\leq\, i\,\leq\, n}\, ||f_i(t)||_\infty$$

■

Some common norms of scalar signals $u(t)$ that are persistent (i.e. $\lim_{t\to\infty} u(t) \neq 0$) are the following:

1. $||u||_{rms} = \left[\lim_{T\to\infty} \frac{1}{T} \int_0^T u^2(t)dt\right]^{1/2}$ which is valid for signals with finite steady-state power.
2. $||u||_\infty = \sup_{t\geq 0} |u(t)$ which is valid for bounded signals but is dependent on outliers.
3. $||u||_a = \lim_{T\to\infty} \frac{1}{T} \int_0^T |u(t)|dt$ which measures the steady-state average resource consumption.

For vector signals, we obtain:

1. $||u||_\infty = \max_{1\leq i\leq n} ||u_i||_\infty$
2. $||u||_{rms} = \left[\lim_{T\to\infty} \frac{1}{T} \int_0^T u^T(t)u(t)dt\right]^{1/2}$
3. $||u||_a = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{i=1}^n |u(t)|dt$

Note that $||u||_\infty \geq \frac{1}{\sqrt{n}}||u||_{rms} \geq \frac{1}{n}||u||_a$.

On the other hand, if signals do not persist, we may find their L_2 or L_1 norms as follows:

1. $||u||_1 = \int_0^\infty |u(t)|dt$ which measures the total resource consumption

2. $||u||_2 = \left[\int_0^\infty u^2(t)dt\right]^{1/2} = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |U(jw)|^2 dw\right]^{1/2}$ which measures the total energy.

EXAMPLE 2.5-6: Function Norms

1. The function $f(t) = e^{-t}$ belongs to L_1. In fact, $\| e^{-t} \|_1 = 1$. The function $f(t) = \frac{1}{t+1}$ belongs to L_2. The sinusoid $f(t) = 2\text{sint}$ belongs to L_∞ since its magnitude is bounded by 2 and $\| 2\text{sint} \|_\infty = 2$.
2. Suppose the vector function $x(t)$ has continuous and real-valued components, i.e.

$$x: \ [a,b] \longrightarrow \ \Re^n$$

where $[a,b]$ is a closed-interval on the real line R. We denote the set of such functions x by $\mathbb{C}^n[a,b]$. Then, let us define the real-valued function

$$||x(.)|| \ = \ \text{sup}_{\text{t}\in[\text{a},\text{b}]} \ ||\text{x(t)}||$$

where $\| x(t) \|$ is any previously defined norm of $x(t)$ for a fixed t. It can be verified that $\| x(.) \|$ is a norm on the set $\mathbb{C}^n[a,b]$ and may be used to compare the size of such functions [Desoer and Vidyasagar 1975]. In fact, it is very important to distinguish between $\| x(t) \|$ and $\| x(.) \|$. The first is the norm of a fixed vector for a particular time t while the second is the norm of a time-dependent vector. It is this second norm (which was introduced in definition 2.5.6) that we shall use when studying the stability of systems.
3. The vector $f(t) = [e^{-t} \ -e^{-t} \ -(1+t)^{-2}]^T$ is a member of L_1^3. On the other hand, $f(t) = [e^{-t} \ -e^{-t} \ -(1+t)^{-1}]^T$ is a member of L_2^3 and L_∞^3.

■

In some cases, we would like to deal with signals that are bounded for finite times but may become unbounded as time goes to infinity. This leads us to define the extended L_p spaces. Thus consider the function

$$f_T(t) = \begin{cases} f(t) & \text{if } t \le T \\ 0 & \text{if } t > T \end{cases} \tag{2.5.1}$$

then, the extended L_p space is defined by

$$L_{pe} = \{f(t) : f_T(t) \in L_p\}$$

where $T < \infty$. We also define the norm on L_pe as

$$||f(.)_T||_p \;=\; ||f(.)||_{Tp}$$

Similar definitions are available for L_p^n and the interested reader is referred to [Boyd and Barratt], [Desoer and Vidyasagar 1975].

