EE5121: Convex Optimization

Assignment #1

NAME: Swathi Shree Narashiman — MARKS: 60

ROLL NO: EE22B149 — DUE: Sep 28, 23:59

Problem 1 — 25 marks

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Assume the linear system $Ax = b$ is consistent and admits multiple solutions.

1. **[3 marks]** (ℓ_0-sparse solution) Among all solutions of $Ax = b$, consider the problem of maximizing sparsity (most elements are zero):

$$\min_{x\in\mathbb{R}^n} \|x\|_0 \quad \text{s.t.} \quad Ax = b,$$

where $\|x\|_0$ counts the number of nonzero components of x. Is this a convex optimization problem? If yes, prove. If no, give a counterexample.

Solution: Consider, the Lo norm, assume n to be an even number

$$x_1 = (1, 1, 1, \ldots, 1, 0, 0, 0, \ldots, 0)$$

where the first half components are 1's and the second half components are 0's. Similarly define,

$$x_2 = (-1, -1, -1, \ldots, -1, 0, 0, 0, \ldots, 0)$$

Counting the number of zeros we get,

$$||x_1||_0 = \frac{n}{2} \quad and \quad ||x_2||_0 = \frac{n}{2}$$

Consider,

$$||\frac{1}{2}x_1 + \frac{1}{2}x_2||_0 = ||0, 0, \ldots, 0||_0 = n$$

But,

$$\frac{1}{2}||x_1||_0 + \frac{1}{2}||x_2||_0 = \frac{n}{2} \leq ||\frac{1}{2}x_1 + \frac{1}{2}x_2||_0 = n$$

This example violates the convexity, hence l_0 norm is not convex. Since, the objective function is not convex, the optimization problem is not a convex optimization problem.

2. **[3 marks] (ℓ_1 proxy)** Consider the ℓ_1-relaxation of the above problem:

$$\min_{x\in\mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \quad Ax = b,$$

where $\|x\|_1 = \sum_{i=1}^n |x_i|$. Is this a convex optimization problem? If yes, prove. If no, give a counterexample.

Solution:

Yes, the minimization problem defined above is a convex optimization problem.

The objective function :
Let,

$$f(x) = ||x||_1$$

Consider two vectors x_1 *and* x_2

$$f(x_1) = ||x_1||_1 = \sum_{i=1}^{n} |x_{1i}| \quad and \quad f(x_2) = ||x_2||_1 = \sum_{i=1}^{n} |x_{2i}|$$

Let $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) = ||\lambda x_1 + (1-\lambda)x_2||_1 = \sum_{i=1}^{n} |\lambda x_{1i} + (1-\lambda)x_{2i}|$$

The modulus function follows the following property (triangle inequality in 1 D),

$$|a+b| \leq |a| + |b|$$

Therefore,

$$f(\lambda x_1 + (1-\lambda)x_2) = \sum_{i=1}^{n} |\lambda x_{1i} + (1-\lambda)x_{2i}| \leq \sum_{i=1}^{n} |\lambda x_{1i}| + |(1-\lambda)x_{2i}| = \sum_{i=1}^{n} \lambda |x_{1i}| + (1-\lambda)|x_{2i}|$$

$$\implies f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda \sum_{i=1}^{n} |x_{1i}| + (1-\lambda) \sum_{i=1}^{n} |x_{2i}|$$

$$\implies f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

Since the choice of λ, x_1, x_2 are arbitrary, the function is convex.

Constraints and Constraint Set : $Ax = b$ is an affine trasnformation, hence it is convex. The Constaint set R^n is also convex.

Therefore, the given optimization problem is a convex optimization problem.

3. **[5 marks] (ℓ_1 as a linear program)** Show that the problem in (b) can be written as a linear program via auxiliary variables $u \in \mathbb{R}^n$:

$$\min_{x,u} \mathbf{1}^T u \quad \text{s.t.} \quad Ax = b, \quad -u \leq x \leq u, \quad u \geq 0,$$

In particular, prove the equivalence of problem (b) and the above linear program.

