CHAPTER -3-

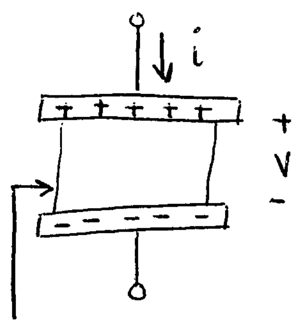
INDUCTANCE AND CAPACITANCE

Resistors convert electrical energy into heat.

Inductors and capacitors are energy-storage elements. They don't generate energy.

Capacitance

Capacitance is the circuit property that accounts for electric-field effects.



Dielectric material

The charged stored by a capacitor:

$$q = C V$$

where q - charge in coulombs (C)

C - capacitance in farad (F)

v - voltage in volts (V)

$$i = \frac{dq}{dt} = \frac{d(cv)}{dt} = c\frac{dV}{dt}$$

$$q(t) = \int_{t_0}^{t} i(t)\,dt + q(t_0)$$

where t_0 is the initial time

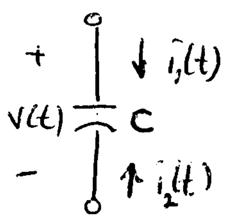
$$q(t) = c\,v(t)$$

$$v(t) = \frac{1}{c}\int_{t_0}^{t} i(t)\,dt + \frac{q(t_0)}{c}$$

$$= \frac{1}{c}\int_{t_0}^{t} i(t)\,dt + v(t_0)$$

usually the initial time $t_0 = 0$.

The circuit symbol for capacitance

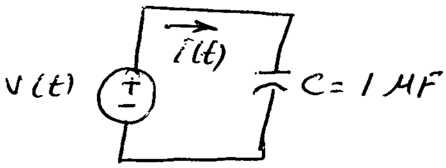


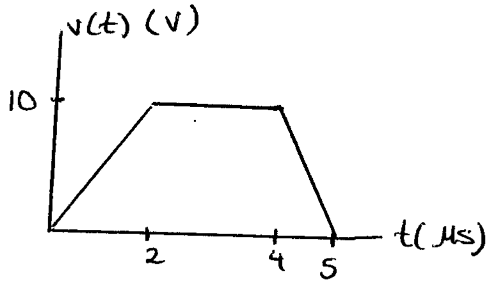
$$i_1(t) = c\,\frac{dv(t)}{dt}$$

$$i_2(t) = -c\,\frac{dv(t)}{dt}$$

If $v(t) =$ constant $\Rightarrow i(t) = 0 \Rightarrow$ A capacitor appears to be an open circuit for a steady dc voltage.

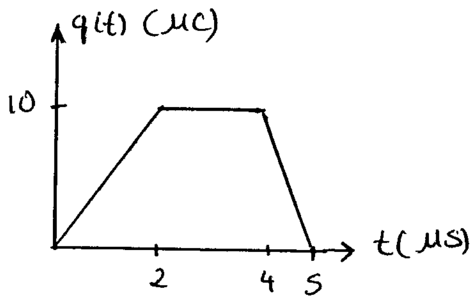
Example: Suppose that the voltage $v(t)$ given below is applied to a $1\,\mu F$ capacitance. Plot stored charge and the current through the capacitance versus time.





Solution:

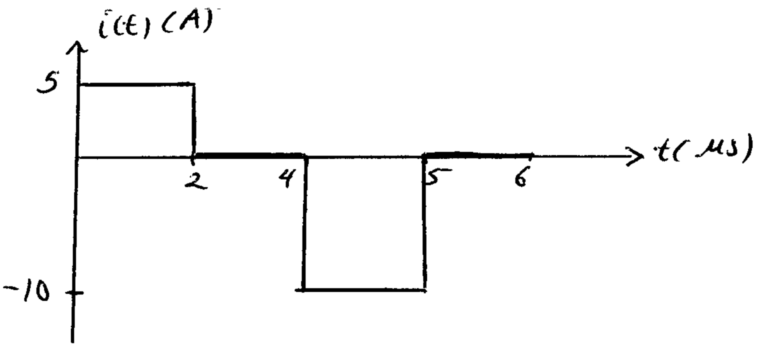
$$q(t) = C\,v(t) = 10^{-6}\,v(t) = v(t)\ \mu C$$



$$i(t) = \frac{dq(t)}{dt} = C\,\frac{dv(t)}{dt} = 10^{-6}\,\frac{dv(t)}{dt}$$

$$v(t) = \begin{cases} 5\times10^{6}\,t & 0 < t < 2\,\mu s \\ 10 & 2\,\mu s < t < 4\,\mu s \\ 10\times10^{6}(5\times10^{-6} - t) & 4\,\mu s < t < 5\,\mu s \end{cases}$$

$$i(t) = \begin{cases} (10^{-6})(5\times10^{6}) = 5 & 0 < t < 2\,\mu s \\ (10^{-6})(0) = 0 & 2\,\mu s < t < 4\,\mu s \\ (10^{-6})(-10\times10^{6}) = -10 & 4\,\mu s < t < 5\,\mu s \end{cases}$$



Notice that as the voltage increases the capacitor charges. For constant voltage, the current is zero and charge is constant. When voltage decreases, the direction of the current reverses and the stored charge is removed from the capacitor.

