Some Issues About Stability

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1. Most Basic- Lyapunov

Let the system be given by

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

Then for any scalar C^1 function $V(x)$ one has

$$\dot{V} = \frac{\partial V^T}{\partial x}\dot{x} = \frac{\partial V^T}{\partial x} f + \frac{\partial V^T}{\partial x} gu \equiv V_x^{\ T} f + V_x^{\ T} gu$$

Completing the squares for any matrix $R>0$ yields

$$\dot{V} = V_x^{\ T} f + V_x^{\ T} gu = V_x^{\ T} f + \tfrac{1}{2}(V_x^{\ T} gR^{-1} + u^T)R(R^{-1}g^T V_x + u) - \tfrac{1}{2}V_x^{\ T} gR^{-1}g^T V_x - \tfrac{1}{2}u^T Ru\,.$$

Now suppose that $V(x)>0$ and satisfies the HJ inequality

$$V_x^{\ T} f + \tfrac{1}{2}h^T h - \tfrac{1}{2}V_x^{\ T} gR^{-1}g^T V_x \le 0\,.$$

Assume the system is locally i/o detectable in the sense that there exists a neighborhood such that

$$u(t) = 0 \quad and \quad y(t) = 0 \quad \forall t \quad \text{implies that} \quad x(t) \to 0$$

Then the closed-loop system is asymptotically stable if one selects the control

$$u = -R^{-1}g^T(x)V_x\,.$$

For note that, according to the HJ equation

$$\dot{V} \le \tfrac{1}{2}(V_x^{\ T} gR^{-1} + u^T)R(R^{-1}g^T V_x + u) - \tfrac{1}{2}h^T h - \tfrac{1}{2}u^T Ru$$

and according to the control selection

$$\dot{V} \le -\tfrac{1}{2}h^T h - \tfrac{1}{2}u^T Ru = -\tfrac{1}{2}y^T y - \tfrac{1}{2}u^T Ru$$

which is negative definite under the i/o detectable assumption. Therefore $V(x)$ is a Lyapunov function with $\dot{V} < 0$.

2. Dissipativity

The closed-loop system is dissipative with respect to the supply rate

$$w(t) = -\tfrac{1}{2}(\|y\|^2 + \|u\|_R^{\ 2}) = -\tfrac{1}{2}(h^T h + u^T Ru)\,.$$

if there exists a non-negative scalar storage function $V(x)$ such that

$$V(x(t)) \le V(x(0)) + \int_0^t w(t)\, dt \quad \text{for all } t$$

An infinitesimal equivalent to this is

$$\frac{dV}{dt} \le -\tfrac{1}{2}\left(h^T h + u^T R u\right)$$

or

$$H(x, V_x, u) \equiv \dot{V} + \tfrac{1}{2} h^T h + \tfrac{1}{2} u^T R u = V_x{}^T f + V_x{}^T g u + \tfrac{1}{2} h^T h + \tfrac{1}{2} u^T R u \le 0$$

Complete the squares to get

$$V_x{}^T f + \tfrac{1}{2} h^T h + \tfrac{1}{2}(V_x{}^T g R^{-1} + u^T) R (R^{-1} g^T V_x + u) - \tfrac{1}{2} V_x{}^T g R^{-1} g^T V_x \le 0$$

Select now the control

$$u = -R^{-1} g^T(x) V_x .$$

So that

$$V_x{}^T f + \tfrac{1}{2} h^T h - \tfrac{1}{2} V_x{}^T g R^{-1} g^T V_x \le 0$$

If this HJ equation has a non-negative solution then the closed-loop system is dissipative.

Note that then,

$$\tfrac{1}{2}\int_0^t (y^T y + u^T R u)\, dt \le V(x(0)) - V(x(t))$$

and the non-negativity of *V(x(t))* shows that

$$\tfrac{1}{2}\int_0^t (y^T y + u^T R u)\, dt \le V(x(0))$$

so the system is L_2 stable. Due to uniform continuity one has $u(t) \to 0 \quad and \quad y(t) \to 0$, and under the i/o detectability assumption one has $x(t) \to 0$.

