

EE 5323 - HW03

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HW03 – Nonlinear System Simulations

EE 5323 – Nonlinear Systems

Dr. Lewis

Exercise 1

Volterra Predator-Prey System

Consider the Volterra predator-prey system,

$$\dot{x}_1 = -x_1 + x_1 x_2$$

$$\dot{x}_2 = x_2 - x_1 x_2$$

Find the equilibrium points and their nature.

Answer

State variable is given as:

$$\dot{x}_1 = -x_1 + x_1 x_2$$

$$\dot{x}_2 = x_2 - x_1 x_2$$

The Volterra predator-prey system has limit cycles therefore the system is at equilibrium when the population of both predator and prey remain constant; thus, the derivative should be zero. To find the equilibrium, I set $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. Solve the system for its roots.

$$\dot{x}_1 = 0 \Rightarrow 0 = -x_1 + x_1 x_2$$

$$\dot{x}_2 = 0 \Rightarrow 0 = x_2 - x_1 x_2$$

$$0 = x_1(\beta x_2 - \alpha) \Rightarrow x_1 = 0; x_2 = \alpha/\beta$$

$$0 = x_2(\gamma - \sigma x_1) \Rightarrow x_1 = \gamma/\sigma; x_2 = 0$$

There are two equilibrium points at (x_1, x_2) ,

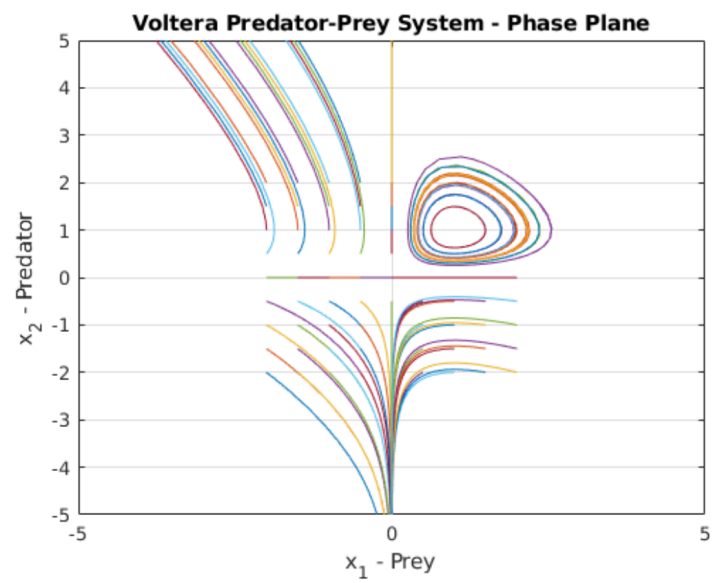
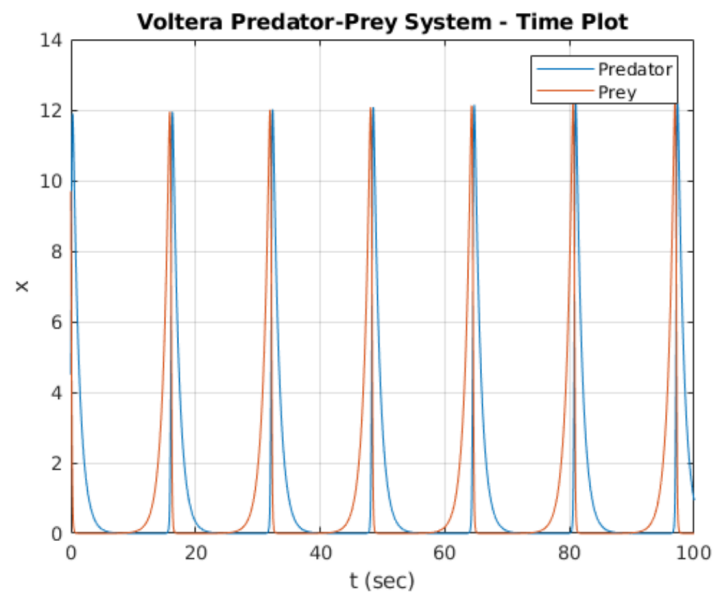
- At zero, $(0, 0)$,
- Any positive pair of integers $(\alpha/\beta, \gamma/\sigma)$

The equilibrium point nature of the zero is a stable center point that is a limit cycle. The other e.p. has a saddle point nature because it is stable in one dimension (goes to zero) and unstable in the other (goes to infinity).