Stored Energy on capacitor

$$w(t) = \int_{t_0}^{t} p(t)\,dt$$

$$p(t) = v(t)\,i(t)$$

$$= v(t)\,\frac{dq(t)}{dt}$$

$$= v(t)\,\frac{d}{dt}\left(C v(t)\right)$$

$$= C\,v(t)\,\frac{dv(t)}{dt}$$

$$w(t) = \int_{t_0}^{t} C\,v(t)\,\frac{dv(t)}{dt}\,dt$$

$$w(t) = \int_{v(t_0)=0}^{v(t)} C\,v(t)\,dv(t)$$

← suppose that we have a capacitor that initially has $v(t_0)=0$

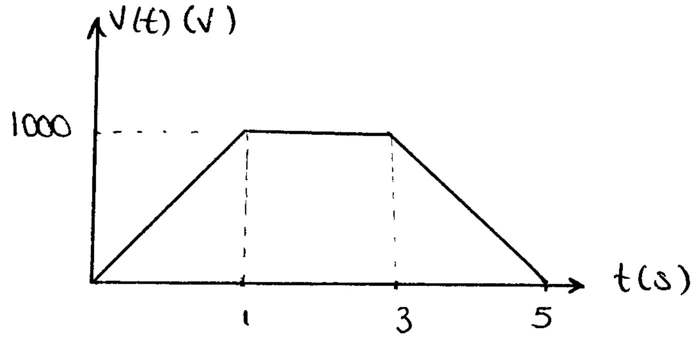
$$w(t) = \frac{1}{2} C v^2(t)$$

Energy stored in the capacitance that can be returned to the circuit.

$$w(t) = \frac{1}{2}\,\overbrace{C\,v(t)}^{q(t)}\,v(t) = \frac{1}{2}\,q(t)\,v(t)$$

$$w(t) = \frac{1}{2} C \left(\frac{q(t)}{C}\right)^2 = \frac{1}{2}\,\frac{q^2(t)}{C}$$

Example: Suppose that the voltage waveform given as below is applied to a 10 μF capacitance. Find and plot the current, the power delivered and the energy stored for time between 0 and 5s.

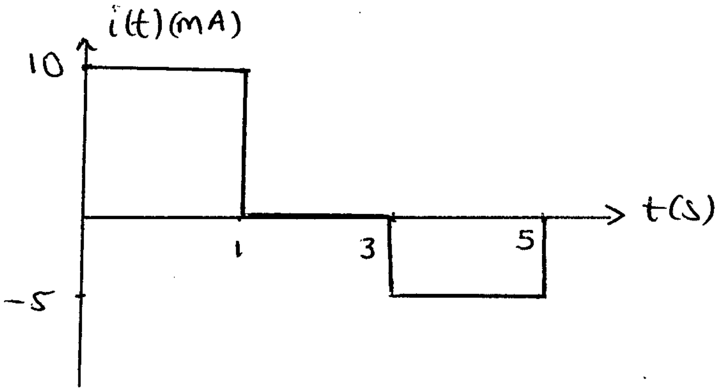


Solution:

$$v(t) = \begin{cases} 1000t & V & 0<t<1s \\ 1000 & V & 1s<t<3s \\ 500(5-t) & V & 3s<t<5s \end{cases}$$

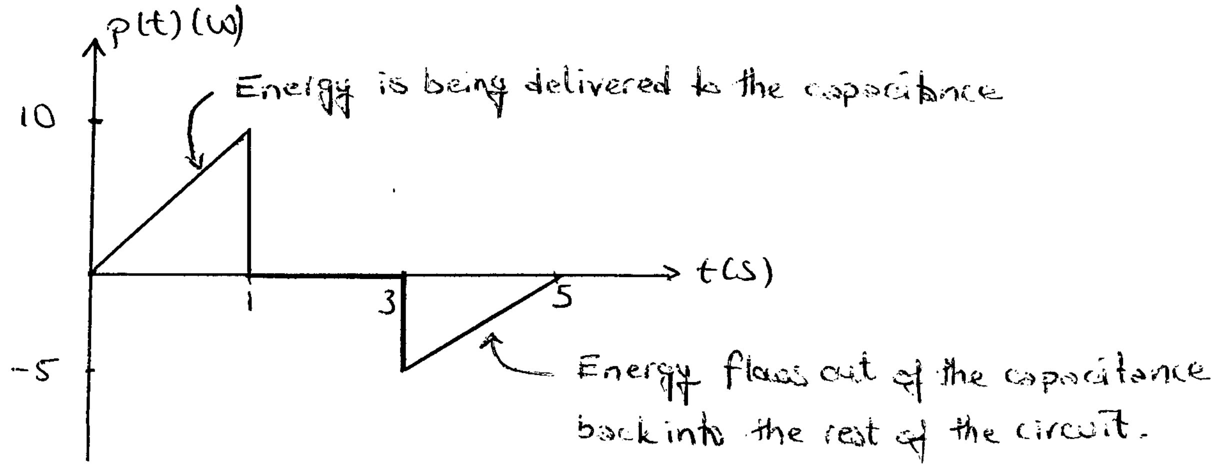
$$i(t) = c\frac{dv(t)}{dt} = 10\times10^{-6}\frac{dv(t)}{dt}$$

$$i(t) = \begin{cases} 10\times10^{-6}(1000) = 10\times10^{-3} = 10\ mA & 0<t<1s \\ 10\times10^{-6}(0) = 0\ A & 1s<t<3s \\ 10\times10^{-6}(-500) = -5\times10^{-3} = -5\ mA & 3s<t<5s \end{cases}$$



