Introduction to Lyapunov functions and their use in robustness analysis

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SHy Seminar
01/04/15

Overview

Context

Essential questions in many control applications:

- Does the system eventually behave as I want it to?

Convergence

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- Does the systems interact properly with other systems?

Convergence | Stability | Robustness | Interconnection

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Examples of GES Systems That can be Driven to Infinity by Arbitrarily Small Additive Decaying Exponentials

A. R. Teel and J. Hespanha

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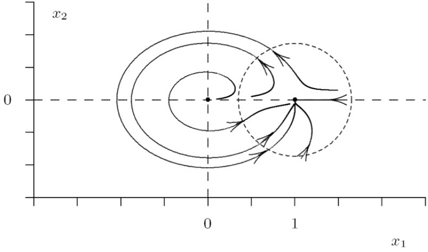

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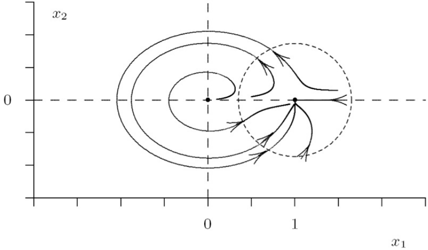
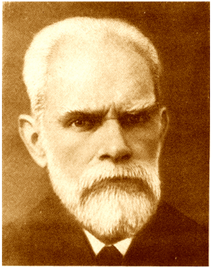


- Key tools: Lyapunov functions and Input-to-State Stability (ISS).

Alexandr Mikhaïlovitch Lyapunov

Short bio:

- Born in 1857 in Yaroslavl, Russian Empire
- Studies at Saint Petersburg
- Ph.D. thesis: The general problem of the stability of motion (1892)
- Chair of mechanics at Kharkiv university (1895)
- Professor in applied mathematics at Saint Petersburg university (1902)
- Member of Russian Academy of Sciences
- Died in 1918, three days after his wife.



State-space representation

- Any dynamical system ruled by an ordinary differential equation with constant coefficients can be written as:

$$\dot{x}(t) = f(x(t), u(t))$$

  - $x(t) \in \mathbb{R}^n$ is the state
  - $u(t) \in \mathbb{R}^m$ is an exogenous input (control or disturbance)
  - $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a function smooth enough to ensure existence and uniqueness of solutions.

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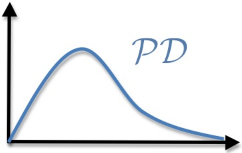
- Linear time-invariant (LTI) case:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

Comparison functions



$$\begin{cases} \alpha \text{ continuous} \\ \alpha(0) = 0 \\ \alpha(s) > 0, \forall s > 0 \end{cases}$$

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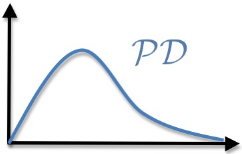

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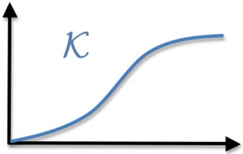

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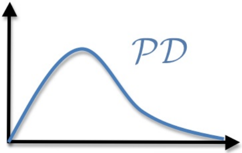

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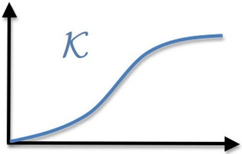

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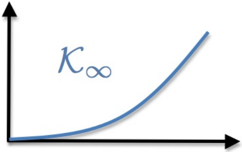

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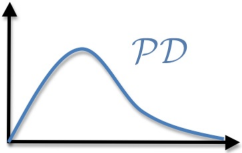

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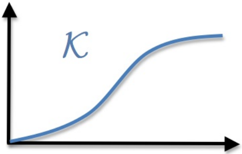

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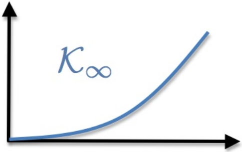

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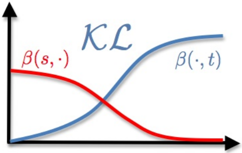

$$\begin{cases} \beta(\cdot,t) \in \mathscr{K}, \forall t \geq 0 \\ \beta(s,\cdot) \text{ nonincreasing}, \forall s \geq 0 \\ \lim_{t\to\infty} \beta(s,t) = 0, \forall s \geq 0 \end{cases}$$

Stability and convergence

We start by considering systems without inputs ($u \equiv 0$).

