EE 5323 - HW05

Bardia Mojra
1000766739
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HW05 – Lyapunov Stability Analysis
EE 5323 – Nonlinear Systems
Dr. Frank Lewis

Exercise 1

Lyapunov's Direct Method - SISL
Consider the system

$$\begin{cases} \dot{x}_1 = x_1 x_2^2 - x_1 \\ \dot{x}_2 = -x_1^2 x_2 \end{cases}$$

Use Lyapunov to study the stability. SISL? AS?

Answer
1) Lyapunov function candidate: $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) > 0$

$$\dot{V} = \frac{\partial V^\top}{\partial x}\dot{x} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \implies$$

$$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2$$

Now we plug in system dynamics to check stability,

$$\dot{V} = x_1(x_1 x_2^2 - x_1) + x_2(-x_1^2 x_2) \implies$$

$$\dot{V} = \cancel{x_1^2 x_2^2} - x_1^2 - \cancel{x_1^2 x_2^2}$$

$$\dot{V} = -x_1^2 \leq 0$$

Thus, the system is *marginally stable* and it is considered *SISL*.

Exercise 2

Lyapunov's Direct Method - AS
Consider the system

$$\begin{cases} \dot{x}_1 = x_1 x_2^2 - x_1 \\ \dot{x}_2 = -x_1^2 x_2 - x_2 \end{cases}$$

Use Lyapunov to study the stability. SISL? AS?

Answer
2) Lyapunov function candidate: $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) > 0$

$$\dot{V} = \frac{\partial V^\top}{\partial x}\dot{x} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \Longrightarrow$$

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$$

Now we plug in system dynamics to check stability,

$$\dot{V} = x_1(x_1 x_2^2 - x_1) + x_2(-x_1^2 x_2 - x_2) \Longrightarrow$$

$$\dot{V} = \cancel{x_1^2 x_2^2} - x_1^2 - \cancel{x_1^2 x_2^2} - x_2^2$$
$$\dot{V} = -x_1^2 - x_2^2 < 0$$

Thus, the system is *asymptotically stable* (AS). Moreover, one can assume it is *global asymptotically stable* (GAS) since the Lyapunov function is always negative.

Exercise 3

Asymptotic Stability Simulation
a) Use Lyapunov to show that the system

$$\begin{cases} \dot{x}_1 = x_1 x_2^2 - x_1(x_1^2 + x_2^2 - 3) \\ \dot{x}_2 = -x_1^2 x_2 - x_2(x_1^2 + x_2^2 - 3) \end{cases}$$

is locally asymptotically stable. Find the Region of Asymptotic Stability.
b) Simulate the system from many uniformly spaced ICs.

Answer
3.a) Lyapunov function candidate: $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) > 0$

$$\dot{V} = \frac{\partial V^\top}{\partial x}\dot{x} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \Longrightarrow$$

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$$

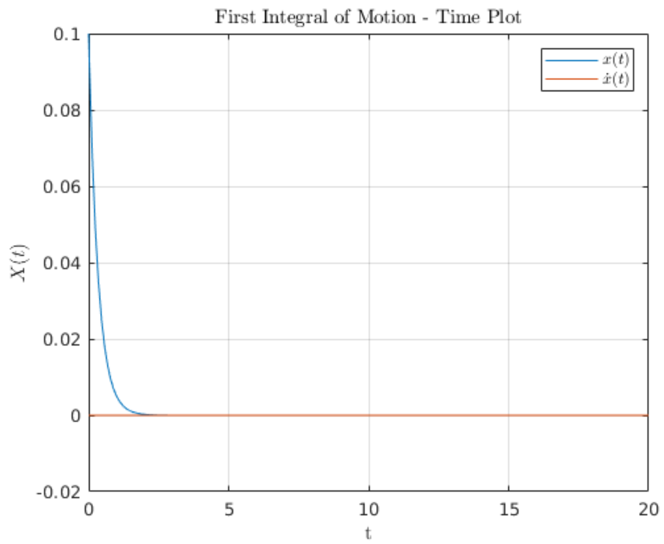
Now we plug in system dynamics to check stability,

