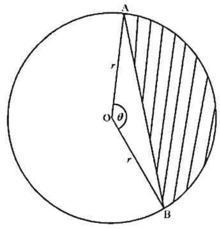
REVISION EXERCISES

P2 1. The following diagram shows a circle centre O, radius r. The angle $\mathrm{A\hat{O}B}$ at the centre of the circle is θ radians. The chord AB divides the circle into a minor segment (the shaded region) and a major segment.



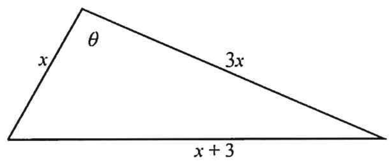
(a) Show that the area of the minor segment is $\frac{1}{2}r^2(\theta - \sin\theta)$.

(b) Find the area of the major segment. [$= \pi r^2 - \frac{1}{2}r^2(\theta - \sin\theta)$ $\left(= r^2\left(\pi - \frac{\theta}{2} + \frac{\sin\theta}{2}\right)\right)$]

(c) Given that the ratio of the areas of the two segments is 2:3, show that $\sin\theta = \theta - \frac{4\pi}{5}$

(d) Hence find the value of θ. [$\theta = 2.82$ radians]

P2 2. The area of the triangle shown below is 2.21 cm^2. The length of the shortest side is x cm and the other two sides are $3x$ cm and $(x + 3)$ cm.



(a) Using the formula for the area of the triangle, write down an expression for $\sin\theta$ in terms of x. [$\sin\theta = \frac{4.42}{3x^2}$]

(b) Using the cosine rule, write down and simplify an expression for $\cos\theta$ in terms of x. [$\frac{3x^2 - 2x - 3}{2x^2}$]

(c) (i) Using your answers to parts (a) and (b), show that,

$$\left(\frac{3x^2 - 2x - 3}{2x^2}\right)^2 = 1 - \left(\frac{4.42}{3x^2}\right)^2$$

(ii) Hence find

(a) the possible values of x; [1.24, 2.94]

(b) the corresponding values of θ, in radians, using your answer to part (b) above. [1.86 radians or $\theta = 0.171$]

P2 3. Consider the triangle ABC where $\mathrm{B\hat{A}C} = 70°$, AB = 8 cm and AC = 7 cm. The point D on the side BC is such that $\frac{\mathrm{BD}}{\mathrm{DC}} = 2$. Determine the length of AD. [6.12 cm]

P1 4. Let $z_1 = a\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$ and $z_2 = b\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$.

Express $\left(\frac{z_1}{z_2}\right)^3$ in the form $z = x + yi$. $\left[\left(\frac{\sqrt{2}}{2}\frac{a^3}{b^3}\right) - \left(\frac{\sqrt{2}}{2}\frac{a^3}{b^3}\right)\mathrm{i}\right]$

P1 5. The complex number z is defined by

$$z = 4\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right) + 4\sqrt{3}\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right).$$

(a) Express z in the form $re^{\mathrm{i}\theta}$, where r and θ have exact values. $\left(z = 8e^{\mathrm{i}\left(\frac{\pi}{3}\right)}\right)$

(b) Find the cube roots of z, expressing in the form $re^{\mathrm{i}\theta}$, where r and θ have exact values. $\left[z^{\frac{1}{3}} = 2e^{\mathrm{i}\left(\frac{\pi}{9}\right)}\left(=2e^{\mathrm{i}20°}\right), z^{\frac{1}{3}} = 2e^{\mathrm{i}\left(\frac{7\pi}{9}\right)}\left(=2e^{\mathrm{i}140°}\right), z^{\frac{1}{3}} = 2e^{\mathrm{i}\left(\frac{13\pi}{9}\right)}\left(=2e^{\mathrm{i}260°}\right)\right]$

P1 6. (a) Express the complex number $1+\mathrm{i}$ in the form $\sqrt{a}e^{\mathrm{i}\frac{\pi}{b}}$, where $a, b \in \mathbb{Z}^+$. $[\sqrt{2}e^{i\frac{\pi}{4}}]$

(b) Using the result from (a), show that $\left(\frac{1+\mathrm{i}}{\sqrt{2}}\right)^n$, where $n \in \mathbb{Z}$, has only eight distinct values.

(c) Hence solve the equation $z^8 - 1 = 0$. $[1, e^{\frac{\mathrm{i}\pi}{4}}, e^{\frac{\mathrm{i}\pi}{2}}, e^{\frac{\mathrm{i}3\pi}{4}}, e^{\pi}, e^{\frac{\mathrm{i}5\pi}{4}}, e^{\frac{\mathrm{i}3\pi}{2}}, e^{\frac{\mathrm{i}7\pi}{4}}]$

P1 7. Consider the complex number $z = \dfrac{\left(\cos\frac{\pi}{4} - \mathrm{i}\sin\frac{\pi}{4}\right)^2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)^3}{\left(\cos\frac{\pi}{24} - \mathrm{i}\sin\frac{\pi}{24}\right)^4}$.