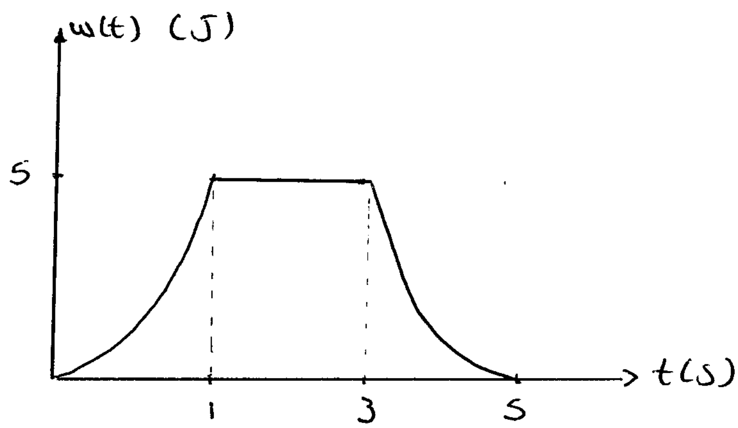
$p(t) = v(t)\,i(t)$

$$p(t) = \begin{cases} (1000t)(10\times10^{-3}) = 10t \ \text{W} & 0 < t < 1\text{s} \\ (1000)(0) = 0 \ \text{W} & 1\text{s} < t < 3\text{s} \\ ((500)(5-t))(-5\times10^{-3}) = 2.5(t-5) \ \text{W} & 3\text{s} < t < 5\text{s} \end{cases}$$



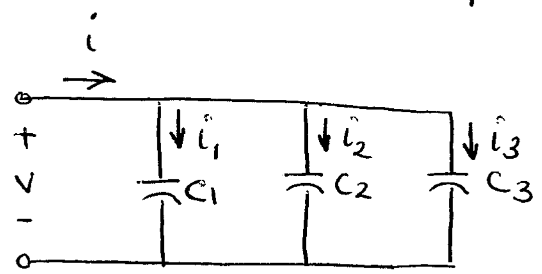
$$w(t) = \frac{1}{2} C\, v^2(t) = \frac{1}{2}\, 10\times10^{-6}\, v^2(t)$$

$$w(t) = \begin{cases} \frac{1}{2}(10\times10^{-6})(1000t)^2 = 5t^2 \ \text{J} & 0 < t < 1\text{s} \\ \frac{1}{2}(10\times10^{-6})(1000)^2 = 5 \ \text{J} & 1\text{s} < t < 3\text{s} \\ \frac{1}{2}(10\times10^{-6})(500(5-t))^2 = 1.25(5-t)^2 \ \text{J} & 3\text{s} < t < 5\text{s} \end{cases}$$



Capacitance in parallel

Consider the following circuit



$$i_1 = C_1 \frac{dv}{dt}$$

$$i_2 = C_2 \frac{dv}{dt}$$

$$i_3 = C_3 \frac{dv}{dt}$$

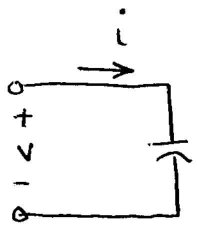
KCL at the top node.

$$i = i_1 + i_2 + i_3$$

$$= C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt}$$

$$= (C_1 + C_2 + C_3) \frac{dv}{dt}$$

$$= C_{eq} \frac{dv}{dt} \qquad C_{eq} = C_1 + C_2 + C_3$$



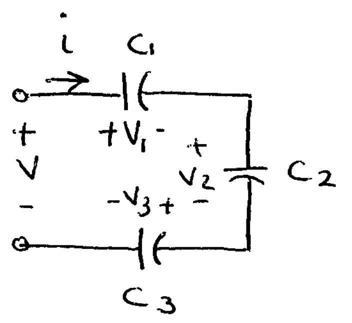
$C_{eq} = C_1 + C_2 + C_3$

In general

$$C_{eq} = \sum_{i=1}^{n} C_i = C_1 + C_2 + \dots + C_n$$

Capacitance in series

Consider the following circuit.



KVL ; $\quad -V + V_1 + V_2 + V_3 = 0$

$$V = V_1 + V_2 + V_3$$

$$V_1 = \frac{1}{C_1}\int_{t_0}^{t} i(t)\,dt + V_1(t_0)$$

$$V_2 = \frac{1}{C_2}\int_{t_0}^{t} i(t)\,dt + V_2(t_0)$$

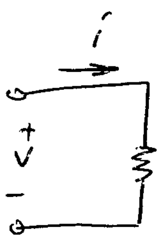
$$V_3 = \frac{1}{C_3}\int_{t_0}^{t} i(t)\,dt + V_3(t_0)$$

$$V = \frac{1}{C_1}\int_{t_0}^{t} i(t)\,dt + V_1(t_0) + \frac{1}{C_2}\int_{t_0}^{t} i(t)\,dt + V_2(t_0) + \frac{1}{C_3}\int_{t_0}^{t} i(t)\,dt + V_3(t_0)$$

$$= \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)\int_{t_0}^{t} i(t)\,dt + V_1(t_0) + V_2(t_0) + V_3(t_0)$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \implies C_{eq} = \frac{1}{1/C_1 + 1/C_2 + 1/C_3}$$

