Antenna Array Theory

The coordinate system referenced for the array factor calculation is illustrated in Figure 1. The angle ϕ refers to azimuthal (x-y) plane which is defined with respect to the +x axis and is rotated in the x-y plane about the z axis for $0 < \phi < 2\pi$. Besides, θ refers to elevation (z) plane which is defined with respect to the +z axis and is defined for $0< \theta < \pi$ [1].

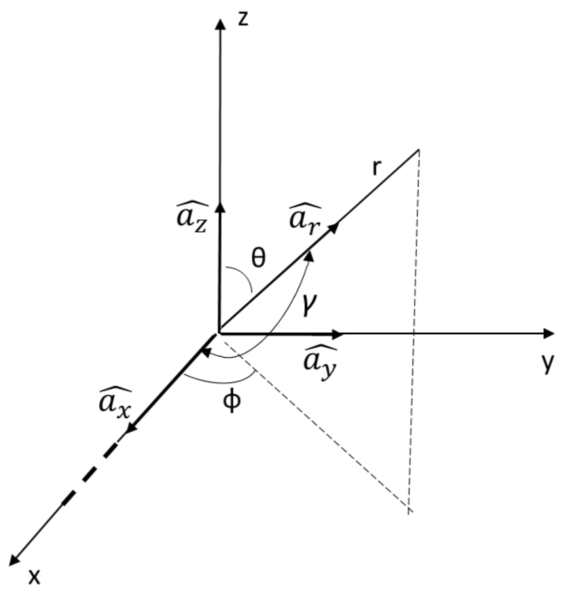

Figure 1. Antenna Coordinate System [1]

In Figure 1, $\widehat{a_r}$ is the unit vector of the position vector r (the vector along the direction of the incoming signal) and γ is the angle between the array axis and the position vector. In this figure, it is given as the angle between x-axis and r as an example.

According to Figure 1,

$$\widehat{a_r} = \sin\theta\cos\phi\widehat{a_x} + \sin\theta\sin\phi\widehat{a_y} + \cos\theta\widehat{a_z} \tag{1}$$

where $\widehat{a_x}$, $\widehat{a_y}$, $\widehat{a_z}$ are unit vectors along the relevant axis.

Hence, for an array along x-axis:

$$\widehat{a_x}\,.\widehat{a_r} = \widehat{a_x}\,(\sin\theta\cos\phi\widehat{a_x} + \sin\theta\sin\phi\widehat{a_y} + \cos\theta\widehat{a_z}) \tag{2}$$

$$\Rightarrow\ \widehat{a_x}\,.\widehat{a_r} = \sin\theta\cos\phi \tag{2.1}$$

For an array along y-axis:

$$\widehat{a_y}\,.\,\widehat{a_r} = \widehat{a_y}\,(sin\,\theta\,cos\,\phi\widehat{a_x} \,+\, sin\,\theta\,sin\,\phi\widehat{a_y} \,+\, cos\,\theta\widehat{a_z}) \tag{3}$$

$$\Rightarrow \widehat{a_y}\,.\,\widehat{a_r} = \,sin\,\theta\,sin\,\phi \tag{3.1}$$

For an array along z-axis:

$$\widehat{a_z}\,.\,\widehat{a_r} = \widehat{a_z}\,(sin\,\theta\,cos\,\phi\widehat{a_x} \,+\, sin\,\theta\,sin\,\phi\widehat{a_y} \,+\, cos\,\phi\widehat{a_z}) \tag{4}$$

$$\Rightarrow \,\widehat{a_z}\,.\,\widehat{a_r} = \,cos\,\theta \tag{4.1}$$

Linear Array Along Z-Axis

In order to give a better physical interpretation of the array theory, N- element array along a line (i.e. z axis) can be considered as shown in Figure 2. The total field of this array, assuming no mutual coupling effect between the elements [2,3], is determined by the vector addition of the fields radiated by the individual elements. Here, r_1, r_2,…,r_N represent the distance of the individual element to the far-field observation point and d_0,d_1,…d_N represent the distance of each array elements from the first (reference) element.

