Solution of sheet 10
Metodi Matematici per l'IA
12-12-2024

Exercise 1:

1. State the Dirichelet–Weierstrass theorem.
2. Let $N \in \mathbb{N}$. Define the Dirichlet kernel D_N and prove that it is an even function.
3. Let $f \in L^1_T(\mathbb{C})$ and let S_N be the N-th partial sum associated with f. Prove that $S_N = f * D_N$.

Solution

1. State the Dirichlet–Weierstrass theorem

The Dirichlet–Weierstrass theorem states (ref. theorem 3.2, lecture notes):
Let $f \in L^1_{\mathbb{C}}(T)$, and let $x_0 \in \mathbb{R}$. Suppose the following four limits exist:

$$\lim_{x \to x_0^+} f(x) = f(x_0^+), \quad \lim_{x \to x_0^-} f(x) = f(x_0^-),$$

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0^+)}{x - x_0}, \quad \lim_{x \to x_0^-} \frac{f(x) - f(x_0^-)}{x - x_0}.$$

Then, the Fourier partial sums $S_N(x_0)$ converge to:

$$\lim_{N \to \infty} S_N(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2}.$$

2. Define the Dirichlet kernel D_N and prove that it is an even function.

From the lecture notes (ref. Definition 3.8 and Lemma 3.9), the Dirichlet kernel is defined as:

$$D_N(x) = \frac{1}{T} \sum_{|n| \leq N} e^{in\omega x},$$

where $\omega = \frac{2\pi}{T}$.

Proof:

$$D_N(-x) = \frac{1}{T} \sum_{|n| \leq N} e^{in\omega(-x)} = \frac{1}{T} \sum_{|n| \leq N} e^{-in\omega x}.$$

Changing the index $n \to -n$, we have:

$$D_N(-x) = \frac{1}{T} \sum_{|n| \leq N} e^{in\omega x} = D_N(x).$$

Thus, $D_N(x)$ is an even function.

3. Prove that $S_N = f * D_N$, where S_N is the N-th partial sum.

From the lecture notes (ref. Proposition 3.21):

The N-th partial sum $S_N(x)$ is given by:

$$S_N(x) = \sum_{|n| \leq N} c_n(f) e^{in\omega x}.$$

Substituting $c_n(f) = \frac{1}{T} \int_{-T/2}^{T/2} f(y) e^{-in\omega y} \, dy$, we get:

$$S_N(x) = \frac{1}{T} \int_{-T/2}^{T/2} f(y) \sum_{|n| \leq N} e^{in\omega(x-y)} \, dy.$$

Using the definition of $D_N(x)$, we have:

$$S_N(x) = \int_{-T/2}^{T/2} f(y) D_N(x-y) \, dy.$$

This is the convolution of f and D_N, so:

$$S_N(x) = f * D_N(x).$$

Esercizio 2:

1. Dimostrare che la successione di funzioni $f_n(x) = e^{-nx^2}$ converge to a zero in $L^1(\mathbb{R})$.
2. Dare un esempio di una successione di funzioni $(\phi_n)_{n \in \mathbb{N}} : \mathbb{R} \to \mathbb{R}^+$ tale che $\phi_n(x) \to 0$ q.o. in $\mathbb{R}$ e tale che $(\phi_n)_{n \in \mathbb{N}}$ non converge a zero in $L^1(\mathbb{R}; \mathbb{R}^+)$.

Solution

1. Prove that the sequence of functions $f_n(x) = e^{-nx^2}$ converges to zero in $L^1(\mathbb{R})$.

The L^1-norm of $f_n(x)$ is:

$$\|f_n\|_{L^1} = \int_{-\infty}^{\infty} e^{-nx^2}\,dx.$$

Using a substitution $u = \sqrt{n}x$, $dx = \frac{du}{\sqrt{n}}$, we get:

$$\|f_n\|_{L^1} = \int_{-\infty}^{\infty} e^{-u^2}\frac{du}{\sqrt{n}} = \frac{1}{\sqrt{n}}\int_{-\infty}^{\infty} e^{-u^2}\,du.$$

The Gaussian integral $\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$, so:

$$\|f_n\|_{L^1} = \frac{\sqrt{\pi}}{\sqrt{n}}.$$

As $n \to \infty$, $\|f_n\|_{L^1} \to 0$. Thus, $f_n(x) \to 0$ in $L^1(\mathbb{R})$.