(a) (i) Find the modulus of z. [1]

(ii) Find the argument of z, giving your answer in radians. $[\frac{2\pi}{3}]$

(a)
(b) Using De Moivre's theorem, show that z is a cube root of one, *ie* $z = \sqrt[3]{1}$.

P1 8. Let $z = \cos\theta + \mathrm{i}\sin\theta$, for $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

(a) (i) Find z^3 using the binomial expansion.

(ii) Use de Moivre's theorem to show that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \text{ and } \sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

(b) Hence prove that $\dfrac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta} = \tan\theta$.

(c) Given that $\sin\theta = \frac{1}{3}$, find the exact value of $\tan 3\theta$. $\left[\tan 3\theta = \frac{23}{10\sqrt{2}}\left(=\frac{23}{20}\sqrt{2}\right)\right]$

P1 9. Given that $z = \cos\theta + \mathrm{i}\sin\theta$ show that $\mathrm{Im}\left(z^n + \frac{1}{z^n}\right) = 0, n \in \mathbb{Z}^+$.

P1 10. For what values of m is the line $y = mx + 5$ a tangent to the parabola $y = 4 - x^2$? $[m = \pm 2]$

P1 11. Consider the function $y = \tan x - 8 \sin x$.

(a) Find $\frac{dy}{dx}$. [$\sec^2 x - 8 \cos x$]

(b) Find the value of $\cos x$ for which $\frac{dy}{dx} = 0$. [$\cos x = \frac{1}{2}$]

P1 12. Consider the tangent to the curve $y = x^3 + 4x^2 + x - 6$.

(a) Find the equation of this tangent at the point where $x = -1$. [$y = -4x - 8$]

(b) Find the coordinates of the point where this tangent meets the curve again. [$(-2, 0)$]

P1 13. Let f be a cubic polynomial function. Given that $f(0) = 2$, $f'(0) = -3$, $f(1) = f'(1)$ and $f''(-1) = 6$, find $f(x)$.

[$f(x) = -\frac{1}{5}x^3 + \frac{12}{5}x^2 - 3x + 2 \left(\text{Accept } a = -\frac{1}{5}, b = \frac{12}{5}, c = -3, d = 2 \right)$]

P2 14. Let $y = e^{3x} \sin(\pi x)$.

(a) Find $\frac{dy}{dx}$. [$3e^{3x} \sin(\pi x) + \pi e^{3x} \cos(\pi x)$]

(b) Find the smallest positive value of x for which $\frac{dy}{dx} = 0$. [$x = 0.743$]

P1 15. Let $y = x \arcsin x$, $x \in]-1, 1[$. Show that $\frac{d^2 y}{dx^2} = \frac{2 - x^2}{(1 - x^2)^{\frac{3}{2}}}$.

P1 16. Let $f(x) = \cos^3(4x + 1)$, $0 \leq x \leq 1$.
(a) Find $f'(x)$. [$-12 \cos^2(4x + 1)(\sin(4x + 1))$]
(b) Find the exact values of the three roots of $f'(x) = 0$.

[$x = \frac{\pi}{8} - \frac{1}{4}, x = \frac{3\pi}{8} - \frac{1}{4}, x = \frac{\pi}{4} - \frac{1}{4}$]

P2 17. The function f is defined by $f : x \mapsto 3^x$.

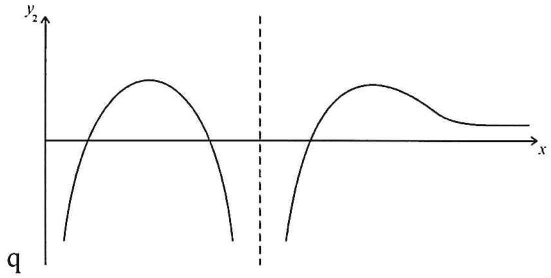
Find the solution of the equation $f''(x) = 2$. [0.460]

P1 18. Given that $3^{x+y} = x^3 + 3y$, find $\frac{dy}{dx}$. [$\frac{dy}{dx} = \frac{3x^2 - (\ln 3)3^{x+y}}{(\ln 3)3^{x+y} - 3}$]

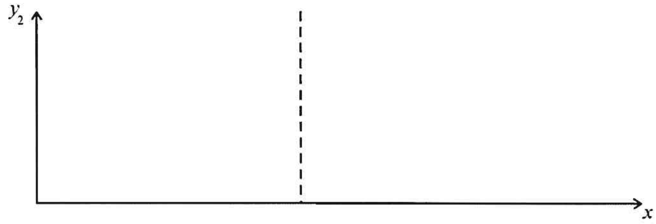
P1 19. The tangent to the curve $y^2 = x^3$ at the point P(1, 1) meets the x-axis at Q and the y-axis at R. Find the ratio PQ : QR. [2 : 1]

P1 20. A normal to the graph of $y = \arctan(x - 1)$, for $x > 0$, has equation $y = -2x + c$, where $c \in \mathbb{R}$. Find the value of c. [$c = 4 + \frac{\pi}{4}$]

P1 21. The diagram below shows the graph of $y_1 = f(x)$.