EXAMPLE 2.5-7: Extended L_p Spaces

The function $f(t) = t$ belongs to L_{pe} for any $p \in [1, \infty]$ but not to L_p.

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System Norms

We would like next to study the effect of a multi-input-multi-output (MIMO) system on a multidimensional signal. In other words, what happens to a time-varying vector $u(t)$ as it passes through a MIMO system H? Let H be a system with m inputs and l outputs, so that its output to the input $u(t)$ is given by

$$y(t) = (Hu)(t)$$

We say that H is L_p stable if Hu belongs to L_p^l whenever u belongs to L_p^m and there exists finite constants $\gamma > 0$ and b such that

$$||Hu||_p \;\leq \gamma||u||_p + b$$

If $p = \infty$, the system is said to be bounded-input-bounded-output (BIBO) stable.

DEFINITION 2.5-7 *The L_p gain of the system H is denoted by $\gamma_p(H)$ and is the smallest γ such that a finite b exists to verify the equation.*

$$||Hu||_p \;\leq\; \gamma||u||_p \;+\; b$$

■

Therefore, the gain γ_p characterizes the amplification of the input signal as it passes through the system. The following lemma characterizes the gains of linear systems and may be found in [Boyd and Barratt].

LEMMA 2.5-2: *Given the linear system H such that an input $u(t)$ results in an output $y(t) = (Hu)(t) = \int_0^t h(t-\tau)u(\tau)d\tau$ and suppose H is BIBO stable, then*

1. $\gamma_p(H)$ *is* $\leq \| h \|_1 \quad \forall p \in [1, \infty]$
2. $\gamma_\infty(H) = \int_0^\infty \mid h(t) \mid dt$
3. $\gamma_2(H) = max_{w \in R} \| H(jw) \| \leq \gamma_\infty(H)$

■

EXAMPLE 2.5-8: System Norms

1. Consider the system

$$H(s) \;=\; \frac{1}{s+2}$$

so that the impulse response is

$$h(t) = \begin{cases} e^{-2t} & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{1}$$

Note that $H(s)$ is BIBO stable. Then

$$\gamma_\infty(H) \;=\; 0.5$$

$$\gamma_2(H) \;=\; 0.5$$

2. Consider the system

$$H(s) \;=\; \begin{bmatrix} \frac{1}{s^2+k_v s+k_p} \\ \frac{s}{s^2+k_v s+k_p} \end{bmatrix}$$

where k_v and k_p are positive constants. The system is therefore BIBO stable. Then

$$\gamma_\infty(H) \;=\; max\{1/k_p,\; 4/\mathrm{e}k_v\}$$

$$\gamma_2(H) \;=\; \frac{\sqrt{1+k_p}}{k_v}$$

where e = 2.7183 is the base of natural logarithms.

■

This concludes our brief review of norms as they will be used in this book.

Inner Products

An inner product is an operation between two vectors of a vector space which will allow us to define geometric concepts such as orthogonality and Fourier series, etc. The following defines an inner product.

DEFINITION 2.5-8 *An inner product defined over a vector space V is a function $< .,. >$ defined from V to F where F is either $\Re$ or $\mathbb{C}$ such that $\forall x, y, z, \in V$*

1. $< x, y > = < y, x >^*$ *where the $< .,. >^*$ denotes the complex conjugate.*
2. $< x, y + z > = < x, y > + < x, z >$
3. $< x, \alpha y > = \alpha < x, y >, \ \forall \alpha \in F$
4. $< x, x > \geq 0$ *where the 0 occurs only for $x = 0_V$*

■

EXAMPLE 2.5-9: Inner Products

The usual dot product in $\Re^n$ is an inner product.

■

We can define a norm for any inner product by

$$\| x \| = \sqrt{< x, x >} \ \forall x \in V \tag{2.5.2}$$

Therefore a norm is a more general concept: A vector space may have a norm associated with it but not an inner product. The reverse however is not true. Now, with the norm defined from the inner product, a complete vector space in this norm (i.e. one in which every Cauchy sequence converges) is known as a Hilbert Space

Matrix Properties

Some matrix properties play an important role in the study of the stability of dynamical systems. The properties needed in this book are collected in this section. We will assume that the readers are familiar with elementary matrix operations and only consider the more advanced concepts of matrix analysis [Strang 1988], [Horn and Johnson 1991].

