EE 5323 - HW04

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HW04 – Vector Fields, Flows, First Integrals
EE 5323 – Nonlinear Systems
Dr. Lewis

Exercise 1

First Integral of Motion for Undamped Oscillator

$$\ddot{x} + x = 0$$

(a) Write position-velocity state space form $\dot{X} = f(X)$.

(b) Plot the trajectories $x(t)$, $\dot{x}(t)$ vs. time. Use initial conditions of $x(0) = 0.1$, $\dot{x}(t) = 0$.

(c) Plot the vector field $f(X)$ in the phase plane $(x_1, x_2) = (x, \dot{x})$. Plot for points spaced in a uniform mesh in the box $x_1 = [-10, 10]$, $x_2 = [-10, 10]$.

(d) Plot the system trajectories (flows or orbits) in the phase plane. Take ICs spaced in a uniform mesh in the box $x_1 = [-10, 10]$, $x_2 = [-10, 10]$.

(e) Derive the First Integral of Motion $F(x_1, x_2)$ as done in class. Plot the FIM as a 3-D surface over the phase plane on the $x_1 = [-10, 10]$, $x_2 = [-10, 10]$.

Answer
a) Write position-velocity state space form $\dot{X} = f(X)$.

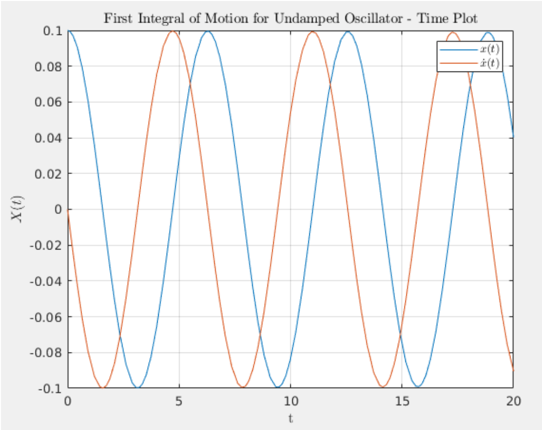
$$\ddot{x} + x = 0 \implies \ddot{x} = -x \implies$$

$$X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \; ; \; \dot{X} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = f(x)$$

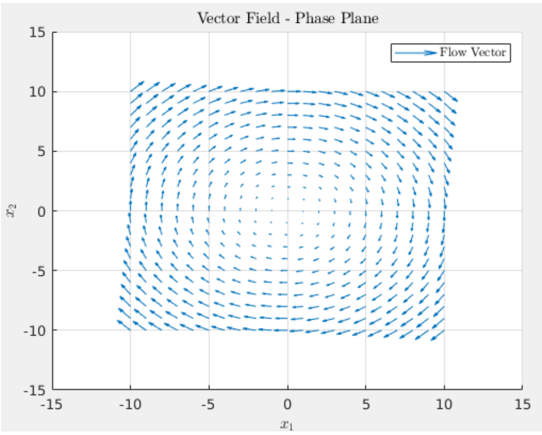
State space form:

$$\dot{X} = AX \implies \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

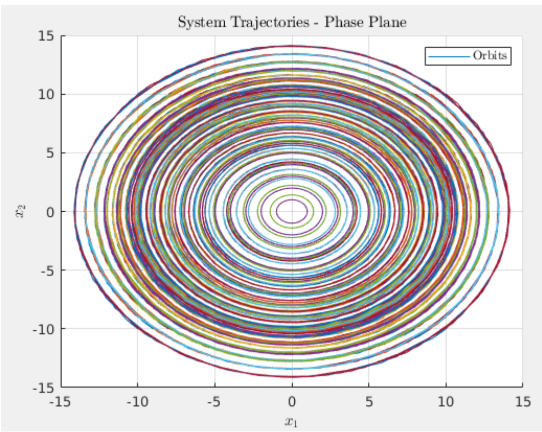
b) Plot the trajectories $x(t)$, $\dot{x}(t)$ vs. time. Use initial conditions of $x(0) = 0.1$, $\dot{x}(t) = 0$.



c) Plot the vector field $f(X)$ in the phase plane $(x_1, x_2) = (x, \dot{x})$. Plot for points spaced in a uniform mesh in the box $x_1 = [-10, 10]$, $x_2 = [-10, 10]$.