On the axes below, sketch the graph of $y_2 = |f'(x)|$.



P2 22. The function f is defined by

$$f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$$

(a) (i) Find an expression for $f'(x)$, simplifying your answer. $\left[\frac{2(x^2-1)}{(x^2+x+1)^2}\right]$

(ii) The tangents to the curve of $f(x)$ at points A and B are parallel to the x-axis.

Find the coordinates of A and of B. [A$\left(1, \frac{1}{3}\right)$, B$(-1, 3)$]

(b) (i) Sketch the graph of $y = f'(x)$.

(ii) Find the x-coordinates of the three points of inflexion on the graph of f.

[$x = 1.53, -0.347, 1.88$]

(c) Find the range of f. $\left[\left[\frac{1}{3}, 3\right]\right]$

P1 23. Consider the curve with equation $x^2 + xy + y^2 = 3$.

(a) Find in terms of k, the gradient of the curve at the point $(-1, k)$. $\left[\frac{dy}{dx} = \frac{2-k}{2k-1}\right]$

(b) Given that the tangent to the curve is parallel to the x-axis at this point, find the value of k. [$k = 2$]

P1 24. Consider the function $f(x) = \dfrac{\ln x}{x}$, $0 < x < e^2$.

(a) (i) Solve the equation $f'(x) = 0$. $[x = e]$

(ii) Hence show the graph of f has a local maximum.

(iii) Write down the range of the function f. $[y \le \dfrac{1}{e}]$

(b) Show that there is a point of inflexion on the graph and determine its coordinates. $[x = e^{\frac{3}{2}}]$

(c) Sketch the graph of $y = f(x)$, indicating clearly the asymptote, x-intercept and the local maximum.

P1 25. A family of cubic functions is defined as $f_k(x) = k^2x^3 - kx^2 + x$, $k \in \mathbb{Z}^+$.

(a) Express in terms of k

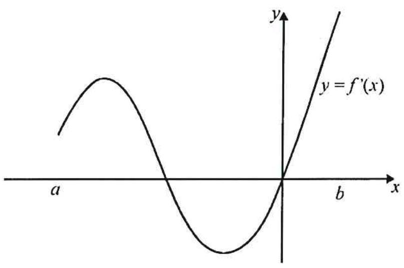
(i) $f'_k(x)$ and $f''_k(x)$; $[f'_k(x) = 3k^2x^2 - 2kx + 1,\ f''_k(x) = 6k^2x - 2k]$

(ii) the coordinates of the points of inflexion P_k on the graphs of f_k. $[\left(\dfrac{1}{3k}, \dfrac{7}{27k}\right)]$

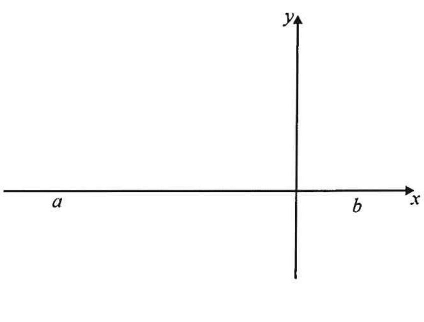
(b) Show that all P_k lie on a straight line and state its equation.

(c) Show that for all values of k, the tangents to the graphs of f_k at P_k are parallel, and find the equation of the tangent lines. $[y = \dfrac{2}{3}x + \dfrac{1}{27k}]$

P1 26. The diagram shows a sketch of the graph of $y = f'(x)$ for $a \le x \le b$.



On the grid below, which has the same scale on the x-axis, draw a sketch of the graph of $y = f(x)$ for $a \le x \le b$, given that $f(0) = 0$ and $f(x) \ge 0$ for all x. On your graph you should clearly indicate any minimum or maximum points, or points of inflexion.



P1 27. The function f is defined by $f(x) = x\,e^{2x}$.

(a) Show that there is one minimum point P on the graph of f, and find the coordinates of P. $\left[\mathrm{P}\left(-\frac{1}{2}, -\frac{1}{2e}\right)\right]$

(b) Show that f has a point of inflexion Q at $x = -1$.

(c) Determine the intervals on the domain of f where f is

(i) concave up; [concave up for $x > -1$]

(ii) concave down. [concave down for $x < -1$]

(d) Sketch f, clearly showing any intercepts, asymptotes and the points P and Q.