The experiments of Birch and Swinnerton-Dyer

Comp-nt Day 1

Monday, 25th October 2021

Introduction

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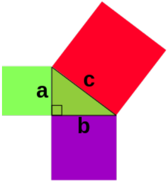

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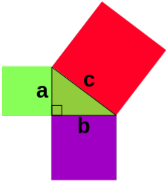

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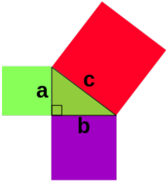

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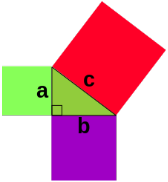

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Do there exist integral right triangles? i.e. $a, b, c \in \mathbb{Z}$ *such that* $a^2 + b^2 = c^2$*?*

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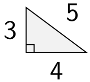

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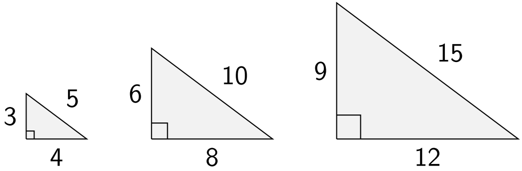

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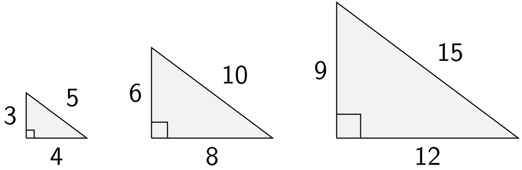

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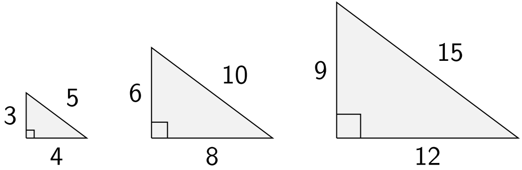

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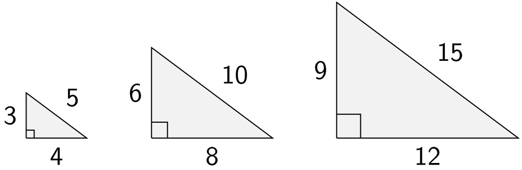

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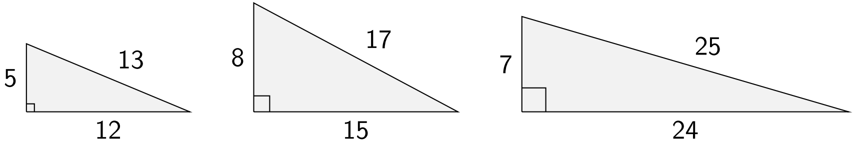
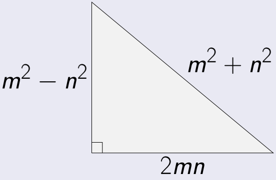


Theorem (Euclid, ca. 200 BC)

Every integral right triangle is of the form



for integers $m > n > 0$.

Euclid, from *The School of Athens* by Raphael, 1511

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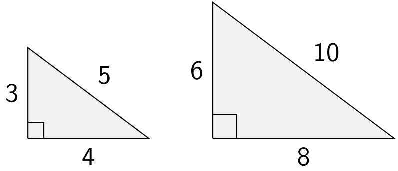

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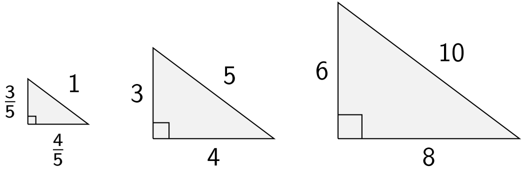

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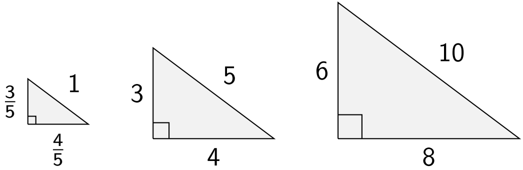

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A right triangle is called rational if its sides have lengths in the field of rationals $\mathbb{Q}$.