Definition: Global Asymptotic Stability

The origin of the system $\dot{x} = f(x,0)$ is GAS if there exists $\beta \in \mathcal{KL}$ such that, for all $x_0 \in \mathbb{R}^n$,

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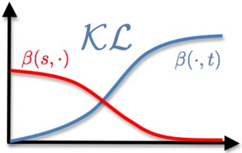

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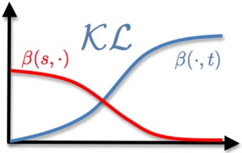

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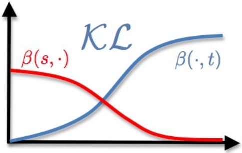

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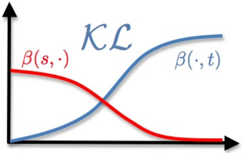

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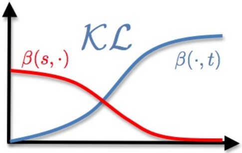

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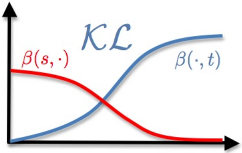

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- $\beta(|x_0|,t) = k_1|x_0|e^{-k_2 t}$: GES implies GAS
- For LTI systems: GAS and GES are equivalent.

ISS and iISS notions

What about systems with inputs ?

Definition: Input-to-State Stability (ISS), [Sontag, 1989]

The system $\dot{x} = f(x,u)$ is ISS if there exist $\beta \in \mathscr{K}\mathscr{L}$ and $\gamma \in \mathscr{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathscr{U}^m$,

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Strengths and weaknesses

ISS and iISS: central tools in analysis and control:

- Theoretical contributions to: output feedback, optimal control, hybrid systems, predictive control, chaotic systems...
- Applications in: robotics, production lines, transportation, bio-chemical networks, control under communication constraints, neuroscience, ...

ISS	iISS
$\dot{x} = f(x,0)$ is GAS	$\dot{x} = f(x,0)$ is GAS
Bounded input $\Rightarrow$ Bounded state	Bounded energy input $\Rightarrow$ Bounded, converging state
Converging input $\Rightarrow$ Converging state	Small inputs may yield unbounded state
Cascade: ISS + ISS $\Rightarrow$ ISS	Converging input $\not\Rightarrow$ Converging state
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In practice, some systems do exhibit robustness for inputs under a certain threshold, but diverge for larger ones.

Strong iISS: halfway between ISS and iISS.

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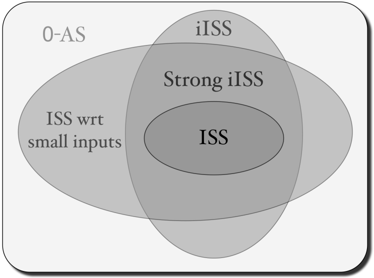

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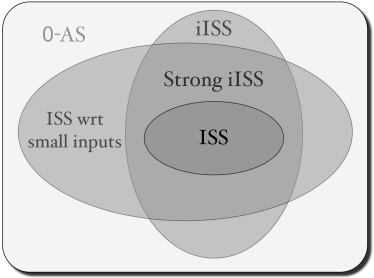

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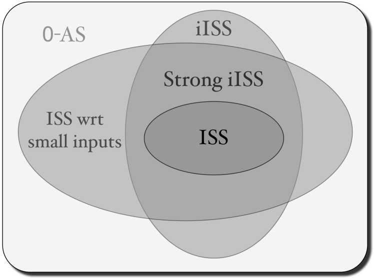