2. Example of a sequence $(\phi_n)_{n\in\mathbb{N}} : \mathbb{R} \to \mathbb{R}^+$:

Let:

$$\phi_n(x) = \begin{cases} n, & \text{if } |x| \leq \frac{1}{n}, \\ 0, & \text{if } |x| > \frac{1}{n}. \end{cases}$$

Pointwise convergence: For any fixed $x \neq 0$, $\phi_n(x) \to 0$ as $n \to \infty$. Thus, $\phi_n(x) \to 0$ almost everywhere.

L^1-norm:

$$\|\phi_n\|_{L^1} = \int_{-\infty}^{\infty} \phi_n(x)\,dx = \int_{-1/n}^{1/n} n\,dx = n \cdot \frac{2}{n} = 2.$$

Since $\|\phi_n\|_{L^1} = 2$ for all n, the sequence does not converge to 0 in $L^1(\mathbb{R})$.

Exercise 3:

1. Given a T-periodic function $f \in C^k$, calculate $c_n(f^{(k)})$ in terms of $c_n(f)$.
2. Suppose that $c_n(f) = O(|n|^{-p})$, $p \geq 0$, and consider $k \in \mathbb{N}$. For which values of p can we say that $f \in C^k$? Prove this assertion.
3. If $p = 1 + k$, can we say that $f \in C^k$? Justify your response.

Solution

1. **Given a T-periodic function $f \in C^k$, calculate $c_n(f^{(k)})$ in terms of $c_n(f)$**

   The Fourier coefficients $c_n(f)$ of a T-periodic function $f(x)$ are given by:

$$c_n(f) = \frac{1}{T}\int_{-T/2}^{T/2} f(x)e^{-in\omega x}dx, \quad \text{where } \omega = \frac{2\pi}{T}.$$

   For the k-th derivative $f^{(k)}(x)$, the Fourier coefficients are:

$$c_n(f^{(k)}) = \frac{1}{T}\int_{-T/2}^{T/2} f^{(k)}(x)e^{-in\omega x}dx.$$

   Using integration by parts, we rewrite the integral for $c_n(f^{(k)})$:

$$\int_{-T/2}^{T/2} f^{(k)}(x)e^{-in\omega x}dx = \left[f^{(k-1)}(x)e^{-in\omega x}\right]_{-T/2}^{T/2} - \int_{-T/2}^{T/2} f^{(k-1)}(x)(-in\omega)e^{-in\omega x}dx.$$

   Since $f(x)$ is T-periodic, the boundary terms vanish, and we get:

$$c_n(f^{(k)}) = (in\omega)^k c_n(f).$$

   Thus, the relationship between the Fourier coefficients of $f^{(k)}$ and f is:

$$c_n(f^{(k)}) = (in\omega)^k c_n(f).$$

2. **Determining the values of p for which $f \in C^k$**

   Decay of Fourier coefficients and regularity:

   - If $c_n(f) = O(|n|^{-p})$, the decay rate of $c_n(f)$ determines the smoothness of f.
   - If $f \in C^k$, then $c_n(f^{(k)})$ must be sufficiently small for the $k-th$ derivative $f^{(k)}$ to exist and be continuous.

   Using $c_n(f^{(k)}) = (in\omega)^k c_n(f)$, we find:

$$|c_n(f^{(k)})| = |n\omega|^k |c_n(f)| = O(|n|^{k-p}).$$

Condition for $f^{(k)} \in C$:
For $f^{(k)}$ to exist and be continuous, we require that the series of Fourier coefficients $c_n(f^{(k)})$ converges absolutely, i.e.,

$$\sum_{n=-\infty}^{\infty} |c_n(f^{(k)})| < \infty.$$

This implies:

$$|c_n(f^{(k)})| = O(|n|^{(k-p)}).$$

For the series $\sum |n|^{k-p}$ to converge, the exponent must satisfy:

$$k - p < -1 \quad \Rightarrow \quad p > k + 1.$$

Conclusion:
The function $f \in C^k$ if and only if:

$$p > k + 1.$$

3. **If $p = 1 + k$, can we say that $f \in C^k$?**
   If $p = 1 + k$, then $c_n(f) = O(|n|^{-(1+k)})$. Substituting into $|c_n(f^{(k)})|$, we get:
   $$|c_n(f^{(k)})| = O(|n|^{k-(1+k)}) = O(|n|^{-1}).$$
   The series $\sum_{n=-\infty}^{\infty} |c_n(f^{(k)})|$ becomes:
   $$\sum_{n=-\infty}^{\infty} |n|^{-1}.$$
   The series $\sum |n|^{-1}$ diverges.

   Conclusion
   If $p = 1 + k$, the Fourier coefficients $c_n(f^{(k)})$ does not converge absolutely, and hence $f \notin C^k$. Therefore:
   $$f \notin C^k \quad \text{if } p = 1 + k.$$

Exercise 4:

1. State the inversion theorem as presented in class.

2. Provide an example of a convolution kernel.
3. Is the normed space $((0,1], |\cdot|)$ complete? Justify your answer by proving completeness in the affirmative case or providing a counterexample in the negative case.

Solution

1. **State the inversion theorem as presented in class.**

   The inversion theorem states (ref. page 38):

   If $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then for almost all $x, \xi \in \mathbb{R}^2$:

   $$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(y) e^{-i\xi y} \, dy \right) e^{ix\xi} \, d\xi,$$

   and equivalently:

   $$\hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x) e^{-i\xi x} \, dx \right) e^{i\eta x} \, dx.$$

   This theorem allows the recovery of a function from its Fourier transform.

2. **Provide an example of a convolution kernel.**

   A convolution kernel is a function $\eta \in L^1(\mathbb{R}; \mathbb{R}^+)$ such that:

   $$\|\eta\|_{L^1} = 1.$$

   Examples of standard convolution kernels (ref. Esempio 7.29, page 40):

   - **Gaussian kernel**:
   $$G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$
   - **Poisson kernel**:
   $$P(x) = \frac{1}{\pi} \frac{1}{1+x^2}.$$
   - **Compactly supported kernel**:
   $$M(x) = \begin{cases} c_0^{-1} e^{-1/(1-x^2)} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

   where c_0 is a normalization constant ensuring $\|M\|_{L^1} = 1$.

3. **Is the normed space** $((0,1], |\cdot|)$ **complete?**

Definition of completeness (ref. pages 28-29):

A metric space (X, d) is complete if every Cauchy sequence converges to a point in X.

Analyzing $((0, 1], |\cdot|)$**:**

- The space $((0, 1], |\cdot|)$ is normed using the absolute value $|x|$, which corresponds to the usual metric $d(x, y) = |x - y|$.
- A sequence $\{x_n\} \subset (0, 1]$ is Cauchy if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ for all $n, m \geq N$.

Completeness check:

- Suppose $x_n \to x$. For x_n to converge in $(0, 1]$, x must also belong to $(0, 1]$.
- If $x = 0$, then $x \notin (0, 1]$, which implies $(0, 1]$ is not closed under limits.

Conclusion:

The space $((0, 1], |\cdot|)$ is **not complete** because a Cauchy sequence in $(0, 1]$ can converge to 0, which is not included in the space.

Examples related to the completeness of $((0, 1], |\cdot|)$

Example 1: A Cauchy sequence in $(0, 1]$ **that converges to 0**
Consider the sequence $x_n = \frac{1}{n}$, where $n \in \mathbb{N}$.

1. **Sequence Analysis:**
   - Each $x_n \in (0, 1]$.
   - The sequence is Cauchy because for any $\epsilon > 0$, there exists N such that $|x_n - x_m| < \epsilon$ for all $n, m \geq N$.
2. **Limit:**
   - The sequence converges to $x = 0$ as $n \to \infty$.
   - However, $0 \notin (0, 1]$.

