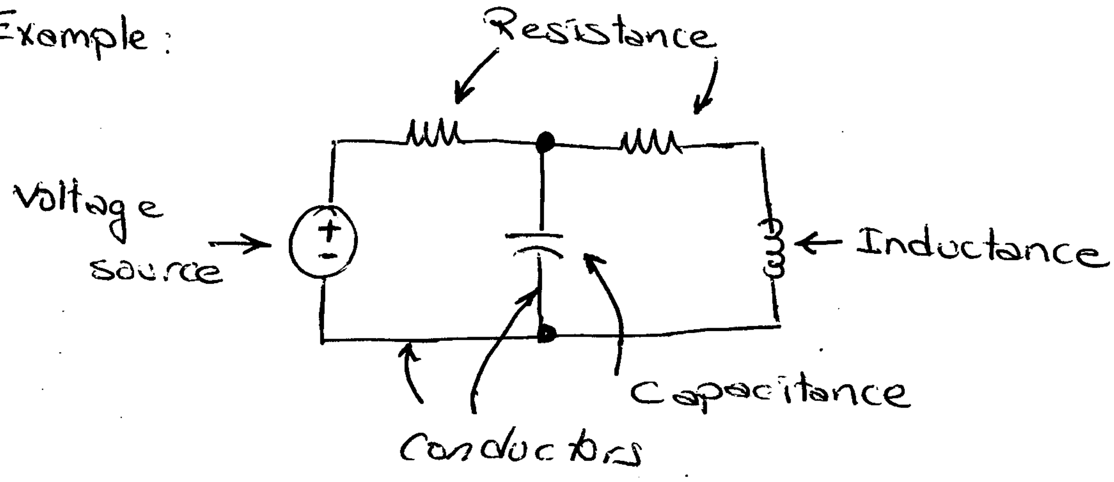
INTRODUCTION

Electrical Circuits

An electrical circuit consists of various types of circuit elements connected by conductors.

Example:



Electrical Current

Electrical current is the time rate of flow of electrical charge through a conductor or circuit element.

The electrical current flowing through the element is given by

$$i(t) = \frac{dq(t)}{dt}$$

where $i(t)$ - current in amperes (A)

$q(t)$ - charge in coulombs (C)

t - time in seconds (s)

if $i(t)$ is constant with time, it is direct current (dc)

if $i(t)$ is changing with time, reversing direction periodically, it is alternating current (ac)

Example: $i_1(t) = 2\ A$ (dc)

$i_2(t) = 2\cos(2\pi t)$ (ac)

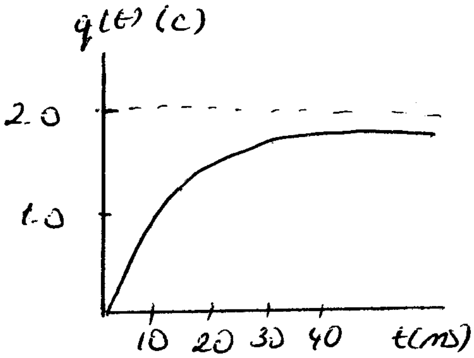
Example: suppose that charge versus time for a given circuit element is given by

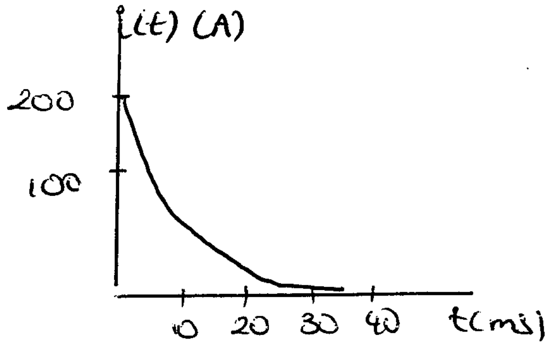
$$q(t) = 0 \qquad \text{for } t<0$$
$$q(t) = 2 - 2e^{-100t} \qquad \text{for } t>0$$

Sketch $q(t)$ and $i(t)$ to scale versus time.

Solution:

$$i(t) = \frac{dq(t)}{dt}$$
$$= 0 \qquad \text{for } t<0$$
$$= 200e^{-100t} \qquad \text{for } t>0$$





P1.3 The current in a given circuit element is given by

$$i(t) = 2e^{-t}$$

Find the net charge that passes through the element in the interval from $t=0$ to $t=\infty$

Solution: $i(t) = \dfrac{dq(t)}{dt}$

$$q(t) = \int_0^\infty i(t)dt$$
$$= \int_0^\infty 2e^{-t}dt$$
$$= [-2e^{-t}]_0^\infty$$
$$= -2(e^{-\infty} - e^{-0}) = -2(0-1) = 2C$$

Electrical Voltage

Electrical voltage is the energy transferred per unit of charge that flows through the element.

$$v(t) = \frac{dw(t)}{dq(t)}$$

where $v(t)$ - voltage in volts (V)
$w(t)$ - energy in joules (J)
$q(t)$ - charge in coulombs (C)

Constant voltages are called dc voltages. Voltages that change in magnitude and alternate in polarity with time are said to be ac voltages.