$$V = \frac{1}{C_{eq}}\int_{t_0}^{t} i(t)\,dt + V_1(t_0) + V_2(t_0) + V_3(t_0)$$

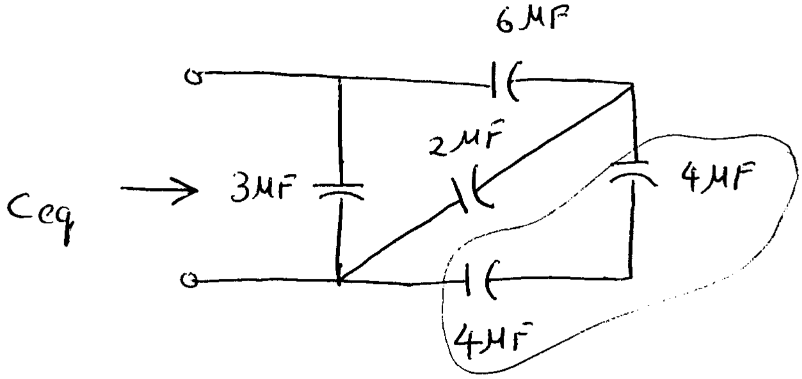


$$C_{eq} = \frac{1}{1/C_1 + 1/C_2 + 1/C_3}$$

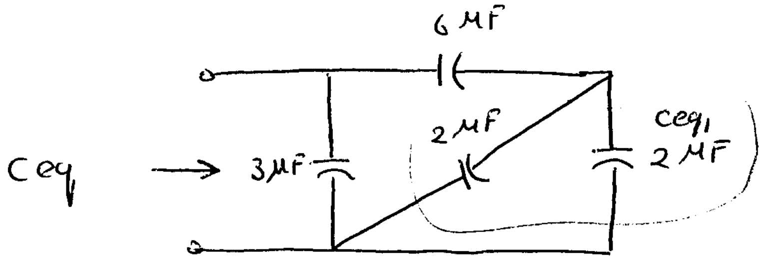
In general

$$\frac{1}{C_{eq}} = \sum_{i=1}^{n} \frac{1}{C_i} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}$$

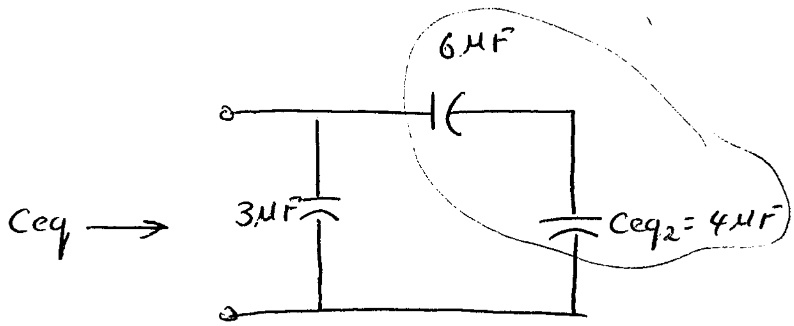
P3.10 Find the equivalent capacitance.



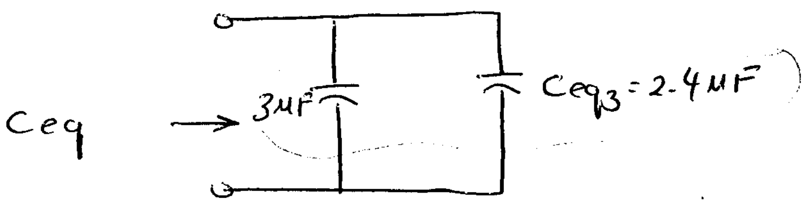
$$C_{eq_1} = \frac{(4\mu)(4\mu)}{4\mu + 4\mu} = 2\mu F$$



$$C_{eq_2} = 2\mu + 2\mu = 4\mu F$$

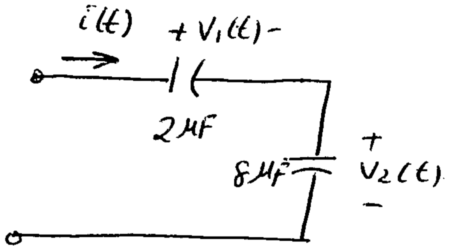


$$C_{eq_3} = \frac{6\,C_{eq_2}}{6 + C_{eq_2}} = \frac{(6)(4)}{6+4} = 2.4\mu F$$



$$C_{eq} = 3 + 2.4 = 5.4\mu F$$

Question: The current at the terminals of the two capacitors shown is $240e^{-10t}\ \mu A$ for $t \geq 0$. The initial values of V_1 and V_2 are -10 and $-5V$ respectively. Calculate the total energy trapped in the capacitors as $t \to \infty$.



Solution:

$$V_1 = \frac{1}{C_1}\int_0^{\infty} i(t)\,dt + V_1(0)$$

$$= \frac{1}{2\mu F}\int_0^{\infty} 240e^{-10t}\mu\, dt + (-10)$$

$$= \frac{1}{2}\left(\frac{240}{-10}\, e^{-10t}\Big|_0^{\infty}\right) - 10$$

$$= -12\,(\overset{0}{e^{-\infty}} - \overset{1}{e^{0}}) - 10$$

$$= 2\,V$$

$$\omega_1 = \frac{1}{2} C_1 V_1^2 = \frac{1}{2}(2\times 10^{-6})(2) = 4\mu J$$

$$V_2 = \frac{1}{C_2}\int_0^{\infty} i(t)\,dt + V_2(0)$$

$$= \frac{1}{8\mu F}\int_0^{\infty} 240e^{-10t}\mu\, dt + (-5)$$

$$= \frac{1}{8}\left(\frac{240}{-10}\, e^{-10t}\Big|_0^{\infty}\right) - 5$$

$$= -3\,(e^{-\infty} - e^{0}) - 5$$

$$= -2\,V$$

$$\omega_2 = \frac{1}{2} C_2 V_2^2 = \frac{1}{2}(8\times 10^{-6})(-2)^2 = 16\mu J$$