DEFINITION 2.5-9 *All matrices in this definition are square and real.*

- Positive Definite: *A real $n \times n$ matrix A is positive definite if $x^T Ax > 0$ for all $x \in \Re^n$, $x \neq 0$.*
- Positive Semidefinite: *A real $n \times n$ matrix A is positive semidefinite if $x^T Ax \geq 0$ for all $x \in \Re^n$.*
- Negative Definite: *A real $n \times n$ matrix A is negative definite if $x^T Ax < 0$ for all $x \in \Re^n$, $x \neq 0$.*
- *Negative Semidefinite: A real $n \times n$ matrix A is negative semidefinite if $x^T Ax \leq 0$ for all $x \in \Re^n$.*
- Indefinite: *A is indefinite if $x^T Ax > 0$ for some $x \in \Re^n$ and $x^T Ax < 0$ for other $x \in \Re^n$.*

■

Note that

$$x^T Ax \;=\; x^T \frac{(A + A^T)}{2} x \;=\; x^T A_s x$$

where A_s is the symmetric part of A. Therefore, the test for the definiteness of a matrix may be done by considering only the symmetric part of A.

THEOREM 2.5-1: *Let $A = [a_{ij}]$ be a symmetric $n \times n$ real matrix. As a result, all eigenvalues of A are real. We then have the following*

- Positive Definite: *A real $n \times n$ matrix A is positive definite if all its eigenvalues are positive.*
- Positive semidefinite: *A real $n \times n$ matrix A is positive definite if all its eigenvalues are nonnegative.*
- Negative Definite: *A real $n \times n$ matrix A is negative definite if all its eigenvalues are negative.*
- Negative Semidefinite: *A real $n \times n$ matrix A is negative semidefinite if all its eigenvalues are non-positive.*
- Indefinite: *A real $n \times n$ matrix A is indefinite if some of its eigenvalues are positive and some are negative.*

■

THEOREM 2.5-2: Rayleigh-Ritz *Let A be a real, symmetric $n \times n$ positive-definite matrix. Let λ_{min} be the minimum eigenvalue and λ_{max} be the maximum eigenvalue of A. Then, for any $x \in \Re^n$,*

$$\lambda_{min}[A]||x||^2 \ \leq \ x^T A x \ \leq \ \lambda_{max}[A]||x||^2$$

■

THEOREM 2.5-3: Gerschgorin *Let $A = [a_{ij}]$ be a symmetric $n \times n$ real matrix. Suppose that*

$$|a_{ii}| \ > \ \sum_{j=1}^{n} |a_{ij}| \ ; \ \text{for all } i \ = \ 1, ..., n \ ; \ j \ != \ i$$

If all the diagonal elements are positive, i.e. $a_{ii} > 0$, then the matrix A is positive definite. ■

EXAMPLE 2.5-10: Positive Definite Matrics

Consider the matrix

$$A = \begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix} \tag{1}$$

Its symmetric part is given by

$$A_s = \begin{bmatrix} 4 & -3 \\ -3 & 6 \end{bmatrix} \tag{2}$$

This matrix is positive-definite since its eigenvalues are both positive (1.8377, 8.1623). Of course, Gershgorin's theorem could have been used since the diagonal elements of A_s are all positive and

$$|a_{11}| \ = \ 4 \ \ |a_{12}| \ = \ 3 \ ; \ |a_{22}| \ = \ 6 \ \ |a_{21}| \ = \ 3$$

On the other hand, consider a vector $x = [x_1 \ x_2]^T$ and its 2-norm, then

$$0.3944(x_1^2 + x_2^2) \ \leq \ 4x_1^2 \ - \ 6x_1x_2 \ + \ 6x_2^2 \ \leq \ 7.6056(x_1^2 + x_2^2)$$

as a result of Rayleigh-Ritz theorem.

■