Solution: Consider the simplified version of problem (b),

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} |x_i| \quad \text{s.t.} \quad Ax = b, \tag{A}$$

and its equivalent,

$$\min_{x,u} \mathbf{1}^T u \quad \text{s.t.} \quad Ax = b, \quad -u \leq x \leq u, \quad u \geq 0, \tag{B}$$

We know that,

$$-|x_i| \le x_i \le |x_i|, \qquad \text{for all } i. \tag{1}$$

Take,

$$u = \big(|x_1|, |x_2|, \ldots, |x_n|\big), \qquad \text{for some } n \in \mathbb{N}. \tag{2}$$

$$\implies \sum_{i=1}^{n} |x_i| = \mathbf{1}^T u \tag{3}$$

From 1 and 2 we can say,

$$-u \le x \le u \quad and \quad u \ge 0 \tag{4}$$

Assume that A^* be the optimal value of A and B^* be the optimal value of B.

For any feasible solution x to A, since the objective functions are equal as shown in (3). Take $u = |x|$, then (x, u) is a feasible solution B. Since, every feasible x in A also has a feasible u in B, takinf infimum for both we get,

$$B^* \le A^* \tag{5}$$

For any feasible u in B, by constraints we know that,

$$u \ge x \;\; and \;\; u \ge -x \quad \implies u \ge |x|$$

Further simplifying we get,

$$\mathbf{1}^T u \ge \sum_{i=1}^{n} |x_i|$$

Hence, taking minimum on both sides we get,

$$B^* \ge A^* \tag{6}$$

From (5) and (6) ,

$$A^* = B^*$$

Therefore, the two optimization problems are equivalent.

4. **[7 marks] (Application)** A color-matching system measures reflectance at three wavelengths (R, G, B). You can synthesize a target reflectance vector by mixing three laboratory pigments P_1, P_2, P_3. The measured response of each pigment at the three wavelengths is the column of a 3×3 matrix A; the target reflectance is $b \in \mathbb{R}^3$. Because two pigments have overlapping spectra, the responses are linearly dependent (so A is singular).

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 0.2 \\ 0.8 \\ 1.0 \end{bmatrix}.$$

$Ax = b$ is feasible and has many solutions.

(a) Pose the fewest-pigments formulation as a sparse recovery problem and solve.

Solution:
Optimization Problem

$$\min_{x\in\mathbb{R}^n} \|x\|_0 \quad \text{s.t.} \quad Ax=b,\ \ x\geq 0,$$

Solution
Consider the augmented matrix as follows,

$$A|b = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0.2 \\ 0 & 1 & 1 & 0.8 \\ 1 & 1 & 2 & 1.0 \end{array}\right]$$

Perform thw row operation $R_3 \leftarrow R_3 - R_2 - R_1$

$$\implies A|b = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0.2 \\ 0 & 1 & 1 & 0.8 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0.2 \\ 0 & 1 & 1 & 0.8 \end{array}\right]$$

This means that the rank of the matrix is 2 and therefore the system has infinitely many solutions.
Let,

$$x_3 = t \implies x_1 = 0.2 - t \quad and \quad x_2 = 0.8 - t$$

Hence,

$$x = (0.2-t, 0.8-t, t)$$

Therefore, the support of the solution set for the above equation is 2. For the solutions to be quantitatively meaninful,

$$t \geq 0 \quad and \quad t \leq 0.2$$

Trying to maximise the sparsity of the solution set, we get two solutions,

$$t = 0 \implies x = (0.2, 0.8, 0)$$

$$t = 0.2 \implies x = (0, 0.6, 0.2)$$

(b) Solve with CVXPY and report: the optimizer $x^\star$, its support $\{i : x_i^\star \neq 0\}$.