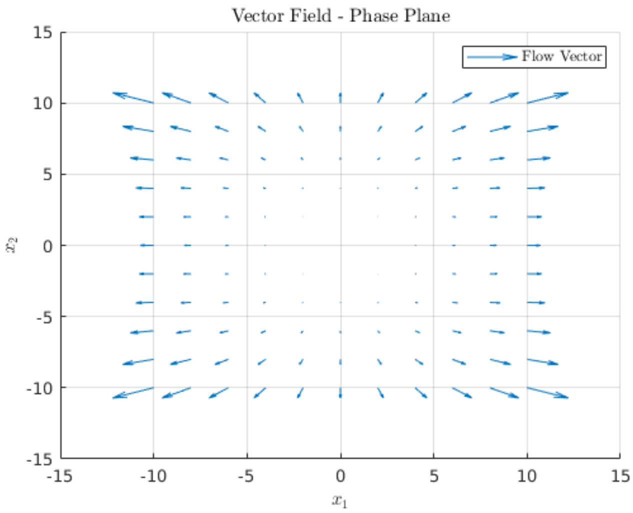
$$\dot{V} = x_1(x_1x_2^2 - x_1(x_1^2 + x_2^2 - 3)) + x_2(-x_1^2x_2 - x_2(x_1^2 + x_2^2 - 3)) \Longrightarrow$$

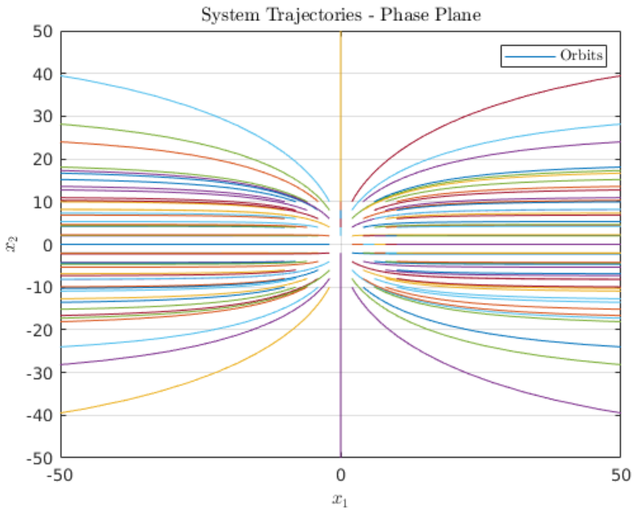
$$\dot{V} = \cancel{x_1^2x_2^2} - x_1^2(x_1^2 + x_2^2 - 3) - \cancel{x_1^2x_2^2} - x_2^2(x_1^2 + x_2^2 - 3)$$
$$\dot{V} = (x_1^2 + x_2^2 - 3)(-x_1^2 - x_2^2) < 0$$

Thus, the system is *asymptotically stable* (AS) and it is bound by a region with radius of $\sqrt{3}$.

3.b) Simulation:







Matlab Code

```
%% HW05 - Q03 - AS
% @author: Bardia Mojra
% @date: 10/28/2021
% @title HW05 - Q03 - Asymptotic Stability Simulation
% @class ee5323 - Nonlinear Systems
% @professor - Dr. Frank Lewis

clc
clear
close all
warning('off','all')
warning

% part a
t_intv= [0 20];
x_0= [0.1, 0]'; % initial conditions for x(t)
```

```
figure
[t,x]= ode23('q03_sys', t_intv, x_0);
plot(t,x)
hold on;
grid on;
title('First Integral of Motion - Time Plot','Interpreter','latex');
ylabel('$X(t)$','Interpreter','latex');
xlabel('t','Interpreter','latex');
legend('$x(t)$', '$\dot{x}(t)$','Interpreter','latex');

% part b
figure();
hold on;
grid on;
mesh = -10:2:10;
[x1,x2] = meshgrid(mesh,mesh);
dx1=[];
dx2=[];
N=length(x1);
for i=1:N
  for j=1:N
    dx = q03_sys(0, [x1(i,j);x2(i,j)]);
    dx1(i,j) = dx(1);
    dx2(i,j) = dx(2);
  end
end
quiver(x1,x2,dx1,dx2);
ylabel('$x_2$','Interpreter','latex');
xlabel('$x_1$','Interpreter','latex');
legend('Flow Vector','Interpreter','latex');
title('Vector Field - Phase Plane','Interpreter','latex');
axis([-15 15 -15 15])

% part c
figure
for i=mesh
  for j=mesh
    init=[i, j];
    [t, x] = ode23(@q03_sys, [0 10], init);
    plot(x(:,1),x(:,2))
    hold on;
  end
end
ylabel('$x_2$','Interpreter','latex');
xlabel('$x_1$','Interpreter','latex');
legend('Orbits','Interpreter','latex');
title('System Trajectories - Phase Plane','Interpreter','latex
```

```
    ');
grid on;
axis([-50 50 -50 50])

%%
%
% function xdot = q03_sys(t,x)
%    xdot = [x(1)*x(2)^2+x(1)*(x(1)^2+x(2)^2-3); -x(1)^2 *x(2)+
    x(2)*(x(1)^2+x(2)^2-3)];
% end
```

Exercise 4

SISL Simulation
a) Use quadratic Lyapunov Function to show this system is locally SISL

$$\begin{cases} \dot{x}_1 = x_2 + x_1(x_1^2 - 2) \\ \dot{x}_2 = -x_1 \end{cases}$$

Find a region within which $V \leq 0$.
b) Simulate the system from many uniformly spaced ICs.

Answer
4.a) Lyapunov function candidate: $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) > 0$

$$\dot{V} = \frac{\partial V^\top}{\partial x}\dot{x} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \implies$$

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$$

Now we plug in system dynamics to check stability,

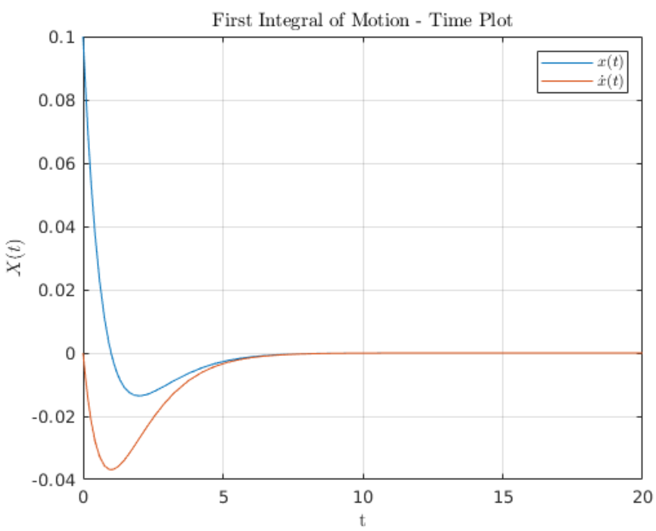
$$\dot{V} = x_1(x_2 + x_1(x_1^2 - 2)) + x_2(-x_1) \implies$$

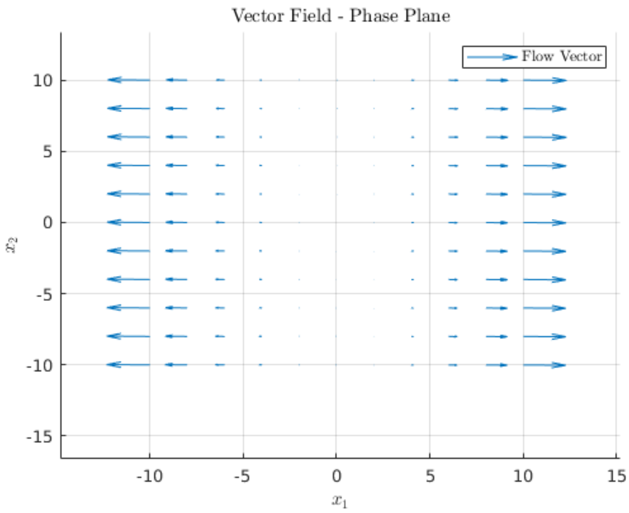
$$\dot{V} = \cancel{x_1x_2} - x_1^2(x_1^2 - 2) - \cancel{x_1x_2}$$