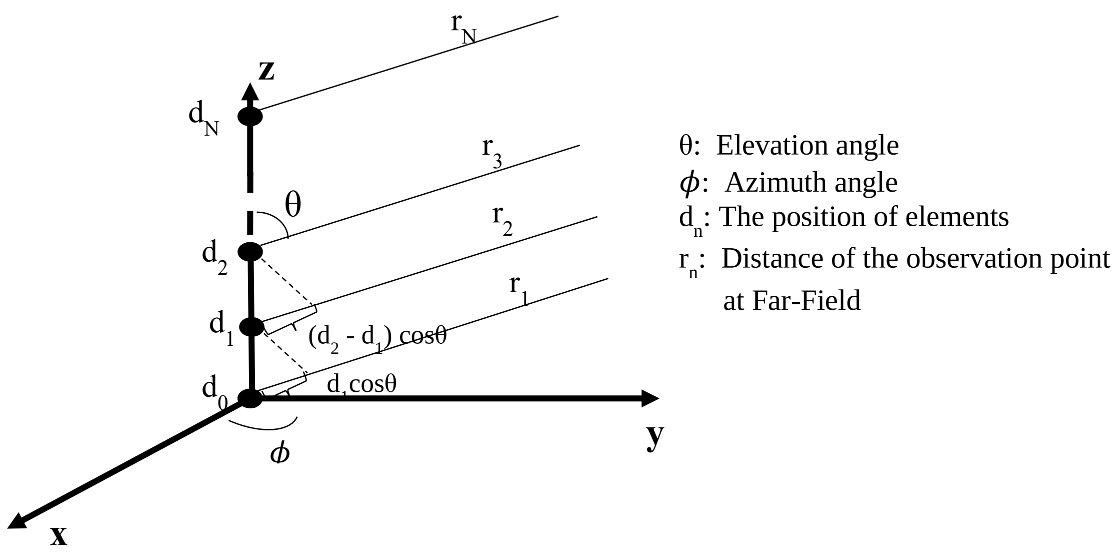

Figure 2. Linear array along z axis

Accordingly, the far electromagnetic fields of the first elements is calculated as follows [2]:

$$E_1 = I_0 \frac{e^{-jkr_1}}{4\pi r_1} \tag{5}$$

where E_1: The field of an isotropic radiator located at the origin, I_0: Excitation of the isotropic radiator, r_1: Distance of the observation point from the origin, k:wave number = 2π/λ, λ: wavelength

In addition to this, the current magnitudes of the array elements are supposed to be equal (uniform excitation) and the current on the array element located at the origin is considered as the phase reference (zero phase). Thus,

$$I_1 = I_0\ ,\quad I_2 = I_0 e^{j\beta_2}\quad , \dots ,\ I_N = I_0 e^{j\beta_N} \tag{6}$$

where β represents the difference in the phase of the signals in adjacent elements. Antenna arrays can be controlled by the relative phase β between the elements.

It is explicitly seen from Figure 2 that the path length of the wave received at element 1 is greater than the path length of element 2. This path length difference effects directly the propagation of wave whether the waves interfere constructively or destructively. Assume the array is in far field, the path length will become approximately parallel that allows to use simple trigonometry to define path difference. Since the field point is in the far field, the vectors from elements are assumed parallel. If the signals arrive from angle θ to the antenna boresight, then according to geometry in Figure 2, the relationship of individual elements' distances from the observation point is formed by the reference distance r_1 as follows:

$$r_2 = r_1 - d_1 cos\theta, \tag{7}$$

$$r_3 = r_2 - (d_2 - d_1)cos\theta \ \Rightarrow r_3 = \ r_1 - d_2 cos\theta \tag{7.1}$$

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$$r_N = r_1 - d_{N-1} cos\theta \tag{7.2}$$

It should be noted that $dcos\theta \ll$ r. Therefore, the subtraction in the denominator of equations (8,8.1,8.2) can be ignored in order to calculate the far electromagnetic fields of other elements.

Hence,

$$E_2 = I_0 e^{j\beta_2} \frac{e^{-jk(r_1 - d_1 cos\theta)}}{4\pi(r_1 - d_1 cos\theta)} \Rightarrow E_2 = E_1 e^{j(\beta_2 + kd_1 cos\theta)} \tag{8}$$

$$E_3 = I_0 e^{j\beta_3} \frac{e^{-jk(r_1 - d_2 cos\theta)}}{4\pi e^{-jk(r_1 - d_2 cos\theta)}} \Rightarrow E_3 = E_1 e^{j(\beta_3 + kd_2 cos\theta)} \tag{8.1}$$

$$\vdots$$

$$E_N = I_0 e^{j\beta_N} \frac{e^{-jk(r_1 - d_{N-1} cos\theta)}}{4\pi(r_1 - d_{N-1} cos\theta)} \Rightarrow E_N = E_1 e^{j(\beta_N + kd_{N-1} cos\theta)} \tag{8.2}$$

Then, the overall array far field is calculated using superposition:

$$E_t = \sum_{m=1}^{N} E_m \Rightarrow E_t = E_1[1 + e^{j(\beta_2 + kd_1 cos\theta)} + \cdots + e^{j(\beta_N + kd_{N-1} cos\theta)}] \tag{9}$$

It is apparent from the equation (9) that the total field of the array is equal to the field of a single element positioned at the origin multiplied by a factor which is known as the **Array Factor.**

$$AF = \left[1 + e^{j(\beta_2 + kd_1 cos\theta)} + \cdots + e^{j(\beta_N + kd_{N-1} cos\theta)}\right] \tag{10}$$