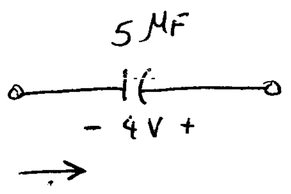
$$\omega = \omega_1 + \omega_2 = 4\mu + 16\mu = 20\mu J$$

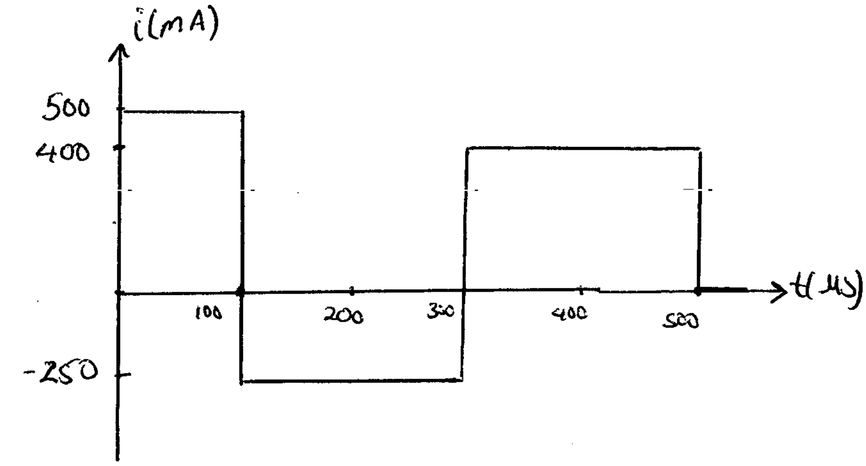
Question:

The initial voltage on the 5 μF capacitor shown in the figure is 4V. The capacitor current has the waveform given in the following figure. How much energy is stored in the capacitor at

(a) $t = 400\,\mu s$

(b) $t = 600\,\mu s$





Solution:

$$w = \frac{1}{2} C V^2$$

$$V = \frac{1}{C} \int_0^t i(t)\,dt + V(0)$$

$V(0) = -4\,V$

(a) $$V = \frac{1}{5\times10^{-6}} \int_0^{400\mu} i(t)\,dt + (-4)$$

$$= \frac{1}{5\times10^{-6}} \left(\int_0^{100\mu} 500\,m\,dt + \int_{100\mu}^{300\mu} (-250\,m)\,dt + \int_{300\mu}^{400\mu} (400\,m)\,dt \right) - 4$$

$$= \frac{1}{5\times10^{-6}} \left(500m(100\mu) - 250m(300-100)\mu + 400m(400-300)\mu \right) - 4$$

$$= \frac{1}{5\times10^{-6}} (40\times10^{-6}) - 4$$

$$= 8 - 4 = 4\,V$$

$$\omega = \frac{1}{2} C V^2$$

$$= \frac{1}{2} (5 \times 10^{-6})(4)^2$$

$$= 40 \mu J$$

(b) $$V = \frac{1}{5 \times 10^{-6}} \left(\int_0^{100\mu} 500m \, dt + \int_{100\mu}^{300\mu} (-250m) \, dt + \int_{300\mu}^{600\mu} (400m) dt \right) - 4$$

$$= \frac{1}{5 \times 10^{-6}} \left(500m(100\mu) - 250m(300-100)\mu + 400m(500-300)\mu \right) - 4$$

$$= \frac{1}{5 \times 10^{-6}} (80 \times 10^{-6}) - 4$$

$$= 16 - 4$$

$$= 12 V$$

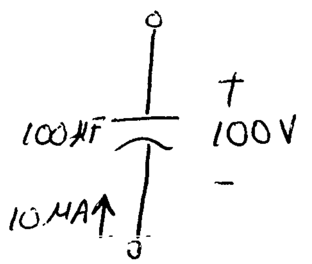
$$\omega = \frac{1}{2} C V^2$$

$$= \frac{1}{2} (5 \times 10^{-6})(12)^2$$

$$= 360 \mu J$$

P3.1 A 100 μF capacitor is initially charged to 100 V. It is discharged by a steady current of 10 μA. How long does it take to discharge the capacitor to 0 V?

Solution:



$$v(t) = -\frac{1}{C}\int_0^t i(t)\,dt + v(0)$$

$$0 = -\frac{1}{100\mu}\int_0^t 10\times10^{-6}\,dt + 100$$

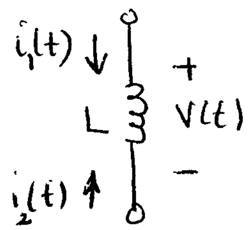
$$0 = \frac{-10t}{100} + 100$$

$$t = 1000 \text{ s.}$$

Inductance

An inductor is constructed by coiling a wire around some type of form. Inductance accounts for magnetic-field effects.

