Kernel trick :

$$J = \|Ax - b\|_2^2 + \lambda \|x - x^d\|_2^2$$

where $x^d \in \mathbb{R}^n$ is given vector.

Note that $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $x^d \in \mathbb{R}^n$ $\Big\}$ given and $\lambda > 0$

$m < n$.

(A is a wide matrix)

$$A = \begin{bmatrix} & & \end{bmatrix}_{m \times n}$$

$\tilde{A} \in \mathbb{R}^{(m+n) \times n}$ corresponding to J

$$\tilde{A} = \begin{array}{c} \underbrace{\begin{bmatrix} A \\ \\ \sqrt{\lambda}\, I \end{bmatrix}}_{n \text{ columns}} \end{array} \begin{array}{l} \} \ m \text{ rows} \\ \\ \} \ n \text{ rows} \end{array}$$

Note that $\tilde{A}$ always has full column rank.

$$\hat{x} = (A^TA + \lambda I)^{-1}(A^Tb + \lambda x^d)$$

$$= (A^TA + \lambda I)^{-1}(A^Tb + (\lambda I + A^TA)x^d - A^TA x^d)$$

$$= (A^TA + \lambda I)^{-1}A^T(b - Ax^d) + x^d$$

Claim: $(A^TA + \lambda I)^{-1}A^T = A^T(AA^T + \lambda I)^{-1}$

observe: $A^T(AA^T + \lambda I) = (A^TA + \lambda I)A^T$

Multiply by $(A^TA + \lambda I)^{-1}$ on left & $(AA^T + \lambda I)^{-1}$ on the right of the above eqn:

$$\boxed{\hat{x} = A^T(AA^T + \lambda I)^{-1}(b - Ax^d) + x^d}$$

$A = [\quad]$ small x large

$A^TA = [$ large x large $]$

$AA^T = [\,]$ small x small

Digression:

No inverse required.

$A \in \mathbb{R}^{n \times n}$ invertible & $b \in \mathbb{R}^{n}$ given.

Solve $Ax = b$

Obtain QR decomposition of A.

$A = QR$ where $Q \in \mathbb{R}^{n \times n}$ is orthogonal $(Q^T Q = Q Q^T = I)$

$R \in \mathbb{R}^{n \times n}$ is an upper triangular matrix.

$Ax = b$

$\Rightarrow QRx = b$

$\Rightarrow Rx = Q^T b$