3. Optimal Control

Select the performance index

$$J = \tfrac{1}{2}\int_0^\infty (h^T h + u^T R u)\, dt$$

and define the Hamiltonian

$$H(x, p, u) = p^T (f(x) + g(x) u) + \tfrac{1}{2}(h^T h + u^T R u)$$

Complete the squares to obtain

$$H(x, p, u) = p^T f + \tfrac{1}{2} h^T h + \tfrac{1}{2}(p^T g R^{-1} + u^T) R (R^{-1} g^T p + u) - \tfrac{1}{2} p^T g R^{-1} g^T p$$

The stationary point $0 = \dfrac{\partial H}{\partial u}$ is given with the control

$$u^* = -R^{-1}g^T(x)p$$

to be

$$H(x,p,u^*) = p^T f + \tfrac{1}{2}h^T h + -\tfrac{1}{2}p^T gR^{-1}g^T p$$

It is easy to show that

$$H(x,p,u) = H(x,p,u^*) + \tfrac{1}{2}(u-u^*)^T R(u-u^*) \tag{3.2}$$

whence

$$H(x,p,u^*) \le H(x,p,u)$$

for all; u(t). This shows that

$$\frac{\partial^2 H}{\partial u^2} > 0$$

at u^* which shows that the optimal control problem has a unique solution and $u^*(t)$ minimizes J.

Now, note that for any C^1 scalar function $V(x)$ one has

$$\dot{V} = \frac{\partial V^T}{\partial x}\dot{x} = \frac{\partial V^T}{\partial x} f + \frac{\partial V^T}{\partial x} gu \equiv V_x{}^T f + V_x{}^T gu$$

so that

$$H(x,V_x,u) = V_x{}^T(f(x)+g(x)u) + \tfrac{1}{2}(h^T h + u^T Ru) = \frac{dV}{dt} + \tfrac{1}{2}(h^T h + u^T Ru)$$

Suppose there exists a positive C^1 scalar function $V(x)$ whose gradient satisfies the HJ inequality

$$H(x,V_x,u^*) = V_x{}^T f + \tfrac{1}{2}h^T h + -\tfrac{1}{2}V_x{}^T gR^{-1}g^T V_x \le 0$$

Then from (3.2) one has

$$H(x,V_x,u) = \frac{dV}{dt} + \tfrac{1}{2}(h^T h + u^T Ru) \le \tfrac{1}{2}(u-u^*)^T R(u-u^*)$$

so that, for the optimal $u(t)$,

$$\frac{dV}{dt} \le -\tfrac{1}{2}(h^T h + u^T Ru)$$

Therefore, $V(x)$ serves as a Lyapunov function. Now the i/o detectability assumption shows that the system is AS.

4. Output Feedback

The system is

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

and the control has the constrained form

$$u = L(y)$$

Let us follow the development in Section 1. Then for any scalar C^1 function $V(x)$ one has

$$\dot{V} = \frac{\partial V^T}{\partial x}\dot{x} = \frac{\partial V^T}{\partial x} f + \frac{\partial V^T}{\partial x} gu \equiv V_x^{\ T} f + V_x^{\ T} gu$$

Completing the squares for any matrix *R>0* yields

$$\dot{V} = V_x^{\ T} f + V_x^{\ T} gu = V_x^{\ T} f + \tfrac{1}{2}(V_x^{\ T} gR^{-1} + u^T)R(R^{-1}g^T V_x + u) - \tfrac{1}{2}V_x^{\ T} gR^{-1}g^T V_x - \tfrac{1}{2}u^T Ru \,. \tag{4.1}$$

One desires now to select the control

$$u = -R^{-1}g^T(x)V_x \,.$$

However, now this equation may have no solution. In fact, it is required that

$$u = L(y) = L(h(x)) = -R^{-1}g^T(x)V_x$$

which has a solution *L(.)* iff

$$g^T(x)V_x = 0 \text{ for every } x \text{ such that } h(x)=0$$

However, there may exist a *G(x)* such that

$$\eta(x) \equiv g^T(x)V_x - G(x) = 0 \text{ for every } x \text{ such that } h(x)=0$$