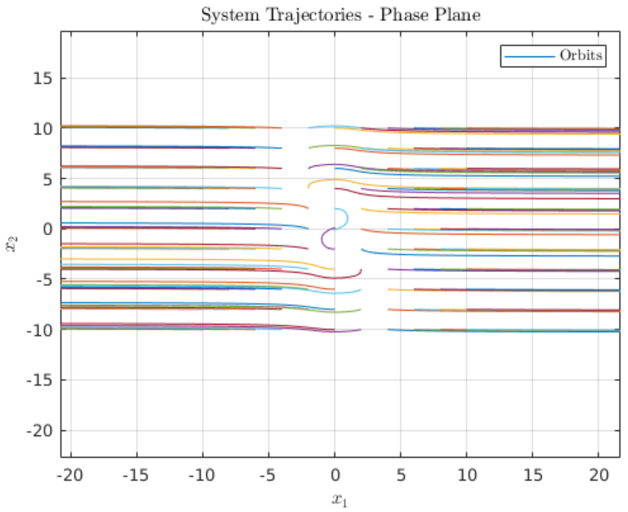
$$\dot{V} = -x_1^2(x_1^2 - 2) \leq 0$$

Thus, the system is *asymptotically stable* (AS) and it is bound by a region with radius of $\sqrt{3}$.

4.b) Simulation:







Matlab Code

```
%% HW05 - Q04 - AS
% @author: Bardia Mojra
% @date: 10/28/2021
% @title HW05 - Q04 - SISL Simulation
% @class ee5323 - Nonlinear Systems
% @professor - Dr. Frank Lewis

clc
clear
close all
warning('off','all')
warning

% part a
t_intv= [0 20];
x_0= [0.1, 0]'; % initial conditions for x(t)
```

```
figure
[t,x]= ode23('q04_sys', t_intv, x_0);
plot(t,x)
hold on;
grid on;
title('First Integral of Motion - Time Plot','Interpreter','latex');
ylabel('$X(t)$','Interpreter','latex');
xlabel('t','Interpreter','latex');
legend('$x(t)$', '$\dot{x}(t)$','Interpreter','latex');

% part b
figure();
hold on;
grid on;
mesh = -10:2:10;
[x1,x2] = meshgrid(mesh,mesh);
dx1=[];
dx2=[];
N=length(x1);
for i=1:N
  for j=1:N
    dx = q04_sys(0, [x1(i,j);x2(i,j)]);
    dx1(i,j) = dx(1);
    dx2(i,j) = dx(2);
  end
end
quiver(x1,x2,dx1,dx2);
ylabel('$x_2$','Interpreter','latex');
xlabel('$x_1$','Interpreter','latex');
legend('Flow Vector','Interpreter','latex');
title('Vector Field - Phase Plane','Interpreter','latex');
axis([-15 15 -15 15])

% part c
figure
for i=mesh
  for j=mesh
    init=[i, j];
    [t, x] = ode23(@q04_sys, [0 10], init);
    plot(x(:,1),x(:,2))
    hold on;
  end
end
ylabel('$x_2$','Interpreter','latex');
xlabel('$x_1$','Interpreter','latex');
legend('Orbits','Interpreter','latex');
title('System Trajectories - Phase Plane','Interpreter','latex
```

```
    ');
grid on;
axis([-50 50 -50 50])

%%
%
% function xdot = q04_sys(t,x)
%    xdot = [x(2) + x(1)*(x(1)^2-2); -x(1)];
% end
```