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SECOND ORDER SYSTEMS

The figures in this lecture are taken from Dorf and Bishop, *Modern Control Systems,* edition 8.

Many useful systems are of second order, and have two complex poles. Even higher-order systems often have a slow complex pair and some faster poles. In a first approximation, the *dominant (low frequency) pole pair* can be analyzed to provide approximate performance and design insight.
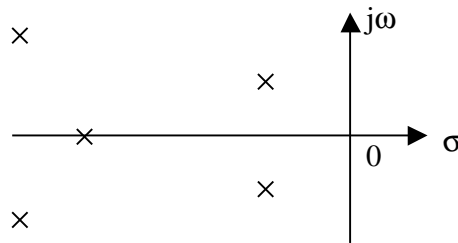
A fifth-order system with a complex pair of dominant (slow) poles

COMPLEX POLE PAIR

A transfer function with a complex pair of poles and no finite zeros can be written as

$$H(s) = \frac{w_n^2}{s^2 + 2as + w_n^2} = \frac{w_n^2}{s^2 + 2zw_n s + w_n^2} \equiv \frac{w_n^2}{\Delta(s)} \ .$$

The numerator is chosen to scale the transfer function so that the DC gain (e.g. set *s=0*) is equal to one. The denominator is the *Characteristic polynomial* which can be written in several natural or *canonical forms*, including

$$\Delta(s) = s^2 + 2as + w_n^2 = s^2 + 2zw_n s + w_n^2 \ . \tag{1}$$

All of the symbols have names and important properties, which we shall soon explore.

1

One may write

$$\Delta(s) = s^2 + 2as + w_n{}^2 = (s+a)^2 + b^2$$

where

$$b^2 + a^2 = w_n{}^2.$$

This shows that, if $a^2 < w_n{}^2$, the polynomial $\Delta(s)$ describes a complex pair of poles at

$$s = -a \pm jb.$$

In this class we do not use 'j', but as engineers prefer to keep the complex pole factors together in the quadratic form (1).

The figure shows the pole locations. The real part of the poles is $-a$ and the imaginary part is jb The norm of the vector from the origin to the pole is w_n, which is known as the *natural frequency*.
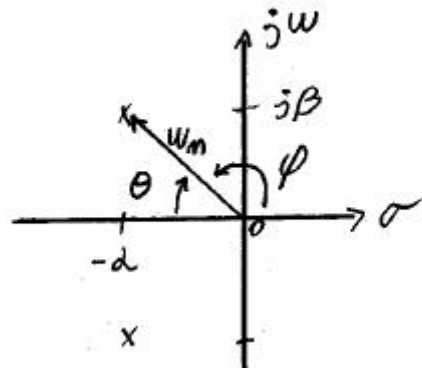
To link the two forms in (1) we define the *damping ratio* as

$$z = \frac{a}{w_n},$$

which defines it as

$$z = \cos q = -\cos j = \frac{a}{w_n}.$$

For complex poles in the left-half plane one has $0 < z < 1$. If $0 > z > -1$ then one has a complex pair in the right-half plane (e.g. unstable complex pair). Note that one may write

$$b = \sqrt{w_n{}^2 - a^2} = w_n \sqrt{1 - z^2}.$$

TIME DOMAIN MEANING OF VARIABLES

The symbols appearing above all have well-defined meanings in the time domain. Referring to the Laplace transform table, the impulse response of the system with transfer function