Example : $v_1(t) = 10\ V$ (dc)

$v_2(t) = 10 \cos(200\pi t)\ V$ (ac)

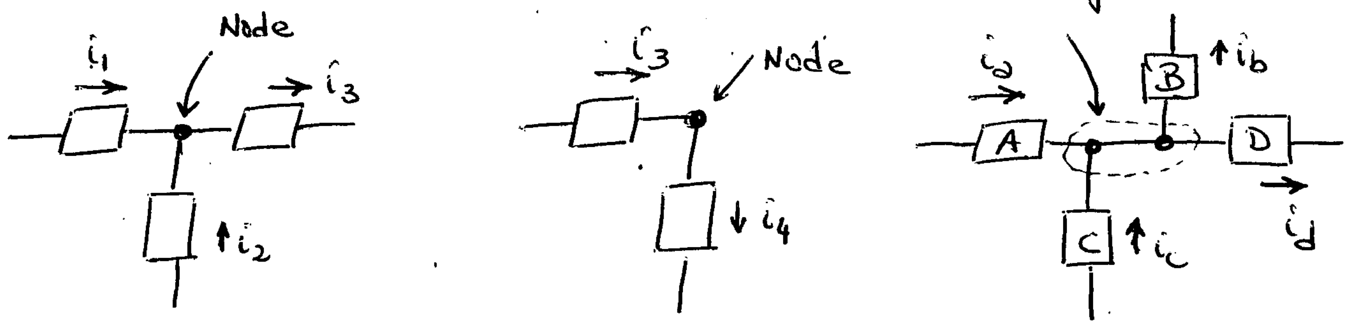
Kirchoff's Current Low (KCL)

A node in an electrical circuit is a point at which two or more circuit elements are joined together.

KCL states that the sum of the currents entering a node equals the sum of the currents leaving

$$\sum i_{entering} = \sum i_{leaving}$$

Example:



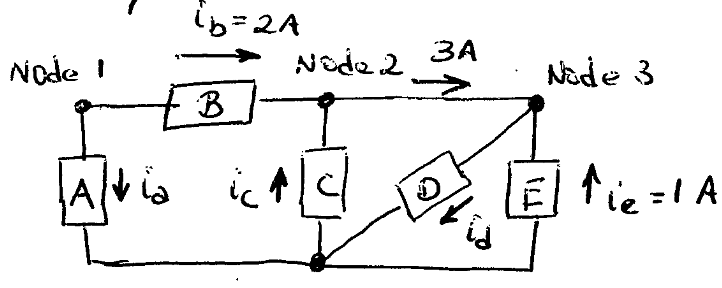
$i_1 + i_2 = i_3$

$i_3 = i_4$

$i_a + i_c = i_b + i_d$

(All points in a circuit that are connected directly by conductors can be considered as a single node)

P1.12 Use KCL to find the values of i_a, i_c and i_d for the following circuit.



Solution:

KCL at node 1:

$i_a + i_b = 0 \Rightarrow i_a = -i_b = -2A$

KCL at node 2:

$i_b + i_c = 3 \Rightarrow i_c = 3 - i_b = 1\,A$

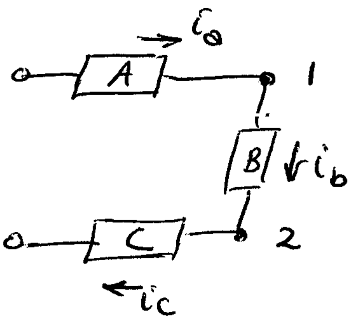
KCL at node 3:

$3 + i_e = i_d \Rightarrow i_d = 4\,A$

Series Circuits

When elements are connected end to end, we say that they are connected in series.

Example.

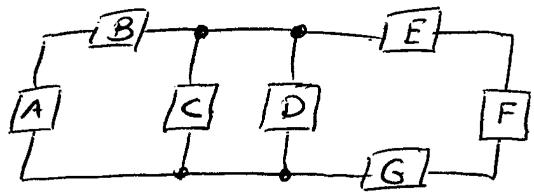


KCL at node 1 : $i_a = i_b$

KCL at node 2 : $i_b = i_c$ $\Rightarrow i_a = i_b = i_c$

All elements in a series circuit have identical currents.

Example: consider the following circuit. Identify the groups of circuit elements that are connected in series.



Solution: A and B are in series

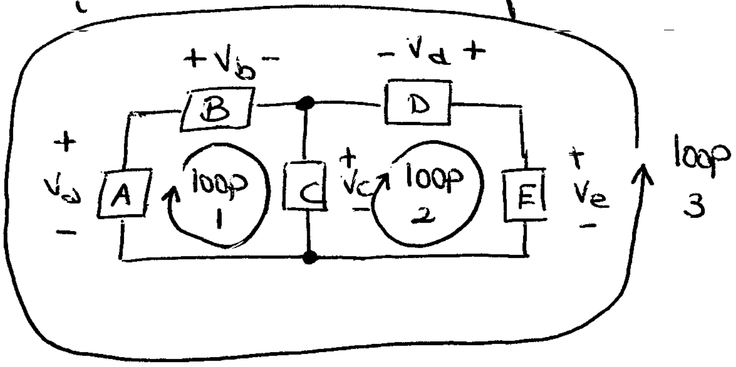
E, F and G are in series.