Then one can solve the equation

$$u = L(y) = L(h(x)) = R^{-1}[G(x) - g^T(x)V_x]$$

Setting, therefore

$$u = R^{-1}[G(x) - g^T(x)V_x]$$

one obtains from (4.1)

$$\dot{V} = V_x^{\ T} f + V_x^{\ T} gu = V_x^{\ T} f + \tfrac{1}{2}G^T(x)R^{-1}G(x) - \tfrac{1}{2}V_x^{\ T} gR^{-1}g^T V_x - \tfrac{1}{2}u^T Ru \tag{4.2}$$

Now suppose that *V(x)>0* satisfies the HJ inequality

$$V_x^{\ T} f + \tfrac{1}{2}h^T h + \tfrac{1}{2}G^T R^{-1}G - \tfrac{1}{2}V_x^{\ T} gR^{-1}g^T V_x \le 0 \,.$$

Then, according to (4.2)

$$\dot{V} \le -\tfrac{1}{2}h^T h - \tfrac{1}{2}u^T Ru$$

which is negative definite under the i/o detectable assumption. Therefore *V(x)* is a Lyapunov function with $\dot{V} < 0$.

5. LTI Output Feedback

Consider the system

$$\dot{x} = Ax + Bu$$

with output

$$y = Cx$$

and static OPFB control

$$u = -Ky = -KCx$$

where $x(t) \in R^n, u(t) \in R^m, y(t) \in R^p$. The closed-loop system is

$$\dot{x} = (A - BKC)u \equiv A_c x .$$

Let us follow the development in Section 4. Then for any scalar C^1 function $V(x)$ one has

$$\dot{V} = \frac{\partial V^T}{\partial x}\dot{x} = \frac{\partial V^T}{\partial x}Ax + \frac{\partial V^T}{\partial x}Bu \equiv V_x{}^T Ax + V_x{}^T Bu$$

Completing the squares for any matrix $R>0$ yields

$$\dot{V} = V_x{}^T Ax + V_x{}^T Bu = V_x{}^T Ax + \tfrac{1}{2}(V_x{}^T BR^{-1} + u^T)R(R^{-1}B^T V_x + u) - \tfrac{1}{2}V_x{}^T BR^{-1}B^T V_x - \tfrac{1}{2}u^T Ru .$$

Assume now that $V(x) = x^T Px$ so that

$$\dot{V} = \tfrac{1}{2}x^T(PA + A^T P)x + \tfrac{1}{2}(x^T PBR^{-1} + u^T)R(R^{-1}B^T Px + u) - \tfrac{1}{2}x^T PBR^{-1}B^T Px - \tfrac{1}{2}u^T Ru \tag{5.1}$$

One desires now to select the control

$$u = -R^{-1}B^T Px .$$

However, now this equation may have no solution. In fact, it is required that

$$u = -Ky = -KCx = -R^{-1}B^T Px$$

which has a solution K iff

$$B^T Px = 0 \text{ for every } x \text{ such that } Cx = 0$$

i.e.

$$B^T P(I - C^+C) = 0$$

with C^+ the Moore-Penrose inverse and $(I - C^+C)$ the projection onto nullspace of C. Assuming C has full row rank p, one has

$$C^+ = C^T(CC^T)^{-1}$$

which is a right inverse for C.

However, there may exist a matrix G such that

$$[B^T P + G](I - C^+C) = 0$$

Then one can solve the equation

$$KC = R^{-1}[G + B^T P] .$$

Setting, therefore

$$u = -R^{-1}[G + B^T P]x$$

one obtains from (5.1)

$$\dot{V} = \tfrac{1}{2}x^T(PA + A^T P + G^T R^{-1}G - PBR^{-1}B^T P)x - \tfrac{1}{2}u^T Ru \tag{5.2}$$

Now suppose that $P>0$ satisfies the Riccati inequality

$$PA + A^T P + C^T C + G^T R^{-1} G - PBR^{-1} B^T P \leq 0$$

Then, according to (5.2)

$$\dot{V} \leq -\tfrac{1}{2} y^T y - \tfrac{1}{2} u^T Ru$$

which is negative definite under the i/o detectable assumption. Therefore *V(x)* is a Lyapunov function with $\dot{V} < 0$.