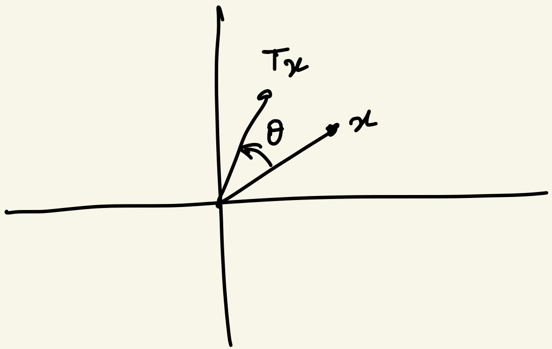
Ex: Rotation in anti-clockwise direction by an angle θ. ($\mathbb{R}^2$)



Action by matrix

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$Te_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} ; \quad Te_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

$$\det(Q - \lambda I) = 0 \Rightarrow \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} = 0$$

$$\Rightarrow (\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\Rightarrow \lambda^2 - 2\cos\theta\,\lambda + 1 = 0$$

$$\Rightarrow \lambda = \cos\theta \pm i\sin\theta$$

Ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\lambda = 1, 1$

$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\dim N_\lambda(A) =$ Geometric multiplicity of λ.

Define $N_\lambda(A) = \{ x \in \mathbb{R}^n \mid (A - \lambda I)x = 0 \} \to$ eigen space.

Ex: for a matrix $A \in \mathbb{R}^{n \times n}$, if $\lambda_1 \neq \lambda_2$ are eigenvalues of A, then v_1 & v_2 are lin. indep. where

$Av_1 = \lambda_1 v_1$

& $Av_2 = \lambda_2 v_2$

are eigenvectors corresponding to eigenvalues λ_1 & λ_2.

Assume $\alpha v_1 + \beta v_2 = 0$. To prove $\alpha = \beta = 0$.

$\underline{v_1 = \alpha v_2}$

$Av_1 = \lambda_1 v_1$

$Av_2 = \lambda_2 v_2$

//

$A \in \mathbb{R}^{n \times n}$

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of A which are all distinct.

$\Rightarrow$ the eigenvectors $v_1, v_2, \dots, v_n$ corresponding these eigenvalues are linearly independent,

$$A v_i = \lambda_i v_i \qquad \text{for } i = 1, 2, \dots, n.$$

$$A \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}_{n \times n} = \begin{bmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \\ | & | & & | \end{bmatrix}_{n \times n}$$

$$= \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}_{n \times n} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}_{n \times n}$$

$\Rightarrow \quad AV = V\Lambda \quad \Rightarrow \quad \boxed{A = V \Lambda V^{-1}}$

Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be distinct eigenvalues of $A \in \mathbb{R}^{n\times n}$ $r < n$.

$\dim N_{\lambda_1}(A) + \dim N_{\lambda_2}(A) + \cdots + \dim N_{\lambda_r}(A) \overset{??}{=} n$

(Note that $N_{\lambda_i}(A) \cap \underset{d_j}{\underset{\shortparallel}{N_{\lambda_j}(A)}} = \{0\}$ for $\lambda_i \neq \lambda_j$)

If $\sum_{i=1}^{r} \dim N_{\lambda_i}(A) = n$,

$A v_i^1 = \lambda_1 v_i^1$ $\quad i = 1, 2, \ldots, d_1$

$$A \overset{V}{\begin{bmatrix} \underbrace{v \cdots}_{d_1} & \underbrace{\quad}_{d_2} & \cdots \end{bmatrix}}_{n\times n} = V \begin{bmatrix} \overbrace{\lambda_1 \cdots \lambda_1}^{d_1} & & & \\ & \overbrace{\lambda_2 \cdots \lambda_2}^{d_2} & & \\ & & \ddots & \\ & & & \overbrace{\lambda_r \cdots \lambda_r}^{d_r} \end{bmatrix}_{n\times n}$$

$d_1 = \dim N_{\lambda_1}(A)$ $\quad d_2 = \dim N_{\lambda_2}(A)$

Ex: $\left\{ \begin{array}{l} A \in \mathbb{R}^{n \times n} \quad \text{symmetric} \\ A = V \Lambda V^T \end{array} \right\}$ Spectral decomposition.