A Brief Overview of Nonlinear Control

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Abstract

Control system design has reached a level of considerable maturity and there exists substantial understanding of "standard" linear control problems. The aim of this paper is to look beyond linear solutions and briefly review recent developments in Nonlinear Control. We adopt a practical viewpoint so as to introduce this field to those who may not be familiar with it.

1 Introduction

1.1 Why Nonlinear Control?

Over the past 5 decades there have been remarkable developments in linear control theory. Indeed, it might be reasonably claimed that this topic has been largely resolved. Moreover, linear theory has been extensively used in applications. It is not uncommon to find practical systems [20] that utilize very sophisticated linear controllers, e.g., based on high order Kalman filters and LQR theory. However, the real world behaves in a nonlinear fashion – at least when considered over wide operating ranges. Not withstanding this observation, linear control design methods have been hugely successful and hence it must be true that many systems can be well approximated by linear models. On the other hand, there are well known examples of nonlinear practical systems, e.g., high performance aircraft operating over a wide range of Mach Numbers, Altitude and Angle of Attack. A common engineering approach to these kinds of problem is to base the design on a set of linearized models valid at a set of representative operating conditions. One can then use some form of "gain scheduling" to change the control law settings based on the current operating condition. Recently there has been significant developments in the supporting theory [27, 33, 43, 56, 60]. Indeed, over the last few years the more general topic of nonlinear control has attracted substantial research interest. The reader is referred to some of the excellent books available on this topic for further information (see for example [35, 3, 42, 53, 61, 67, 29]). The aim of this paper is to briefly outline some of these developments. We will aim to emphasise engineering aspects rather than mathematical rigour. In particular, we will summarize a selection of the available approaches to nonlinear control and to relate these, where possible, to practical problems.

1.2 Classification of Nonlinear Problems

To give a broad classification it is convenient to distinguish smooth and non-smooth nonlinearities [20]. Smooth nonlinearities are commonly met. They include products and other "power law" type functions as well as other continuous nonlinear operators. Nonsmooth nonlinearities are also very commonly met in practice. Well known examples include stiction and backlash in valves. Another type of nonsmooth nonlinearity met in practice is that of input slew rate and amplitude constraints. Unless compensated, these effects can have a dramatic influence on the final control system performance.

1.3 Smooth Nonlinearities

Smooth nonlinearities are conceptually easier to deal with than nonsmooth nonlinearities. This is these nonlinearities are, in a sense, invertible [20]. Indeed, there exist a class of smooth nonlinear problems which are "close to" linear in the sense that they can be converted, via feedback, into linear problems. The associated design strategy is known as "feedback linearization". We will briefly outline the essential ideas below.

1.4 Nonsmooth Nonlinearities

Nonsmooth nonlinearities are typically more difficult to deal with because they lack a globally definable inverse [20]. One of the most common forms of nonsmooth nonlinearity is that of actuator amplitude and/or slew rate limits. Approaches to this problem can be classified [10] as being *serendipitous, cautious, evolutionary, and tactical.* In the *serendipitous* approach, one allows occasional violation of the constraints but no

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particular precautions are taken to mitigate the consequences of the constraints. This approach constitutes the most naive design. In the *cautious* approach, one reduces the performance demands to the point where the constraints are no longer encountered. An example of this approach is the low-gain control law [47] in which the objective is to enforce closed-loop asymptotic stability and to ensure that the domain of attraction of the closed-loop system contains an *a priori* given bounded set. This is achieved by decreasing the gain of the controller (low-gain) until control saturation is never met in the given set. It can be clearly seen that this avoids the problem but at the cost of incurring a performance penalty. In the *evolutionary* approach, one begins with a standard (possible linear) design but then adds embellishments and enhancements so as to ensure that the deleterious effects of the constraints are mitigated. A well known example of this approach is the class of control laws known as "anti-windup controllers". These are widely used in industry. See [38] for an exposition of several of the anti-windup strategies that exist. Finally, in the *tactical* approach, one includes the constraints in the problem formulation ab initio. A well known example of such an approach is Model Predictive Control [2], [10], [51], [52], [50]. This design method aims to optimize performance subject to the presence of constraints. This method has been widely used in industry especially for chemical process control problems [54].

1.5 Stability Issues

Whether we are dealing with smooth or nonsmooth problems, a major concern is that of stability. In practice, one would presumably never deliberately design an unstable system. However, one needs to be careful about what kind of stability is important. For example, it is relatively easy to guarantee local stability, e.g., by designing a linear controller based on a local linearized model. However, global stability is a very demanding concept and is, anyway, only relevant to special types of problems. We will see later that the available methods for achieving stability in nonlinear feedback design problems are frequently associated with being able to find an "output" that has stable zero dynamics associated with it. We will find that this is a recurring theme in many of the design methods.

1.6 Passivity

Closely related to having stable zero dynamics is the concept of "*passivity*" [26]. Not surprisingly, *passivity* plays an important role in many existing approaches to the global stabilization of nonlinear systems [58], [57]. A system with input $u(t)$ and output $z(t)$ is *passive* if there exists a nonnegative function $U(x(T), T)$ (storage function) such that

$$\int_0^T u(t)z(t)dt \geq U(x(T), T) - U(x(0), 0) \qquad (1)$$

The term, "passive", has its origins in classical circuit theory [7], where circuits containing only R, L and C elements have this property which here is roughly equivalent to the statement that they absorb power. Note that, when viewed in a general light, passivity is a very strong requirement. For a linear system, it basically, implies [35] that the Nyquist plot of the system is restricted to the closed-right-half plane and this, in turn, means that the system is stable, minimum phase and has relative degree 1. Of course, these are very restrictive requirements. The reason that passivity is such a powerful tool for nonlinear stability is that there exists a close connection between *passivity* and *stability*. Specifically, if a system is passive (plus some other technical requirements), then the feedback $u(t) = -z(t)$ renders the system asymptotically stable. In particular, if we take U, as defined in (1), as a Lyapunov like function, then

$$\dot{U} \leq uz = -z^2 \leq 0 \qquad (2)$$

This idea can be used to establish that the equilibrium plant is globally uniformly asymptotically stable.

With this as background, the search for stabilizing control can be reformulated as a search for an output $z = h(x)$ and a feedback transformation $u = \alpha(x) + \beta(x)\upsilon$ with $\beta(x)$ invertible such that the system connecting υ to z is passive. We then say that the system is feedback passive [58] and it follows that asymptotic stability is achieved by the feedback $\upsilon = -z$.

The crucial limitation of the passivation design is that the chosen "output" z, must have two key properties; (i) it must have relative degree 1, and (ii) it must be stably invertible.

The search for a passivation output is unfortunately nontrival. To aid in the construction of such outputs, recursive designs can sometimes be used for systems possessing special structure. Specifically, for "lower-triangular" nonlinear systems, we can use *Backstepping* and for "upper-triangular" nonlinear systems we can use *Forwarding*. These two methods can be used to overcome the key restrictions of passivation, namely backstepping overcomes the relative degree 1 problem and forwarding overcomes the stable invertibility problem [57]. We will discuss backstepping briefly below.

1.7 Nonlinear Model Predictive Control

Model Predictive Control is a well established technique for dealing with constraints [10, 51, 52]. There also exist [2, 50] nonlinear versions of the Model Predictive Control Strategies. These strategies come with a ready made Lyapunov function, provided one can force the trajectory associated with the receding horizon optimization problem into a suitable terminal set having certain properties [51]. This converts the question of stability into one of verifying *feasibility* of the control law.

