CS6170: Randomized Algorithms
Problem Set #4

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MARKS: 30
DUE: April 30, 23:59

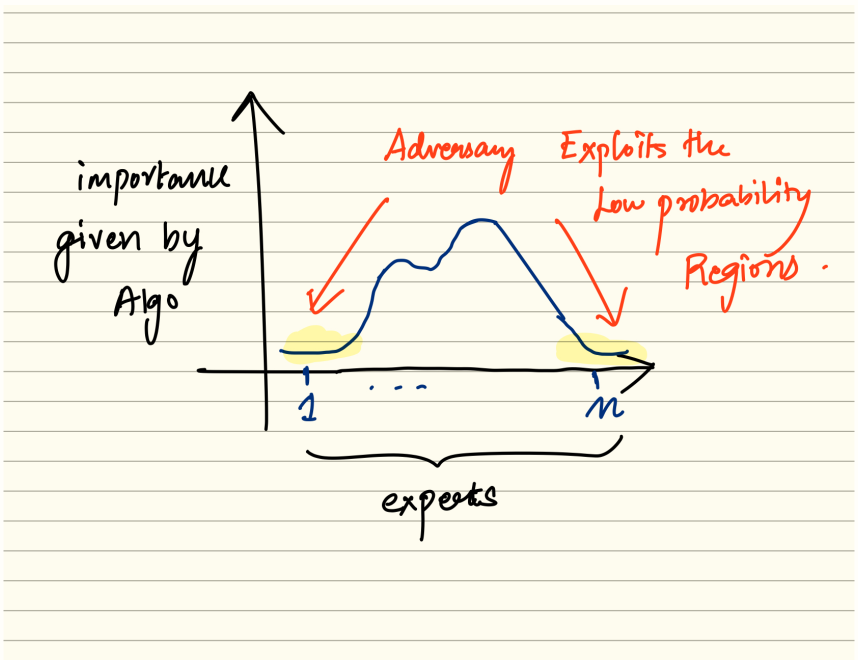
Problem 1 **3 marks**

We saw in class that if there is an omniscient expert, then there is an online decision making algorithm that has at most $\log n$ regret in the presence of n experts. Show that every deterministic algorithm will have $\Omega(\log n)$ regret if there is exactly one omniscient expert among the n experts.

Solution:
Every deterministic algorithm tries to catch hold of the omniscient expert at the earliest to minimize its regret. However we construct a worst case adversary that does the following:

1. Ask questions in a way that at each step more than half the experts give wrong ansewr
2. The agent incurrs a loss at every step before identifying the omniscient expert

If any deterministic algorithm doesnot give uniform (equal) importance to every expert, then the adversary can place the omniscient expert in the low probability region and can make the algorithm incurr loss at every step. So the best deterministic algorithm gives no bias to any expert and performs majority voting.



The deterministic algorithm conducts a majority vote among the experts and makes its decision. This means that more than half the experts are eliminated in each step.

$$\# \textit{ of experts after i steps} = \frac{n}{2^i}$$

When we identify the omniscient expert,

$$\frac{n}{2^i} = 1$$

$$\implies i = \log_2 n$$

Therefore the algorithm has to error in at least $log_2 n$ steps, therefore

$$regret = \Omega(\log n)$$

Problem 2 **5 marks**

Consider the following cat and mouse game on a connected, undirected, non-bipartite graph G. The cat and mouse start at different vertices in G. At every step, the cat and the mouse independently choose a neighbor at random and move to that vertex. If the cat and the mouse ever reach the same vertex, then the cat eats the mouse. If the graph has n vertices and m edges, show that the cat catches the mouse in $O(m^2 n)$ steps.

Solution:
Both Tom and the Jerry follow an independent random walk. In the markov chain each state represents the state of the markov chain. For random walks on undirected graphs the stationary distribution for each vertex is given by

$$\pi(v_i) = \frac{deg(v_i)}{2m}$$

Let us denote a combined markov chain for the states of both Tom and Jerry. Let (u,v) denote the state of Tom and Jerry at any instant of time,

$$Pr[(u,v) \longrightarrow (u',v')] = Pr[u \longrightarrow u'] * Pr[v \longrightarrow v']$$

Hence, the TPM of this combined chain is given by the tensor product,

$$P' = P \otimes P$$

This combined markov chain has n^2 vertices and $O(m^2)$ edges. We use the following lemma (stated without proof) to solve the problem.