3. **Conclusion:** This shows that $((0,1], |\cdot|)$ is not complete because the limit of a Cauchy sequence does not belong to the space.

Example 2: A Cauchy sequence in $(0,1]$ that converges within the space

Consider the sequence $x_n = 1 - \frac{1}{n}$, where $n \in \mathbb{N}$.

1. **Sequence Analysis:**
   - Each $x_n \in (0,1]$.
   - The sequence is Cauchy because for any $\epsilon > 0$, there exists N such that $|x_n - x_m| < \epsilon$ for all $n, m \geq N$.
2. **Limit:**
   - The sequence converges to $x = 1$ as $n \to \infty$.
   - The limit $1 \in (0,1]$.
3. **Conclusion:** This shows that not all Cauchy sequences fail to converge within $(0,1]$. However, the space is not complete because some Cauchy sequences, like in example 1, have limits outside the space.

Information

While $((0,1], |\cdot|)$ contains some Cauchy sequences that converge within the space, the existence of Cauchy sequences that converge to points outside the space (e.g., 0) proves that the space is not complete.

Verification of Convolution kernels provided in exercise 2

Definition of a Convolution Kernel

A convolution kernel $\eta(x)$ must satisfy the following:

1. $\eta(x) \in L^1(\mathbb{R})$, meaning $\int_{\mathbb{R}} |\eta(x)| dx < \infty$.
2. $\|\eta\|_{L^1} = 1$, meaning $\int_{\mathbb{R}} \eta(x) dx = 1$.

1. Gaussian Kernel

The Gaussian kernel is defined as:

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Verification:

1. $G(x) \in L^1(\mathbb{R})$:
$$\int_{\mathbb{R}} G(x)dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$
The integral evaluates to 1 because the Gaussian function is normalized:
$$\int_{\mathbb{R}} e^{-x^2/2} dx = \sqrt{2\pi}.$$

2. $\|G\|_{L^1} = 1$: Since the Gaussian kernel integrates to 1, it satisfies the normalization condition.

Thus, $G(x)$ is a valid convolution kernel.

2. Poisson Kernel

The Poisson kernel is defined as:
$$P(x) = \frac{1}{\pi} \frac{1}{1+x^2}.$$

Verification:

1. $P(x) \in L^1(\mathbb{R})$:
$$\int_{\mathbb{R}} P(x)dx = \int_{\mathbb{R}} \frac{1}{\pi} \frac{1}{1+x^2} dx.$$
The integral is finite because $\frac{1}{1+x^2}$ is the density function of the Cauchy distribution, and its integral evaluates to:
$$\int_{\mathbb{R}} \frac{1}{1+x^2} dx = \pi.$$

2. $\|P\|_{L^1} = 1$: After integrating, we normalize:
$$\frac{1}{\pi} \cdot \pi = 1.$$

Thus, $P(x)$ is a valid convolution kernel.

3. Compactly Supported Kernel

The compactly supported kernel is defined as:
$$M(x) = \begin{cases} c_0^{-1} e^{-1/(1-x^2)} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$
where c_0 is a normalization constant ensuring $\int_{\mathbb{R}} M(x)dx = 1$.

Verification:

1. Compact support: By definition, $M(x)$ is zero outside $|x| < 1$, so the integral over $\mathbb{R}$ reduces to $\int_{-1}^{1} M(x)dx$, which is finite.

2. Normalization constant c_0: Compute c_0 as:

$$c_0 = \int_{-1}^{1} e^{-1/(1-x^2)}dx.$$

   Since c_0 is finite, we normalize $M(x)$ by dividing by c_0.

3. $M(x) \in L^1(\mathbb{R})$: By construction, $M(x)$ integrates to 1:

$$\int_{\mathbb{R}} M(x)dx = \frac{1}{c_0} \int_{-1}^{1} e^{-1/(1-x^2)}dx = 1.$$

Thus, $M(x)$ is a valid convolution kernel.

Conclusion

All three examples (Gaussian kernel, Poisson kernel, and compactly supported kernel) satisfy the properties of convolution kernels because they are integrable, non-negative, and normalized to 1.