Matlab Code

```
1 %% HW03 - Q01 - Voltera Predator-Prey System
2 % @author: Bardia Mojra
3 % @date: 09/28/2021
4 % @title HW03 - Q01 - Voltera Predator-Prey System
5 % @class ee5323 - Nonlinear Systems
6 % @professor - Dr. Frank Lewis
7
8 clc
9 close all
10 warning('off','all')
11 warning
12
13 x0_set = -2:.5:2;
14 t_intv= [0 100];
15 x_0= [4.5, 9.7]'; % initial conditions for x(t)
16
17 figure
18 [t,x]= ode23('Voltera', t_intv, x_0);
19 plot(t,x)
20 hold on;
21 grid on;
22 title('Voltera Predator-Prey System - Time Plot');
23 ylabel('x');
24 xlabel('t (sec)');
25 legend('Predator', 'Prey');
26 t_intv= [0 10];
27
28 figure
29 for i=x0_set
30     for j=x0_set
31         x0 = [i; j];
32         [t,x]= ode45('Voltera', t_intv, x0);
33         plot(x(:,1),x(:,2))
34         hold on;
35     end
36 end
37 title('Voltera Predator-Prey System - Phase Plane');
38 ylabel('x_2 - Predator');
39 xlabel('x_1 - Prey');
40 axis([-5 5 -5 5]);
41 grid on;
42
43 function xdot = Voltera(t,x)
44     xdot = [-x(1)+x(1)*x(2); x(2)-x(1)*x(2)];
45 end
```

Figures



Exercise 2

Equilibrium points and linearization

Consider the following system,

$$\dot{x}_1 = x_2(-x_1 + x_2 - 1)$$

$$\dot{x}_2 = x_1(x_1 + x_2 + 1)$$

- (a) Find all equilibrium points
- (b) Find Jacobian
- (c) Find the nature of all e.p.s

Answer

a. Find all e.p.s

At equilibrium points, all states reach their minimal energy state; therefore, the derivative of the state should equal zero. Then, we solve for the roots of the obtained characteristic equation.

$$f(x_1, x_2) = \dot{X} = 0 \Rightarrow \begin{cases} \dot{x}_1 = 0 \Rightarrow x_2(-x_1 + x_2 - 1) = 0 \\ \dot{x}_2 = 0 \Rightarrow x_1(x_1 + x_2 + 1) = 0 \end{cases}$$

There are four possible cases for state derivative, \dot{X} , to equal zero, $\dot{X} = 0$; where the system is at an equilibrium point.

Case 1

$$\begin{cases} (x_2 = 0)(\cancel{-x_1 + x_2 - 1}) = 0 \\ (x_1 = 0)(\cancel{-x_1 + x_2 + 1}) = 0 \end{cases}$$

e.p. at $(x_1, x_2) = (0, 0)$

Case 2

$$\begin{cases} (x_2 = 0)(\cancel{-x_1 + x_2 - 1}) = 0 \\ (\cancel{x_1})(x_1 + x_2 + 1 = 0) = 0 \Rightarrow x_1 = -1 \end{cases}$$

e.p. at $(x_1, x_2) = (-1, 0)$

Case 3

$$\begin{cases} (\cancel{x_2})(-x_1 + x_2 - 1 = 0) = 0 \Rightarrow x_2 = +1 \\ (x_1 = 0)(\cancel{-x_1 + x_2 + 1 = 0}) = 0 \end{cases}$$

e.p. at $(x_1, x_2) = (0, +1)$

Case 4

$$\begin{cases} (\cancel{x_2})(-x_1 + x_2 - 1 = 0) = 0 \\ (\cancel{x_1})(x_1 + x_2 + 1 = 0) = 0 \end{cases} \\ \Rightarrow x_2 = 0; \Rightarrow x_1 = -1 \text{ as in Case 2 or Case 3}$$

b. Find the Jacobian

$$f(x_1, x_2) = \begin{cases} x_2(-x_1 + x_2 - 1) \\ x_1(x_1 + x_2 + 1) \end{cases} = \dot{X} = AX$$

