Linearization + eps

Example 1

A second-order differential equation of the sort occurring in robotic systems is

$$m\ddot{q} + mL\dot{q}^2 + mgL \sin q = \tau$$

where q(t) is an angle and τ(t) is an input torque. By defining the state $x = [x_1 \quad x_2]^T$ as

$$x_1 = q(t), \quad x_2 = \dot{q}(t)$$

one may write the state equation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -Lx_2^2 - gL \sin x_1 + \frac{1}{m}u$$

where the control input is u(t)= τ(t). To solve the second-order differential equation one requires two initial conditions, e.g. $q(0), \dot{q}(0)$. Thus, there are two state components. The state components correspond to energy storage variables. For instance, in this case one could think of potential energy mgh (the third term in the differential equation, which involves $q(t)$), and rotational kinetic energy $m\omega^2$ (the second term, which involves $\omega = \dot{q}(t)$).

This is a nonlinear state equation. One can place it into the form (1) simply by noting that $x = [x_1 \quad x_2]^T$ and writing

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -Lx_2^2 - gL \sin x_1 + \frac{1}{m}u \end{bmatrix} \equiv f(x,u)$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

This defines *f(x,u)* as the given nonlinear function 2-vector.

By computing the Jacobians, the linear SV representation is found to be

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -gL \cos x_1 & -2Lx_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u = Ax + Bu. \quad , A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}$$

Evaluating this at a nominal equilibrium point of x=0, u=0 yields the linear time-invariant state description

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -gL & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u = Ax + Bu$$

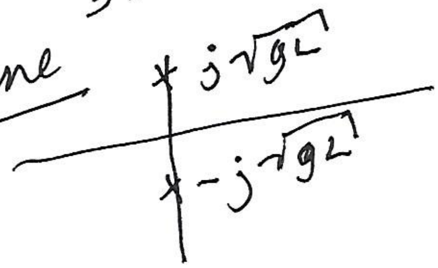
which describes small excursions about the origin.

$$\Delta(s) = |sI - A| = \begin{vmatrix} s & -1 \\ gL & s \end{vmatrix} = s^2 + gL = 0$$

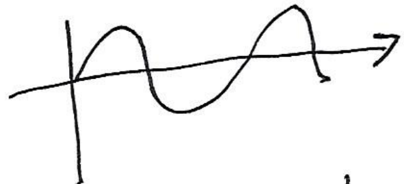
$$s^2 = -gL$$

$$s = \pm j\sqrt{gL} = \pm j\beta$$

s-plane



time Domain

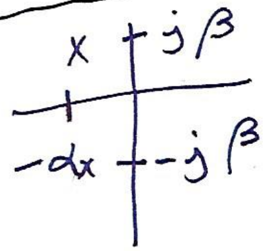


$$k_1 \sin\beta t + k_2 \cos\beta t$$

$$\cancel{e^{-\alpha t}} \sin\beta t = \sin\frac{2\pi t}{T}$$

$$T = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{gL}}$$

s plane



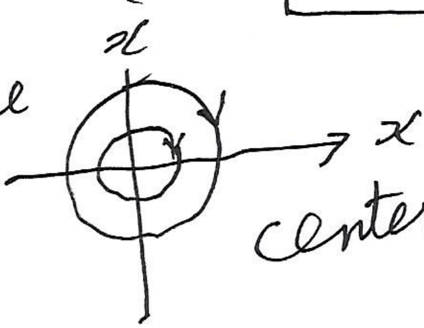
review

$$(s+\alpha)^2 + \beta^2 = 0$$

$$s^2 + 2\alpha s + \underbrace{\alpha^2 + \beta^2}_{w_n^2} = 0$$

$$s^2 + 2\alpha s + w_n^2$$

phase plane



center pt

EE 5321

$x = x_1 e^{-\lambda_1 t} + x_2 e^{-\lambda_2 t}$

$J = M^{-1} A M$

$A = M J M^{-1}$

$[v_1 \; v_2] \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix}$

$x(t) = \sum_{i=1}^{n} v_i e^{\lambda_i t} w_i^T x(0)$

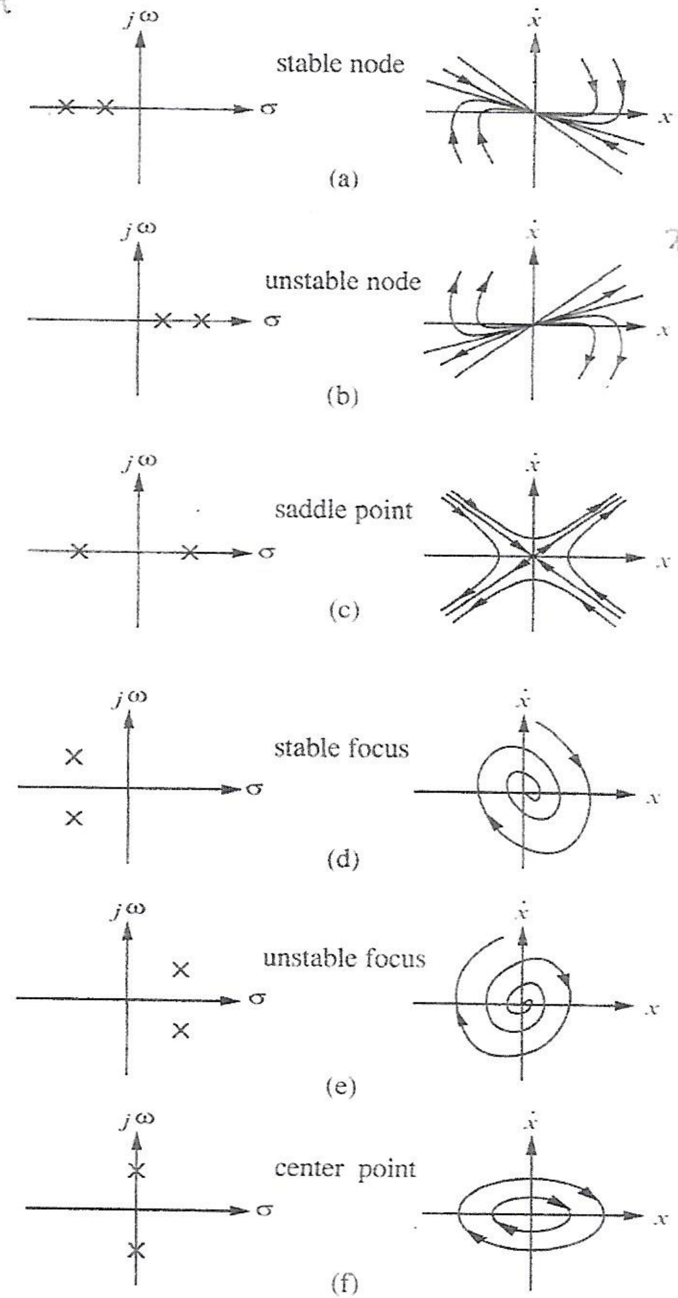


Figure 2.9 : Phase-portraits of linear systems