Considering eq. 10, the array factor of a linear array along z axis can also be obtained as:

$$AF = \sum_{n=0}^{N-1} e^{j(kd_n cos\theta + \beta_n)} \tag{11}$$

where k: the wave propagation constant (k=2π/λ), d_n: distance of n^{th} element from the reference. N: Total number of elements in the array, β: Phase Delay

Considering all of the above, the array factor for an array along x-axis, y-axis and z- axis are calculated as follows:

Array Factor of a linear array along x axis : $AF = \sum_{n=0}^{N-1} e^{j(kd_n sin\theta cos\phi + \beta_n)}$ (12)

Array Factor of a linear array along y axis : $AF = \sum_{n=0}^{N-1} e^{j(kd_n sin\theta sin\phi + \beta_n)}$ (13)

Array Factor of a linear array along z axis : $AF = \sum_{n=0}^{N-1} e^{j(kd_n cos\theta + \beta_n)}$ (14)

where k: the wave propagation constant (k=2π/λ), d_n: distance of n^{th} element from the reference. N: Total number of elements along the relevant axis, β: Phase Delay

In general, array factor can be written as:

$$\boxed{AF = \sum_{n=0}^{N-1} e^{j(\psi)}}$$

where ψ: Phase function, the real part of the exponential function that varies depending upon the axis. n: n^{th} element from the reference
N: Total number of elements along the axis (15)

To have a better visualisation, array factor plots are generally obtained by means of normalized array factor with dB unit. Therefore, equation (16) is used to generate the relevant plots in the user interface.

$$\boxed{AF_n = \frac{1}{N} \sum_{n=0}^{N-1} e^{j(\psi)}}$$

where ψ: Phase function, the real part of the exponential function that varies depending upon the axis. n: n^{th} element from the reference
N: Total number of elements along the axis (16)

As it is described, phase function (ψ) depends both θ and ϕ angles for an array along x- or y-axis. In this case, one of these angles must be kept constant with respect to the plane (E or H) while other is varied over an appropriate range. This means:

For array along x axis: $\psi_x = \ kd_n sin\theta cos\phi + \beta_n$

- x-y (H plane): $\theta = 90 \Rightarrow \psi_x = \ kd_n cos\phi + \beta_n$
- y-z (E plane): $\phi = 0 \quad \Rightarrow \psi_x = \ kd_n sin\theta + \beta_n$

For array along y axis: $\psi_y = \ kd_n sin\theta sin\phi + \beta_n$

- x-y (H plane): $\theta = 90 \Rightarrow \psi_y = \ kd_n sin\phi + \beta_n$
- y-z (E plane): $\phi = 90 \ \Rightarrow \psi_y = \ kd_n sin\theta + \beta_n$

It should be noted that phase delay β_n may be calculated with respect to the method of steering a phased array which can be phase shifting or True Time Delay (TTD). Accordingly,

$\beta_n = k_0 d_n \mathrm{u}$ for phase shifting

$\beta_n = k d_n \mathrm{u}$ for TTD

where u may be $cos\phi_0$, $sin\theta_0$, $sin\phi_0$ depending on the axis and plane. $k_0 = 2\pi/\lambda_0$, $k = 2\pi/\lambda$.
(λ: The wavelength where TTD is based on, λ_0: The wavelength the phase shift is based on.

Planar Arrays

In order to obtain the normalized array factor pattern of a planar array, the multiplication of the linear arrays in the relevant axes is required [2]. For instance, for a planar array composed of the elements positioned along x and y axis, the array factor will be:

$$\boxed{\begin{aligned} &AF = AF_x * AF_y \\ &\Rightarrow AF = \frac{1}{N_x} \sum_{n=0}^{N_x-1} e^{j(\psi_x)} \ * \ \frac{1}{N_y} \sum_{n=0}^{N_y-1} e^{j(\psi_y)} \end{aligned}} \qquad (17)$$

where, N_x: Number of elements along x axis, N_y: Number of elements along y axis,
$\psi_x = \ kd_n sin\theta cos\phi + \beta_n, \ \psi_y = \ kd_n sin\theta sin\phi + \beta_n$

Figure 3 illustrates the geometry of a planar array placed on x-y plane.