Solution:
Find attached the link to the code files: Collab

$$\mathbf{x}^* = \begin{bmatrix} 5.07929345 \times 10^{-7} \\ 0.600000510 \\ 0.199999490 \end{bmatrix}$$

Support (1-based indices): $\{2,3\}$

Proof : l-1 can be the surrogate objective function for non-convex l-0 norm
Consider two optimization problems

$$\min_{x\in\mathbb{R}^n} \|x\|_0 \quad \text{s.t.} \quad Ax=b, \tag{A}$$

$$\min_{x\in\mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \quad Ax = b, \tag{B}$$

Intutitively, minimizing l-0 norm tries to find the sparse solutions in the affine set, $Ax = b$, but it is non-convex. However, the l-1 norm is convex and tractable. It has a polytope geometry with sparse solutions located around the corners. Minimising the l-1 norm therefore retrieves these sparse solutions.
This idea is extensively used for eg: [1] uses a sparse recovery framework established by Dunaho and Huo [2], who proved the exact equivalence between the problems A and B.
Based on my reading of proofs to relax the l-0 norm to l-1 norm, I present here a short (not very rigrous) proof of how the relaxation works for sparse recovery on bounded domain.
Proposition. On the bounded domain $[-1,1]^n$, the ℓ_1 norm is the convex envelope of the ℓ_0 "norm":

$$\mathrm{conv}_{[-1,1]^n} \|x\|_0 = \|x\|_1.$$

Proof. Consider $f(x) = \|x\|_0 = \sum_{i=1}^n \mathbf{1}_{\{x_i\neq 0\}}$, which is separable. On $[-1,1]^n$, the convex envelope of a separable function is the sum of the 1D envelopes. For $g(t) = \mathbf{1}_{\{t\neq 0\}}$ on $[-1,1]$, the largest convex function $h(t)$ satisfying $h(t) \le g(t)$ is $h(t) = |t|$. Summing over all components yields

$$\mathrm{conv}_{[-1,1]^n} f(x) = \sum_{i=1}^n |x_i| = \|x\|_1.$$

□

5. **[7 marks]** Same laboratory setup, but the target reflectance arises from a slightly different device, so an exact match may not exist.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 0.2 \\ 0.8 \\ 0.95 \end{bmatrix}.$$

$Ax = b$ is infeasible.

(a) Modify the formulation intelligently to account for model error/measurement noise.

Solution:
We can assume that the target reflectances contain some errors dude to the usage of a slightly different device now,

$$b_{true} = b + N$$

Hence, we now account for the least squares approximate of x as a constraint,

$$\|Ax - b\|_2 \le \epsilon$$

or equivalently, we can incorporate this in the objective function itself as a penalization.
We can thus rewrite our optimization problem as ,

$$\min_{x\in\mathbb{R}^n} \|x\|_1 + \|Ax - b\|_2$$

Note the usage of L1 norm as the surrogate objective function for the non-convex L0 norm above.

(b) Solve the chosen convex program and report the optimizer, its support, and the achieved residual $\|Ax^{\star} - b\|_2$.

Solution:
Find attached the link to the code files: Collab
Optimal solution:

$$x^{\star} = \begin{bmatrix} 5.11 \times 10^{-11} \\ 0.55000178 \\ 0.1999992 \end{bmatrix}, \quad \text{Support of } x^{\star} = \{2, 3\}, \quad \|Ax^{\star} - b\|_2 = 0.05$$

Problem 2 **13 marks**

Let $A \in \mathbb{R}^{m \times p}$, $X \in \mathbb{R}^{p \times n}$, and $B \in \mathbb{R}^{m \times n}$. Assume the linear matrix equation $AX = B$ is consistent and admits multiple solutions (A is rank-deficient or $p > m$ so that the solution set is an affine space of positive dimension).