1.8 Output Feedback Issues in Nonlinear Control

A point that has been glossed over in the above brief overview is the issue of the type of information on which the control law is based. Many of the nonlinear design methods assume that the full state is available. Clearly this is rarely the case in practice. Thus, one usually needs to add provisions for state estimation, including disturbances, in practical nonlinear control. The reader will recall that in linear control the issue of state estimation presents no major difficulties [20]. Indeed, it is well known that one can invoke "Certainty Equivalence" i.e., one can treat the state estimates as if they were the true states for the purpose of control system design. Stability of the composite system follows by ensuring that the state estimator and full state feedback controller are separately stable. Also, in the linear case, one can be rather casual about how one deals with disturbances since these do not affect the stability properties. Alas, the situation for nonlinear systems is far more complex. For example,

- Certainty Equivalence no longer holds in general. Indeed, there is usually a cross coupling between estimation and control. Specifically, the control law may need to be "cautious" to account for state uncertainty and, conversely, the control policy may incorporate "learning" i.e., the input signal can enhance the state estimation accuracy [63].

- Stability does not follow by separately guaranteeing that the state estimator and state feedback control law are stable. Indeed, there exist examples of systems where an exponentially convergent state estimator can still destabilize an exponentially stable state feedback closed-loop system [35].

- Disturbances can present a difficulty in nonlinear control problems. On the one hand, these can be conceptually dealt with by simply modelling them along with other system states. However, there are some hidden traps in nonlinear problems. For example, if a step-type disturbance is modelled as if it were at the system output, but it actually occurs at the system input, or vice versa, then this change can effect performance and may be potentially destablizing [21]. This is due to the inherent nonlinear nature of the problem which means that superposition does not hold. Thus, the nature of the system dynamics can, at least in theory, be dramatically altered by the point of injection and nature of disturbances.

1.9 Complex Behaviour of Nonlinear Systems

The response of certain nonlinear systems can be exceedingly complex when viewed in a general setting. We recall that linear systems can be classified very easily as being either stable, marginally stable or unstable. However, nonlinear systems can exhibit quite bizarre behaviour. For example, one can easily generate chaotic behaviour where the trajectories are arbitrarily sensitive to the initial conditions. In case the reader should feel that chaos is a theoretical concept that would never be met in practice, one can readily think of real world nonlinear systems that do indeed exhibit this kind of behaviour. Two examples that come to mind are

- simple adaptive control laws [49] - this is a little surprising since one would not expect such bizarre behaviour by simply connecting together a parameter estimation with a feedback controller. However, we remind the reader that this is fundamentally a nonlinear problem and thus "certainty equivalence" can not be expected to hold in general.

- on-off controllers - these are very prevalent in practical control but the exact behaviour can be very complex including the onset of chaos.

1.10 Overview of the Remainder of the Paper

The remainder of the paper will expand on the above ideas. Wherever possible, we will link the ideas back to the linear case. We will also use simple examples to illustrate the ideas. The layout of the remainder of the paper is as follows: *Section 2* - Gives a brief description of a practical nonlinear control problem to act as motivation, *Section 3* - Explains how linear control can be extended to the case where actuators are subject to hard limits, *Section 4* - Gives a brief overview of nonlinear model predictive control, *Section 5* - Outlines feedback linearization, *Section 6* - Gives a simple generalization of the idea of feedback linearization to include systems which are not necessarily stably invertible, *Section 7* - Describes the idea of flatness and relates it to feedback linearization, *Section 8* - Gives a brief introduction to the idea of backstepping, *Section 9* - Describes several simple nonlinear control problems and illustrates the complex behaviour that can arise even from simple systems, and *Section 10* - Draws conclusions.

2 A Motivational Example – *Continuous Metal Casting*

A continuous caster is a common means by which molten liquid metal is solidified. The molten metal is poured into a mould which is a rectangular container open at each end. (See Figure 1.) On the system with which we are familiar (BHP Steel Works Newcastle, Australia), the mould level sustained a continuous oscillation [23]. The amplitude of this oscillation was relatively small ($\pm 10mm$) but resulted in serious degradation in quality of the caste material.

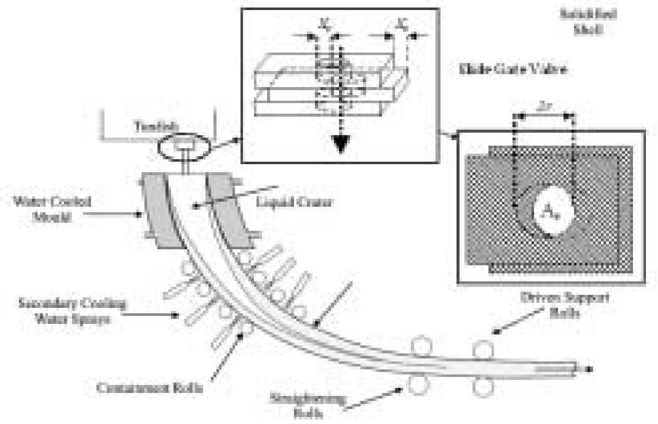

Figure 1: Continuous caster schematic with slide gate valve

As in most real world control problems, it is not difficult to pin-point a raft of nonlinearities associated with this problem. For example, the flow of molten metal through the slide gate valve (SGV) depends on the type of material being cast which can vary from "sticky" to "freely flowing". This suggests that we may need "gain scheduling" or adaptation to adjust the controller settings. Also, the geometry of the SGV itself introduces nonlinear behaviour since it compromises two intersecting circular regions (See Figure 1). Fortunately, it is relatively easy to compensate for this kind of smooth nonlinearity in the design [23]. Alas, our observation has been that these steps did little, or nothing, to mitigate the mould level oscillation problem. This suggests that one should expand one's view of the problem to include the possibility of other nonlinear phenomena, including nonsmooth nonlinearities. Further examination revealed that the SGV did indeed have a fundamental problem. On the one hand, it must be sized to deal with the gross material flow. However, it then experiences slip-stick friction when dealing with the finer regulation needed to maintain the mould level. If this problem involved a material such as water, then it would be feasible to address this problem by using two valves to achieve a dual range solution. However, this is not a feasible option with molten metal. In the case with which we are familiar, a high frequency dither signal was applied to the valve. This kept the valve in motion thus avoiding the slip-stick phenomena. The result was that the oscillations were immediately eliminated! Of course, as in all real world designs, there is a price to be paid for the improved output quality, namely the dither causes increased wear of the SGV. However, the cost of more frequent valve replacements was more than off-set [32] by the achieved improvements in product quality (estimated at millions of dollars annually).

Engineers involved in practical control problems may well see common elements from this example in problems that they have dealt with. Indeed, it is the authors' experience that one doesn't have to search far to find significant nonlinear issues in all real world control system design problems. With this as motivation, we proceed to examine some approaches to nonlinear control system design.

3 Dealing with Actuator Constraints

Arguably, the most commonly met nonlinearity is that of actuator amplitude and slew rate limits. Indeed, most industrial controllers will invariably incorporate some form of anti-integral windup protection to deal with actuator constraints [48, 15, 25, 5, 38, 65, 66]. There are many ways that can be used to describe anti-windup strategies. A systematic formulation of many schemes is given in [38]. There exists substantial work on this topic, see for example [48, 15, 25, 5, 38, 65, 66]. Many of the schemes can be understood as mechanisms for "informing" the control law that the constraints have been met so that appropriate modifications to the future control actions can be taken.