$$H(s) = \frac{w_n{}^2}{(s^2 + a)^2 + b^2}$$

is given by

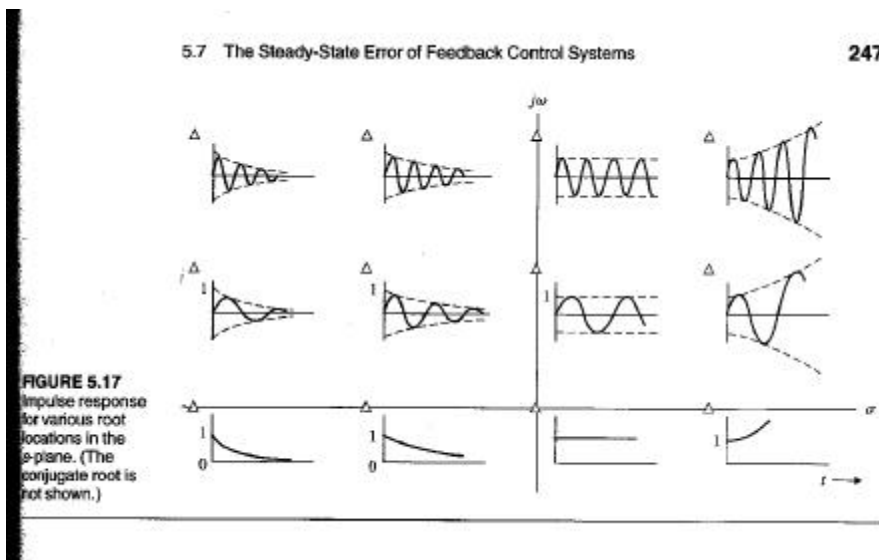$$\frac{w_n^{\,2}}{b}\,e^{-at}\,\sin b\;u_{-1}(t)\,,$$

which is plotted for $0 < z < 1$ in the figure. This is known as the *underdamped* case. The figure clearly shows the meaning in the time domain of the real part $-a$ of the poles, which provides the exponential decay term. Having in mind the standard form for sinusoids $\sin \frac{2pt}{T}$, the period of the oscillation is given by

$$T = \frac{2p}{b}.$$

The variable β is known as the *oscillation frequency*.
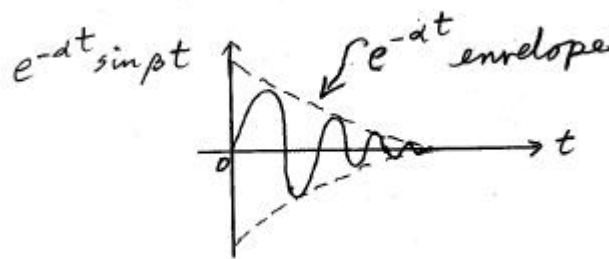
Refer to the above equations. If $z = 0$ then $a = 0, b = w_n$ and the poles are at $s = \pm jb$ on the imaginary axis. This is known as the undamped case. If $z = 1$ then $b = 0, a = w_n$ and the poles are on the real axis, both at $s = -a$. In this *overdamped case*, the impulse response has the form te^{-at}. All these various cases are shown below in the very nice figure from Dorf and Bishop.

FIGURE 5.17
Impulse response for various root locations in the s-plane. (The conjugate root is not shown.)

If $z > 1$ then there are two real poles and we can split the quadratic factor $\Delta(s) = s^2 + 2as + w_n^2$ into two real linear factors.
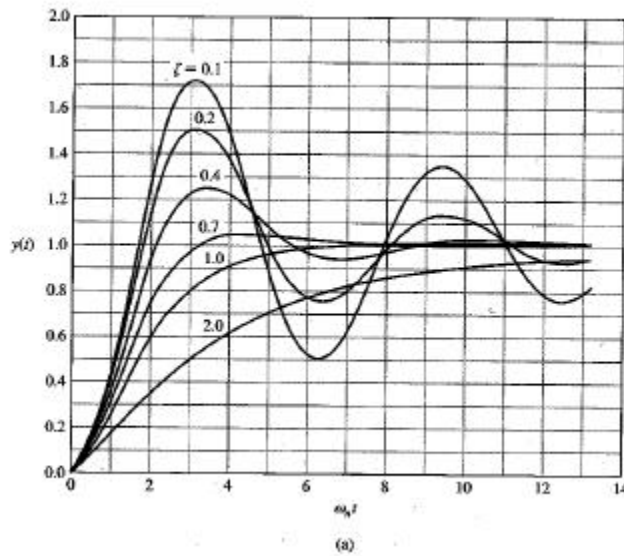
STEP RESPONSE AND POV

The step response is given by inverse transforming

$$\frac{H(s)}{s} = \frac{w_n^2}{s[(s^2 + a)^2 + b^2]}$$

to obtain

$$r(t) = 1 - e^{-at}[\cos b + \frac{a}{b}\sin b] = 1 - \frac{w_n}{b}e^{-at}\sin(b + q)u_{-1}(t)$$

where the angle $q = \arctan \frac{b}{a}$ is shown in the figure above. This is shown in the figure from Dorf and Bishop for various values of damping ratio.


