

Torsion groups of elliptic curves over quadratic fields $\mathbb{Q}(\sqrt{d})$ for $|d| < 800$

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Modular curves and Galois representations

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Introduction

Mazur's Torsion Theorem



Barry C. Mazur

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This was conjectured by Beppo Levi in 1908 (in his Rome ICM address), then again by Andrew Ogg in 1970.

Kamienny-Kenku-Momose Torsion Theorem



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Monsur A. Kenku



Fumiyuki Momose

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For K a quadratic field, $E(K)_{\text{tors}}$ is one of the following 26 groups:

$$\begin{array}{ll} \mathbb{Z}/N\mathbb{Z} & 1 \leq N \leq 16 \text{ or } N = 18 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z} & 1 \leq N \leq 6 \\ \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3N\mathbb{Z} & 1 \leq N \leq 2 \\ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} & \end{array}$$



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Moreover, as K varies, each group occurs infinitely often.



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i.e. which of the 26 groups from the KKM classification arise for a particular K ?

Quadratic Cyclotomic fields



Filip Najman

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Quadratic Cyclotomic fields



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- ② Let E be an elliptic curve over $\mathbb{Q}(\sqrt{-3})$. Then $E(\mathbb{Q}(\sqrt{-3}))_{\text{tors}}$ is isomorphic to one of the groups from Mazur's theorem, or $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$.

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For K a quadratic field that is not $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, which of the 8 groups

$$\mathbb{Z}/11\mathbb{Z}$$

$$\mathbb{Z}/14\mathbb{Z}$$

$$\mathbb{Z}/15\mathbb{Z}$$

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$$

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$$

$$\mathbb{Z}/13\mathbb{Z}$$

$$\mathbb{Z}/16\mathbb{Z}$$

$$\mathbb{Z}/18\mathbb{Z}$$

arise as a possible torsion group over K ?

Question (Motivating question of the talk, v3)

For K a quadratic field that is not $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, which of the 8 modular curves

<u>genus 1</u>	<u>genus 2</u>
$X_1(11)$	
$X_1(14)$	$X_1(13)$
$X_1(15)$	$X_1(16)$
$X_1(2, 10)$	$X_1(18)$
$X_1(2, 12)$	

admit a noncuspidal K -rational point?

Elliptic cases



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Elliptic cases

Theorem (Kamienny-Najman, 2012)

Let $K \neq \mathbb{Q}(\sqrt{-7}), \mathbb{Q}(\sqrt{-15})$ be a quadratic field. If any of the 5 genus 1 modular curves X from the motivating question admit a noncuspidal K -rational point, then $\text{rk}(X(K))$ is positive.



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SLOGAN

To deal with the 5 elliptic modular curves, you ‘just’ need to compute their rank over K



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Theorem

For E/\mathbb{Q} ,

$$\mathrm{rk}(E(\mathbb{Q}(\sqrt{d}))) = \mathrm{rk}(E(\mathbb{Q})) + \mathrm{rk}(E_d(\mathbb{Q})).$$

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SLOGAN

To deal with the 5 elliptic modular curves, you ‘just’ need to compute the \mathbb{Q} -rank of their twists!

Genus 2 cases

$$X_1(13) : y^2 = f_{13}(x) := x^6 - 2x^5 + x^4 - 2x^3 + 6x^2 - 4x + 1$$

$$X_1(16) : y^2 = f_{16}(x) := x(x^2 + 1)(x^2 + 2x - 1)$$

$$X_1(18) : y^2 = f_{18}(x) := x^6 + 2x^5 + 5x^4 + 10x^3 + 10x^2 + 4x + 1$$

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David Krumm

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Theorem (Krumm, 2013)

If X admits a noncuspidal $\mathbb{Q}(\sqrt{d})$ -point, then the x -coordinate of that point is in \mathbb{Q} ; i.e. it yields a \mathbb{Q} -point on the d -twist X^d .



David Krumm

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Theorem (Krumm, 2013)

- 1 $Y_1(13)(\mathbb{Q}(\sqrt{d})) \neq \emptyset \iff X_1^d(13)(\mathbb{Q}) \neq \emptyset$
- 2 $Y_1(16)(\mathbb{Q}(\sqrt{d})) \neq \emptyset \iff X_1^d(16)(\mathbb{Q})$ contains a point with nonzero y coordinate
- 3 $Y_1(18)(\mathbb{Q}(\sqrt{d})) \neq \emptyset \iff X_1^d(18)(\mathbb{Q}) \neq \emptyset$

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SLOGAN

This reduces the problem to determining the existence of \mathbb{Q} -points on specific genus 2 curves over \mathbb{Q} (or for $X_1(16)$, determining all \mathbb{Q} -points).