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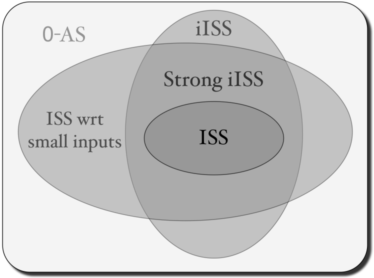

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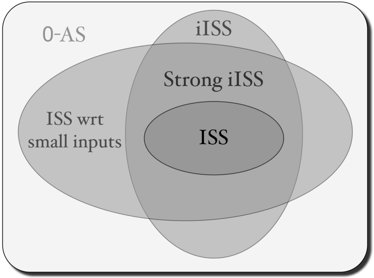

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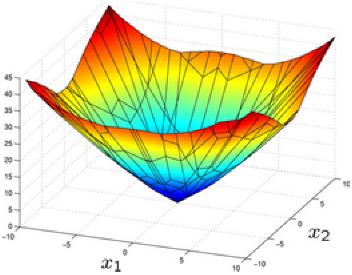
Lyapunov function candidate

- Lyapunov second method for stability analysis
- Lyapunov function candidate (LFC):
  - $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuously differentiable
  - $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$
  - $V(x) \to \infty$ whenever $|x| \to \infty$.
- LFC essentially behaves like a norm:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),$$

with $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.



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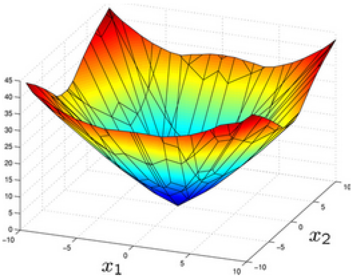

Lyapunov functions

GAS characterization

The origin of the system $\dot{x} = f(x,0)$ is GAS if and only if there exist a LFC V and $\alpha \in \mathscr{PD}$ such that

$$\frac{\partial V}{\partial x}(x) f(x,0) \leq -\alpha(|x|), \quad \forall x \in \mathbb{R}^n.$$

- $\dot{V} < 0$ as long as the origin is not reached
- Purely algebraic condition: no need to know about solutions
- Necessary and sufficient condition.

LTI systems: Lyapunov equation

Let $A \in \mathbb{R}^{n \times n}$ be a Hurwitz matrix. Then, given any $Q = Q^T > 0$, there exists $P = P^T > 0$ such that

$$A^T P + PA = -Q.$$

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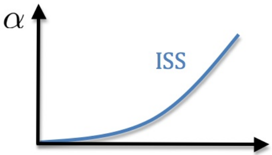

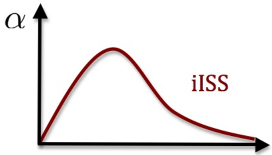

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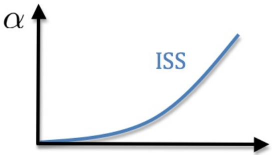

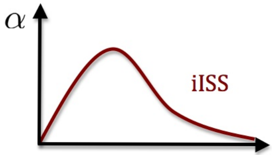

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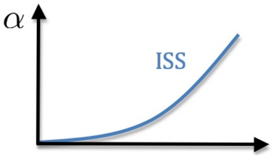

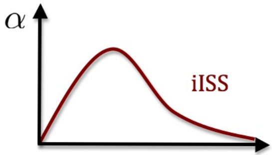

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Lyapunov conditions for Strong iISS

Theorem: $\mathscr{K}$ dissipation rate $\Rightarrow$ Strong iISS [Chaillet, Angeli, Ito 2014]

If there exists a LFC $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$,

$$\frac{\partial V}{\partial x}(x)f(x,u) \leq -\alpha(|x|) + \gamma(|u|).$$

where $\alpha \in \mathscr{K}$ and $\gamma \in \mathscr{K}_\infty$, then the system $\dot{x} = f(x,u)$ is Strongly iISS with input threshold $R = \gamma^{-1} \circ \alpha(\infty)$.