d) Plot the system trajectories (flows or orbits) in the phase plane. Take ICs spaced in a uniform mesh in the box $x_1 = [-10, 10]$, $x_2 = [-10, 10]$.



e) Derive the First Integral of Motion $F(x_1, x_2)$ as done in class. Plot the FIM as a 3-D surface over the phase plane on the $x_1 = [-10, 10]$, $x_2 = [-10, 10]$.

$$\ddot{x} + x = 0 \Longrightarrow \ddot{x}\dot{x} + x\dot{x} = 0 \;\; ;$$

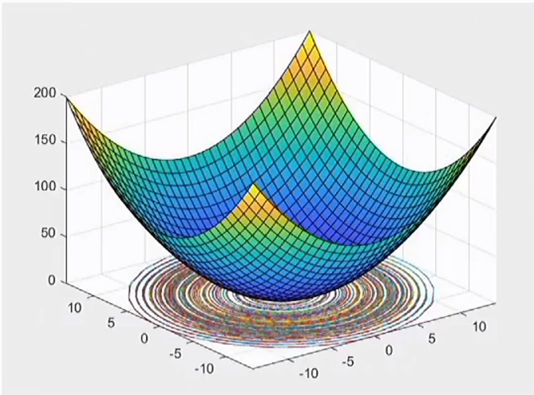
Where

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2\right) = 0 \Longrightarrow \qquad F(x) = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 \;\; ;$$

$$\dot{F}(x) = \frac{\partial F}{\partial x}\dot{x} \quad ; \qquad \frac{\partial F}{\partial x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \quad ; \qquad \dot{x} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$$

Hence, we have FIM as

$$\dot{F}(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = 0$$



Matlab Code

```
%% HW04 - Q01 - Vector Fields, Flows, First Integrals
% @author: Bardia Mojra
% @date: 10/14/2021
% @title HW03 - Q01 - First Integral of Motion for Undamped
    Oscillator
% @class ee5323 - Nonlinear Systems
% @professor - Dr. Frank Lewis

clc
clear
close all
warning('off','all')
warning

% part b
t_intv= [0 20];
x_0= [0.1, 0]'; % initial conditions for x(t)
figure
[t,x]= ode23('q01_sys', t_intv, x_0);
plot(t,x)
hold on;
grid on;
title('First Integral of Motion for Undamped Oscillator - Time
    Plot','Interpreter','latex');
ylabel('$X(t)$','Interpreter','latex');
xlabel('t','Interpreter','latex');
legend('$x(t)$', '$\dot{x}(t)$','Interpreter','latex');

% part c
figure
hold on;
grid on;
[x1,x2] = meshgrid(-10:1:10,-10:1:10);
dx1=[];
dx2=[];
N=length(x1);
for i=1:N
  for j=1:N
    dx = q01_sys(0, [x1(i,j);x2(i,j)]);
    dx1(i,j) = dx(1);
    dx2(i,j) = dx(2);
  end
end
quiver(x1,x2,dx1,dx2);
ylabel('$x_2$','Interpreter','latex');
xlabel('$x_1$','Interpreter','latex');
```

```
legend('Flow Vector','Interpreter','latex');
title('Vector Field - Phase Plane','Interpreter','latex');

% part d
figure
hold on
for i=-10:1:10
  for j=-10:1:10
    init=[i j];
    [t, x] = ode23('q01_sys', [0 10], init);
    figure(3)
    plot(x(:,1),x(:,2))
    hold on;
  end
end
ylabel('$x_2$','Interpreter','latex');
xlabel('$x_1$','Interpreter','latex');
legend('Orbits','Interpreter','latex');
title('System Trajectories - Phase Plane','Interpreter','latex');
grid on;

% part e
figure
hold on
for i=-10:1:10
  for j=-10:1:10
    init=[i j];
    [t, x] = ode23('q01_sys', [0 10], init);
    N=length(x);
    figure(4)
    hold on;
    plot3(x(:,1), x(:,2), zeros(N,1))
    hold on;
    syms x1 x2
    fsurf(0.5* x1^2 +0.5* x2^2)
    hold on;
  end
end
ylabel('$x_2$','Interpreter','latex');
xlabel('$x_1$','Interpreter','latex');
xlabel('$V$','Interpreter','latex');
legend('Potential','Interpreter','latex');
title('First Integral of Motion - Phase Plane','Interpreter','latex');
grid on;

%%
```

```
%
% function xdot = q01_sys(t,x)
%    xdot = [x(2); -x(1)];
% end
%
```

Exercise 2

First Integral of Motion for Undamped Oscillator

$$\ddot{x} + x = 0$$

(a) Write position-velocity state space form $\dot{X} = f(X)$.