The circuit symbol of inductance



$$v(t) = L\frac{di_1(t)}{dt}$$ where L is inductance in henry (H)

$$v(t) = -L\frac{di_2(t)}{dt}$$

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(t)\,dt + i(t_0)$$

If $i(t) = \text{constant} \Rightarrow v(t) = 0 \Rightarrow$ An inductor appears to be a short circuit for a steady dc current.

Stored Energy on inductor

$$w(t) = \int_{t_0}^{t} p(t)\, dt$$

$$p(t) = i(t)\, v(t)$$

$$= i(t)\, L \frac{di(t)}{dt}$$

$$w(t) = \int_{t_0}^{t} L\, i(t) \frac{di(t)}{dt} \cdot dt$$

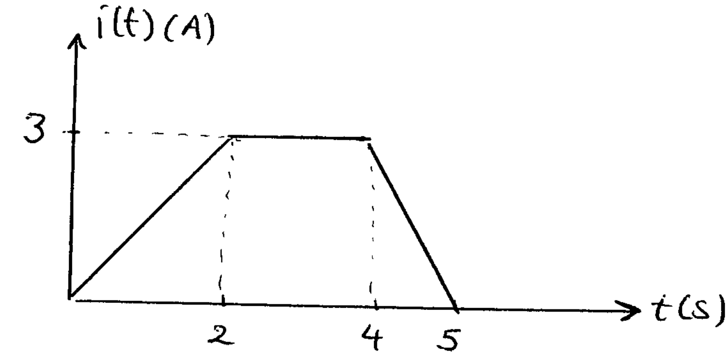
$$= \int_{i(t_0)=0}^{i(t)} L\, i(t)\, di(t)$$

← Consider an inductor having an initial current $i(t_0) = 0$

$$= \frac{1}{2} L\, i^2(t)$$

This represents energy stored in the inductance that is returned to the circuit if the current changes back to zero.

Example: The current through a 5 H inductance is shown below. Plot the voltage, power and stored energy to scale versus time for t between 0 and 5 s.

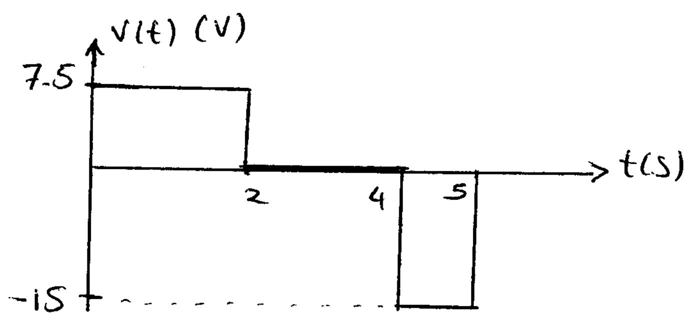


Solution:

$$v(t) = L\frac{di(t)}{dt} = 5\frac{di(t)}{dt}$$

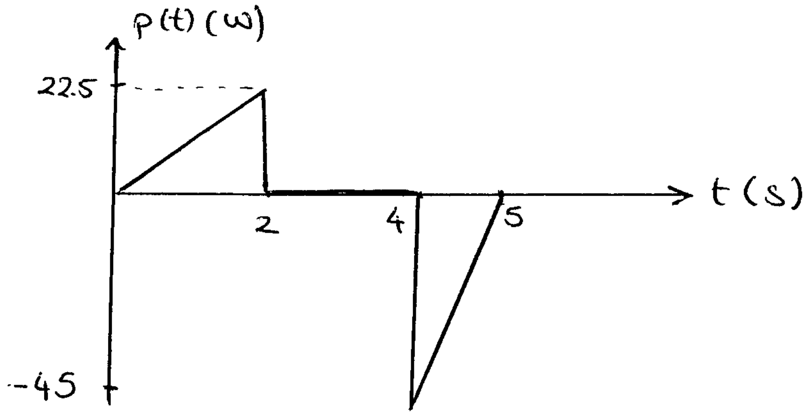
$$i(t) = \begin{cases} 3/2\, t \quad A & 0 < t < 2s \\ 3 \quad A & 2s < t < 4s \\ -3t + 15 \quad A & 4s < t < 5s \end{cases}$$

$$v(t) = \begin{cases} 5\,(3/2) = 7.5\,V & 0 < t < 2s \\ 5\,(0) = 0 \quad V & 2s < t < 4s \\ 5\,(-3) = -15\,V & 4s < t < 5s \end{cases}$$



$$p(t) = V(t)\, i(t)$$

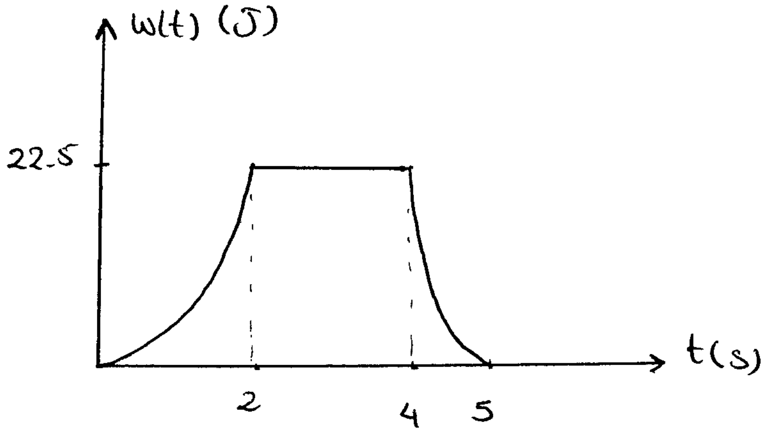
$$p(t) = \begin{cases} (7.5)(3/2\,t) = 11.25\,t \text{ W} & 0<t<2\text{ s} \\ (0)(3) = 0 \text{ W} & 2\text{s}<t<4\text{s} \\ (-15)(-3t+15) = 45t-225 \text{ W} & 4\text{s}<t<5\text{s} \end{cases}$$