Kirchhoff's Voltage Law (KVL)

KVL states that the algebraic sum of voltages for a closed path (loop) must be zero.

$$\sum_i V_i = 0 \quad \text{for any loop.}$$

Example:

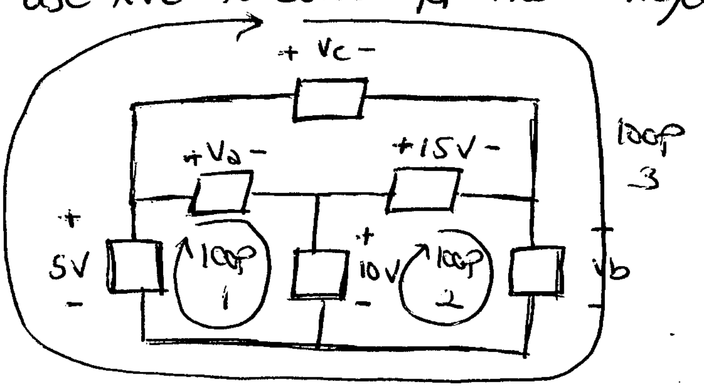


KVL for loop 1: $-V_a + V_b + V_c = 0$

KVL for loop 2: $-V_c - V_d + V_e = 0$

KVL for loop 3: $-V_e + V_d - V_b + V_a = 0$

P1.15 Use KVL to solve for the voltages V_a, V_b and V_c.



Solution:

KVL for loop 1: $-5 + V_a + 10 = 0 \Rightarrow V_a = -5V$

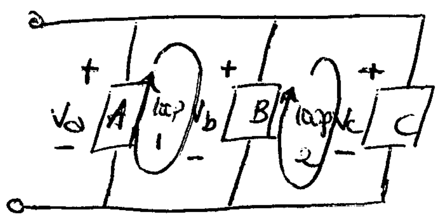
KVL for loop 2: $V_b - 10 + 15 = 0 \Rightarrow V_b = -5V$

KVL for loop 3: $V_c + V_b - 5 = 0 \Rightarrow V_c = 10V$

Parallel Circuits

We say that two circuit elements are connected in parallel if both ends of one element are connected directly to corresponding ends of the others.

Example:



KVL for loop 1: $-V_a + V_b = 0 \implies V_a = V_b$

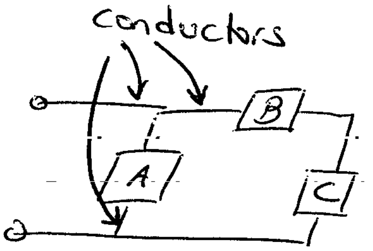
KVL for loop 2: $-V_b + V_c = 0 \implies V_b = V_c$

$\implies V_a = V_b = V_c$

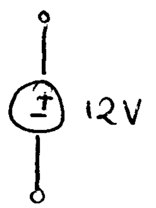
The voltages across parallel elements are equal in magnitude and have the same polarity.

Circuit Elements

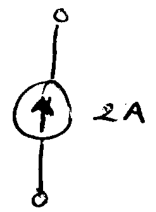
conductors : The line that connects the elements.



Independent sources:

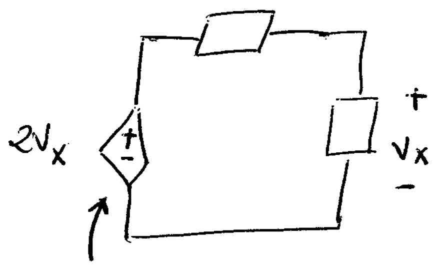


independent voltage source

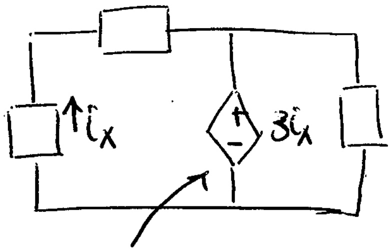


independent current source

① Dependent (Controlled) Voltage Sources

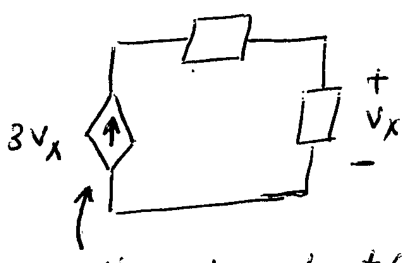


voltage dependent (controlled) voltage source

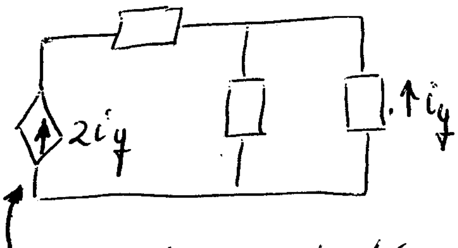


current dependent (controlled) voltage source

Dependent (controlled) Current Sources



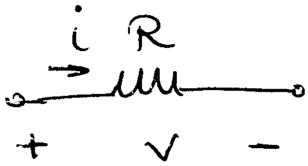
voltage dependent (controlled) current source



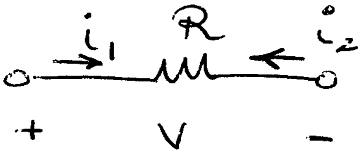
current dependent (controlled) current source

Resistors

The voltage V accross an ideal resistor is proportional to the current i through the resistor. Unit of resistor is ohms(Ω)



Ohm's Law



$V = i_1 R$

$V = -i_2 R$

conductance

$i = \left(\frac{1}{R}\right) V$

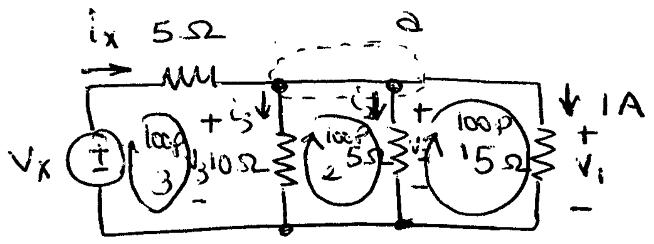
↳ conductance $= G = \frac{1}{R}$

Conductances have the units of inverse ohms (Ω^{-1}) which are called Siemens (S)

Thus we can write Ohm's law as

$i = GV$

P1.28 Consider the circuit shown below. Use Ohm's Law, KVL and KCL to find V_x and i_x.