To illustrate, say that we are given a minimum phase bi-proper SISO controller transfer function $C(s)$. We can expand the transfer function in the form

$$C(s)^{-1} = H_\infty + H(s) \tag{3}$$

where H_∞ is the high frequency gain and $H(s)$ is strictly proper. We can then implement the controller as shown in Figure 2. Note that, if the gain is set to unity, then the transfer function from e to u is

$$\frac{U(s)}{E(s)} = \frac{H_\infty^{-1}}{1 + H_\infty^{-1}H(s)} = \frac{1}{H_\infty + H(s)} = \frac{1}{C(s)^{-1}} = C(s)$$

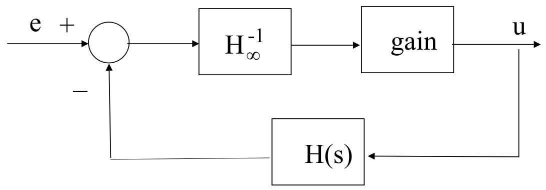

Figure 2: Anti-windup scheme

We also see from Figure 2, that all of the dynamics of the controller are contained in the transfer function $H(s)$ and hence are "informed" about the true plant input. This suggests that an *evolutionary* solution to the constrained control problem could be to simply replace the gain element in Figure 2 by the required nonlinear operation (saturation and/or slew rate limits). Actually, this simple idea underlies many of the available schemes for anti-windup control [20]. The idea generally works well in practice provided precautions are taken. For example, it is important that one does not call for too much oversaturation i.e., the unsaturated control should not exceed the available limits by more than say, 2 or 3 times.

The same idea can be used with multiple-input multiple-output controllers. Here, however, additional issues arise with respect to the gain block (here a multivariable gain) in Figure 2. Specifically, when one input saturates, then the relative proportioning of the other inputs can be perturbed [20]. To retain good performance in the presence of constraints, the relativities between the different inputs usually need to be preserved. Methods for doing this include replacing the gain block in Figure 2 by some form of input (or error) scaling to meet the required constraints. A discussion of some of the options is given in [20].

As mentioned above, this kind of strategy performs well when the demanded level of "oversaturation" is not too severe (say a factor of 2:1 from the linear response). This suggests that one might be able to improve performance by using multiple linear designs of different "gain" so that the control action is typically scaled back by something like 2:1 under all conditions. Related methods include the multi-mode regulator [36] and the piecewise linear control (PLC) [68]. In these methods, a number of controllers is precomputed and, at each time, the best controller such that the constraints are not violated, is used. A further improvement of these methods is to recognize that performance can be improved by forcing the controls into saturation. Two alternative methods that modify the schemes described evolve from this idea; i.e., the controller proposed in [11], which incorporates the concept of *oversaturation* to measure the extent to which the demanded input exceeds the saturation level. The alternative controller of [46] combines the ideas of PLC and of the low-and-high gain controller [45]. Finally, the stability of these kinds of schemes can be addressed using piecewise constant Lyapunov functions, as used in [11, 46]. This exploits the inherent switched nature of the control law.

4 Nonlinear Model Predictive Control

Model Predictive Control is a control algorithm based on solving a constrained *optimal* control problem. Since, constraints can be included from the beginning, this is an example of a *tactical* approach to constrained nonlinear control. A *receding horizon* approach is typically used, which can be summarized in the following steps:

(i) At time k and for the current state $x(k)$, solve, on-line, an open-loop optimal control problem over some future interval, taking into account the *current* and *future* constraints.

(ii) Apply the first step in the optimal control sequence.

(iii) Repeat the procedure at time $(k+1)$, using the (new) current state $x(k+1)$.

The solution is converted into a closed-loop strategy by using the measured value of $x(k)$ as the current state. When $x(k)$ is not directly measured, then one can obtain a closed-loop policy by replacing $x(k)$ by an estimate provided by some form of observer. The method can be summarized as follows: Given a model for a nonlinear system,

$$x(\ell+1) = f(x(\ell), u(\ell)), \qquad x(k) = x \tag{4}$$

the MPC at event (x, k) is computed by solving a constrained optimal control problem:

$$\mathcal{P}_N(x): \qquad V_N^o(x) = \min_{U \in \mathcal{U}_N} V_N(x, U) \tag{5}$$

where

$$U = \{u(k), u(k+1), \ldots, u(k+N-1)\} \tag{6}$$

$$V_N(x, U) = \sum_{\ell=k}^{k+N-1} L(x(\ell), u(\ell)) + F(x(k+N)) \tag{7}$$

$\mathcal{U}_N$ is the set of U that satisfy the constraints over the entire interval $[k, k+N-1]$:

$$\begin{aligned} u(\ell) \in \mathbb{U} \qquad \ell &= k, k+1, \ldots, k+N-1 \quad (8) \\ x(\ell) \in \mathbb{X} \qquad \ell &= k, k+1, \ldots, k+N \quad (9) \end{aligned}$$

together with the terminal constraint

$$x(k+N) \in W \tag{10}$$

Usually, $\mathbb{U} \subset \mathbb{R}^m$ is convex and compact, $\mathbb{X} \subset \mathbb{R}^n$ is convex and closed, and W is a set that can be selected appropriately to achieve stability. In the above formulation, the model and cost function are time invariant. Hence, one obtains a time-invariant feedback control law. In particular, we can set $k = 0$ in the open-loop control problem without loss of generality. Then, at event (x, k), we solve

$$\mathcal{P}_N(x): \qquad V_N^o(x) = \min_{U \in \mathcal{U}_N} V_N(x, U) \tag{11}$$

where

$$\begin{aligned} U &= \{u(0), u(1), \ldots, u(N-1)\} \quad (12) \\ V_N(x, U) &= \sum_{\ell=0}^{N-1} L(x(\ell), u(\ell)) + F(x(N)) \quad (13) \end{aligned}$$

subject to the appropriate constraints. Standard optimization methods are used to solve the above problem.

Let the minimizing control sequence be

$$U_x^o = \{u_x^o(0), u_x^o(1), \ldots, u_x^o(N-1)\} \tag{14}$$

Then the actual control applied at time k is the first element of this sequence, i.e.,

$$u = u_x^o(0) \tag{15}$$

Time is then stepped forward one instant, and the above procedure is repeated for another N-step-ahead optimization horizon. The first input of the new N-step-ahead input sequence is then applied. The above procedure is repeated endlessly.

The above MPC strategy *implicitly* defines a time-invariant control policy $h(\cdot)$, i.e., a static mapping $h : \mathbb{X} \to \mathbb{U}$ of the form

$$h(x) = u_x^o(0) \tag{16}$$

It is worth recalling that MPC, originally, arose in industry [54] as a response to the need to deal with constraints. In fact, it can be said that *constraints* are the *raison d'être* of MPC.

Until recently, MPC was restricted to relatively "slow" processes due to the need to carry out an on-line optimization to find the control policy $u = h(x)$, for the current state, x. However, recent work [59, 4] has obviated the need for this on-line calculation, at least for certain classes of systems. In particular, for the case of constrained control of linear systems, it turns out that the time invariant control policy $h(\cdot)$ can be *explicitly* characterised, making on-line computation unnecessary. This has many advantages beyond simplification of the required on-line computations; e.g., the policy can be characterised off-line leading to better acceptance in practice; the range of performance can be examined and verified prior to inplementation and; it has the potential to offer alternative mechanisms for analyzing stability. Moreover, these studies reveal a relationship between these optimality-based control policies and the heuristically-based anti-windup strategies [9]. Indeed, this body of work seems to open the door to widespread application of these methods to high speed, high performance and mission critical applications.