1. **[3 marks] (Rank minimization)**
   Among all matrices X satisfying $AX = B$, consider

$$\min_{X \in \mathbb{R}^{p \times n}} \ \text{rank}(X) \quad \text{s.t.} \quad AX = B.$$

   Is this a convex optimization problem? If yes, prove. If no, give a counterexample.

Solution:
No, the above problem is not a convex optimization problem.

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \implies Rank(X_1) = 1$$

$$X_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \implies Rank(X_2) = 1$$

$$\frac{1}{2}X_1 + \frac{1}{2}X_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \implies Rank(\frac{1}{2}X_1 + \frac{1}{2}X_2) = 2$$

Clearly,

$$Rank(\frac{1}{2}X_1 + \frac{1}{2}X_2) = 2 \geq \frac{1}{2} * 1 + \frac{1}{2} * 1 = \frac{1}{2}Rank(X_1) + \frac{1}{2}Rank(X_2)$$

Therefore, the objective function, $Rank(X)$ is not a convex function and the optimization problem is hence non-convex.

2. **[3 marks] (Nuclear norm minimization)**
   Consider the following nuclear-norm minimization problem

$$\min_{X \in \mathbb{R}^{p \times n}} \ \|X\|_* \quad \text{s.t.} \quad AX = B,$$

where $\|X\|_* = \sum_i \sigma_i(X)$ (sum of singular values). Prove that $\|X\|_*$ is convex by using the representation

$$\|X\|_* = \sup_{\|Y\|_2 \leq 1} \langle X, Y \rangle,$$

where $\|\cdot\|_2$ is the spectral norm and $\langle X, Y \rangle = \text{Trace}(X^T Y)$.

Solution: Given,

$$f(X) = \|X\|_* = \sup_{\|Y\|_2 \leq 1} Trace(X^T Y)$$

$$f(\lambda X_1 + (1-\lambda)X_2) = \|\lambda X_1 + (1-\lambda)X_2\|_* = \sup_{\|Y\|_2 \leq 1} Trace((\lambda X_1 + (1-\lambda)X_2)^T Y)$$

$$Trace((\lambda X_1 + (1-\lambda)X_2)^T Y) = Trace(\lambda X_1^T Y + (1-\lambda)X_2^T Y)$$

Due to linearity of trace,

$$Trace(A+B) = Trace(A) + Trace(B) \quad and \quad Trace(\alpha A) = \alpha * Trace(A)$$

$$\implies Trace(\lambda X_1^T Y + (1-\lambda)X_2^T Y) = Trace(\lambda X_1^T Y) + Trace((1-\lambda)X_2^T Y)$$

$$Trace(\lambda X_1^T Y + (1-\lambda)X_2^T Y) = \lambda * Trace(X_1^T Y) + (1-\lambda)Trace(X_2^T Y) \tag{1}$$

Now consider the properties of supremeum function,

$$\sup_x(\lambda f(.)) = \lambda \sup_x f(.) \tag{2}$$

$$\max(a+b) \leq max(a) + max(b)$$

$$\implies \sup_x(f+g) \leq \sup_x f + \sup_x g \tag{3}$$

From (2) and (3),

$$\sup_x(\lambda f + (1-\lambda)g) \leq \lambda \sup_x f + (1-\lambda)\sup_x g \tag{4}$$

From (1) and (4),

$$\sup_{\|Y\|_2 \leq 1} Trace((\lambda X_1 + (1-\lambda)X_2)^T Y) = \sup_{\|Y\|_2 \leq 1} \lambda * Trace(X_1^T Y) + (1-\lambda)Trace(X_2^T Y)$$

$$\sup_{\|Y\|_2 \leq 1} Trace((\lambda X_1 + (1-\lambda)X_2)^T Y) \leq \lambda \sup_{\|Y\|_2 \leq 1} Trace(X_1^T Y) + (1-\lambda) \sup_{\|Y\|_2 \leq 1} Trace(X_2^T Y)$$

$$\implies \|\lambda X_1 + (1-\lambda)X_2\|_* \leq \lambda \|X_1\|_* + (1-\lambda)\|X_2\|_*$$

Hence, the objective function in nuclear norm minimization is convex. The constraint, $AX = B$ is an affine transformation. Hence, the optimization problem is a convex optimization problem.

