Multi-objective least squares problem.

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2 \qquad \left.\begin{array}{l} A \in \mathbb{R}^{m \times n} \\ b \in \mathbb{R}^m \end{array}\right\} \text{ given}$$

$$\downarrow$$

$$J = \lambda_1 J_1 + \lambda_2 J_2 + \cdots + \lambda_k J_k \qquad \lambda_1, \ldots, \lambda_k > 0$$

$$J_i = \|A_i x - b_i\|_2^2 \qquad \text{for } i = 1, 2, \ldots, k$$

$$\left[\begin{array}{l} A_i \in \mathbb{R}^{m_i \times n} \\ b_i \in \mathbb{R}^{m_i} \end{array} \quad \text{for } i = 1, 2, \ldots, k \right] \leftarrow \text{ given}$$

$$\lambda_1, \ldots, \lambda_k > 0$$

We want to compute $\hat{x}$ such that

$$\hat{x} = \operatorname*{argmin}_{x \in \mathbb{R}^n} \lambda_1 J_1 + \lambda_2 J_2 + \cdots + \lambda_k J_k = \operatorname*{argmin}_{x \in \mathbb{R}^n} \left(\lambda_1 \|A_1 x - b_1\|_2^2 + \lambda_2 \|A_2 x - b_2\|_2^2 + \cdots + \lambda_k \|A_k x - b_k\|_2^2 \right)$$

Ex:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$J = \lambda_1 J_1 + \lambda_2 J_2 = \underbrace{\|A_1 x - b_1\|_2^2}_{J_1} + \underbrace{\|A_2 x - b_2\|_2^2}_{J_2} \quad \text{for } \lambda_1 = \lambda_2 = 1,$$

$$\hat{x}_{J_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\hat{x}_{J_2} = \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix} \qquad (A_2^T A_2)^{-1} A_2^T b_2 = \hat{x}_{J_2}$$

$x \in \mathbb{R}^{m_1}$, $\quad y \in \mathbb{R}^{m_2}$

$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{m_1+m_2}$ $\qquad \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_2^2 = \|x\|_2^2 + \|y\|_2^2$

$J_1 = \|A_1 x - b_1\|_2^2$, $\quad J_2 = \|A_2 x - b_2\|_2^2$, $\quad \dots$, $\quad J_k = \|A_k x - b_k\|_2^2$

$$J_1 + J_2 + \cdots + J_k = \left\| \begin{bmatrix} A_1 x - b_1 \\ A_2 x - b_2 \\ \vdots \\ A_k x - b_k \end{bmatrix} \right\|_2^2$$

$$= \left\| \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} x - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \right\|_2^2$$

$= \|\tilde{A} x - \tilde{b}\|_2^2$ $\quad$ where $\quad \tilde{A} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{pmatrix} \in \mathbb{R}^{(m_1+\cdots+m_k)\times n}$

$\tilde{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix} \in \mathbb{R}^{\sum m_i}$

If columns of $\tilde{A}$ are linearly independent, then there is a unique minimizer to the problem

$$\|\tilde{A}x - \tilde{b}\|_2^2 \quad \text{which is} \quad \hat{x} = (\tilde{A}^T\tilde{A})^{-1}\tilde{A}^T\tilde{b}.$$

$$J = \lambda_1 J_1 + \lambda_2 J_2 + \cdots + \lambda_k J_k$$

$$= \lambda_1\|A_1x - b_1\|_2^2 + \lambda_2\|A_2x - b_2\|_2^2 + \cdots + \lambda_k\|A_kx - b_k\|_2^2$$

$$= \|\sqrt{\lambda_1}A_1x - \sqrt{\lambda_1}b_1\|_2^2 + \|\sqrt{\lambda_2}A_2 - \sqrt{\lambda_2}b_2\|_2^2 + \cdots + \|\sqrt{\lambda_k}A_kx - \sqrt{\lambda_k}b_k\|_2^2$$

$$= \left\| \begin{bmatrix} \sqrt{\lambda_1}A_1 \\ \sqrt{\lambda_2}A_2 \\ \vdots \\ \sqrt{\lambda_k}A_k \end{bmatrix} x - \begin{bmatrix} \sqrt{\lambda_2}b_1 \\ \sqrt{\lambda_2}b_2 \\ \vdots \\ \sqrt{\lambda_k}b_k \end{bmatrix} \right\|_2^2$$