$$\dot{X} = \begin{bmatrix} \frac{df}{dx_1} & \frac{df}{dx_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \dot{X} = \begin{bmatrix} \frac{d}{dx_1}(-x_2) & \frac{d}{dx_2}(x_2 - 1) \\ \frac{d}{dx_1}(x_1 + 1) & \frac{d}{dx_2}(x_1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow$$

$$\dot{X} = AX \Rightarrow \dot{X} = \begin{bmatrix} -x_2 & 0 \\ 0 & x_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow$$

$$A = \begin{bmatrix} -x_2 & 0 \\ 0 & x_1 \end{bmatrix}$$

c. Find the nature of all e.p.s,

Compute the poles of the system using Eigen values,

$$\Delta(s) = |SI - A| = \begin{vmatrix} s + x_2 & 0 \\ 0 & s - x_1 \end{vmatrix} \Rightarrow$$

$$(s + x_2)(s - x_1) = 0 \Rightarrow \begin{cases} s = x_1 \\ s = -x_2 \end{cases}$$

$$(s + x_2)(s - x_1) = 0 \Rightarrow s^2 - x_1 x_2 s - x_1 x_2 = 0$$

Standard form for characteristic equation,

$$s^2 + 2\alpha s + \alpha^2 + \beta^2 = s^2 + 2\alpha s + \omega_n^2 = 0; \quad \beta = j\omega$$

Case 1, at (0, 0); $s^2 = 0 \Rightarrow s = 0$, center point with poles at the origin on the s-plane.

Case 2, at (-1, 0); $s = 1$ and $s = 0$, center point with poles at the origin and +1 on the s-plane.

Case 3, at (0, +1); $s = 0$ and $s = -1$, center point with poles at -1 and the origin on the s-plane.

Exercise 3

System simulation

Consider the following system,

$$\dot{x}_1 = x_2(-x_1 + x_2 - 1)$$

$$\dot{x}_2 = x_1(x_1 + x_2 + 1)$$

Simulate the system using MATLAB for various initial conditions for the two cases:

- (a) Take ICs spaced in a uniform mesh in the box $x_1=[-10,10]$, $x_2=[-10,10]$. Make one phase plane plot with all the trajectories on it. Plot phase plane on square $[-15,15] \times [-15,15]$.
- (b) Take ICs spaced in a uniform mesh in the box $x_1=[-3,3]$, $x_2=[-3,3]$. Make one phase plane plot with all the trajectories on it. Plot phase plane on square $[-5,5] \times [-5,5]$.

Answer

- a. The phase plane rotation for all test cases are counter-clockwise.