3. **[7 marks] Multi-experiment system identification**
You conduct $k = 5$ experiments on the same unknown, discrete-time, linear time-invariant (LTI) system, each driven by the same input sequence of length 5. Let $x_j \in \mathbb{R}^5$ denote the (length-5) FIR impulse response of the j-th experiment (e.g., different sensors/outputs or slightly different operating points). Stack the impulse responses as columns of $X = [x_1 \ x_2 \ \cdots \ x_5] \in \mathbb{R}^{5 \times 5}$. With the common input Toeplitz matrix $A \in \mathbb{R}^{5 \times 5}$ built from the

input $u = (u_0, \ldots, u_4)$, the measured output records form $B \in \mathbb{R}^{5\times 5}$ via

$$B = AX, \quad \text{where } A = \begin{bmatrix} u_0 & 0 & 0 & 0 & 0 \\ u_1 & u_0 & 0 & 0 & 0 \\ u_2 & u_1 & u_0 & 0 & 0 \\ u_3 & u_2 & u_1 & u_0 & 0 \\ u_4 & u_3 & u_2 & u_1 & u_0 \end{bmatrix}.$$

In many such settings, experiments for different sensors or different operating conditions can have similar responses, so X is (approximately) low rank. This motivates estimating X by rank minimization and understanding which experiments could be avoided in the future.

Given data: Use the fixed input $u = (1, 0.8, -0.2, 0.5, 0)$, so

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.8 & 1 & 0 & 0 & 0 \\ -0.2 & 0.8 & 1 & 0 & 0 \\ 0.5 & -0.2 & 0.8 & 1 & 0 \\ 0 & 0.5 & -0.2 & 0.8 & 1 \end{bmatrix}.$$

The measured outputs (five experiments, five time samples each) are

$$B = \begin{bmatrix} 1.0 & -1.0 & 0.0 & 0.0 & 1.0 \\ 2.8 & -0.8 & 1.0 & -1.0 & 0.8 \\ 2.4 & 1.2 & 1.8 & -1.8 & -1.2 \\ 2.9 & -1.7 & 0.6 & -0.6 & 1.7 \\ 0.4 & -1.8 & -0.7 & 0.7 & 1.8 \end{bmatrix}.$$

(a) Solve with CVXPY and report the optimizer.

Solution:
Find attached the code file here
The optimizer is given below.

$$X_{\text{opt}} = \begin{bmatrix} 1.00000003 & -1.00000001 & 9.06 \cdot e^{-9} & -9.06 \cdot e^{-9} & 1.00000001 \\ 1.99999994 & 2.47 \cdot e^{-8} & 0.999999981 & -0.999999981 & -2.47 \cdot e^{-8} \\ 1.00000008 & 0.999999978 & 1.00000003 & -1.00000003 & -0.999999978 \\ 1.99999988 & -1.99999996 & -4.09 \cdot e^{-8} & 4.09 \cdot e^{-8} & 1.99999996 \\ -1.99999985 & -5.65 \cdot e^{-8} & -0.999999952 & 0.999999952 & 5.65 \cdot e^{-8} \end{bmatrix}$$

Proof : Nuclear Norm can be the surrogate objective function for non-convex rank

Consider the two optimization problems,

$$\min_{X \in \mathbb{R}^{p\times n}} \ \text{rank}(X) \quad \text{s.t.} \quad AX = B.$$

$$\min_{X \in \mathbb{R}^{p\times n}} \ \|X\|_* \quad \text{s.t.} \quad AX = B,$$

