EE 5323- Take Home Exam 2

Fall 2021

This exam has 6 pages in all. There are 4 problems.

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Almost all questions require numerical calculations to arrive at the answers. To obtain full credit, show all your work. No partial credit will be given without the supporting work.

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Pledge of honor:

"On my honor I have neither given nor received aid on this examination."

Signature: ______________________

1. **Lyapunov Function**

Use Lyapunov function to examine the stability of the following systems. Be clear and show all steps.

a.

$$\dot{x}_1 = x_2 \sin x_1 - x_1$$
$$\dot{x}_2 = -x_1 \sin x_1 - x_2$$

$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) > 0$

$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2 = 0 \Rightarrow$

$x_1(x_2 \sin x_1 - x_1) + x_2(-x_1 \sin x_1 - x_2) = 0 \Rightarrow$

$\cancel{x_1 x_2 \sin x_1} - x_1^2 - \cancel{x_1 x_2 \sin x_1} - x_2^2 = 0 \Rightarrow -(x_1^2 + x_2^2) = 0$

e.p. $(0,0)$ only

$\dot{V} \leq 0$ then <u>GAS</u>

b.

$$\dot{x}_1 = x_2 \sin x_1 - x_1$$
$$\dot{x}_2 = -x_1 \sin x_1$$

$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2 = 0 \Rightarrow$

$x_1(x_2 \sin x_1 - x_1) + x_2(-x_1 \sin x_1) = 0$

$\cancel{x_1 x_2 \sin x_1} - x_1^2 - \cancel{x_1 x_2 \sin x_1} = 0$

$\dot{V} = -x_1^2$; $\dot{V} = 0$; $x_1 = 0$

$\dot{V} \leq 0$ ✓

<u>SISL</u>

marginally stable

because it only depends

on x_1

2. LaSalle's Extension

$\ddot{x} + k_1\dot{x} + k_2\dot{x}^3 + k_3x^5 = 0$ $\Rightarrow$ $\ddot{x} = -k_1\dot{x} - k_2\dot{x}^3 - k_3x^5$

a. Use Lyapunov to check the stability. Hint: Use the energy as the Lyapunov function. Take the potential energy as

$$PE = \int_0^x (k_2\dot{x}^3 + k_3x^5)dx$$

$V = \frac{1}{2}\dot{x}^2 + \int_0^x C(y)dy = K + U;$

$\dot{V} = \dot{x}\ddot{x} + \frac{d}{dt}\int_0^x C(y)dy \Rightarrow$ $\boxed{\text{Per Leibnaz's theorem: } \frac{d}{dt}\int_\alpha^\beta F(x,t)dx = \dot{\beta}F(\beta,t) - \dot{\alpha}F(\alpha,t) + \int_\alpha^\beta \frac{d}{dt}F(x,t)\,dx}$

$\Rightarrow \dot{x}(-k_1\dot{x} - \cancel{k_2\dot{x}^3} - \cancel{k_3x^5}) + \dot{x}(\cancel{k_2\dot{x}^3} + \cancel{k_3x^5}) - \cancel{(0)(\ )} + (0+0)\Big|_{\alpha=0}^{\beta=x}$

$\Rightarrow \dot{V} = -k_1\dot{x}^2 \leq 0$ if $k_1 > 0$;

$\dot{V} = 0 \Rightarrow \dot{x} \to 0$ or $k_1 = 0$ | SISL

↳ hints at use of LaSalle's extention

we plug this in dynamics equations.

b. Use LaSalle's extension to find a stronger type of stability for the system.

if $\dot{x} \to 0 \Rightarrow \ddot{x} \to 0$; Plug in sys. dynamic

$\underset{0}{\cancel{\ddot{x}}} + k_1\underset{0}{\cancel{\dot{x}}} + k_2\underset{0}{\cancel{\dot{x}^3}} + k_3x^5 = 0 \Rightarrow k_3x^5 = 0 \Rightarrow x = 0$

$\Rightarrow$ thus the system is dissapative & reaches equelibrium at $x=0$; thus it is considered Global Asymptotic Stable (GAS)