5 Feedback Linearization

A well known and conceptually simple technique for the control of nonlinear systems where the nonlinearities are smooth is Feedback Linearization [41]. This method is restricted to certain classes of nonlinear systems and stable invertibility is required [41]. However, Feedback Linearization has strong practical appeal because of its simplicity (see, for example, [34, 44]). The notion of Feedback Linearization was formally addressed for the first time by Brockett [6] in 1978, where he solved the problem of finding a smooth local change of coordinates $z = \phi(x)$ and a smooth feedback control law $u = \alpha(x) + \beta(x)v$ in order to transform a general nonlinear system $\dot{x} = f(x) + g(x)u$ into a linear system $\dot{z} = Az + Bv$. A closely related problem formulation was previously considered by Krener [39, 40] where the linearization of a general affine nonlinear control system was achieved by means of a local change of coordinates only. In later years, a more general version of the Feedback Linearization problem studied by Brockett was solved using different approaches (see, for example, Korobov [37], Jacubczyk and Respondek [31], Sommer [62], Hunt et al. [28] and Su [64]). A detailed description of the evolution of Feedback Linearization and its influence over some important developments in nonlinear system theory can be reviewed in [41].

One noticeable drawback of feedback linearization is that, in order for it to be applicable, a certain involutivity condition has to hold [29, 41], thus only a limited class of nonlinear systems can be feedback linearizable.

An important limitation of feedback linearization is that it fails to ensure stability when the nonlinear system has unstable zero dynamics or, in analogy with linear systems, when the nonlinear system is "nonminimum phase". It is possible to extend feedback linerization to nonminimum phase systems. One possible approach is via selecting another output of the system with respect to which the system has minimum phase characteristics. A different method [12, 1], is based on the construction of a minimum phase approximation of the original model using an inner-outer factorization.

We present below a brief review of Feedback Linearization. Consider the single-input single-output nonlinear state space system

$$\begin{aligned} \dot{x}(t) &= f(x) + g(x)u(t) \\ y(t) &= h(x) \end{aligned} \tag{17}$$

Assume that $x = 0$ is an equilibrium point of (17) i.e., $f(0) = 0$, and that the nonlinear system (17) has relative degree r defined in a certain neighbourhood U of $x = 0$. Next, consider a stable linear differential operator $p(\rho)$ of degree r

$$p(\rho) = p_r\,\rho^r + p_{r-1}\,\rho^{r-1} + \ldots + 1$$

Then $p(\rho)$, applied to the system output $y(t)$ can be written as

$$p(\rho)y(t) = b(x) + a(x)u(t). \tag{18}$$

where $b(x)$ and $a(x)$ are suitable nonlinear functions of the system states. Also, we have that $a(x) \neq 0\ \forall x \in U$, since the nonlinear system (17) has relative degree r in U. From (18) it is clear that if we apply the control law,

$$u(t) = \frac{y^*(t) - b(x)}{a(x)} \tag{19}$$

we will be able to transform the original nonlinear system into a linear system of the form

$$p(\rho)y(t) = y^*(t) \tag{20}$$

where $y^*(t)$ can be any external signal.

The control law defined in (19) is known as Input-Output Feedback Linearization [30, 29]. One advantage of Feedback Linearization is that it is simple and it allows a straightforward design of the differential operator $p(\rho)$, since the roots of $p(\rho)$ determine the dynamic behaviour of the output of the closed-loop system. Input to state stability issues are addressed, for example in [35].

6 Generalized Feedback Linearization

A nonlinear control strategy related to Feedback Linearization is known as Generalized Feedback Linearization (GFL) [22]. This method aims to retain the essential simplicity of the Feedback Linearization scheme whilst overcoming some of the restrictions. We will see later that this objective will be achievable at the expense of having to compromise the linear behaviour of the closed-loop system. Therefore the "Generalized Feedback Linearization" name reflects the origin of the idea, rather than describing the properties of the resulting system.

In order to introduce the Generalized Feedback Linearization strategy we re-examine equality (20) which leads to the usual Feedback Linearization control policy (19). Previously, the order of the linear differential operator $p(\rho)$ was r i.e., the relative degree of the nonlinear system. We now allow the degree of $p(\rho)$ to be $n_p \geq r$. Notice, at this stage, that from the choice made for n_p the equality $p(\rho)y(t) = b(x) + a(x)u(t)$ will no longer hold, since the relative degree of the nonlinear system has not changed.

As mentioned above, the Feedback Linearization strategy which achieves (20) suffers from a lack of internal stability in the presence of unstable zero dynamics, i.e., the input can diverge. It thus seems heuristically reasonable to match (20) by a similar requirement on the input $u(t)$ of the nonlinear system. Thus, we might ask that the input satisfies a linear dynamic model of the form

$$l(\rho)u(t) = u^*(t) \tag{21}$$

where $l(\rho)$ is a differential operator of degree $n_l = n_p - r$

$$l(\rho) = l_{n_l}\,\rho^{n_l} + l_{n_l-1}\,\rho^{n_l-1} + \cdots + 1 \tag{22}$$

and $u^*(t)$ is the steady state input signal which makes the steady state value of the output $y(t)$ to be equal to $y^*(t)$. The control law expressed by (21) is clearly an open loop strategy. Thus, it is not suitable for open loop unstable systems. Also, conditions (20) and (21) are not, in general, simultaneously compatible. This suggests that we determine the control signal $u(t)$ by combining both conditions in some way. A possible choice is to use a linear combination of (20) and (21), yielding

$$(1-\lambda)\Big[p(\rho)y(t) - y^*(t)\Big] + \lambda\Big[l(\rho)u(t) - u^*(t)\Big] = 0 \tag{23}$$

with $\lambda \in [0,1]$. Equation (23) constitutes the *Generalized Feedback Linearization* (GFL) control law, which can also be written as

$$z(t) \stackrel{\triangle}{=} p'(\rho)y(t) + l(\rho)u(t) = \bar{z}(t) \tag{24}$$

where

$$\begin{aligned} p'(\rho) &= \frac{1-\lambda}{\lambda}\,p(\rho) \\ \bar{z}(t) &= \frac{1-\lambda}{\lambda}\,y^*(t) + u^*(t) \end{aligned}$$

Notice that (24) implicitly defines an improper linear control law which becomes a nonlinear proper control law when the state space model is used to evaluate $p'(\rho)y(t)$. It is clear that this strategy reverts to the usual Feedback Linearization strategy; by taking $\lambda = 0$ in (23). The strategy can handle all stable systems, whether or not they are stably invertible, by taking $\lambda = 1$. By continuity, various combinations of stable and stably invertible dynamics will also be able to be stabilized by this class of control law, depending upon the design of the differential operators $p(\rho)$ and $l(\rho)$, as well as the value of the parameter λ.