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Fix $B > 0$. For $|d| < B$, can you determine the torsion groups that occur over $\mathbb{Q}(\sqrt{d})$?

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Definition

For $B > 0$ and $N \in \{13, 16, 18\}$, define

$$T_B(N) := \left\{ |d| < B \text{ squarefree} : \mathbb{Z}/N\mathbb{Z} \text{ is a torsion group over } \mathbb{Q}(\sqrt{d}) \right\}.$$

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Theorem (Krumm, 2013)

$$\begin{aligned} \{17, 113, 193, 313, 481\} \subseteq T_{1000}(13) &\subseteq \{17, 113, 193, 313, 481\} \cup \{257, 353, 601, 673\} \\ \{33, 337, 457\} \subseteq T_{1000}(18) &\subseteq \{33, 337, 457\} \cup \{681\}. \end{aligned}$$

Theorem (Trbović, 2018)

$$\{10, 15, 41, 51, 70, 93\} \subseteq T_{100}(16) \cap \mathbb{Z}_{\geq 1} \subseteq \{10, 15, 41, 51, 70, 93\} \\ \cup \{26, 31, 47, 58, 62, 74, 78, 79, 82, 87, 94\}$$



Antonela Trbović

Statement of results

Theorem (B.-Derickx, 2023)

$$T_{10,000}(13) = \{17, 113, 193, 313, 481, 1153, 1417, \\ 2257, 3769, 3961, 5449, 6217, 6641, 9881\}$$

$$T_{10,000}(18) = \{33, 337, 457, 1009, 1993, 2833, 7369, 8241, 9049\}$$

Theorem (B.-Derickx, 2023)

$$T_{800}(16) = \{-671, -455, -290, -119, -15, 10, 15, 41, 51, \\ 70, 93, 105, 205, 217, 391, 546, 609, 679\}.$$

Corollary (B.-Derickx, 2023)

We solve the Quadratic Torsion Challenge for $B = 800$.

$X_1(13)$ and $X_1(18)$

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We're only going to show $X_1(13)$ because the two cases are basically identical.

ELS

Lemma

If $X_1^d(13)(\mathbb{Q}) \neq \emptyset$, then it is everywhere locally soluble.

Krumm's filter

Theorem (Krumm, 2013)

If $X_1^d(13)(\mathbb{Q}) \neq \emptyset$, and $d \neq -3$, then

- 1 $d > 0$;
- 2 $d \equiv 1 \pmod{8}$.

Introduction
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Results
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$X_1(13)$ and $X_1(18)$
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$X_1(16)$
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Todo
○○

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Proof.

For $\widetilde{p} \geq 5$, $p \neq 13$, the torsion subgroup $J_1(13)(K)_{tors}$ injects into $J_1(13)(\mathbb{F}_{p^2})$.

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For $\underbrace{p \geq 5, p \neq 13}$, the torsion subgroup $J_1(13)(K)_{tors}$ injects into $J_1(13)(\mathbb{F}_{p^2})$. By computing this latter group for $p = 5$ and 7 , one sees that it must be a subgroup of $\mathbb{Z}/19\mathbb{Z}$.

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For $\underline{p \geq 5, p \neq 13}$, the torsion subgroup $J_1(13)(K)_{tors}$ injects into $J_1(13)(\mathbb{F}_{p^2})$. By computing this latter group for $p = 5$ and 7 , one sees that it must be a subgroup of $\mathbb{Z}/19\mathbb{Z}$. OTOH, the torsion over \mathbb{Q} is $\mathbb{Z}/19\mathbb{Z}$. □

Proposition

Let $K = \mathbb{Q}(\sqrt{d})$. If $X_1(13)(K) \neq X_1(13)(\mathbb{Q})$, then $J_1(13)(K)$ and hence $J_1^d(13)(\mathbb{Q})$ has positive rank.

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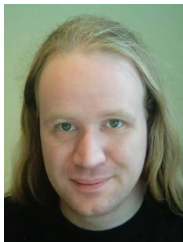
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Corollary

If $X_1^d(13)(\mathbb{Q}) \neq \emptyset$, then $J_1^d(13)$ has positive \mathbb{Q} -rank.