3. Lyapunov Equation for Linear Systems

Use Lyapunov Equation to check the stability of the linear systems

a. $\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ 0 & -6 \end{bmatrix} x$

not stable

undetermined

$$A^T P + PA = -Q \Rightarrow \begin{bmatrix} a_1 & a_3 \\ a_2 & a_3 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} = -Q$$

$$\begin{bmatrix} a_1P_1 + a_3P_3 + a_1P_1 + a_3P_2 & a_1P_2 + a_3P_3 + a_2P_1 + a_4P_2 \\ a_2P_1 + a_4P_2 + a_1P_2 + a_3P_3 & a_2P_2 + a_4P_3 + a_2P_2 + a_4P_3 \end{bmatrix} = -Q;$$

where I test the following

Q matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, [\ldots$$

& a unique solution

DNE

b. $\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x$

stable w/

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$m_{11} = .8\bar{3}$

$m_{22} = .1667$

$\begin{bmatrix} .8\bar{3} & -.5 \\ -.5 & .5 \end{bmatrix}$

$\begin{bmatrix} \text{Same} \\ a_x P_y \text{ matrix} \end{bmatrix} = -Q$

From 3.a.

$m_{22} = (.8\bar{3} \cdot .5) - (.25)$

4. UUB

Use Lyapunov to show that the system

$$\dot{x}_1 = x_1 x_2^2 - x_1(x_1^2 + x_2^2 - 3)$$
$$\dot{x}_2 = -x_1^2 x_2 - x_2(x_1^2 + x_2^2 - 3)$$

is uniformly ultimately bounded UUB. That is, show that the Lyapunov derivative is NEGATIVE OUTSIDE A BOUNDED REGION. Find the radius of the bounded region outside which $\dot{V} < 0$. Any states outside this region are attracted towards the origin.

$$V(x_1, x_2) = \tfrac{1}{2}(x_1^2 + x_2^2 - 3)^2 > 0 \Rightarrow \dot{V} = \tfrac{2}{2}(x_1^2 + x_2^2 - 3)(2x_1\dot{x}_1 + 2x_2\dot{x}_2)$$

$$\Rightarrow \dot{V} = (x_1^2 + x_2^2 - 3)\left(2x_1\left(x_1x_2^2 - x_1(x_1^2 + x_2^2 - 3)\right) + 2x_2\left(-x_1^2x_2 - x_2(x_1^2 + x_2^2 - 3)\right)\right)$$

$$= (x_1^2 + x_2^2 - 3) \quad 2x_1(-x_1^3 - x_1x_2^2 + 3x_1) + 2x_2(-x_1^2x_2 - x_2^3 + 3x_2)$$

$$\left(-2x_1^2(x_1^2 + x_2^2 - 3) - 2x_2^2(x_1^2 + x_2^2 - 3)\right)$$

$$\Rightarrow \dot{V} = \underbrace{(x_1^2 + x_2^2 - 3)^2}_{>0}\underbrace{(-2x_1^2 - 2x_2^2)}_{\leq 0} = 0 \rightarrow \text{Per Cauchy Schwarz}$$

$$\Rightarrow \|\dot{V}\| \leq \|x_1^2 + x_2^2 - 3\|^2 \, \|(-2)\| \, \|x_1^2 + x_2^2\|$$

$|x^T \cdot y| \leq \|x\| \cdot \|y\|$ given an orthonormal space

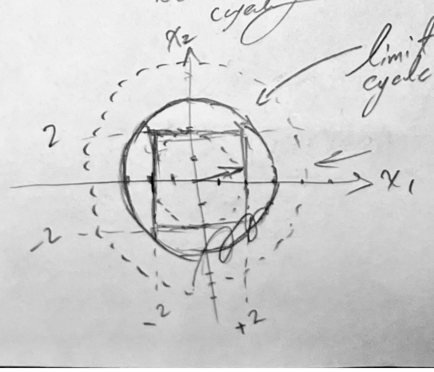
creates boundary cycle ($\sqrt{3}$)

≥ 0

Cannot do!

forces future state to the limit cycle

limit cycle w/ $r = \sqrt{3}$



thus

$$-2(x_1^2 + x_2^2 - 3)^2(x_1^2 + x_2^2) \ldots \leq \dot{V} \leq +2(x_1^2 + x_2^2 - 3)^2(x_1^2 + x_2^2)$$

$C = 2$

Ultimately bounded by around limit cycle