To develop the control law implicitly defined in (24), we introduce a dummy variable $\bar{u}(t)$, as follows:

$$\bar{u}(t) = l(\rho)\,u(t) \tag{25}$$

In this way, if the original nonlinear system has relative degree r, then the nonlinear system that relates $\bar{u}(t)$ to $y(t)$ has relative degree n_p, which is equal to the order of the $p'(\rho)$ operator. Hence, $p'(\rho)y(t)$ will depend again explicitly on $\bar{u}(t)$ and we are able to write $p'(\rho)y(t)$ as a nonlinear function of the states, i.e.,

$$p'(\rho)\,y(t) = b(\xi) + a(\xi)\,\bar{u}(t). \tag{26}$$

where $b(\cdot)$ and $a(\cdot)$ are not necessarily the same functions as in (18) and $\xi = [x^T\ \mu^T]^T$ are the states of the extended system formed by the combination of the linear system defined in (25) and the original nonlinear system (17), as follows:

$$\begin{aligned} \dot{x} &= f(x) + g(x)\,\mu_1 \\ \dot{\mu} &= A_l\,\mu + B_l\,\bar{u}(t) \\ y &= h(x) \end{aligned} \tag{27}$$

where, without loss of generality, we have used a controller canonical form for the n_l-order state space linear model given by A_l and B_l

$$A_l = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\frac{1}{l_{n_l}} & -\frac{l_1}{l_{n_l}} & \cdots & -\frac{l_{n_l-1}}{l_{n_l}} \end{bmatrix}; \quad B_l = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{l_{n_l}} \end{bmatrix}$$

The input $u(t)$ of the original nonlinear system becomes now a state of the extended system (27); in particular $\mu_1 = u$. Substituting (25) and (26) into expression (24) which defines the GFL control strategy, we finally obtain the following nonlinear control law:

$$\bar{u}(t) = \frac{\bar{z}(t) - b(\xi)}{1 + a(\xi)} \tag{28}$$

Comparing the GFL control law (28) with the Feedback Linearization control law (19), we can see that there are close connections between the two strategies. The additional term in the denominator of the GFL control law (28) is the one responsible for preventing the controller from cancelling the system zero dynamics. The resulting closed loop obtained from the application of the GFL strategy is depicted in Figure 3.

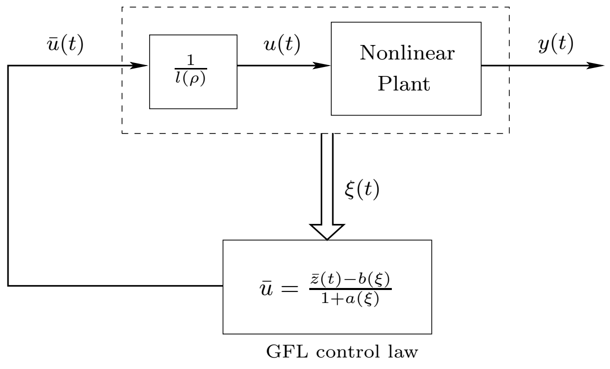

Figure 3: General scheme of the application of the GFL strategy to a nonlinear system

To give further insight into the GFL strategy we next examine its application to the special case of linear systems. Consider the single-input single-output linear system defined by the following ordinary differential equation:

$$A(\rho)y(t) = B(\rho)u(t) \tag{29}$$

where

$$\begin{aligned} A(\rho) &= \rho^n + a_{n-1}\rho^{n-1} + \ldots + a_0 \\ B(\rho) &= b_m\rho^m + b_{m-1}\rho^{m-1} + \ldots + b_0\ ;\ b_m \neq 0 \\ r &= n - m \end{aligned} \tag{30}$$

and where $A(\rho)$ and $B(\rho)$ are assumed relatively prime.

In the sequel, we will be interested in the linear version of (24). In particular, we will show how, for the linear case, explicit dependence on the states can be transformed into dependence on filtered values of the input and output. Towards this end, we present the following:

Lemma 1 *Consider the linear system given in* (29)*, then the output predictor used in* (24) *can always be expressed in terms of the system input and output signals. We call this a Generalized Predictor for the system. In this case we need to choose $n_p = n$, that is the order of the linear system, for reasons that will become clear later. Thus, we can write*

$$p'y(t) = \frac{M}{E}\,y(t) + \frac{GB}{E}u(t) \tag{31}$$

where E and G are differential operators of degree r with E stable, and M is a differential operator of degree $n-1$ satisfying the following identity:

$$Ep' = M + GA \tag{32}$$

Proof: *Starting from the system o.d.e. in* (29) *and multiplying both sides by G we obtain:*

$$GAy(t) = GBu(t)$$

then using (32) *to replace GA, we obtain*

$$(Ep' - M)y(t) = GBu(t) \tag{33}$$

which establishes (31) *after solving for $p'y(t)$.* ■

We next show how the predictor presented in lemma 1 can be used to define a proper feedback control law. This control law will be linear in the case of linear systems. Consider again the GFL control law introduced in equation (24). Our aim is to express this control law as a casual feedback relationship in terms of $u(t)$ and $y(t)$. Substituting expression (31) for the generalized predictor of lemma 1, into equation (24), we have that

$$\frac{M}{E}y(t) + \frac{GB + El}{E}u(t) = \bar{z}(t) \tag{34}$$

which clearly defines a linear proper controller, namely

$$u(t) = -\frac{M}{L}y(t) + \frac{E}{L}\bar{z}(t) \tag{35}$$

where

$$L = GB + El$$

It can be easily verified that the degree of L is equal to n and therefore, in the linear case, the GFL control strategy is equivalent to a standard output feedback linear controller. The closed-loop characteristic polynomial can readily be seen to be $E(Al + Bp')$. Hence, if we choose l and p' such that $Al + Bp' = A^*$ is stable, then the above designs leads to a stable closed-loop system. It is also interesting to examine the implicit "output" defined by

$$z = p'y + lu \tag{36}$$

It is easily seen that this output has stable zero dynamics since

$$Az = Ap'y + Alu = A^*u \tag{37}$$

where A^* is stable by choice of p' and l.

The above special linear case suggests that in the nonlinear case we might choose p' and l so that $[A_o l + B_o p']$ is stable where A_o, B_o denote the denominator and numerator of a linearized model at the required operating point. In this way, we ensure that z has stable zero dynamics locally to the desired operating point. Indeed, via this route, it is relatively easy to establish [55] that the GFL strategy is, at least, locally stabilizing for a broad class of nonlinear systems. Global Stability results for systems of Luré type are also studied in [55].

7 Flatness

Another concept related to that of stable zero dynamics is that of flatness [18]. The main features of differential flatness is the presence of a fictitious output z (called a flat or linearizing output) such that

- every system variable may be expressed as a function of the components of z and of a finite number of its time derivatives.
- z may be expressed as a function of the system variables and of a finite number of their time derivatives; and
- the number of independent components of z is equal to the value of independent input elements.

Usually the search for linearizing outputs is based on physical insights into the problem [17]. To gain insight into the idea we will reexamine the special case of linear single-input single-output systems. Consider a linear – not necessarily minimum phase system having transfer function

$$G(s) = \frac{B(s)}{A(s)} \tag{38}$$

where $B(s)$, $A(s)$ are relatively prime. Since A and B are relatively prime, we can choose l and p, such that $Al + Bp' = 1$. We now define a new system variable by

$$z = p'y + lu \tag{39}$$

Notice that

$$Az = u, \ Bz = y$$

Thus z qualifies as a flat output. Note that stabilizing control can be easily obtained by use of the control law implicitly defined in equation (24) i.e.,

$$z = \bar{z}$$

where $\bar{z}$ depends on the desired operating point.

We note that there is a close connection between finding a flat output and generalized feedback linearization. Specifically, in generalized feedback linearization we use the linear form (39) to define a variable z. The key point in GFL is that we design p and l' such that the zero dynamics associated with z are locally stable. However, when using flatness, we seek some other variable z (defined via a nonlinear relationship) which has stable zero dynamics associated with it.

8 Backstepping

As mentioned in the introduction, backstepping is a method that can be used on systems of special structure to find an output having a passivity property, including relative degree one and stable zero dynamics. The notion of backstepping was probably first used in the context of Adaptive Control to extend known results to the (relative degree)> 1 case [16]. The method was developed and used for general nonlinear systems see [57, 58]. To simplify the presentation we will first consider the simplest case of second order systems having a special form.

