Torsion groups of elliptic curves over quadratic fields $\mathbb{Q}(\sqrt{d})$ for $|d| < 800$

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Boston University

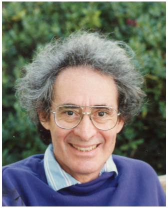
Modular curves and Galois representations
Zagreb, Croatia
Thursday 21st September 2023
`https://tinyurl.com/quadratic-torsion`

Sveučilište u Zagrebu
Prirodoslovno-matematički fakultet
MATEMATIČKI ODSJEK

BOSTON UNIVERSITY

Introduction

Mazur's Torsion Theorem



Barry C. Mazur

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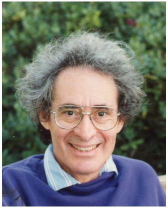

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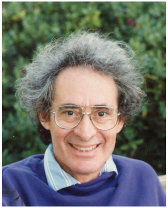

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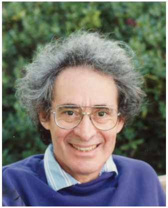
$$\begin{aligned} \mathbb{Z}/N\mathbb{Z}, &\qquad 1 \leq N \leq 10 \text{ or } N = 12 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}, &\qquad 1 \leq N \leq 4. \end{aligned}$$



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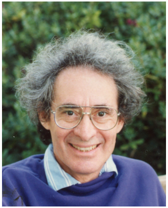
Moreover, each group occurs infinitely often.



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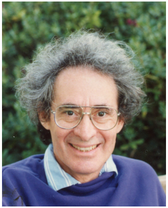

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This was conjectured by Beppo Levi in 1908 (in his Rome ICM address), then again by Andrew Ogg in 1970.

Kamienny-Kenku-Momose Torsion Theorem

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Monsur A. Kenku

Fumiyuki Momose

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Moreover, as K varies, each group occurs infinitely often.

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i.e. which of the 26 groups from the KKM classification arise for a particular K?

Quadratic Cyclotomic fields

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For K a quadratic field that is not $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, which of the 8 groups

$$\begin{array}{cc} \mathbb{Z}/11\mathbb{Z} & \\ \mathbb{Z}/14\mathbb{Z} & \mathbb{Z}/13\mathbb{Z} \\ \mathbb{Z}/15\mathbb{Z} & \mathbb{Z}/16\mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z} & \mathbb{Z}/18\mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} & \end{array}$$

arise as a possible torsion group over K?

Question (Motivating question of the talk, v3)

For K a quadratic field that is not $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, which of the 8 modular curves

genus 1	*genus* 2
$X_1(11)$	
$X_1(14)$	$X_1(13)$
$X_1(15)$	$X_1(16)$
$X_1(2,10)$	$X_1(18)$
$X_1(2,12)$	

admit a noncuspidal K-rational point?

Elliptic cases

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Theorem (Kamienny-Najman, 2012)

Let $K \neq \mathbb{Q}(\sqrt{-7}), \mathbb{Q}(\sqrt{-15}), \mathbb{Q}(\sqrt{5})$ be a quadratic field. If any of the 5 genus 1 modular curves X from the motivating question admit a noncuspidal K-rational point, then $\mathrm{rk}(X(K))$ is positive.

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SLOGAN

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Theorem

For $E/\mathbb{Q}$,

$$\mathrm{rk}(E(\mathbb{Q}(\sqrt{d}))) = \mathrm{rk}(E(\mathbb{Q})) + \mathrm{rk}(E_d(\mathbb{Q})).$$

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SLOGAN

To deal with the 5 elliptic modular curves, you ‘just’ need to compute the $\mathbb{Q}$-rank of their twists!

Genus 2 cases

$$X_1(13) : y^2 = f_{13}(x) := x^6 - 2x^5 + x^4 - 2x^3 + 6x^2 - 4x + 1$$
$$X_1(16) : y^2 = f_{16}(x) := x(x^2 + 1)(x^2 + 2x - 1)$$
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Theorem (Krumm, 2013)

1. $Y_1(13)(\mathbb{Q}(\sqrt{d})) \neq \emptyset \iff X_1^d(13)(\mathbb{Q}) \neq \emptyset$
2. $Y_1(16)(\mathbb{Q}(\sqrt{d})) \neq \emptyset \iff X_1^d(16)(\mathbb{Q})$ *contains a point with nonzero y coordinate*
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SLOGAN

This reduces the problem to determining the existence of $\mathbb{Q}$-points on specific genus 2 curves over $\mathbb{Q}$ (or for $X_1(16)$, determining all $\mathbb{Q}$-points).

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Fix $B > 0$. For $|d| < B$, can you determine the torsion groups that occur over $\mathbb{Q}(\sqrt{d})$?