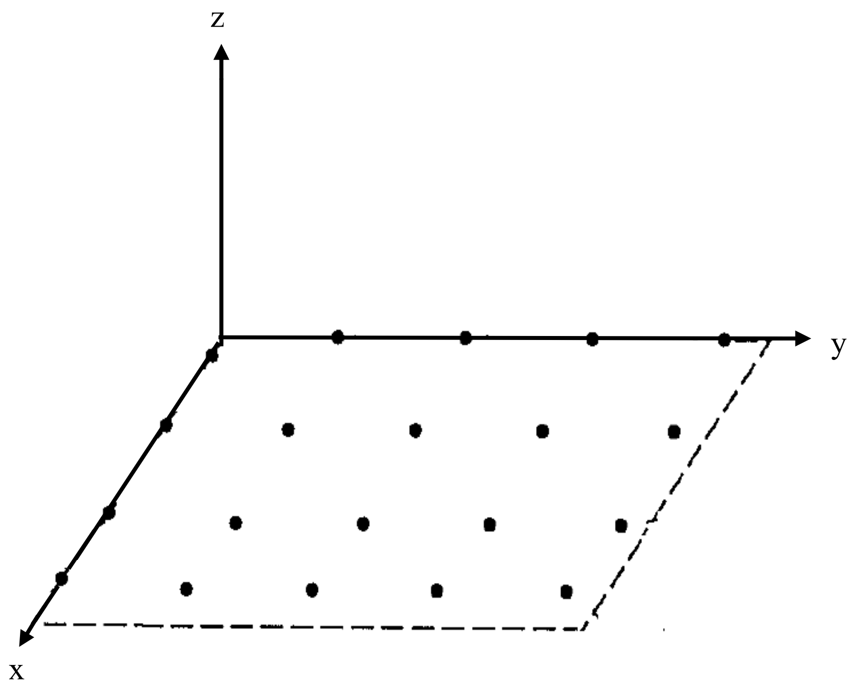
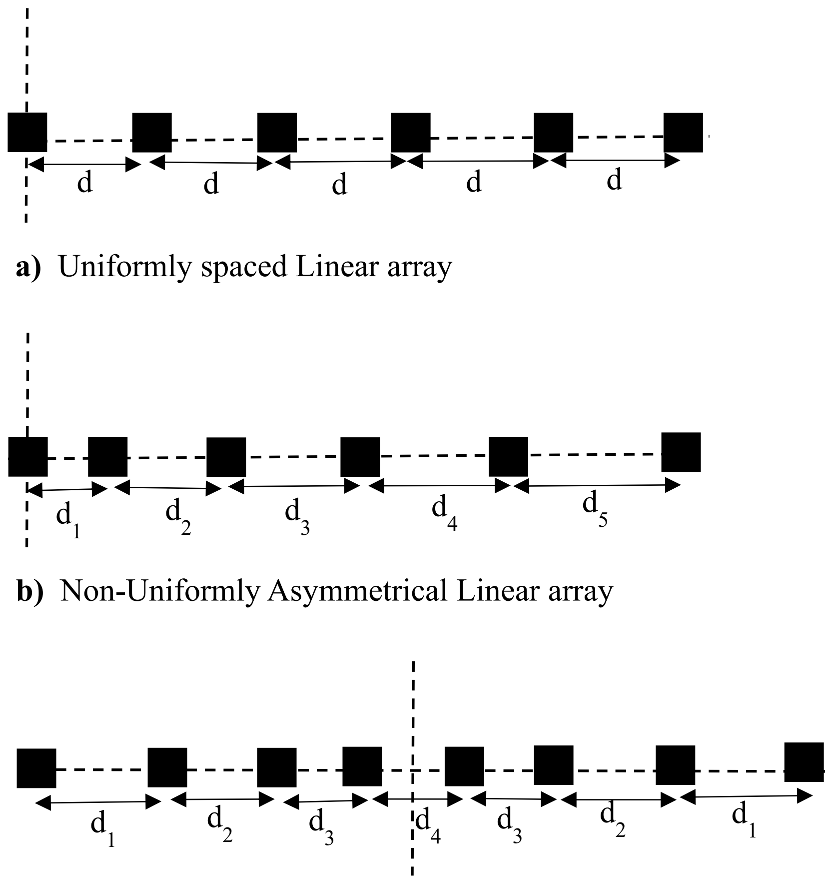


Figure 3. Two-dimensional Planar Array

Array Layout

In the user interface, it is possible to investigate the array factor pattern for three different geometrical configurations which are uniformly spaced array, non-uniformly asymmetrical spaced array and non-uniformly symmetrical array as shown by Figure 4.



a) Uniformly spaced Linear array

b) Non-Uniformly Asymmetrical Linear array

c) Non-Uniformly Symmetrical Linear array

Figure 4. Different Geometrical Configurations

The relation of each distance for non-uniform configurations are associated with '**increase rate'** which is mathematically the differences between two successive distances:

$$\text{increase rate} = d_n - d_{n-1}$$

References

[1] Connor J.D. Antenna Array Synthesis Using the Cross Entropy Method.2008

[2] C. A. Balanis. Antenna Theory: Analysis and Design (3rd edition). 2005

[3] Merrill I. Skolnik, Introduction to Radar Systems (2nd edition), 1980