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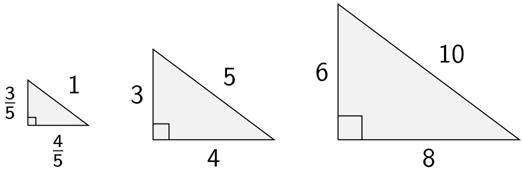

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Remark

By clearing denominators, every rational right triangle is essentially the same as an integral right triangle. But working with rational triangles instead of integral triangles is often easier because $\mathbb{Q}$ *is a field, whereas* $\mathbb{Z}$ *is not.*

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Which numbers are congruent?

Example

(3,4,5) $\triangle$ $\Rightarrow$ 6 is congruent.

(5,12,13) $\triangle$ $\Rightarrow$ 30 is congruent.

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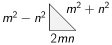
(clearing denominators) $\Leftrightarrow Dw^2 = \text{Area}\Big($ 

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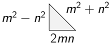
(**clearing denominators**) $\Leftrightarrow Dw^2 = \text{Area}\Big($ 

$\Big)$ for $w, m, n \in \mathbb{Z}^+, \;\; m > n > 0$

$$\Leftrightarrow Dw^2 = mn(m^2 - n^2) \;\; \text{for } w, m, n \in \mathbb{Z}^+, \;\; m > n > 0$$

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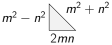
(**clearing denominators**) $\Leftrightarrow Dw^2 = \text{Area}\Big($ 

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(**setting** $x := \frac{Dm}{n}, y = \frac{D^2 w}{n}$) $\Leftrightarrow y^2 = x^3 - D^2 x$ for $x, y \in \mathbb{Q},\ x \neq 0 \neq y$

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$$(\textbf{clearing denominators}) \Leftrightarrow Dw^2 = \text{Area}\left({}^{m^2-n^2}\triangle^{m^2+n^2}_{2mn} \right) \text{ for } w, m, n \in \mathbb{Z}^+,\ m > n > 0$$

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$$\left(\textbf{setting } x := \tfrac{Dm}{n}, y = \tfrac{D^2 w}{n}\right) \Leftrightarrow y^2 = x^3 - D^2 x \text{ for } x, y \in \mathbb{Q},\ x \neq 0 \neq y$$

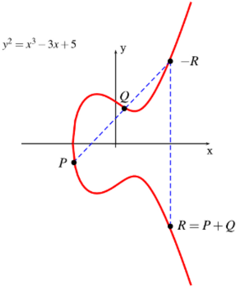
This equation $E_D : y^2 = x^3 - D^2 x$ is an example of an **elliptic curve**, which in general has equation

$$y^2 = x^3 + Ax + B$$

for $A, B \in K$, where K is (almost) *any* field.

The chord and tangent process

Elliptic curves are remarkable because their set of solutions form a group under the chord and tangent process:



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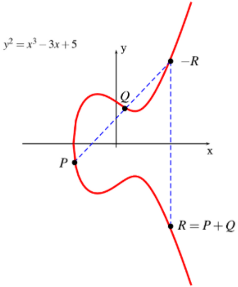

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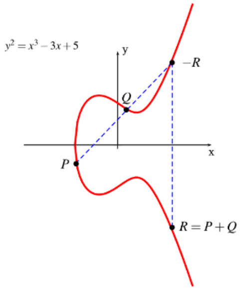

The identity of the group law is the point at infinity O_E, infinitely far up the y-axis. Most[1] projective curves do not admit a group structure with a geometrical interpretation, making elliptic curves a rather special class among all curves.

[1] If the genus of the curve is ≥ 2

The Mordell-Weil Theorem

Theorem (Mordell-Weil)

Let K be a global field (e.g. a number field), and let E/K be an elliptic curve. Then the group of K-rational points $E(K)$ is a finitely generated abelian group:

$$E(K) \cong E(K)_{tors} \oplus \mathbb{Z}^r.$$

The integer r is called the rank of E over K.

Back to congruent numbers ...