8.1 Backstepping in a Simple Case

Consider a system comprising a chain of integrators, i.e.,

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \tag{40}$$
$$\dot{x}_2 = u \tag{41}$$

Notice that the first equation does not depend explicitly on u. To ensure passivity, we require z to be of relative degree one. It suffices that z depends on x_2 via an equation of the form

$$z = x_2 - \alpha(x_1) \tag{42}$$

Next, we choose $\alpha(x_1)$ such that the system has stable zero dynamics. Setting $z = o$, shows that the zero dynamics satisfy

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\alpha(x_1) \tag{43}$$

Hence, we have another stabilization problem. However, this stabilization problem is for a lower dimensional system (here of order 1) in which the state $x_2 = \alpha(x_1)$ is input. Indeed making the special choice

$$\alpha = \frac{-[x_1 + f_1]}{g_1} \tag{44}$$

would yield the zero dynamics as $\dot{x}_1 = -x_1$ which is clearly stable. We next express the original problem in terms of x_1 and z. Specifically, from (40), (42) and (44) we have

$$\dot{x}_1 = -x_1 + g_1 z \tag{45}$$

Also, from (42) and (41)

$$\begin{aligned} \dot{z} &= \dot{x}_2 - \frac{d\alpha}{dx_1}\dot{x}_1 && (46) \\ &= u - \frac{d\alpha}{dx_1}(-x_1 + g_1 z) && (47) \end{aligned}$$

We now introduce the Lyapunov function

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}z^2 \tag{48}$$

Then

$$\begin{aligned} \dot{V} &= x_1\dot{x}_1 + z\dot{z} \\ &= x_1[-x_1 + g_1 z] + z[u - \frac{d\alpha}{dx_1}(-x_1 + g_1 z)] \end{aligned}$$

It is then clear, that if we choose u such that

$$x_1 g_1 + [u - \frac{d\alpha}{dx_1}(-x_1 + g_1 z)] = -z \tag{49}$$

Then

$$\dot{V} = -x_1^2 - z^2 \tag{50}$$

This line of argument can be used [57, 7] to establish that the feedback control law defined via (49), (44) and (42) is globally asymptotically stabilizing. We see again that finding and "output" z having stable zero dynamics plays a key role in this design.

Indeed, the closed-loop system resulting from this control law is

$$\begin{aligned} \dot{x}_1 &= -x_1 + g_1 z && (51) \\ \dot{z} &= -z - g_1 x_1 && (52) \end{aligned}$$

Of course, since z has stable zero dynamics associated with it, we could also attempt to apply feedback linearization. Indeed, choosing $p(\rho) = \rho + 1$, then

$$\begin{aligned} \dot{z} + z &= \dot{x}_2 - \frac{d\alpha}{dx_1}\dot{x}_1 + z && (53) \\ &= u - \left[\frac{d\alpha}{dx_1}\right][f_1 + g_1 x_2] + x_2 - \alpha && (54) \end{aligned}$$

Then the alternative control law is as follows:

$$u = \frac{d\alpha}{dx_1}[f_1 + g_1 x_2] - x_2 - \alpha \tag{55}$$

which yields the following closed-loop system:

$$\begin{aligned} \dot{z} &= -z && (56) \\ \dot{x}_1 &= -x_1 + g_1 z && (57) \end{aligned}$$

Note that this alternative system is input-output stable and locally input-to-state stable. However, global input-state stability requires additional restrictions, e.g., linear growth conditions.

8.2 An Algebraic view of Backstepping

Here we present a more general formulation of backstepping due to Clements and Jiang [8]. It is instructive to first examine the linear case where the system is in lower triangular form

$$\dot{x} = Nx + Ax + bu \tag{58}$$

where

N is a zero matrix save for 1's on the upper diagonal
b is a zero vector save for the last entry which is unity
A is lower triangular

This represents a string of integrators together with some feedback defined via A. We now define a new state vector z by

$$z = x + Hx \tag{59}$$

and a new input ν by

$$\nu = u + k^T x \tag{60}$$

where H is strictly lower triangular, k is any row vector. The closed-loop system with input ν and state z satisfies

$$\dot{z} = Nz + \tilde{A}z + B\nu \tag{61}$$

where

$$\tilde{A} = (I + H)[(N + A) - bk^T](I + H)^{-1} - N \tag{62}$$

which also has the same structure as A. The associated design problem is to find k and H that yield a particular $\tilde{A}$. Rearranging (62), we have

$$NH + bk^T = A + H(N + A) - \tilde{A}(I + H) \tag{63}$$

Now, let $h_{1\bullet}, \ldots, h_{n\bullet}$ denote the rows of H
$a_{1\bullet}, \ldots, a_{n\bullet}$ denote the rows of A
$\tilde{a}_{1\bullet}, \ldots, \tilde{a}_{n\bullet}$ denote the rows of $\tilde{A}$

Then, equation (63) can be written row by row for $i = 1, \ldots, n-1$ as

$$h_{i+1,\bullet} = a_{i\bullet} + h_{i\bullet}(N + A) - \tilde{a}_{i\bullet} - \sum_{j=1}^{i-1} \tilde{a}_{ij} h_{j\bullet} \tag{64}$$

initialized by $h_{1\bullet}$ and from the last row

$$k^T = a_{n\bullet} + h_{n\bullet}(N + A) - \tilde{a}_{n\bullet} - \sum_{j=1}^{n} \tilde{a}_{nj} h_{j\bullet} \tag{65}$$

Setting $h_{1\bullet} = 0$, these equations uniquely define k and a strictly lower triangular H. Equations (64)and (65) are the (linear version) of the backstepping equations. We next develop the nonlinear version of this idea. Let $\vec{x}_i$ denote the first i components of x. A matrix function f is said to be lower triangular dependent (strictly) on x if the i^{th} row is a function of $\vec{x}_i(\vec{x}_{i-1})$. We also say that a function f is lower triangular (strictly) if all the entries above (above and on) the diagonal are zero.

Consider the nonlinear system

$$\dot{x} = Nx + \phi(x) + bu \tag{66}$$

where $\phi(x)$ is lower triangularly dependent on x. This system is said to be in "strict feedback form". We then define a new state variable

$$z = x + N^T f(x) \tag{67}$$

and an auxiliary input

$$\upsilon = u + b^T f(x) \tag{68}$$

defined in terms of one function f which we take to be lower triangular dependent on x. Due to the structure of $N^T f(x)$ and $b^T f(x)$, the above transformations are globally invertible: i.e., there exists a function $\tilde{f}(z)$ which is lower triangularly dependent on z such that

$$x = x + N^T \tilde{f}(z) \tag{69}$$

$$u = \upsilon + b^T \tilde{f}(z) \tag{70}$$

Also note that

$$(Nz + bv) = Nx + bu + f(x) \tag{71}$$

since $NN^T + bb^T = I$. Also,

$$N^T[\nabla_x f(x)]b = 0 \tag{72}$$

The desired loop dynamics for z are then simply seen to satisfy

$$\begin{aligned}
\dot{z} &= \dot{x} + N^T[\nabla_x f(x)]\dot{x} \\
&= Nx + bu + \phi(x) + N^T[\nabla_x f(x)](Nx + \phi(x)) \\
&= Nz + bv - f(x) + \phi(x) + N^T[\nabla_x f(x)](Nx + \phi(x)) \\
&= Nz - f(x) + \phi(x) + N^T[\nabla_x f(x)](Nx + \phi(x)) + bv \\
&= Nz + \tilde{\phi}(z) + bv
\end{aligned}$$

where

$$\tilde{\phi}(z) = [-f(x) + \phi(x) + N^T[\nabla_x f(x)](Nx + \phi(x))]_{x=z+N^T\tilde{f}(z)} \tag{73}$$

It is easy to see that $\tilde{\phi}(z)$ is again lower triangularly dependent on z. Now, as in the linear case, we seek to find an f such that we achieve a given $\tilde{\phi}$. Equation (73) can be rewritten as

$$f(x) = \phi(x) + N^T[\nabla_x f(x)](Nx + \phi(x)) - \tilde{\phi}(x + N^T f(x)) \tag{74}$$

Now based on the triangular dependence of the various quantities, this equation can be readily solved for f in a row by row fashion. This yields the backstepping equations for systems in strict feedback form.