Matlab Code

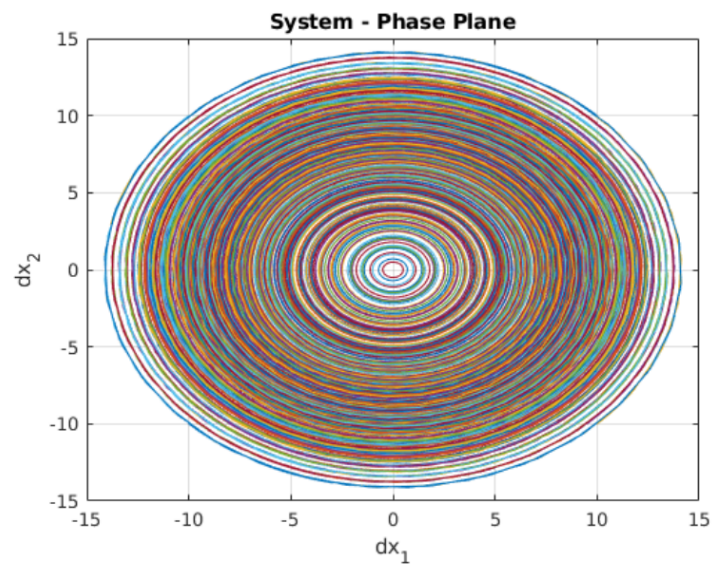
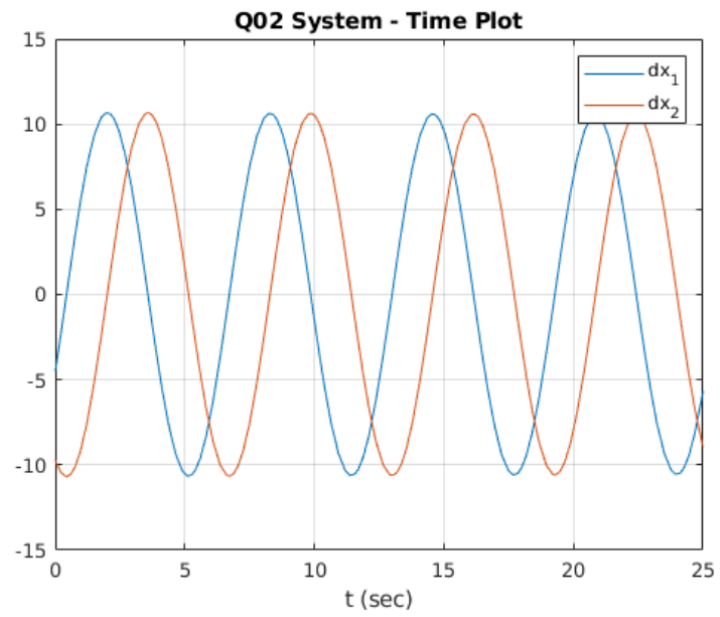
```
1 %% HW03 - Q03a - System
2 % @author: Bardia Mojra
3 % @date: 09/28/2021
4 % @title HW03 - Q02 - System
5 % @class ee5323 - Nonlinear Systems
6 % @professor - Dr. Frank Lewis
7
8 clc
9 close all
10 %warning('off','all')
11 %warning
12
13 x0_set = -10:.5:10;
14 t_intv = [0 25];
15 x_0 = [-4.5; -9.7]; % initial conditions for x(t)
16 %t = t_intv;
17
18 figure
19 [t, x] = ode23(@System, t_intv, x_0);
20 plot(t, x)
21 hold on;
22 grid on;
23 title('Q02 System - Time Plot');
24 xlabel('t (sec)');
25 legend('dx_1', 'dx_2');
26
27 t_intv = [0 25];
28 figure
29 for i=x0_set
```

```

30     for j=x0_set
31         x0 = [ i ; j ];
32         [t,x]= ode45(@System, t_intv , x0);
33         plot(x(:,1),x(:,2))
34         hold on;
35     end
36 end
37 title('System - Phase Plane');
38 ylabel('dx_2');
39 xlabel('dx_1');
40 axis([-15 15 -15 15]);
41 grid on;
42
43 %%
44 %
45 % function xdot = System(t,x)
46 %     xdot = [-x(2); x(1)];
47 % end

```

Figures



b. The phase plane rotation for all test cases are counter-clockwise.

Matlab Code

```
1 %% HW03 - Q03b - System
2 % @author: Bardia Mojra
3 % @date: 09/28/2021
4 % @title HW03 - Q02 - System
5 % @class ee5323 - Nonlinear Systems
6 % @professor - Dr. Frank Lewis
7
8 clc
9 close all
10 %warning('off','all')
11 %warning
12
13 x0_set = -3:.5:3;
14 t_intv = [0 25];
15 x_0 = [-4.5; -9.7]; % initial conditions for x(t)
16 %t = t_intv;
17
18 figure
19 [t, x] = ode23(@System, t_intv, x_0);
20 plot(t, x)
21 hold on;
22 grid on;
23 title('Q02 System - Time Plot');
24 xlabel('t (sec)');
25 legend('dx_1', 'dx_2');
26
27 t_intv = [0 25];
28 figure
29 for i = x0_set
30     for j = x0_set
31         x0 = [i; j];
32         [t, x] = ode45(@System, t_intv, x0);
33         plot(x(:,1), x(:,2))
34         hold on;
35     end
36 end
37 title('System - Phase Plane');
38 ylabel('dx_2');
39 xlabel('dx_1');
40 axis([-5 5 -5 5]);
41 grid on;
42
43 %%
44 %
45 % function xdot = System(t, x)
```

```
46 %    xdot = [ -x(2) ; x(1) ];  
47 % end
```

Figures