$$\Rightarrow \min_x J = \min_{x\in\mathbb{R}^n} \|\tilde{A}x - \tilde{b}\|_2^2 \quad \text{where} \quad \tilde{A} = \begin{bmatrix} \sqrt{\lambda_1}A_1 \\ \vdots \\ \sqrt{\lambda_k}A_k \end{bmatrix} \in \mathbb{R}^{(m_1+\cdots+m_k)\times n}$$

$$\& \quad \tilde{b} = \begin{bmatrix} \sqrt{\lambda_1}b_1 \\ \vdots \\ \sqrt{\lambda_k}b_k \end{bmatrix} \in \mathbb{R}^{m_1+\cdots+m_k}$$

If columns of $\tilde{A}$ are linearly independent, we get unique solution $\hat{x}$ as:

$$\hat{x} = (\tilde{A}^T \tilde{A})^{-1} \tilde{A}^T b$$

$$= \left(\begin{bmatrix} \sqrt{\lambda_1} A_1^T & \sqrt{\lambda_2} A_2^T & \cdots & \sqrt{\lambda_k} A_k^T \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} A_1 \\ \sqrt{\lambda_2} A_2 \\ \vdots \\ \sqrt{\lambda_k} A_k \end{bmatrix} \right)^{-1} \left(\sqrt{\lambda_1} A_1^T \; \sqrt{\lambda_2} A_2^T \cdots \sqrt{\lambda_k} A_k^T \right) \begin{pmatrix} \sqrt{\lambda_1} b_1 \\ \sqrt{\lambda_2} b_2 \\ \vdots \\ \sqrt{\lambda_k} b_k \end{pmatrix}$$

$$\hat{x} = (\lambda_1 A_1^T A_1 + \lambda_2 A_2^T A_2 + \cdots + \lambda_k A_k^T A_k)^{-1} (\lambda_1 A_1^T b_1 + \lambda_2 A_2^T b_2 + \cdots + \lambda_k A_k^T b_k)$$

Case $k=2$:

$$J = \lambda_1 J_1 + \lambda_2 J_2 \qquad \lambda_1, \lambda_2 > 0$$

$$J = J_1 + \underbrace{\frac{\lambda_2}{\lambda_1}}_{\lambda} J_2$$

$$J = J_1 + \lambda J_2 \qquad \text{for } \lambda > 0$$

$$\min_{x \in \mathbb{R}^n} J = \min_{x \in \mathbb{R}^n} (J_1 + \lambda J_2) \qquad \lambda > 0$$

$$\vec{x}(\lambda) = \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} (J_1 + \lambda J_2) \qquad \text{for } \lambda > 0$$

Ex:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$J = J_1 + \lambda J_2 = \|A_1 x - b_1\|_2^2 + \lambda \|A_2 x - b_2\|_2^2$$

$$\hat{x}(\lambda) = (A_1^T A_1 + \lambda A_2^T A_2)^{-1} (A_1^T b_1 + \lambda A_2^T b_2)$$

$$\boxed{\hat{x}(\lambda) = \frac{1}{3\lambda^2 + 4\lambda + 1} \begin{bmatrix} 2\lambda^2 + 6\lambda + 1 \\ 2\lambda^2 + \lambda \end{bmatrix}}$$

Inversion Problem / Reconstruction / control design $x \in \mathbb{R}^n$

$$y = Ax + \eta$$

$$J_1 + \lambda J_2$$

$$J_1 : \|y - Ax\|_2^2$$

Additionally (prior knowledge)

i) $J_2 = \|x\|_2^2$

ii) $J_2 = \|x - x^{prior}\|_2^2$

iii) $J_2 = \|Dx\|_2^2$ where $D = \begin{bmatrix} -1 & 1 & 0 & \cdots & & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ & & \ddots & & \ddots & \\ & & & \ddots & & \\ & & & & -1 & 1 \end{bmatrix}_{(n-1)\times n}$

Consider

$$J = J_1 + \lambda J_2$$

$$\underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \; J = \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \left(\|Ax - b\|_2^2 + \lambda \|x\|_2^2 \right)$$

$$\hat{x} = (A^T A + \lambda I)^{-1} (A^T b)$$