(b) Plot the trajectories $x(t)$, $\dot{x}(t)$ vs. time. Use initial conditions of $x(0) = 0.1$, $\dot{x}(t) = 0$.

(c) Plot the vector field $f(X)$ in the phase plane $(x_1, x_2) = (x, \dot{x})$. Plot for points spaced in a uniform mesh in the box $x_1 = [-10, 10]$, $x_2 = [-10, 10]$.

(d) Plot the system trajectories (flows or orbits) in the phase plane. Take ICs spaced in a uniform mesh in the box $x_1 = [-10, 10]$, $x_2 = [-10, 10]$.

(e) Derive the First Integral of Motion $F(x_1, x_2)$ as done in class. Plot the FIM as a 3-D surface over the phase plane on the $x_1 = [-10, 10]$, $x_2 = [-10, 10]$.

Answer

a) Write position-velocity state space form $\dot{X} = f(X)$.

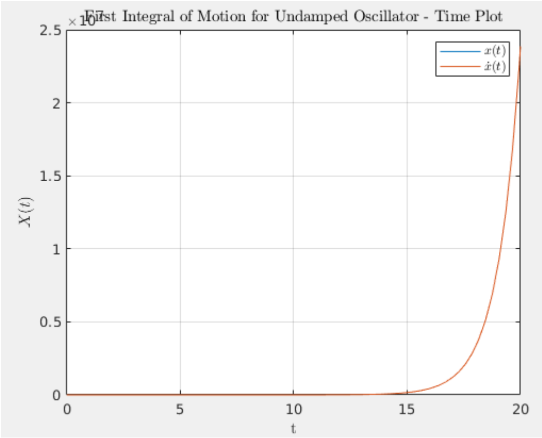
$$\ddot{x} - x = 0 \implies \ddot{x} = +x \implies$$

$$X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ ; \ \dot{X} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = f(x)$$

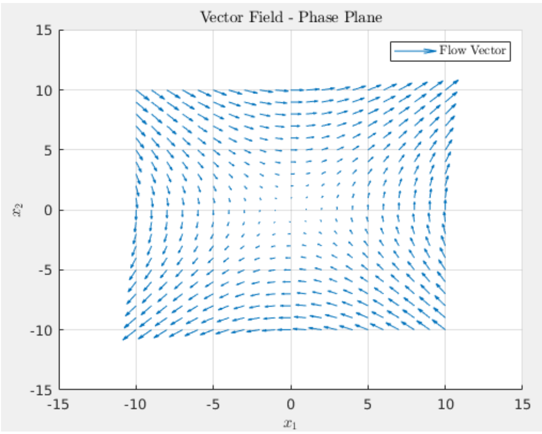
State space form:

$$\dot{X} = AX \implies \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

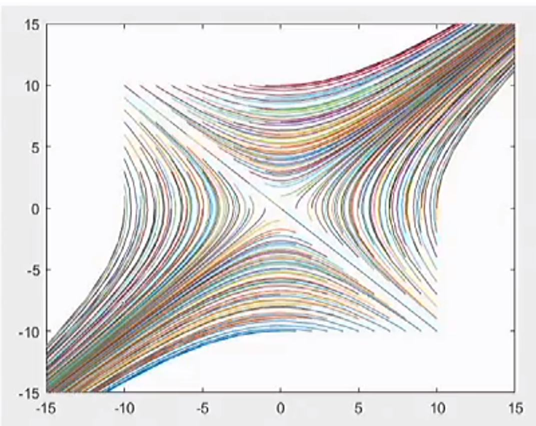
b) Plot the trajectories $x(t)$, $\dot{x}(t)$ vs. time. Use initial conditions of $x(0) = 0.1$, $\dot{x}(t) = 0$.



c) Plot the vector field $f(X)$ in the phase plane $(x_1, x_2) = (x, \dot{x})$. Plot for points spaced in a uniform mesh in the box $x_1 = [-10, 10]$, $x_2 = [-10, 10]$.