$$w(t) = \frac{1}{2} L\, i^2(t)$$

$$w(t) = \frac{1}{2}(5)\, i^2(t)$$

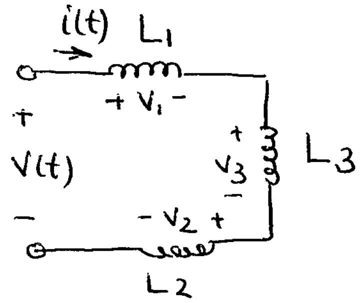
$$w(t) = \begin{cases} \frac{1}{2}(5)(\frac{3}{2}t)^2 = \frac{45}{8}t^2 \text{ J} & 0<t<2\text{s} \\ \frac{1}{2}(5)(3)^2 = \frac{45}{2} \text{ J} & 2\text{s}<t<4\text{s} \\ \frac{1}{2}(5)(-3t+15)^2 = \frac{5}{2}(9t^2-90t+225) \text{ J} & 4\text{s}<t<5\text{s} \end{cases}$$



As current magnitude increases, power is positive and stored energy accumulates. When the current is constant, the voltage is zero, power is zero and the stored energy is constant. When the current magnitude falls toward zero, the power is negative, showing that energy is being returned to other parts of the circuit.

Inductance In Series

Consider the following circuit.



KVL: $-V(t) + V_1 + V_3 + V_2 = 0$

$V(t) = V_1 + V_2 + V_3$

$$V_1 = L_1 \frac{di(t)}{dt}$$

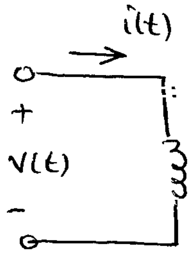
$$V_2 = L_2 \frac{di(t)}{dt}$$

$$V_3 = L_3 \frac{di(t)}{dt}$$

$$V = L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} + L_3 \frac{di(t)}{dt}$$

$$= (L_1 + L_2 + L_3) \frac{di(t)}{dt}$$

$$= L_{eq} \frac{di(t)}{dt} \qquad L_{eq} = L_1 + L_2 + L_3$$



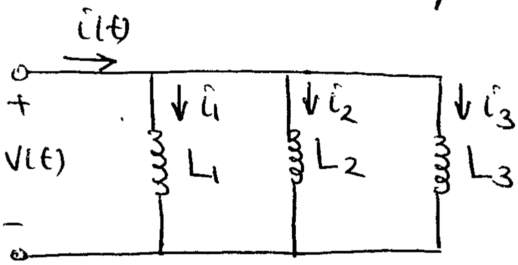
$L_{eq} = L_1 + L_2 + L_3$

In general

$$L_{eq} = \sum_{i=1}^{n} L_i = L_1 + L_2 + \dots + L_n$$

Inductance In Parallel

Consider the following circuit



$$i_1 = \frac{1}{L_1}\int_{t_0}^{t} v(t)\,dt + i_1(t_0)$$

$$i_2 = \frac{1}{L_2}\int_{t_0}^{t} v(t)\,dt + i_2(t_0)$$

$$i_3 = \frac{1}{L_3}\int_{t_0}^{t} v(t)\,dt + i_3(t_0)$$

KCL at the top node

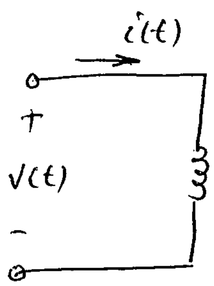
$$i(t) = i_1 + i_2 + i_3$$

$$= \frac{1}{L_1}\int_{t_0}^{t} v(t)dt + i_1(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v(t)dt + i_2(t_0) + \frac{1}{L_3}\int_{t_0}^{t} v(t)dt + i_3(t_0)$$

$$= \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)\int_{t_0}^{t} v(t)\,dt + \underbrace{i_1(t_0) + i_2(t_0) + i_3(t_0)}_{i(t_0)}$$

$$= \frac{1}{L_{eq}}\int_{t_0}^{t} v(t)\,dt + i(t_0)$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \implies L_{eq} = \frac{1}{\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}}$$



$$L_{eq} = \frac{1}{1/L_1 + 1/L_2 + 1/L_3}$$

In general

$$\frac{1}{L_{eq}} = \sum_{i=1}^{n} \frac{1}{L_i} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}$$

Question:

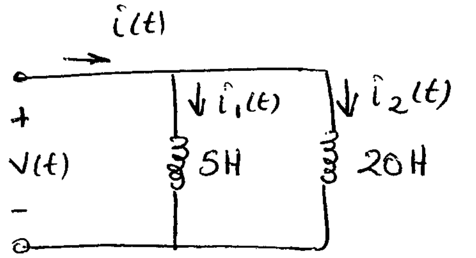
The initial values of i_1 and i_2 in the following circuit are -2 and $4\,A$, respectively. The voltage at the terminals of the parallel inductors for $t \geq 0$ is $-40e^{-5t}$ V.