Lemma: $\forall (u,v) \in E, \quad E[\textit{time to reach } v \textit{ starting from } u] \leq 2|E|$

Therefore, starting from (u,v) to reach any vertex (k,k) where Tom and Jerry can independently traverse through the graph. We can break open the graph as a spanning tree, from u to k Tom takes O(n) steps and from v to k Jerry takes O(n) steps.

Therefore, there exists a path that requires O(n) steps. Each step requires $O(m^2)$ steps (with high probability) as discussed in the previous lemma.

Hence,

$$E[\# \textit{ Steps Tom catches Jerry }] = O(n) * O(m^2) = O(m^2 n)$$

Collaborator: Nitin G

Problem 3 **6 marks**

Consider the following balls and bins game: there are n balls numbered 1 to n and two bins, a and b. Initially the balls are distributed among the two bins. At every step, a ball is chosen uniformly at random and with probability 1/2, it is moved from its current bin to the other bin. Suppose we model this process as a Markov chain where the state of the chain is the number of balls in bin a.

(a) (2 marks) Describe the transition matrix of this Markov chain.

Solution: The states of the markov chain $S = \{0, 1, 2, ...\frac{n}{2}, ...n\}$. Let X_i denote the current state of the chain, we can define the process as follows,

$$X_{i+1} = \begin{cases} X_i + 1, & \textit{if ball is chosen from bin b and moved} \\ X_i - 1, & \textit{if ball is chosen from bin a and moved} \\ X_i \quad , & \textit{if ball is not moved} \end{cases}$$

Threfore,

$$X_{i+1} = \begin{cases} X_i + 1, & \textit{with prob } \frac{n-X_i}{2n} \\ X_i - 1, & \textit{with prob } \frac{X_i}{2n} \\ X_i \quad , & \textit{with prob } \frac{1}{2} \end{cases}$$

Therefore, the T.P.M is given as $P_{(n+1)\times(n+1)}$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & ....0 & 0 & 0 \\ \frac{1}{2n} & \frac{1}{2} & \frac{n-1}{2n} & 0 & 0 & 0 & ....0 & 0 & 0 \\ 0 & \frac{2}{2n} & \frac{1}{2} & \frac{n-2}{2n} & 0 & 0 & ....0 & 0 & 0 \\ . & & & & & & & & \\ . & & & & & & & & \\ . & & & & & & & & \\ . & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & ....\frac{n-1}{2n} & \frac{1}{2} & \frac{1}{2n} \\ 0 & 0 & 0 & 0 & 0 & 0 & ....0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(b) (4 marks) Is this chain aperiodic and irreducible? Explain your answer. If so, describe what its stationary distribution is.

Solution:
The chain is periodic and irreducible.

Periodic:
The chain is periodic because, we can return back to the same state in even number of steps (there are no self loops as well). We need to take equal number of forward and backward steps to return back to the same state.

$$periodicity = 2$$

Irreduible:
From every state we can go to every other state in a finite number of steps, hence the chain is irreducible. That is, $i \longleftrightarrow j$ for all i, j.

Therefore, there exists a stationary distribution satisfying, $\pi P = \pi$

$$\pi = \begin{bmatrix} \pi(0) & \pi(1) & \pi(2) & \ldots\ldots & \pi(n) \end{bmatrix}$$

Substituting, value of P from part (a) we get,

$$\frac{1}{2}\pi(0) + \frac{1}{2n}\pi(1) = \pi(0)$$

$$\frac{1}{2}\pi(0) + \frac{1}{2}\pi(1) + \frac{2}{2n}\pi(2) = \pi(1)$$

$$\frac{n-1}{2n}\pi(1) + \frac{1}{2}\pi(2) + \frac{3}{2n}\pi(3) = \pi(2)$$

and so on.....
Solving these equations, we get

$$\pi(1) = n\pi(0)$$

$$\pi(2) = \frac{n(n-1)}{2}\pi(0)$$

$$\pi(3) = \frac{n(n-1)(n-2)}{3*2}\pi(0)$$

Therefore, for $i \geq 1$,

$$\pi(i) = \binom{n}{i}\pi(0)$$

Sum of all probabilities is 1,

$$\sum_{i=0}^{n}\binom{n}{i}\pi(0) = 1$$

$$2^n\pi(0) = 1 \implies \pi(0) = 2^{-n}$$

Therefore, the stationary distribution is given by,

$$\pi(i) = \binom{n}{i}2^{-n}$$

The derived stationary distribution is same as the probability of selecting i balls and placing them in bin a.