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Strong iISS for bilinear systems

Special case: bilinear systems

$$\dot{x} = \left(A + \sum_{i=1}^{m} u_i A_i \right) x + Bu \,,$$

for which iISS was established in [Sontag 1998, Theorem 5].

Theorem: Strong iISS for bilinear systems [Chaillet, Angeli, Ito 2014]

The above system is Strongly iISS if and only if A is Hurwitz. Moreover, let $P = P^T > 0$ and $Q = Q^T > 0$ be such that $A^T P + PA \leq -Q$ then an estimate of the input threshold is

$$R = \frac{\lambda_0 p_m}{(2 + p_m)\,|P|\max\{b_M ; \sum_{i=1}^{m} |A_i|\}} \,,$$

where λ_0 is the smallest eigenvalue of $\det(Q - \lambda P) = 0$, p_m is the smallest eigenvalue of P and b_M is the largest singular value of B.

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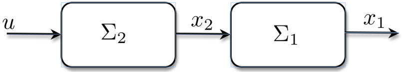
Cascades of ISS systems

Cascade structure:



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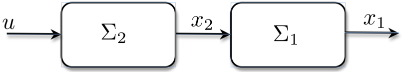

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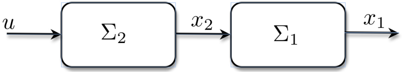

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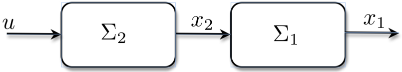

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Let V_1 and V_2 be two LFC. Assume that there exist $\nu_1, \gamma_1, \gamma_2 \in \mathcal{K}$, and $\alpha_1, \alpha_2 \in \mathcal{PD}$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $u \in \mathbb{R}^m$,

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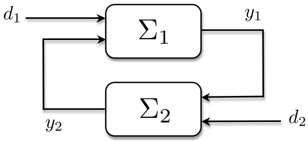
Feedback interconnection of ISS systems



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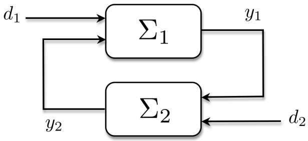

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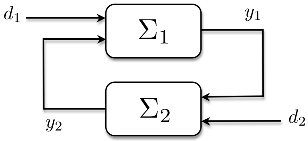

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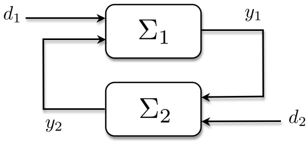

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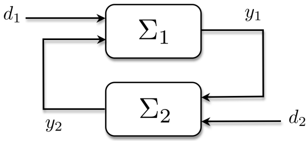

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Theorem: Small gain for ISS [Jiang et al. 1996]

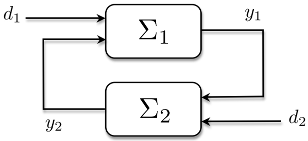
For each $i \in \{1,2\}$, assume that there exist a LFC $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ and $\alpha_i, \gamma_i, \chi_i \in \mathcal{K}_\infty$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $(d_1, d_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$,

$$\begin{aligned} V_1(x_1) \geq \max\{\chi_1(V_2(x_2)); \gamma_1(|d_1|)\} \quad &\Rightarrow \quad \frac{\partial V_1}{\partial x_1}(x_1) f_1(x_1, x_2, d_1) \leq -\alpha_1(|x_1|) \\ V_2(x_2) \geq \max\{\chi_2(V_1(x_1)); \gamma_2(|d_2|)\} \quad &\Rightarrow \quad \frac{\partial V_2}{\partial x_2}(x_2) f_2(x_1, x_2, d_2) \leq -\alpha_2(|x_2|). \end{aligned}$$

If $\chi_1 \circ \chi_2(s) < s$ for all $s > 0$, then the feedback system is ISS.

- Several variants: e.g. [Jiang et al. 1994, Teel 1996, Karafyllis & Tsinias 2004]
- Extension to multiple loops: [Dashkovskiy et al. 2010, Liu et al. 2011]
- Extension to Strong iISS systems: [Ito & Jiang 2009].