(a)

A great advantage of MATLAB is the ease of displaying data in multidimensions and colors to obtain increased engineering insight. The following figure from Dorf and Bishop shows this very nicely.

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(b)
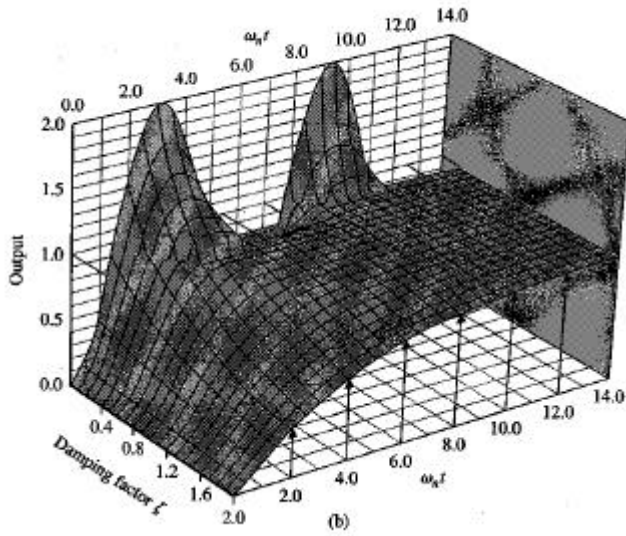
An important quantity for characterizing the performance of systems is the *percent overshoot (POV)* in the step response. This is defined as

$$POV = \frac{r_{max} - r_{ss}}{r_{ss}} \times 100\%$$

where r_{max} is the maximum value of the step response and r_{ss} is its steady-state value.
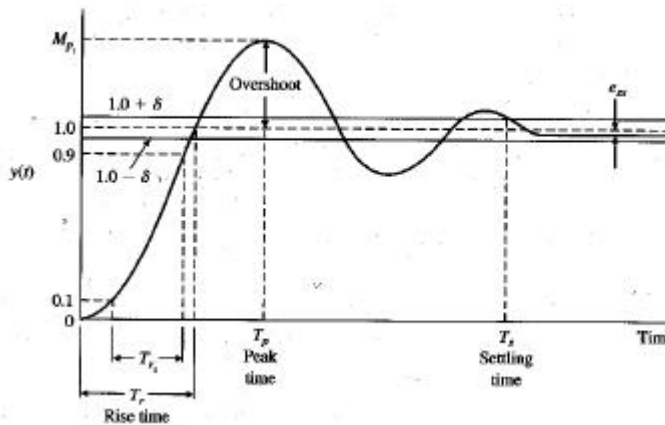
Also useful are the *rise time t_r*, which is the time required for the step response to rise from 0.1 to 0.9 of its steady-state value. The *settling time t_s* is the time required for the signal to effectively reach its steady-state value.

For a pure exponential $r(t) = 1 - e^{-t/t}$ the quantity t is known as the *time constant*. One has $t = \frac{1}{a}$. For the pure exponential one has

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$$t_r = 2.2t$$
$$t_s = 5t \qquad \text{(some take } t_s = 4t \text{)}.$$

For the damped pole pair, one still has $t = \frac{1}{a}$ and one may take the settling time as this value, but the signal rises faster and one may approximate for $0.3 \leq z \leq 0.8$ using

$$t_r = \frac{2.16z + 0.6}{w_n}.$$

The percent overshoot is a function of damping ratio, as shown in the graph from Dorf and Bishop. In fact one has

$$POV = 100e^{-pz/\sqrt{1-z^2}}$$

and conversely

$$z = \left[\frac{\ln^2\left(POV/100\right)}{\ln^2\left(POV/100\right) + p^2} \right]^{1/2}.$$
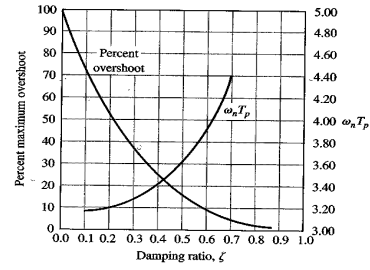
FIGURE 5.8
Percent overshoot and normalized peak time versus damping ratio ζ for a second-order system (Eq. 5.8).