Problem 4 — 5 marks

In class we saw how to convert an FPRAS for counting to an FPAUS. In this question, we will try to apply a similar idea to sample a satisfying assignment for a DNF formula ϕ. Recall the algorithm for counting DNF assignment that we saw in class. Use the idea of rejection sampling to design an FPAUS for the satisfying assignment of a DNF formula.

Solution:
We define our DNF formula as follows

$$\phi = C_1 \vee C_2 \vee C_3 \vee \ldots. \vee C_m$$

where each clause is defined as

$$C_i = T_{1i} \wedge T_{2i} \wedge T_{3i} \wedge \dots T_{ki}$$

Any assignment that satisfies 1 clause is enough to satisfy the formula as well, let us define some satisfying sets below.

$$S_i = \{a \in \{0,1\}^n | C_i(a) = 1\}$$

Let,

$$S = \{(a,i) | C_i(a) = 1\} \implies |S| = \sum_i |S_i|$$

However, this set S involves double counting of certain assignments that satisfies multiple clauses. We go with a similar approach as FPRAS to sample uniformly from S but nullifying the effect of double counting using rejection sampling.

Basically to satisfy one clause C_i, we set k variables to 1 and rest n-k can take either 0 or 1. Hence,

$$|S_i| = 2^{n-k}$$

Algorithm

- Choose a clause C_i uniformly at random

$$Pr[C_i] = \frac{1}{m}$$

- Choose a random clause a^* from the set S_i

$$Pr[a^* | C_i] = \frac{1}{|S_i|}$$

- But some clauses can be over-represented in S, let us define **recurrence**

$$R(a^*) = \{C_i | a^* \in S_i\}$$

- We randomly sample $u \sim U[0,1]$, we accept the assignment a^* if

$$u \leq \frac{1}{|R(a*)|}$$

Hence, the probability of accepting an assignment a^* is given by

$$Pr[a^*] = \sum_{i \in R(a*)} Pr[C_i] * Pr[a^* | C_i] * Pr[acceptance] = \sum_{i \in R(a*)} \frac{1}{m} * \frac{1}{|S_i|} * \frac{1}{|R(a*)|}$$

$$Pr[a^*] = \frac{1}{m|R(a*)|} \sum_{i \in R(a*)} \frac{1}{|S_i|}$$

We can see that, as the recurrence increases the number of terms inside the summation increases, this is nullified by $|R(a^*)|$ in the denominator thus giving an approximate uniform sampler.

Collaborator: Nitin G

Problem 5 **5 marks**

Suppose that we have a deck 52 playing cards in new deck order. Now, as we saw in class we choose a position uniformly at random and move the card at that position to the top of the deck. We also note down the card that we have moved to the top. We keep doing this until the first instance we have noted down all the cards at least once. What is the probability that the deck of cards will still be in the new deck order?

Solution:

We can choose the same card many times, but to get the same order as before, the last time each card was chosen should be in the descending order. That is, the last time 52^{nd} card is chosen should be before the last time we choose 51^{st} which in turn should be before the last time we choose 50^{th} card and so on.

Let the initial ordering of the cards be $\{C_1, C_2, C_3, ..., C_{51}, C_{52}\}$
Based on the coupon collector problem we know that the expected numbed of shuffles (T) is given by the harmonic sum,

$$E[T] = H_{52}$$

Let us take only the last occurences of each of these cards, denote them as $\{L_1, L_2, L_3, ..., L_{52}\}$.