We know that,

$$Rank = \#\ of\ non-zero\ singular\ values$$

Intutitively, rank minimization is a NP hard problem since its non-convex, however minimization of nuclear norm forces certain singular values towards zero thus minimizing rank.
This idea is used extensively for eg [4] uses nuclear norm relaxation drawing conclusions from the work of Fazel's PhD thesis [3], which proves the equivalence between problems A and B.
We can re-imagine this problem to $l-0 \longrightarrow l-1$ relaxation,

$$Rank(X) = \|\sigma(X)\|_0$$

$$\|X\|_* = \|\sigma(X)\|_1 = \sum_i |\sigma(X)_i|$$

From a similar argument we used to represent the l-0 to l-1 relaxation, we can draw the following preposition,
Proposition. The nuclear norm $\|X\|_*$ is the convex envelope of the rank function rank(X).

(b) Now replace B as $B+N$, where $N \in \mathbb{R}^{5\times5}$ and $N[i,j] \sim \mathcal{N}(0, 0.1)$ if $i = j$ and 0 if $i \neq j$. Modify your formulation to account for $AX \neq B$ and solve using CVXPY and report the optimizer.

Solution:
Here, due to the injection of noise,

$$AX \neq B$$

Therefore, we modify the optimization problem to simultaneously account for the least square estimate of X in $AX = B$,

$$\min_{X \in \mathbb{R}^{p\times n}} \ \|X\|_* + \ \|AX - B\|_F$$

Where, F denotes the Frobenius norm, simply the L2 minimization elementwise for the matrix.
Find attached the code file here

$$X_{\text{opt}} = \begin{bmatrix} 1.01778694 & -0.66891118 & 0.16566168 & -0.1265087 & 0.69761865 \\ 0.96692993 & 0.02914004 & 0.51029141 & -0.47606793 & -0.03129642 \\ 1.13564044 & 0.16575665 & 0.66916957 & -0.62958753 & -0.17409436 \\ 0.66487668 & -0.78365724 & -0.07588163 & 0.10299873 & 0.81774568 \\ -0.12530155 & -0.41795971 & -0.2860638 & 0.28348805 & 0.4365562 \end{bmatrix}$$

Problem 3 — 22 marks

Let $G = (V, E)$ be a simple undirected graph with $|V| = n$. A K-coloring assigns to each vertex $v \in V$ a color in $\{1, \ldots, K\}$ so that adjacent vertices receive different colors. The smallest K is the chromatic number $\chi(G)$.

1. **[4 marks] Feasibility via a 0–1 formulation.** Introduce variables $x_{vk} \in \{0, 1\}$ indicating

that vertex v uses color k. Consider the feasibility system for a fixed K:

$$\sum_{k=1}^{K} x_{vk} = 1 \qquad (\forall v \in V), \qquad x_{uk} + x_{vk} \leq 1 \qquad ((u,v) \in E,\ \forall k), \qquad x_{vk} \in \{0,1\}.$$

(a) Is the above constraint set convex? If yes, prove. If not, give a counterexample.

Solution: No, the constraint set is non-convex.
Consider the trivial case having 2 vertices and k=2. We can write the solutions as 2×2 matrix $X = [x_{vk}]$

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad and \quad X_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A convex combination of the above two matrices gives us ,

$$\tilde{X} = \frac{1}{2}X_1 + \frac{1}{2}X_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

We find that $\tilde{X}$ satisfies all the constraints except $x_{vk} \in \{0,1\}$, therefore, $\tilde{X}$ doesnt not belong to the constraint set.
Hence the constraint set is non-convex.

(b) If you drop the binary requirement and only require $0 \leq x_{vk} \leq 1$, is the set convex? If yes, prove. If not, give a counterexample.

Solution:
Yes, the constraint set is not convex.