d) Plot the system trajectories (flows or orbits) in the phase plane. Take ICs spaced in a uniform mesh in the box $x_1 = [-10, 10]$, $x_2 = [-10, 10]$.



e) Derive the First Integral of Motion $F(x_1, x_2)$ as done in class. Plot the FIM as a 3-D surface over the phase plane on the $x_1 = [-10, 10]$, $x_2 = [-10, 10]$.

$$\ddot{x} + x = 0 \Longrightarrow \ddot{x}\dot{x} - x\dot{x} = 0 \ ;$$

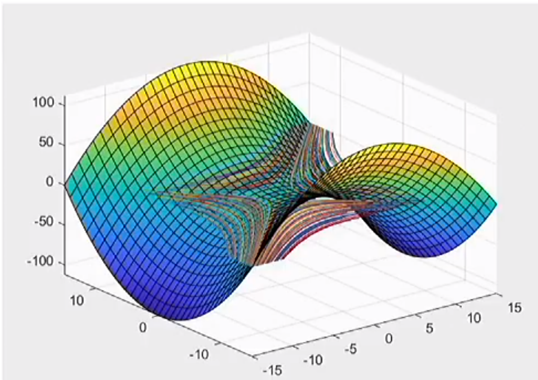
Where

$$\frac{d}{dt}\left(-\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2\right) = 0 \Longrightarrow \quad F(x) = -\frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 \ ;$$

$$\dot{F}(x) = \frac{\partial F}{\partial x}\dot{x} \quad ; \quad \frac{\partial F}{\partial x} = \begin{bmatrix} -x_1 & x_2 \end{bmatrix} \quad ; \quad \dot{x} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

Hence, we have FIM as

$$\dot{F}(x) = \begin{bmatrix} -x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = 0$$



Matlab Code

```
%% HW04 - Q02 - Vector Fields, Flows, First Integrals
% @author: Bardia Mojra
% @date: 10/14/2021
% @title HW03 - Q01 - First Integral of Motion for Undamped
    Oscillator
% @class ee5323 - Nonlinear Systems
% @professor - Dr. Frank Lewis

clc
clear
close all
warning('off','all')
warning

% part b
t_intv= [0 20];
x_0= [0.1, 0]'; % initial conditions for x(t)
figure
[t,x]= ode23('q02_sys', t_intv, x_0);
plot(t,x)
hold on;
grid on;
title('First Integral of Motion for Undamped Oscillator - Time
    Plot','Interpreter','latex');
ylabel('$X(t)$','Interpreter','latex');
xlabel('t','Interpreter','latex');
legend('$x(t)$', '$\dot{x}(t)$','Interpreter','latex');

% part c
figure
hold on;
grid on;
[x1,x2] = meshgrid(-10:1:10,-10:1:10);
dx1=[];
dx2=[];
N=length(x1);
for i=1:N
  for j=1:N
    dx = q02_sys(0, [x1(i,j);x2(i,j)]);
    dx1(i,j) = dx(1);
    dx2(i,j) = dx(2);
  end
end
quiver(x1,x2,dx1,dx2);
ylabel('$x_2$','Interpreter','latex');
xlabel('$x_1$','Interpreter','latex');
```

```
legend('Flow Vector','Interpreter','latex');
title('Vector Field - Phase Plane','Interpreter','latex');

% part d
figure
hold on
for i=-10:1:10
  for j=-10:1:10
    init=[i j];
    [t, x] = ode23('q02_sys', [0 10], init);
    figure(3)
    plot(x(:,1),x(:,2))
    hold on;
  end
end
ylabel('$x_2$','Interpreter','latex');
xlabel('$x_1$','Interpreter','latex');
legend('Orbits','Interpreter','latex');
title('System Trajectories - Phase Plane','Interpreter','latex
   ');
grid on;

% part e
figure
hold on
for i=-10:1:10
  for j=-10:1:10
    init=[i j];
    [t, x] = ode23('q02_sys', [0 10], init);
    N=length(x);
    figure(4)
    hold on;
    plot3(x(:,1), x(:,2), zeros(N,1))
    hold on;
    syms x1 x2
    fsurf(-0.5* x1^2 -0.5* x2^2)
    hold on;
  end
end


%%
%
% function xdot = q02_sys(t,x)
%   xdot = [x(2); x(1)];
% end
```