Solution:

Ohm's Law:

$V_1 = (1)(5) = 5V$

KVL at loop 1: $V_1 - V_2 = 0 \Rightarrow V_2 = V_1 = 5V$

KVL at loop 2: $V_2 - V_3 = 0 \Rightarrow V_3 = V_2 = 5V$

Ohm's Law:

$V_2 = 5i_2 \Rightarrow i_2 = \frac{V_2}{5} = \frac{5}{5} = 1A$

$V_3 = 10i_3 \Rightarrow i_3 = \frac{V_3}{10} = \frac{5}{10} = 0.5A$

KCL at node a: $i_x = i_2 + i_3 + 1$

$= 1 + 0.5 + 1$

$= 2.5A$

KVL at loop 3: $-V_x + 5i_x + V_3 = 0$

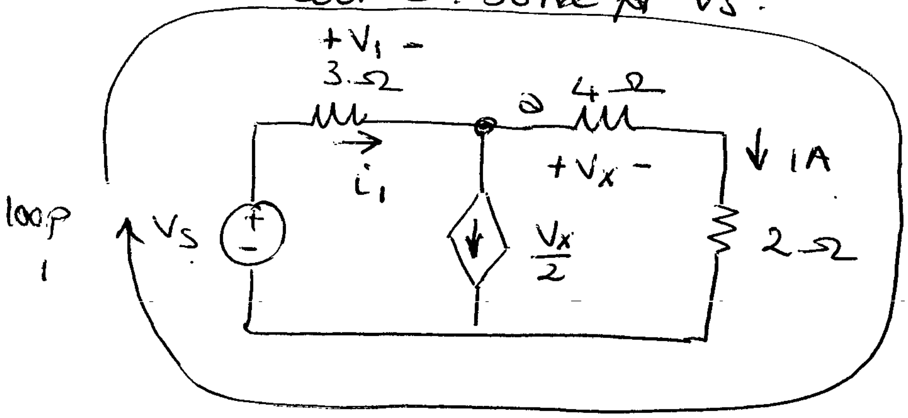
$V_x = 5i_x + V_3$

$= 5(2.5) + 5$

$= 12.5 + 5$

$= 17.5V$

P1.31 The circuit shown below contains a voltage-controlled current source. Solve for V_s.



Solution:

$V_x = (4)(1) = 4\ V$ (Ohm's Law)

$i_1 = 1 + \frac{V_x}{2} = 1 + \frac{4}{2} = 3A$ (KCL at node a)

$V_1 = (3)(i_1) = (3)(3) = 9\ V$ (Ohm's Law)

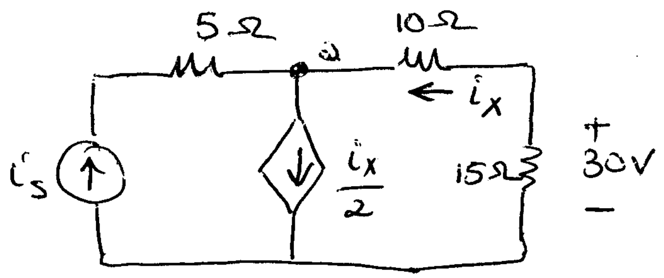
$-V_s + V_1 + V_x + (1)(2) = 0$ (KVL at loop 1)

$V_s = V_1 + V_x + 2$

$= 9 + 4 + 2$

$= 15\ V$

P1-32 For the following circuit solve for i_s.

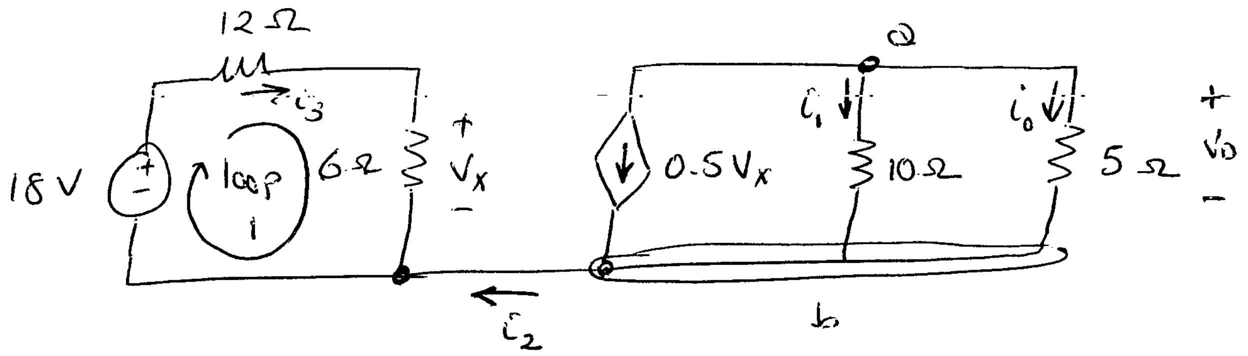


Solution: $30 = -i_x(15) \Rightarrow i_x = \frac{-30}{15} = -2A$

KCL at node a: $i_s + i_x = i_x/2$

$i_s = \frac{i_x}{2} - i_x = -\frac{i_x}{2} = -\frac{(-2)}{2} = 1\ A$

Question: Find (a) i_1, (b) i_o and (c) i_2 in the circuit



Solution:

KVL for loop 1: $-18 + 12 i_3 + 6 i_3 = 0$

$$18 i_3 = 18$$

$$i_3 = 1A$$

$V_x = 6 i_3 = 6(1) = 6\ V$

KCL at node a: $0.5 V_x + i_1 + i_o = 0$

$$i_1 = \frac{V_o}{10}$$

$$i_o = \frac{V_o}{5}$$

$$0.5(6) + \frac{V_o}{10} + \frac{V_o}{5} = 0$$

$$3 + \frac{V_o}{10} + \frac{V_o}{5} = 0$$

$$\frac{3 V_o}{10} = -3$$

$$V_o = -10 V$$

(b) $i_o = \frac{V_o}{5} = \frac{-10}{5} = -2A$

(a) $i_1 = \frac{V_o}{10} = \frac{-10}{10} = -1A$

(c) KCL at node b: $i_2 = 0.5 V_x + i_1 + i_o$

$$= 0.5(6) + (-1) + (-2)$$

$$= 3 - 1 - 2$$

$$= 0 A$$

Power

$$P = Vi$$

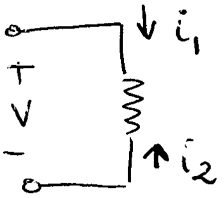
$$= i^2 R \qquad (V = iR)$$

$$= \frac{V^2}{R} \qquad (i = V/R)$$

The units of powes are Watts (W)

If $P > 0$ => Energy is being absorbed by the element

If $P < 0$ => Element is supplying energy to other parts of the circuit.

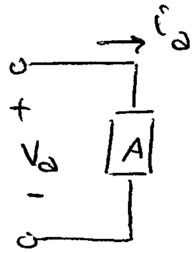


$P = Vi_1$

$P = -Vi_2$

Example:

(a)



$V_a = 12V$

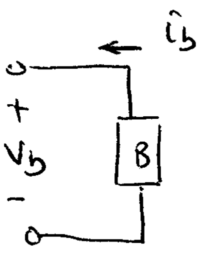
$i_a = 2A$

$P_a = V_a i_a = (12)(2) = 24W$

$P_a > 0$ => Energy is absorbed by A

If A is a battery it is being charged.

(b)



$V_b = 12V$

$i_b = 1A$

$P_b = -V_b i_b = -(12)(1) = -12W$

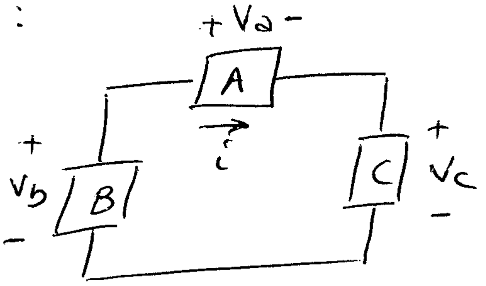
$P_b < 0$ => Energy is supplied by B.

If B is a battery it is being discharged.

At a given instant, the sum of the powers for all of the elements in a circuit must be zero.

$$\sum_i P_i = 0 \quad \text{for all elements}$$

Example:



$$P_a + P_b + P_c = 0$$

substituting for the powers, we have

$$V_a i - V_b i + V_c i = 0$$

cancelling the current i, we obtain

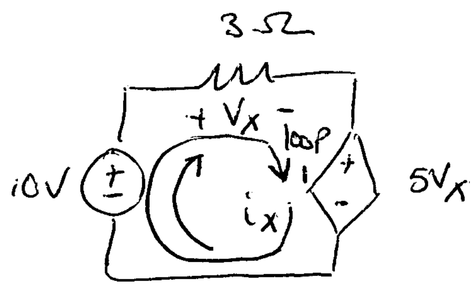
$$V_a - V_b + V_c = 0 \quad \text{(KVL)}$$

So KVL is a consequence of the law of energy conservation.

P1.29 (a) use KVL to write an equation relating the voltages and solve for V_x

(b) Use Ohm's law to find the current i_x

(c) Find the power for each element in the circuit and verify that power is conserved.



solution:

(a) KVL for loop 1:

$-10 + V_X + 5V_X = 0$

$6V_X = 10$

$V_X = 10/6 \text{ V}$

(b) $V_X = i_X(3) \implies i_X = \frac{V_X}{3} = \frac{10/6}{3} = \frac{10}{18} \text{ A}$