(a) If parallel inductors are replaced by a single inductor, what is its inductance?

(b) What is the initial current and its reference direction in the equivalent inductor?

(c) use the equivalent inductor to find $i(t)$

(d) Find $i_1(t)$ and $i_2(t)$ and show that $i(t) = i_1(t) + i_2(t)$



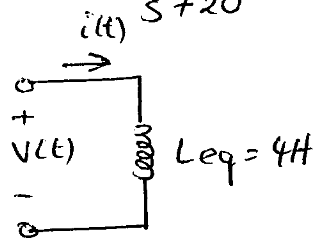
Solution:

$i_1(0) = -2A$

$i_2(0) = 4A$

$v(t) = -40e^{-5t}$ V $\quad t \geq 0$

(a) $\quad L_{eq} = \dfrac{(5)(20)}{5+20} = 4H$



(b) $\quad i(0) = i_1(0) + i_2(0) = -2 + 4 = 2A$

(c) $$i(t) = \frac{1}{L_{eq}} \int_0^t v(t)\,dt + i(0)$$

$$= \frac{1}{4} \int_0^t (-40e^{-5t})\,dt + 2$$

$$= \frac{-40}{4(-5)} \, e^{-5t} \Big|_0^t + 2$$

$$= 2(e^{-5t} - e^{0}) + 2$$

$$= 2e^{-5t} - 2 + 2$$

$$= 2e^{-5t} \quad A \qquad t \geq 0$$

(d) $$i_1(t) = \frac{1}{L_1} \int_0^t v(t)\,dt + i_1(0)$$

$$= \frac{1}{5} \int_0^t (-40e^{-5t})\,dt + (-2)$$

$$= \frac{-40}{5(-5)} \, e^{-5t} \Big|_0^t - 2$$

$$= 1.6(e^{-5t} - e^{0}) - 2$$

$$= 1.6e^{-5t} - 3.6 \quad A \qquad t \geq 0$$

$$i_2(t) = \frac{1}{L_2} \int_0^t v(t)\,dt + i_2(0)$$

$$= \frac{1}{20} \int_0^t (-40e^{-5t})\,dt + (4)$$

$$= \frac{-40}{20(-5)} \, e^{-5t} \Big|_0^t + 4$$

$$= 0.4(e^{-5t} - e^{-0}) + 4$$

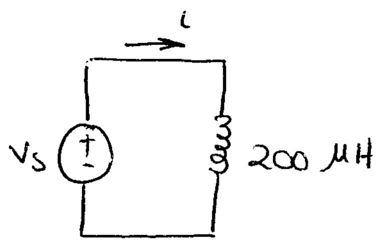
$$= 0.4e^{-5t} + 3.6 \quad A \qquad t \geq 0$$

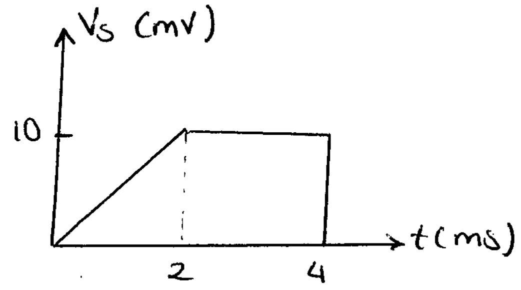
$$i(t) \stackrel{?}{=} i_1(t) + i_2(t)$$
$$= (1.6e^{-5t} - 3.6) + (0.4e^{-5t} + 3.6) = 2e^{-5t} \quad A \quad t \geq 0.$$

Question:

The voltage at the terminals of a 200 μH inductor is given in the figure. The inductor current i is known to be 0.1 A for $t<0$.

Derive the expression for i for $t \geq 0$.





Solution:

$i(0) = 0.1\ A$

$$V_s(t) = \begin{cases} 5t \ \ V & 0 \leq t \leq 2ms \\ 10 \times 10^{-3} \ \ V & 2ms \leq t \leq 4ms \\ 0 & 4\ ms \leq t \leq \infty \end{cases}$$

$$i(t) = \frac{1}{L} \int_{t_0}^{t} v(t)\,dt + i(t_0)$$

$0 \leq t \leq 2ms$:
$$i(t) = \frac{1}{200 \times 10^{-6}} \int_{0}^{t} 5t\,dt + i(0)$$

$$= 25 \times 10^{3} t^2 + 0.1 \ \ A$$

$i(2ms) = 0.2\ A$

$2ms \leq t \leq 4ms$:
$$i(t) = \frac{1}{200 \times 10^{-6}} \int_{2 \times 10^{-3}}^{t} (10 \times 10^{-3})\,dt + i(2ms)$$

$$= 50(t - 2 \times 10^{-3}) + 0.2$$

$$= 50t - 0.1 + 0.2$$

$$= 50t + 0.1 \ \ A$$

$i(4ms) = 0.3\ A$

$$4ms \leqslant t \leqslant \infty : \quad i(t) = \frac{1}{200 \times 10^{-6}} \int_{4\times 10^{-3}}^{\infty} 0 \, dt + i(4m)$$

$$i(t) = 0 + 0.3$$

$$i(t) = 0.3 \ A$$

$$i(t) = \begin{cases} 25 \times 10^{3} t^{2} + 0.1 & A \quad 0 \leqslant t \leqslant 2ms \\ 50t + 0.1 & A \quad 2ms \leqslant t \leqslant 4ms \\ 0.3 & A \quad t \geqslant 4ms \end{cases}$$