The above development has been purely structural. However, since we can generate an arbitrary $\tilde{\phi}(z)$ it is now reasonable to ask how we might choose $\tilde{\phi}(z)$ to insure (Lyapunov) stability. For example, we could choose $\tilde{\phi}(z)$ as

$$\tilde{\phi}(z) = -N^T z - C(z,t)z \tag{75}$$

$C(z,t)$ is lower triangularly dependent on z and is positive semi-definite. Then consider the Lypunov function

$$V(z) = \frac{1}{2}z^T z \tag{76}$$

Then

$$\begin{aligned}
\dot{V}(z) &= z^T(Nz + \tilde{\phi}(z)) \\
&= z^T(N - N^T - C(z,t))z \\
&= -z^T C(z,t)z \\
&\leq 0
\end{aligned} \tag{77}$$

which insures that the system is Lyapunov stable.

9 Illustrative Examples

Finally, we present some simple nonlinear control system design problems. These are motivated by physical systems. They could be used, for example, as bench mark problems to test new nonlinear design methods.

9.1 Boost Converter

We begin with an example taken from [19] of a dc-dc power converter. The average behaviour of a modulated boost converter circuit may be described by

$$\begin{aligned}
\dot{x}_1 &= -C_1 u x_2 + C_2 \\
\dot{x}_2 &= C_3 u x_1 - C_4 x_2
\end{aligned}$$

where x_1 is the average inductor current, x_2 is the average output capacitor voltage and u is the duty cycle taking values in the interval $[0,1]$, and constituting the input. Note that x_2 has unstable zero dynamics. It is shown in [19] that a suitable flat output , z, is given by the average stored energy.

$$z = \frac{1}{2C_1}x_1^2 + \frac{1}{2C_3}x_2^2 \tag{78}$$

It can be shown that (as required by flatness)

$$\begin{aligned}
x_1 &= f_1(z, \dot{z}) &(79)\\
x_2 &= f_2(z, \dot{z}) &(80)\\
u &= f_3(z, \dot{z}, \ddot{z}) &(81)
\end{aligned}$$

where $f_1(\cdot),\ f_2(\cdot),\ f_3(\cdot)$ are particular nonlinear functions. Now, say that we want z to satisfy the following linear dynamics:

$$\ddot{z} + a_1\dot{z} + a_o z = a_o z^* \quad (82)$$

In this case, we have from the (81) that a stabilizing feedback loop around the set point can be obtained by setting

$$u = f_3(z, \dot{z}, -a_1\dot{z} - a_o z - a_o z^*) \quad (83)$$

In the simulations presented below we compare the performance of GFL and the controller based on flatness described above. As in [19], we choose $C_1 = 50,\ C_2 = 750,\ C_3 = 50 * 10^3,\ C_4 = 667$. For the GFL design, we linearized about the operating point $[u_o = 0.25,\ y_o = 60]$ and set $A^*(s) = (s+300)^2(s+600)$ to design p' and l which are then converted into nonlinear feedback via the state equations. Also, we choose $\lambda = 0.37$. For the design based on flatness we choose $a_1 = 360,\ a_o = 4,000$.

Figure 4 shows two responses based on a flatness design. The faster trajectory (having greater undershoot) is defined as in (83). The slower response (with less undershoot) uses the alternative smooth trajectory as defined in [19]. Note that, in a similar fashion to the linear case [20], there exists a connection between the extent of undershoot and the rise time. An open research question is how to formally quantify this relationship in the nonlinear case.

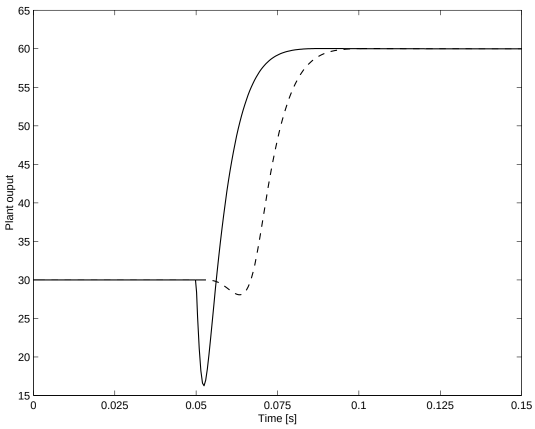

Figure 4:

Figure 5 shows response obtained form the GFL design. Our experience was that the designs based on flatness were easier to tune once one had found the flat output. Choosing p' and l in the GFL design took time. We also tested a purely linear design obtained from the linearized model. This performed well provided the linearization point was chosen carefully but was less robust to the point of linearization than the GFL strategy.

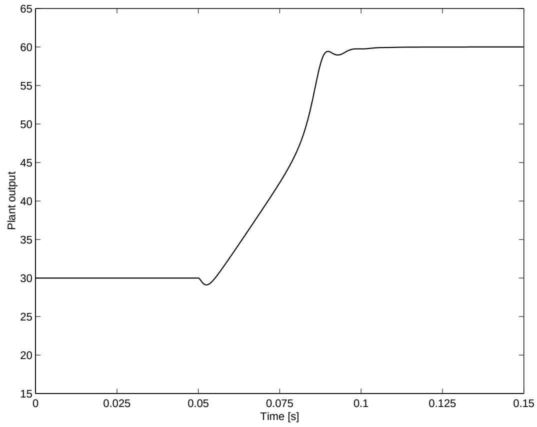

Figure 5:

9.2 Cyclopentenol Synthesis

This system is typical of many chemical and biochemical systems [14, 13]. Other related examples are discussed in [24]. We assume that the volume of the system is constant and that isothermal conditions apply. We consider a set of reactions, in which an input component A reacts as follows:

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$
$$2A \xrightarrow{k_3} E$$

Let $x_1,\ x_2,\ d$ denote the concentration of A, concentration of B and input concentration respectively. Then, if follows that a suitable model can be written as

$$\dot{x}_1 = -k_1x_1 - k_3x_1^2 + (d - x_1)u \quad (84)$$
$$\dot{x}_2 = k_1x_1 - k_2x_2 - x_2u \quad (85)$$
$$y = x_2 \quad (86)$$

where u is the incoming (and outgoing) flow rate. This system has relative degree 1 and unstable zero dynamics. In order to highlight the system zero dynamics we need to transform the system state space representation (84), (85) into an appropriate *normal form* [35]. We suggest to use the following change of variables:

$$z = \Phi(x) \Longrightarrow \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{d-x_1}{x_2} \end{bmatrix} \quad (87)$$

which defines a global diffeomorphism, since its inverse is well defined $\forall x, z \in \mathbb{R}^2$

$$x = \Phi^{-1}(z) \Longrightarrow \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] = \left[\begin{array}{c} d - z_1 z_2 \\ z_1 \end{array} \right] \quad (88)$$