FREQUENCY DOMAIN POINT OF VIEW

The variables mean something in the frequency domain as well, particularly the damping ratio z and the natural frequency w_n. In fact, the *quality factor*

$$Q = \frac{1}{2z} = \frac{w_n}{2a}$$

measures the sharpness of the resonant peak in the Bode plot, as shown in the figure below from Dorf and Bishop. Note that this is effectively determined solely by the damping ratio. The poles are complex if $Q > 1/2$.

In terms of the quality factor one may write the characteristic polynomial in the non-dimensional form

$$\Delta(s) = \left(\frac{s}{w_n}\right)^2 + \frac{1}{Q}\left(\frac{s}{w_n}\right) + 1$$

The resonant frequency is given for $z \leq 0.7$ by

$$w_r = w_n\sqrt{1 - 2z^2}.$$

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The maximum value of the Bode plot at resonance is given by

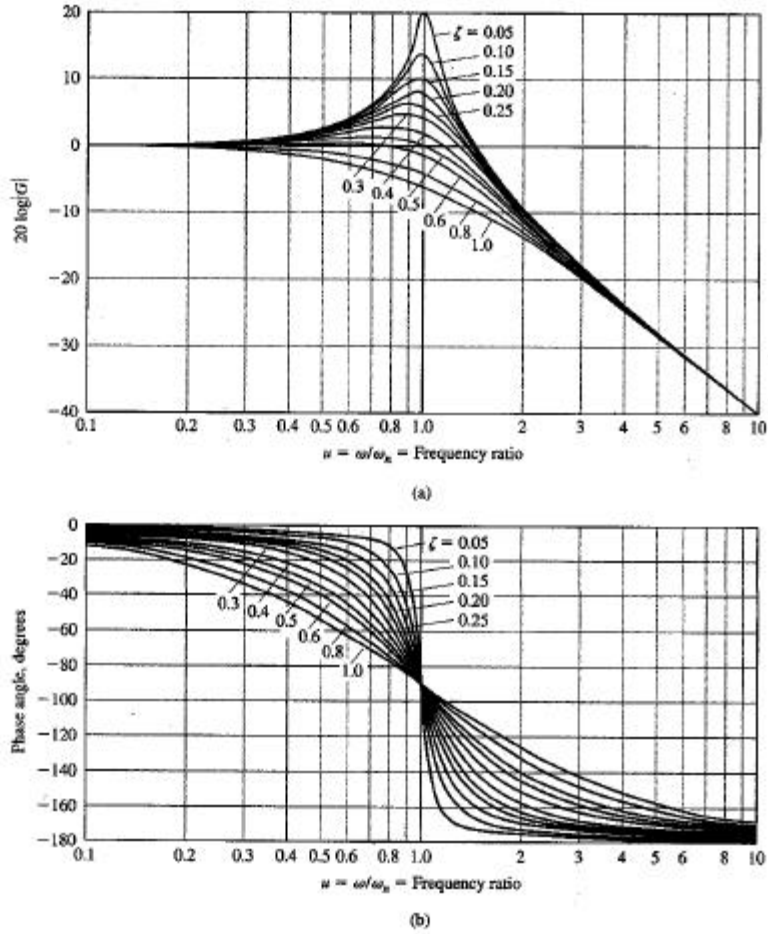$$M_{pw} = \frac{1}{2z\sqrt{1-z^2}} \ .$$

(a)

(b)

FIGURE 8.10
Bode diagram for $G(j\omega) = [1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2]^{-1}$.