Consider $X_1, X_2 \in [0,1]^{n \times k}$,

$$\sum_{k=1}^{K} x_{1vk} = 1 \qquad (\forall v \in V), \qquad x_{1uk} + x_{1vk} \leq 1 \qquad ((u,v) \in E,\ \forall k), \qquad x_{1vk} \in [0,1] \tag{I}$$

$$\sum_{k=1}^{K} x_{2vk} = 1 \qquad (\forall v \in V), \qquad x_{2uk} + x_{2vk} \leq 1 \qquad ((u,v) \in E,\ \forall k), \qquad x_{2vk} \in [0,1] \tag{II}$$

$$Let,\ \tilde{X} = \lambda X_1 + (1-\lambda) X_2$$

Constraint 1

$$\sum_{k=1}^{K} \tilde{x}_{vk} = \sum_{k=1}^{K} (\lambda x_{1vk} + (1-\lambda) x_{2vk}) = \lambda \sum_{k=1}^{K} x_{1vk} + (1-\lambda) \sum_{k=1}^{K} x_{2vk}$$

from (I) and (II),

$$\sum_{k=1}^{K} \tilde{x}_{vk} = \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1 \tag{1}$$

Constraint 2

$$Let,\ (u,v) \in E,\ \forall k, \quad \tilde{x}_{uk} + \tilde{x}_{vk} = \lambda (x_{1uk} + x_{1vk}) + (1-\lambda)(x_{2uk} + x_{2vk})$$

from (I) and (II),

$$\tilde{x}_{uk} + \tilde{x}_{vk} \leq \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1 \tag{2}$$

Constraint 3

$$0 \leq \lambda x_{1vk} \leq \lambda \qquad 0 \leq (1-\lambda) x_{2vk} \leq 1-\lambda$$

Adding both we get,

$$0 \leq \lambda x_{1vk} + (1-\lambda) x_{2vk} \leq 1 \implies 0 \leq \tilde{x}_{vk} \leq 1 \tag{3}$$

From (1), (2) and (3), we can conclude that the constraint set is convex.

2. **[8 marks] SDP relaxation (vector K-coloring).** Associate each vertex v with a unit vector u_v and let $G \succeq 0$ be the Gram matrix with $G_{uv} = u_u^\top u_v$. For a fixed $K \geq 2$, the vector K-coloring feasibility SDP is

$$\text{find } G \in \mathbb{S}^n \quad \text{s.t.} \quad G \succeq 0, \quad G_{vv} = 1 \ (\forall v \in V), \quad G_{uv} \leq -\frac{1}{K-1} \ (\forall (u,v) \in E).$$

(a) Explain in one line what it means if the set is feasible? What does it mean if it is not feasible?

Solution: The feasibility of the set implies that, each vertex can be allocated a unique unit vector such that the vertices are well seperated (simplex geometry in R^{n-1} dimensions). This further means that the graph can be coloured in at most K colors.
Infeasibility means there exists no such Gram matrix satisfying the constraints of the vector K-coloring. In other words, the vertices donot form a well seperated convex polytope . Then, at least K+1 colours are required to color the graph.

(b) Show how the above feasibility problem is equivalent to the optimization problem minimizing the edge-wise inner-product bound ρ:

$$\min_{G,\rho} \ \rho \quad \text{s.t.} \quad G \succeq 0, \ G_{vv} = 1, \ G_{uv} \leq \rho \ (\forall (u,v) \in E).$$

Solution:
Consider the two problems,
Feasibility problem

$$\text{find } G \in \mathbb{S}^n \quad \text{s.t.} \quad G \succeq 0, \quad G_{vv} = 1 \ (\forall v \in V), \quad G_{uv} \leq -\frac{1}{K-1} \ (\forall (u,v) \in E) \tag{A}$$

Minimization problem

$$\min_{G,\rho} \ \rho \quad \text{s.t.} \quad G \succeq 0, \ G_{vv} = 1, \ G_{uv} \leq \rho \ (\forall (u,v) \in E) \tag{B}$$