How to efficiently determine positive rank?

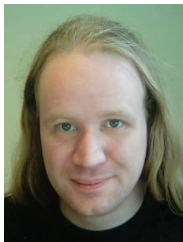
Determining whether the Jacobian of a modular curve has positive analytic rank or not can be done efficiently via a modular symbols computation involving the **twisted winding element**, a method that goes back to Johan Bosman's PhD thesis.



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Johan Bosman

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CHAPTER 2. COMPUTATIONS WITH MODULAR FORMS

The element $\sum_{v=0}^{l-1} \chi(-v) \left\{ \infty, \frac{v}{l} \right\}$ of $M_k(\Gamma_1(N)) \otimes \mathbb{Z}[\chi]$ or of some other modular symbols space where it is well-defined is called a *twisted winding element* or, more precisely the **χ -twisted winding element**. Because of formula (2.7), we can calculate the pairings of newforms in $S_2(\Gamma_1(N))$ with twisted winding elements quite efficiently as well.


```
def is_rank_of_twist_zero(G, d):  
    M = ModularSymbols(G)  
    S = M.cuspidal_subspace()  
    phi = S.rational_period_mapping()  
    chi = kronecker_character(d)  
    w = phi(M.twisted_winding_element(0, chi))  
    return w != 0
```

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The pullback $\pi^*(C)$ yields an unramified cover that has a rational point mapping to P .

Definition

An **n -cover** is any unramified cover geometrically isomorphic to one of the above form. Equivalently, it is an unramified cover D/C over K such that

$$\mathrm{Aut}_{\overline{K}}(D/C) \cong J[n](\overline{K})$$

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$$\mathrm{Sel}^{(n)}(C/K) = \emptyset \Rightarrow C(K) = \emptyset$$

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Nils Bruin



Michael Stoll

This gives us a way to compute the fake Selmer-set explicitly.

define FakeSelmerSet(f):

1. $A := k[x]/(f(x))$
2. Let S be the set of primes of k described above.
3. **if** $2 \mid \deg(f)$:
4. $G := A(2, S)/k(2, S)$
5. **else** :
6. $G := A(2, S)$
7. $W := \{g \in G : N_{A/k}(g) \in f_n k^{*2}\}$. **if** $W = \emptyset$: **return** \emptyset
8. $T := S \cup$ “small” primes, as in Lemma [4.3](#)
9. **for** $p \in T$:
10. $A_p := A \otimes k_p$; $H'_p := A_p^*/A_p^{*2}$.
11. $W'_p := \text{LocalImage}(f_p) \subset H'_p$ or, if $p \mid \infty$, use Section [5](#) to compute W'_p .
12. **if** $2 \mid \deg(f)$:
13. $H_p := H'_p/k_p^*$; $W_p :=$ image of W'_p in H_p
14. **else** :
15. $H_p := H'_p$; $W_p := W'_p$
16. Determine $\rho_p : G \rightarrow H_p$.
17. $W := \{w \in W : \rho(w) \in W_p\}$.
18. **return** W

```
> R<x> := PolynomialRing(Rationals());  
> //y^2=f is isomorphic to  $X_1(13)$   
> f := R![1, 2, 1, 2, 6, 4, 1];  
> d := 7;  
> C := HyperellipticCurve(d*f);  
> TwoCoverDescent(C);  
{}
```


Corollary

If $X_1^d(13)(\mathbb{Q}) \neq \emptyset$, then the fake 2-Selmer set is nonempty.

```
R<x> := PolynomialRing(Rationals());
//y^2=f is isomorphic to X_1(13)
f := R![1, 2, 1, 2, 6, 4, 1];

B:= 10000
output := [];

for d in [-B..B] do
  if IsSquarefree(d) then
    if d > 0 and d mod 8 eq 1 then // Krumm filter
      if HasPointsEverywhereLocally([d*f,2]) then // ELS filter
        if IsRankOfTwistPositive(Gamma1(13),d) then // Rank filter
          C := HyperellipticCurve(d*f);
          if #TwoCoverDescent(C) > 0 then // Two cover descent filter
            Append(~output, d);
          end if;
        end if;
      end if;
    end if;
  end if;
end for;

output;
```

17, 113, 193, 313, 481, 673, 1153, 1417, 1609, 1921, 2089, 2161,
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These are dealt with via the Mordell-Weil sieve.