Feedback interconnection of ISS systems



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Switched systems in triangular form

Consider a switched system in triangular form:

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_i \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f^1_\sigma(x_1, x_{2\to n}) \\ \vdots \\ f^i_\sigma(x_i, x_{i+1\to n}) \\ \vdots \\ f^{n-1}_\sigma(x_{n-1}, x_n) \\ f^n_\sigma(x_n) \end{pmatrix}, \tag{2}$$

- $\sigma : \mathbb{R}_{\geq 0} \to \{1, \ldots, p\}$ is the switching signal
- x_i is scalar and depends on the driving states $x_{i\to n} := (x_i, \ldots, x_n)$.

Theorem: Switched systems in triangular form [Sene, Chaillet, Balde, 2015]

Assume that each subsystem $\dot{x}_i = f^i_k(x_i, u)$ is Strongly iISS and that each subsystem $\dot{x}_n = f_k(x_n)$ is GAS. Then the switched system (2) is GAS under arbitrary switching.

- Conditions on non-switched dynamics only.

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Switched bilinear systems in triangular form

Application: switched system whose modes are all bilinear systems in triangular form

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_i \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} -a_\sigma^1 x_1 + \sum_{j=2}^n b_\sigma^{1,j} x_1 x_j \\ \vdots \\ -a_\sigma^i x_i + \sum_{j=i+1}^n b_\sigma^{i,j} x_i x_j \\ \vdots \\ -a_\sigma^{n-1} x_{n-1} + b_\sigma^{n-1,n} x_{n-1} x_n \\ -a_\sigma^n x_n \end{pmatrix}, \qquad (3)$$

- a_k^i and $b_k^{i,j}$, $i,j \in \{1,\dots,n\}$, $k \in \{1,\dots,p\}$, are real constants
- each state x_i depends only on the driving states $x_i,\dots,x_n$.

Proposition: switched bilinear systems in triangular form

The system (3) is GAS under arbitrary switching if and only if $a_k^i > 0$ for all $k \in \{1,\dots,p\}$ and all $i \in \{1,\dots,n\}$.

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What we have discussed about:

- Introduction to GAS, ISS, iISS and Strong iISS and their Lyapunov characterization
- Useful tools for stability, robustness and interconnection analyses
- Illustration: switched dynamics is triangular form.

	iISS	Strong iISS	ISS
0-GAS	Yes	Yes	Yes
BIBS	No	For $\|u\| < R$	Yes
CICS	No	Yes	Yes
Cascade preservation	Under growth cond.	Yes	Yes

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Perspective:

- Lyapunov characterization for Strong iISS is still missing
- Envisioned applications:
  - Control with quantized measurements
  - Event-triggered systems
  - Saturated control.

Some reading[1]:

- Book on Lyapunov theory:
  - [Khalil 1996. Nonlinear Systems]
- Survey on ISS and iISS:
  - [Sontag 2008. Input to state stability: Basic concepts and results]
- Strong iISS:
  - [Chaillet, Angeli, Ito 2014. Combining iISS and ISS with respect to small inputs: the Strong iISS property]
- Systems in cascade:
  - ISS: [Sontag, Teel 1995. Changing supply functions in input/state stable systems]
  - iISS: [Chaillet, Angeli 2008. Integral Input-to-State Stable systems in cascade]
  - Strong iISS: [Chaillet, Angeli, Ito 2014. Strong iISS is preserved under cascade interconnection]
- Small-gain theorems:
  - Single loop: [Jiang, Mareels, Wang 1996. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems]
  - Multiple loops: [Dashkovskiy, Rüffer, Wirth 2010. Small gain theorems for large scale systems and construction of ISS Lyapunov functions]
  - Strong iISS: [Ito, Jiang 2009. Necessary and sufficient small gain conditions for integral input-to-state stable systems: A Lyapunov perspective].

[1] This list includes shameless auto-promotion.