(c) $P_{10V} = -(10)(10/18) = -100/18 \text{ W}$

$P_{3\Omega} = (10/6)(10/18) = 100/108 \text{ W}$

$P_{5V_X} = 5(10/6)(10/18) = 500/108 \text{ W}$

$P_{10V} + P_{3\Omega} + P_{5V_X} \stackrel{?}{=} 0$

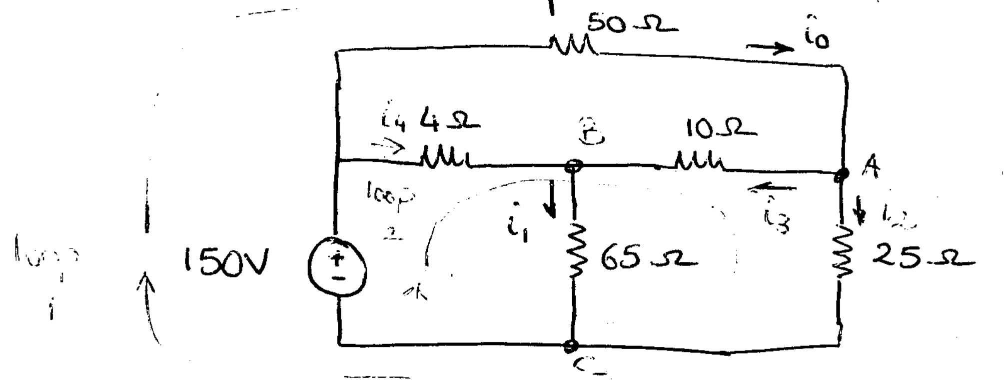
$\frac{-100}{18} + \frac{100}{108} + \frac{500}{108} = 6\left(\frac{-100}{108}\right) + \frac{100}{108} + \frac{500}{108} = 0$ ✓

Question: The current i_o in the circuit is 1 A.

a) Find i_1.

b) Find the power dissipated in each resistor and power delivered by voltage source.

c) Verify that the total power dissipated in the circuit equals the power developed by the 150-V source.



Solution:

(a) KVL for loop 1:

$$-150 + 50 i_o + 25 i_2 = 0$$

$$-150 + 50(1) + 25 i_2 = 0 \implies i_2 = 4\ A$$

KCL at node A:

$$i_o = i_2 + i_3$$

$$i_3 = i_o - i_2 = 1 - 4 = -3\ A$$

KVL for loop 2:

$$-150 + 4 i_4 - 10 i_3 + 25 i_2 = 0$$

$$-150 + 4 i_4 - 10(-3) + 25(4) = 0 \implies i_4 = 5\ A$$

KCL at node B:

$$i_1 = i_3 + i_4$$

$$= (-3) + (5) = 2\ A$$

(b) $P_{50\Omega} = i_0^2(50)$

$= (1)^2 50$

$= 50\ W$

$P_{65\Omega} = i_1^2(65)$

$= (2)^2 65$

$= 260\ W$

$P_{25\Omega} = i_2^2(25)$

$= (4)^2(25)$

$= 400\ W$

$P_{10\Omega} = i_3^2(10)$

$= (-3)^2(10)$

$= 90\ W$

$P_{4\Omega} = i_4^2(4)$

$= (5)^2(4)$

$= 100\ W$

$P_{150V} = -i_5 V$

KCL at node c :

$i_5 = i_1 + i_2 = (2) + (4) = 6\ A$

$P_{150V} = -(6)(150) = -900\ W$

(c) $P_{50\Omega} + P_{65\Omega} + P_{25\Omega} + P_{10\Omega} + P_{4\Omega} + P_{150V} \stackrel{?}{=} 0$

$50 + 260 + 400 + 90 + 100 + (-900) = 0$ ✓

Energy Calculations

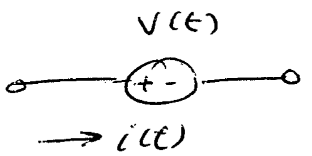
$$p(t) = v(t)\, i(t) = \frac{dw(t)}{dq(t)} \cdot \frac{dq(t)}{dt} = \frac{dw(t)}{dt}$$

$$w(t) = \int_{t_1}^{t_2} p(t)\, dt$$

where $p(t)$ - power in watts (W)

$w(t)$ - energy in Joules (J)

Example: Find the expression for the power for the voltage source given below. compute the energy for the interval from $t_1 = 0$ to $t_2 = \infty$.



$v(t) = 12V$

$i(t) = 2e^{-t} A$

Solution: $p(t) = v(t)\, i(t)$

$= (12)(2e^{-t})$

$= 24e^{-t}$ W

$$w(t) = \int_{t_1}^{t_2} p(t)\, dt$$

$$= \int_0^{\infty} 24e^{-t}\, dt$$

$$= -24e^{-t} \Big|_0^{\infty}$$

$$= -24(e^{-\infty} - e^{-0})$$

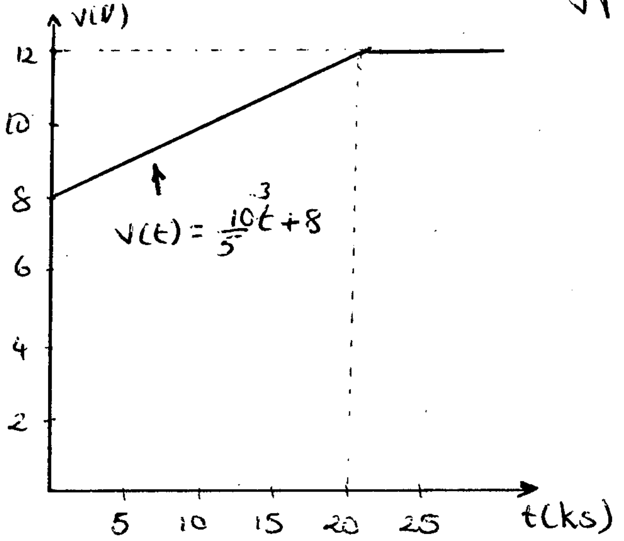
$$= 24 \text{ J}$$

Because the energy is positive, it is absorbed by the source

Question: The voltage and current at the terminals of an automobile battery during a charge cycle are shown below.