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Definition

For $B > 0$ and $N \in \{13, 16, 18\}$, define

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Theorem (Krumm, 2013)

$$\{17, 113, 193, 313, 481\} \subseteq T_{1000}(13) \subseteq \{17, 113, 193, 313, 481\} \cup \{257, 353, 601, 673\}$$
$$\{33, 337, 457\} \subseteq T_{1000}(18) \subseteq \{33, 337, 457\} \cup \{681\}.$$

Theorem (Trbović, 2018)

$$\{10, 15, 41, 51, 70, 93\} \subseteq T_{100}(16) \cap \mathbb{Z}_{\geq 1} \subseteq \{10, 15, 41, 51, 70, 93\} \cup \{26, 31, 47, 58, 62, 74, 78, 79, 82, 87, 94\}$$

Antonela Trbović

Statement of results

Theorem (B.-Derickx, 2023)

$$T_{10,000}(13) = \{17, 113, 193, 313, 481, 1153, 1417, 2257, 3769, 3961, 5449, 6217, 6641, 9881\}$$

$$T_{10,000}(18) = \{33, 337, 457, 1009, 1993, 2833, 7369, 8241, 9049\}$$

Theorem (B.-Derickx, 2023)

$$T_{800}(16) = \{-671, -455, -290, -119, -15, 10, 15, 41, 51, 70, 93, 105, 205, 217, 391, 546, 609, 679\}.$$

Corollary (B.-Derickx, 2023)

We solve the Quadratic Torsion Challenge for $B = 800$.

$X_1(13)$ and $X_1(18)$

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We're only going to show $X_1(13)$ because the two cases are basically identical.

ELS

Lemma

If $X_1^d(13)(\mathbb{Q}) \neq \emptyset$, then it is everywhere locally soluble.

Krumm's filter

Theorem (Krumm, 2013)

If $X_1^d(13)(\mathbb{Q}) \neq \emptyset$, and $d \neq -3$, then

1. $d > 0$;
2. $d \equiv 1 \pmod 8$.

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For $p \geq 5, p \neq 13$, the torsion subgroup $J_1(13)(K)_{tors}$ injects into $J_1(13)(\mathbb{F}_{p^2})$.

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Proposition

Let $K = \mathbb{Q}(\sqrt{d})$. If $X_1(13)(K) \neq X_1(13)(\mathbb{Q})$, then $J_1(13)(K)$ and hence $J_1^d(13)(\mathbb{Q})$ has positive rank.

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Corollary

If $X_1^d(13)(\mathbb{Q}) \neq \emptyset$, then $J_1^d(13)$ has positive $\mathbb{Q}$-rank.

How to efficiently determine positive rank?

Determining whether the Jacobian of a modular curve has positive analytic rank or not can be done efficiently via a modular symbols computation involving the twisted winding element, a method that goes back to Johan Bosman's PhD thesis.

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52 *CHAPTER 2. COMPUTATIONS WITH MODULAR FORMS*

The element $\sum_{\nu=0}^{l-1}\chi(-\nu)\left\{\infty,\frac{\nu}{\ell}\right\}$ of $\mathbb{M}_k(\Gamma_1(N))\otimes\mathbb{Z}[\chi]$ or of some other modular symbols space where it is well-defined is called a *twisted winding element* or, more precisely the *χ-twisted winding element*. Because of formula (2.7), we can calculate the pairings of newforms in $S_2(\Gamma_1(N))$ with twisted winding elements quite efficiently as well.

```
def is_rank_of_twist_zero(G, d):

    M = ModularSymbols(G)
    S = M.cuspidal_subspace()
    phi = S.rational_period_mapping()
    chi = kronecker_character(d)
    w = phi(M.twisted_winding_element(0, chi))
    return w != 0
```

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The pullback $\pi^*(C)$ yields an unramified cover that has a rational point mapping to P.

Definition

An *n-cover* is any unramifed cover geometrically isomorphic to one of the above form. Equivalently, it is an unramified cover D/C over K such that

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$$\mathrm{Sel}^{(n)}(C/K) = \emptyset \Rightarrow C(K) = \emptyset$$

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Nils Bruin

Michael Stoll

This gives us a way to compute the fake Selmer-set explicitly.
define FakeSelmerSet(f):

$A := k[x]/(f(x))$
Let S be the set of primes of k described above.
if $2 \mid \deg(f)$:
    $G := A(2,S)/k(2,S)$
else :
    $G := A(2,S)$
$W := \{g \in G : N_{A/k}(g) \in f_n k^{*2}\}$. **if** $W = \emptyset$: **return** $\emptyset$
$T := S \cup$ "small" primes, as in Lemma 4.3
for $p \in T$:
    $A_p := A \otimes k_p$; $H'_p := A^*_p/A^{*2}_p$.
    $W'_p := \mathsf{LocalImage}(f_p) \subset H'_p$ or, if $p \mid \infty$, use Section 5 to compute W'_p.
    if $2 \mid \deg(f)$:
        $H_p := H'_p/k^*_p$; $W_p :=$ image of W'_p in H_p
    else :
        $H_p := H'_p$; $W_p := W'_p$
    Determine $\rho_p : G \to H_p$.
    $W := \{w \in W : \rho(w) \in W_p\}$.
return W

```
> R<x> := PolynomialRing(Rationals());
> //y^2=f is isomorphic to  X_1(13)
> f := R![1, 2, 1, 2, 6, 4, 1];
> d := 7;
> C := HyperellipticCurve(d*f);
> TwoCoverDescent(C);
{}
```