The normal form obtained from the application of the change of variables (87) is given by

$$\begin{aligned} \dot{z}_1 &= k_1(d - z_1 z_2) - k_2 z_1 - z_1 u \\ \dot{z}_2 &= \frac{k_3(d - z_1 z_2)^2 + k_1(1 - z_2)(d - z_1 z_2) + k_2 z_1 z_2}{z_1} \\ y &= z_1 \end{aligned}$$

Assuming we want to keep the system output $y(t)$ fixed at the value $y(t) = z_1^0$, the system *zero dynamics* are defined by

$$\dot{z}_2 = \frac{k_3(d - z_1^0 z_2)^2 + k_1(1 - z_2)(d - z_1^0 z_2) + k_2 z_1^0 z_2}{z_1^0} \quad (89)$$

In the sequel we take $k_1 = 39.58,\ k_2 = 39.58,\ k_3 = 5.43,\ d = 5.1$. The system shows either a minimum or nonminimum-phase behaviour depending upon the operating point around which the plant is evolving. The linearized dynamics are further explored in Figure 6 which shows the steady state response in x_1 and x_2 versus the input u, the location of the (linearized) zero z versus the input u^o (at the local operating point) and the location of the (linearized) poles versus the input u^o (at the local operating point).

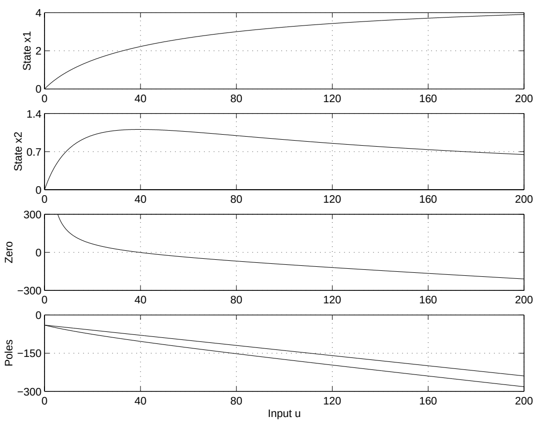

Figure 6:

Note that the linearized gain to x_2 is positive for $u_o \in [0, 40]$ and negative for $u_o \in [40, \infty]$. Similarly, the system is nonminimum phase for $u_o \in [0, 40]$ and minimim phase for $u_o \in [40, \infty]$. These characteristics make control of this seemingly simple system challenging. In the sequel, we assume that the input u must lie in the range $[0, 200]$.

(i) GFL: To apply the Generalized Feedback Linearization strategy we use a differential operator $p(\rho)$ of order $n_p = n = 2$ and a $l(\rho)$ differential operator of order $n_l = n_p - r = 1$

$$\begin{aligned} p(\rho) &= p_2\, \rho^2 + p_1\, \rho + 1, \\ l(\rho) &= l_1\, \rho + 1. \end{aligned} \quad (90)$$

We know that using the auxilliary variable $\bar{u}(t) = l(\rho)\, u(t)$ the expression for $p(\rho)\, y(t)$ can be written as

$$p(\rho)\, y(t) = b(\xi) + a(\xi)\, \bar{u}(t). \quad (91)$$

where ξ represents the states of the extended system obtained by the inclusion of the dynamics of $\frac{1}{l(s)}$.

For the system described in (84) to (86) we have

$$\begin{aligned} b(\xi) &= p_2 \left[k_1 \dot{x}_1 - (k_2 + u)\dot{x}_2 + \frac{1}{l_1} x_2 u \right] + \\ &\quad\ p_1 \left[k_1 x_1 - k_2 x_2 - x_2 u \right] + x_2 \\ a(\xi) &= -\frac{p_2}{l_1} x_2 \end{aligned} \quad (92)$$

Since, in this problem, the steady state gain changes sign, we were not able to use the same GFL design for wildly differing operating points. Figure 7 shows the response in the nonminimum phase region for different values of λ. Notice that as λ is decreased the response becomes faster but the undershoot greater.

(ii) A Simple Switching Control Law: Finally, we adopt an "engineering" approach to this problem. Figure 6 indicates that x_1 uniquely defines the system steady state operating point. Since the input is constrained to lie in the range $[0, 200]$, some ad-hoc reasoning leads to the following simple switching control law (implemented with hysteresis of 0.05).

$$\begin{aligned} u(t) &= 200 \text{ if } x_1 < x_1^* \\ u(t) &= 0 \text{ if } x_1 > x_1^* \end{aligned}$$

where x_1^* defines the desired steady state value of x_1. Figure 8 shows the response of x_2 when x_1^* is changed

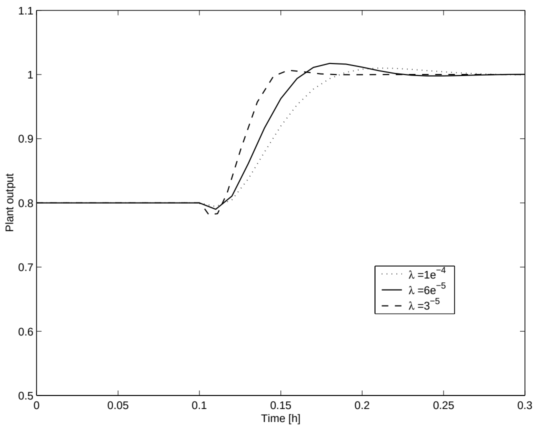

Figure 7:

from 1 to 3 (i.e., from the operating point $[u_o = 10,\ x_2 = 0.7]$ to the operating point $[u_o = 80,\ x_2 = 1]$). The "inverse-response" characteristic of x_2 is evident in this plot. The final limit cycle is due to the switching nature of the controller and could be removed by use of a local controller.

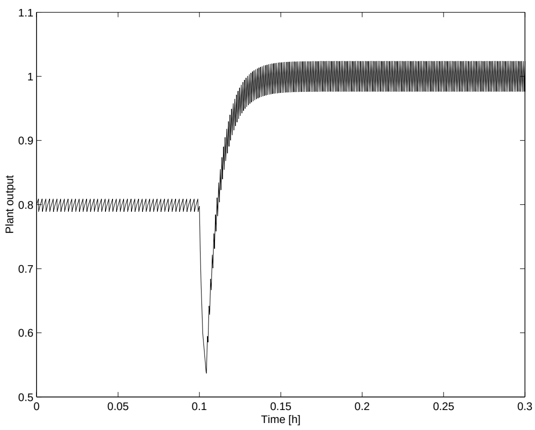

Figure 8:

The responses obtained here is quite good. The moral is that one should not be afraid of nonlinear problems and that, sometimes, an acceptable control strategy can be obtained by ad-hoc design provided it captures the key features of the problem under study.

10 Conclusions

Nonlinearities are rarely "too far from the surface" in real world control problems. Indeed, it might be argued that certain kinds of nonlinear behaviour (e.g., actuator amplitude and slew rate constraints) are ubiquitous issues in control. This paper has given a brief overview of some of the simpler strategies for nonlinear control. Some of these methods, e.g., feedback linearization and backstepping can be seen to be very close to linear methods and, as such, are easy to use in practice. Other methods are more subtle and require specialist knowledge to understand. Our introduction to nonlinear control has necessarily been brief – in part limited by space but perhaps, more truthfully, by our own expertise.

An important question not answered here is, "How can one distinguish those systems where nonlinear control will give genuine performance enhancements from those systems where a simple linear control will do just as well?" This question is, to the best of the authors' knowledge, largely open. Notwithstanding this observation, nonlinear thinking can be of major benefit in practical problems and we encourage readers not to be afraid to try these techniques which can, if the circumstances are right, lead to major benefits.

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