A $\implies$ B :
Let G be a solution to the feasibility problem

$$\implies G_{uv} \leq \frac{-1}{K-1}$$

Hence we can take this G and $\rho = \frac{-1}{K-1}$ as a feasible solution to the minimization problem. If ρ^* is the optimal solution for the minimization problem,

$$\rho^* \leq \frac{-1}{K-1} \tag{1}$$

B $\implies$ A :
If ρ^* is the optimal solution of the minimization problem,

$$G_{uv} \leq \rho^*$$

$$\rho^* \leq \frac{-1}{K-1} \implies \textit{G is a feasible solution to A} \tag{2}$$

From (1) and (2) we can conclude that,

$$\rho^* \leq \frac{-1}{K-1} \iff Feasbility\ of\ A$$

Therefore, minimizing the edge wise inner product can be a test to identify the existance of a feasible solution to the vector K coloring problem.

(c) Explain in 2 lines what you can say about the chromatic number of the graph from the solution of (2).

Solution:
If the optimal value test passes then graph admits vector K-coloring, i.e.

$$\rho^* \leq \frac{-1}{K-1} \implies \chi(G) \leq K$$

If the optimal value test fails then no vector K-coloring exists, hence,

$$\rho^* > \frac{-1}{K-1} \implies \chi(G) > K$$

3. **[10 marks] Application in your field.** Pick a real-world application that maps naturally to graph coloring (e.g., conflict-free scheduling of maintenance tasks; frequency/channel assignment with interference constraints; exam timetable preparation). Explain the application and reduce it to the formulation in (b).

Solution:
Frequency Assignment in Wireless Networks :
In a wireless communication network, nearby transmitters (e.g., cell towers, Wi-Fi access points, or radio stations) interfere with each other if they use the same frequency channel. The goal is to assign frequency channels to each transmitter so that no two interfering transmitters share the same frequency.
Reduction to Graph Coloring:

- Each transmitter corresponds to a vertex $v \in V$.

- An edge $(u, v) \in E$ is drawn if transmitters u and v interfere (i.e., they are geographically close enough that same-channel use would cause conflict).
- Assigning a frequency channel is equivalent to assigning a color to the vertex.
- The chromatic number of the graph corresponds to the minimum number of frequency channels needed for interference-free communication.

Connection to (b):
The SDP relaxation provides a vector K-coloring feasibility test:

$$\text{find } G \in \mathbb{S}^n \quad \text{s.t.} \quad G \succeq 0, \quad G_{vv} = 1 \ (\forall v \in V), \quad G_{uv} \leq -\frac{1}{K-1} \ [\forall (u,v) \in E]$$

or equivalently, the minimization problem ,

$$\min_{G,\rho} \ \rho \quad \text{s.t.} \quad G \succeq 0, \ G_{vv} = 1, \ G_{uv} \leq \rho \ (\forall (u,v) \in E)$$

$$\implies \rho^* \leq \frac{-1}{K-1} \implies \chi(G) \leq K$$

This formulation checks whether the network can be supported with K frequency channels. Feasibility means K channels suffice, infeasibility means at least $K+1$ channels are required.

References

[1] Emmanuel Candes and Terence Tao. Decoding by linear programming, 2005.

[2] D.L. Donoho and X. Huo. Uncertainty principles and ideal atomic decomposition. *IEEE Transactions on Information Theory*, 47(7):2845–2862, 2001.

[3] Maryam Fazel. *Matrix Rank Minimization with Applications*. Ph.d. dissertation, Stanford University, 2002.

[4] Benjamin Recht, Maryam Fazel, and Pablo A. Parrilo. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM Review*, 52(3):471–501, January 2010.

Collaboration

I would like to thank Aravind Ramana V (EP23B003) for the useful discussion and ideation on solving the problems in this assignment. However, the work presented in this submission is original to the best of my knowledge.