Corollary

If $X_1^d(13)(\mathbb{Q}) \neq \emptyset$, then the fake 2-Selmer set is nonempty.

```
R<x> := PolynomialRing(Rationals());
//y^2=f is isomorphic to  X_1(13)
f := R![1, 2, 1, 2, 6, 4, 1];

B:= 10000
output := [];

for d in [-B..B] do
    if IsSquarefree(d) then
        if d gt 0 and d mod 8 eq 1 then  // Krumm filter
            if HasPointsEverywhereLocally(d*f,2) then  // ELS filter
                if IsRankOfTwistPositive(Gamma1(13),d) then // Rank filter
                    C := HyperellipticCurve(d*f);
                    if #TwoCoverDescent(C) gt 0 then  // Two cover descent filter
                        Append(~output, d);
                    end if;
                end if;
            end if;
        end if;
    end if;
end for;

output;
```

$17, 113, 193, 313, 481, 673, 1153, 1417, 1609, 1921, 2089, 2161,$
$2257, 3769, 3961, 5449, 6217, 6641, 8473, 8641, 9689, 9881$

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These are dealt with via the Mordell-Weil sieve.

Mordell-Weil sieve

$$
\begin{tikzcd}
X(\mathbb{Q}) \arrow[r, hook, "\iota"] \arrow[d, "\alpha"] & J(\mathbb{Q}) \arrow[d, "\alpha"] \\
\prod_p X(\mathbb{Q}_p) \arrow[r, "\tilde{\iota}"] & \prod_p J(\mathbb{Q}_p)
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These are infinite groups and sets, so the intersection can't be computed. Instead one works with a finite approximation.

$$\begin{tikzcd}
X(\mathbb{Q}) \arrow[r, hook, "\iota"] \arrow[d, "\alpha"] & J(\mathbb{Q})/NJ(\mathbb{Q}) \arrow[d, "\alpha"] \\
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Here N is a positive integer, and S a finite set of primes. Now we can compute the intersection. Heuristically, if $X(\mathbb{Q}) = \emptyset$, then the intersection will be empty if S and N are large enough.

Theorem (B.-Derickx, 2023)

$$T_{10,000}(13) = \{17, 113, 193, 313, 481, 1153, 1417, 2257, 3769, 3961, 5449, 6217, 6641, 9881\}$$

$$T_{10,000}(18) = \{33, 337, 457, 1009, 1993, 2833, 7369, 8241, 9049\}$$

$X_1(16)$

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Proposition (B.-Derickx, 2023)

Let $K = \mathbb{Q}(\sqrt{d})$. If $\mathbb{Z}/16\mathbb{Z}$ arises as a possible torsion group over K, then $\mathrm{rk}(J_1^d(16)) > 0$.

Using the twisted winding element method from before, we compute the squarefree values of d with $|d| < 10{,}000$ for which $\mathrm{rk}(J_1^d(16)) > 0$; this yields 674 values.

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We do a point search on these; 55 of them have extra points.
How to deal with the remaining 619 values?

Elliptic Curve Chabauty

Theorem (Bruin-Stoll, souped-up version of Chevalley-Weil)

Every rational point on a hyperelliptic curve X lifts to a rational point on some $D \in$ `TwoCoverDescent(X)`*.*

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SUMMARY: If, for every D, there is a degree 3 factor $g \in L[x]$ s.t. $E_D : \gamma_D y^2 = g(x)$ has $\mathrm{rk}(E_D(L)) < [L : \mathbb{Q}]$, then we're done.

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For us, $f(x) = dx(x^2 + 1)(x^2 - 2x - 1)$, so L will always be quite small.

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The remaining 38 values to be dealt with are:

$$-8259, -7973, -7615, -7161, -7006, -6711, -6503, -6095,$$
$$-6031, -6005, -4911, -4847, -4773, -4674, -4371, -4191,$$
$$-4074, -3503, -3199, -1810, -1749, -815, 969, 1186,$$
$$3215, 3374, 3946, 4633, 5257, 5385, 7006, 7210,$$
$$7733, 8459, 8479, 8569, 9709, 9961$$

Todo

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- What about cubic torsion? i.e. for a fixed cubic field K, which of the 26 groups in the cubic torsion classification (due to Derickx-Etropolski, van Hoeij, Morrow, Zureick-Brown) arise as torsion subgroups for that K?