(a) Calculate the total charge transferred to the battery.

(b) Calculate the total energy transferred to the battery.



$y - y_1 = m(x - x_1)$

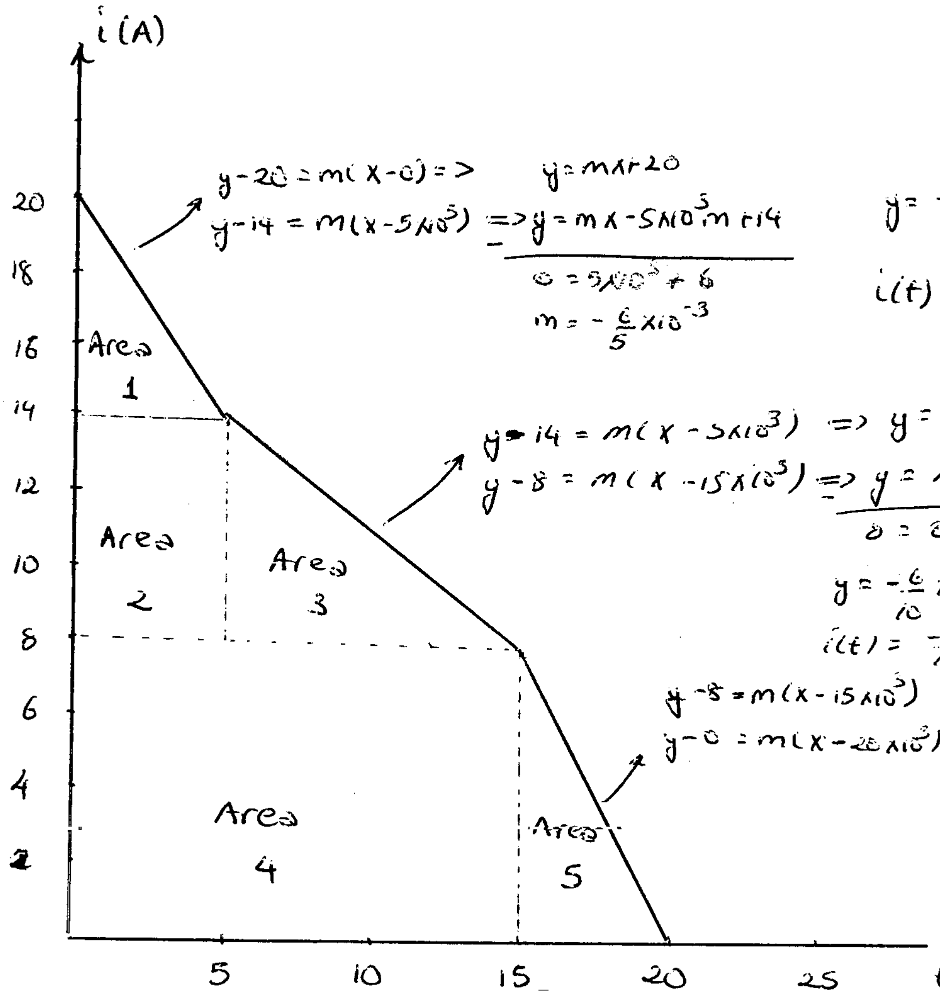
$y - 8 = m(x - 0) \Rightarrow y = mx + 8$

$y - 12 = m(x - 20\times10^{3}) \Rightarrow y = mx - 20\times10^{3}m + 12$

$0 = 20\times10^{3} m - 4$

$m = \frac{1}{5}\times10^{-3}$

$y = \frac{1}{5}\times10^{-3}x + 8$



$y - 20 = m(x - 0) \Rightarrow y = mx + 20$

$y - 14 = m(x - 5\times10^{3}) \Rightarrow y = mx - 5\times10^{3}m + 14$

$0 = 5\times10^{3} + 6$

$m = -\frac{6}{5}\times10^{-3}$

$y = -\frac{6}{5}\times10^{-3}x + 20$

$i(t) = -\frac{6}{5}\times10^{-3}t + 20$

$y - 14 = m(x - 5\times10^{3}) \Rightarrow y = mx - 5\times10^{3}m + 14$

$y - 8 = m(x - 15\times10^{3}) \Rightarrow y = mx - 15\times10^{3}m + 8$

$0 = 0 + 10\times10^{3}m + 6 \Rightarrow m = -\frac{6}{10}\times10^{-3}$

$y = -\frac{6}{10}\times10^{-3}x - 15\left(-\frac{6}{10}\times10^{-3}\right) + 8 = -\frac{6}{10}\times10^{-3}x + 17$

$i(t) = -\frac{6}{10}\times10^{-3}t + 17$

$y - 8 = m(x - 15\times10^{3}) \Rightarrow y = mx - 15\times10^{3}m + 8$

$y - 0 = m(x - 20\times10^{3}) \Rightarrow y = mx - 20\times10^{3}m -$

$0 = 0 + 5\times10^{3}m + 8$

$m = -\frac{8}{5}\times10^{-3}$

$y = -\frac{8}{5}\times10^{-3}x - 20\left(-\frac{8}{5}\times10^{3}\right)$

$y = -\frac{8}{5}\times10^{-3}x + 32$

$i(t) = -\frac{8}{5}\times10^{-3}t + 32$

Solution:

(a) $i(t) = \dfrac{dq(t)}{dt}$

$$q(t) = \int_0^{20} i(t)\, dt = \text{Area under } i(t) \text{ curve}$$

$$= \text{Area } 1 + \text{Area } 2 + \text{Area } 3 + \text{Area } 4 + \text{Area } 5$$

$$= \frac{6 \times 5 \times 10^3}{2} + 6 \times 5 \times 10^3 + \frac{6 \times 10 \times 10^3}{2} + 8 \times 15 \times 10^3 + \frac{8 \times 5 \times 10^3}{2}$$

$$= 15 \times 10^3 + 30 \times 10^3 + 30 \times 10^3 + 120 \times 10^3 + 20 \times 10^3$$

$$= 215 \times 10^3$$

$$= 215 \text{ kC}$$

(b) $w(t) = \int_0^{\infty} p(t)\, dt$

$$p(t) = v(t)\, i(t)$$

$$v(t) = \begin{cases} \frac{1}{5} \times 10^{-3} t + 8 & , \ t < 20\text{ks} \\ 12 & , \ \text{otherwise} \end{cases}$$

$$i(t) = \begin{cases} -\frac{6}{5} \times 10^{-3} t + 20 & , \ t < 5\text{ks} \\ -\frac{6}{10} \times 10^{-3} t + 17 & , \ 5\text{ks} \leq t < 15\text{ks} \\ -\frac{8}{5} \times 10^{-3} t + 32 & , \ 15\text{ks} \leq t < 20\text{ks} \\ 0 & , \ t \geq 20\text{ks} \end{cases}$$

$$w(t) = \int_0^{5k} \left(-\tfrac{6}{5} \times 10^{-3} t + 20\right)\left(\tfrac{1}{5} \times 10^{-3} t + 8\right) dt \ +$$

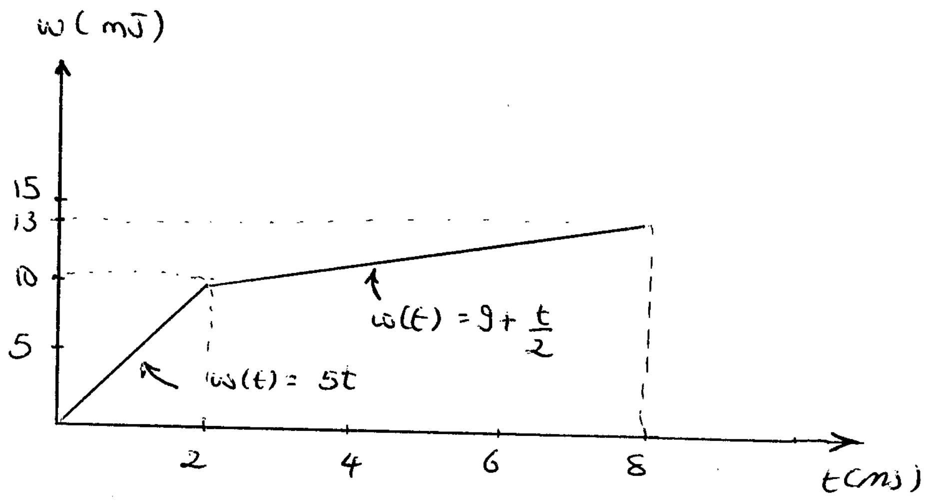
$$\int_{5k}^{15k} \left(-\tfrac{6}{10} \times 10^{-3} t + 17\right)\left(\tfrac{1}{5} \times 10^{-3} t + 8\right) dt \ +$$

$$\int_{15k}^{20k} \left(-\tfrac{8}{5} \times 10^{-3} t + 32\right)\left(\tfrac{1}{5} \times 10^{-3} t + 8\right) dt \ +$$

$$\int_{20k}^{\infty} (0)(12)\, dt$$

$$= 2036\ 67 \text{ kJ}$$

Example: A two terminal element absorbs w milijoules of energy as shown in the figure. If the current entering the positive terminal is $i(t) = 100\cos(1000\pi t)$ mA, find the element voltage at $t = 1$ms and $t = 4$ms.



$$P = \frac{dw(t)}{dt}$$

$$P = Vi \implies V = \frac{P}{i}$$

when $t = 1$ms

$$P = \left.\frac{dw(t)}{dt}\right|_{t=1ms} = \left.\frac{5t}{dt}\right|_{t=1ms} = 5W$$

$$i(1ms) = 100\cos(1000\pi \times 1\times10^{-3}) \text{ mA}$$

$$i(1ms) = 100\cos(\pi) \text{ mA}$$

$$= 100(-1) \text{ mA}$$

$$= -100 \text{ mA}$$

$$V = \frac{P}{i} = \frac{5}{-100\times10^{-3}} = -50V$$

when $t = 4$ms

$$P = \left.\frac{dw(t)}{dt}\right|_{t=4ms} = \left.\frac{9+t/2}{dt}\right|_{t=4ms} = \frac{1}{2}W$$

$$i(4ms) = 100\cos(1000\pi \times 4\times10^{-3}) \text{ mA}$$

$$= 100\cos(4\pi) \text{ mA}$$

$$= 100\,(1) \text{ mA}$$

$$= 100 \text{ mA}$$

$$V = \frac{P}{i} = \frac{1/2}{100\times10^{-3}} = 5V$$