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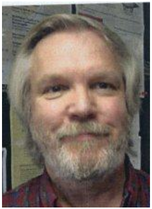
The rank of an elliptic curve is an elusive quantity, so this description doesn't immediately solve the congruent number problem.

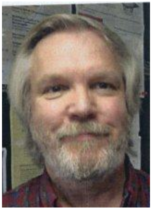


Jerrold B. Tunnell

Theorem (Tunnell, 1983)

For $D \in \mathbb{Z}$, define

$$\begin{aligned} P_D &= \#\left\{(x,y,z) \in \mathbb{Z}^3 | D = 2x^2 + y^2 + 32z^2\right\} \\ Q_D &= \#\left\{(x,y,z) \in \mathbb{Z}^3 | D = 2x^2 + y^2 + 8z^2\right\} \\ R_D &= \#\left\{(x,y,z) \in \mathbb{Z}^3 | D = 8x^2 + 2y^2 + 64z^2\right\} \\ S_D &= \#\left\{(x,y,z) \in \mathbb{Z}^3 | D = 8x^2 + 2y^2 + 16z^2\right\}. \end{aligned}$$



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Then, assuming the Birch-Swinnerton-Dyer conjecture for elliptic curves,

$$E_D \text{ has positive rank } \Leftrightarrow \begin{cases} 2P_D = Q_D & \text{if } D \text{ is odd} \\ 2R_D = S_D & \text{if } D \text{ is even.} \end{cases}$$

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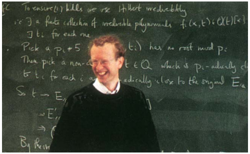
Tunnell's theorem uses the theory of modular forms of half-integral weight. There are many other deep connections between elliptic curves and modular forms.

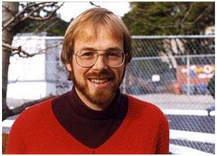
Modularity Theorem (Wiles, Taylor-Wiles, 1995)

Every elliptic curve $E/\mathbb{Q}$ arises from a (weight-2 cuspidal of level $\Gamma_0(N)$) modular form f_E such that the L-functions coincide:

$$L(E, s) = L(f_E, s).$$



Andrew J. Wiles



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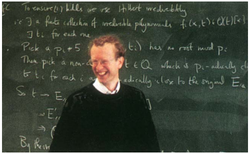
Corollary

Fermat's Last Theorem is true: for $n > 2$, the equation

$$x^n + y^n = z^n$$

admits only the trivial solutions (i.e. when $xyz = 0$).



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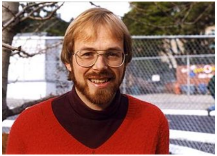

Richard L. Taylor

What did Birch and Swinnerton-Dyer actually do?

Reduction mod p

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For now, let's not worry about what happens if we take $p = 17$ ...

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Until the mid-50s, actually computing these numbers required "lots of pencils and paper" (essentially a quote from Swinnerton-Dyer).

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He was super-excited to be able to do this for all primes $p \leq 1000$ on some of the E_D elliptic curves from earlier.

It turns out that $N_p = p + 1 - a_p$ for some "error" term a_p, so to get a sense of "how often N_p differs from p" as one varies over all primes p, Birch and Swinnerton-Dyer considered the function

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Question

How does $f(x)$ grow as $x \to \infty$?

Based on this, they conjectured

$$\prod_{p \leq x} \frac{N_p}{p} \to C(\log(x))^r \quad \text{as} \quad x \to \infty$$

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Conjecture (The Birch and Swinnerton-Dyer conjecture (weak))

Let E/K be an elliptic curve over a number field, and $L(E/K, s)$ its L-function. Then

$$ord_{s=1}(L(E/K, s) = rank(E(K)).$$

Based on this, they conjectured

$$\prod_{p \leq x} \frac{N_p}{p} \to C(\log(x))^r \quad \text{as} \quad x \to \infty$$

for C a constant.

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(The strong version relates the leading coefficient of the Taylor expansion of $L(E/K, s)$ to arithmetic data of E/K.)