Since the cards are picked uniformly at random, the last occurrences are equally likely to be in any order.

$$No\ of\ permutations = 52!$$

For, the deck to retain the original order the required condition is,

$$L_{52} < L_{51} < L_{50} < .... < L_3 < L_2 < L_1$$

The process stops when the last unseen card is card C_1, this is because this should be on the top of the deck and when process ends the last unseen card occupies the top.

This is satisfied only by one permutation, hence by the symmetry argument,

$$P[Deck\ in\ same\ order] = \frac{1}{52!}$$

Problem 6 **6 marks**

A proper k-coloring of a graph $G(V, E)$ is an assignment $c : V \to [k]$ such that for all $(u, v) \in E$, $c(u) \neq c(v)$. Suppose we have the case that $k \geq \Delta + 2$ where Δ is the maximum degree of G. Using the ideas we discussed in class, show that if there is an FPAUS for proper k-colorings then there exists an FPRAS for counting proper k-colorings.

Solution:

To prove the existance of an FPRAS, we use an FPAUS. Let us define some notations below:

1. G(V, E) = The given graph with Vertex set V and Edge set E, let $|V| = n$ & $|E| = m$
2. $\Omega(G)$ = Set of all possible k colourings
3. for a given order of edges $e_1, e_2, e_3, ..., e_m$ let us define the following

$$G_m = G \quad , G_{m-1} = G_m/\{e_1\} \quad , G_{m-2} = G_{m-1}/\{e_2\} \quad , ..., \quad G_0 = \{\}$$

We know that G_0 is an empty graph so each vertex can take any of the k colours,

$$\Omega(G_0) = k^n$$

By recursion we define,

$$|\Omega(G_m)| = \frac{|\Omega(G_m)|}{|\Omega(G_{m-1})|} \frac{|\Omega(G_{m-1})|}{|\Omega(G_{m-2})|} \frac{|\Omega(G_{m-2})|}{|\Omega(G_{m-3})|} \cdots \frac{|\Omega(G_1)|}{|\Omega(G_0)|} |\Omega(G_0)|$$

If we have an FPAUS to sample uniformly from $|\Omega(G_{i-1})|$ we can use Monte carlo sampling to compute the ratio $\frac{|\Omega(G_i)|}{|\Omega(G_{i-1})|}$ if the ratio is large enough. We sample colourings from $|\Omega(G_{i-1})|$ and check if it is also a colouring for $|\Omega(G_i)|$.

The following observations can be made:

1. G_{i-1} is obtained from G_i by removing an edge e thus removing a constraint. Hence every colouring in G_i is also a colouring in G_{i-1}
2. $$|\Omega(G_{i-1})| = |\Omega(G_i)| + |\Omega(G_{i-1})/\Omega(G_i)|$$
3. $\Omega(G_{i-1})/\Omega(G_i)$ will have all the colourings that have the same colour for the vertices (u,v) constituting edge e that is, c(u)=c(v).
4. **We can define an injective map as follows:**
   Take a map M and change the colour of vertex c(v) such that it doesnt have the same colour as any of its neighbours ($k \geq \Delta + 2$), then we get a map that satisfies $\Omega(G_i)$.

   $$\Omega(G_{i-1})/\Omega(G_i) \longrightarrow \Omega(G_i)$$

   Take two colourings M and M' in $\Omega(G_{i-1})/\Omega(G_i)$, we need to prove that they are one-one. If c(u)≠c'(u) is the only mismatch in the colourings, then after changing c(v) and c'(v) we still preserve $M^* \neq M^{*\prime}$ (which are the corresponding images). If c(u)=c'(u)=c(v)=c'(v), there exists some w for which $c(w) \neq c'(w)$ and this mismatch is preserved while mapping.
   Since we have defined a one-one (injective) map,

   $$|\Omega(G_{i-1})/\Omega(G_i)| \leq |\Omega(G_i)|$$

   Hence,

   $$|\Omega(G_{i-1})| \leq 2|\Omega(G_i)| \implies \frac{|\Omega(G_i)|}{|\Omega(G_{i-1})|} \geq \frac{1}{2}$$

   Since the ratio is large enough, by monte carlo sampling if we have an approximate uniform sampler FPAUS, we can construct an FPRAS.
   Hence proved.