Mordell-Weil sieve

$$\begin{array}{ccc} X(\mathbb{Q}) & \xhookrightarrow{\iota} & J(\mathbb{Q}) \\ \downarrow \alpha & & \downarrow \alpha \\ \prod_p X(\mathbb{Q}_p) & \xrightarrow{\tilde{\iota}} & \prod_p J(\mathbb{Q}_p) \end{array}$$

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If the images of α and $\tilde{\iota}$ do not intersect, then $X(\mathbb{Q})$ is empty.

These are infinite groups and sets, so the intersection can't be computed. Instead one works with a finite approximation.

$$\begin{array}{ccc} X(\mathbb{Q}) & \xhookrightarrow{\iota} & J(\mathbb{Q})/NJ(\mathbb{Q}) \\ \downarrow \alpha & & \downarrow \alpha \\ \prod_{p \in S} X(\mathbb{Q}_p) & \xrightarrow{\tilde{\iota}} & \prod_{p \in S} J(\mathbb{Q}_p)/NJ(\mathbb{Q}_p) \end{array}$$

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Here N is a positive integer, and S a finite set of primes. Now we can compute the intersection. Heuristically, if $X(\mathbb{Q}) = \emptyset$, then the intersection will be empty if S and N are large enough.

Theorem (B.-Derickx, 2023)

$$T_{10,000}(13) = \{17, 113, 193, 313, 481, 1153, 1417, \\ 2257, 3769, 3961, 5449, 6217, 6641, 9881\}$$

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$X_1(16)$

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Proposition (B.-Derickx, 2023)

Let $K = \mathbb{Q}(\sqrt{d})$. If $\mathbb{Z}/16\mathbb{Z}$ arises as a possible torsion group over K , then $\text{rk}(J_1^d(16)) > 0$.

Using the twisted winding element method from before, we compute the squarefree values of d with $|d| < 10,000$ for which $\text{rk}(J_1^d(16)) > 0$; this yields 674 values.

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We do a point search on these; 55 of them have extra points.

How to deal with the remaining 619 values?

Elliptic Curve Chabauty

Theorem (Bruin-Stoll, souped-up version of Chevalley-Weil)

Every rational point on a hyperelliptic curve X lifts to a rational point on some $D \in \text{TwoCoverDescent}(X)$.

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For simplicity assume $X : y^2 = f(x)$ with $\deg(f) = 5$.

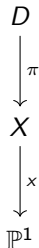
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Can construct elliptic curve quotients by taking degree 3 factors g of f over a number field L :

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$$E_D : \gamma_D y^2 = g(x)$$

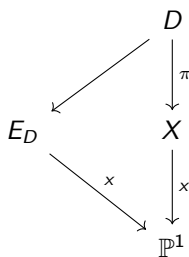
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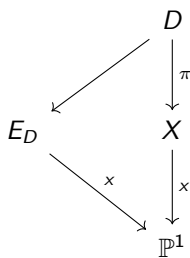
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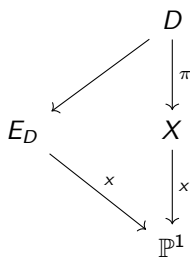
FACT: If $\text{rk}(E_D(L)) < [L : \mathbb{Q}]$, then $x(E_D(L)) \cap \mathbb{P}^1(\mathbb{Q})$ is finite and computable by an algorithm of Nils Bruin.

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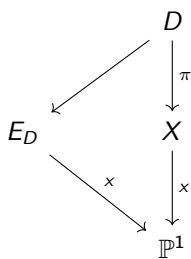
SUMMARY: If, for every D , there is a degree 3 factor $g \in L[x]$ s.t. $E_D : \gamma_D y^2 = g(x)$ has $\text{rk}(E_D(L)) < [L : \mathbb{Q}]$, then we're done.

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For us, $f(x) = dx(x^2 + 1)(x^2 - 2x - 1)$, so L will always be quite small.

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The remaining 38 values to be dealt with are:

– 8259, –7973, –7615, –7161, –7006, –6711, –6503, –6095,
– 6031, –6005, –4911, –4847, –4773, –4674, –4371, –4191,
– 4074, –3503, –3199, –1810, –1749, –815, 969, 1186,
3215, 3374, 3946, 4633, 5257, 5385, 7006, 7210,
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Todo

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- What about cubic torsion? i.e. for a fixed cubic field K , which of the 26 groups in the cubic torsion classification (due to Derickx-Etropolski, van Hoeij, Morrow, Zureick-Brown) arise as torsion subgroups for that K ?