1) Two matrices A & B are called as similar, if there exists a matrix $P \in \mathbb{R}^{n \times n}$ s.t.

$$A = P^{-1} B P$$

2) For any two matrices A & B, non-zero eigenvalues of AB are same as non-zero eigenvalues of BA.

Let λ be a non-zero eigenvalue of AB with corresponding eigenvector v.

$$ABv = \lambda v$$

Consider $\lambda(Bv) = B(\lambda v) = B(ABv) = (BA)(Bv)$

3) $A \in \mathbb{R}^{n\times n}$; Let λ be any eigenvalue of A with eigenvector v.

$$Av = \lambda v$$

$$\lambda(Av) = A(\lambda v) = A(Av) = A^2 v$$

$$\lambda(Av) = \lambda(\lambda v) = \lambda^2 v$$

Let us assume that A has distinct eigenvalues $\lambda_1, \dots, \lambda_n$

$v_1, \dots, v_n$ are lin. indep.

for any $x \in \mathbb{R}^n$

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

$$Ax = \alpha_1\lambda_1 v_1 + \alpha_2\lambda_2 v_2 + \dots + \alpha_n\lambda_n v_n$$

for $x \neq 0$

$x, Ax, A^2x, A^3x, A^4x, \dots$

$$\frac{x^T S x}{x^T x} \leftarrow \text{Rayleigh quotient}$$

S: real symmetric matrix $(S = S^T)$

First, let us prove that the eigenvalues and eigenvectors of a real symmetric matrix are real.

$$Sx = \lambda x$$

Let $\bar{x}^T$ be the conjugate & transpose vector of $x \in \mathbb{C}^n$.

Then $\lambda \bar{x}^T x = \bar{x}^T (\lambda x) = \bar{x}^T S x \Rightarrow \lambda = \dfrac{\bar{x}^T S x}{\bar{x}^T x}$

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix} ; \quad x = \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix}, \quad \bar{x}^T = \begin{pmatrix} a_1 - ib_1 & a_2 - ib_2 \end{pmatrix}$$

$$\bar{x}^T S x = \begin{pmatrix} a_1 - ib_1 & a_2 - ib_2 \end{pmatrix} \begin{pmatrix} S_{11}(a_1 + ib_1) + S_{12}(a_2 + ib_2) \\ S_{12}(a_1 + ib_1) + S_{22}(a_2 + ib_2) \end{pmatrix}$$

For a sysmmetric matrix $\dfrac{\bar{x}^T S x}{\bar{x}^T x}$ is always real.

$\Rightarrow \lambda$ is real.

$\Rightarrow$ eigenvalues & eigenvectors of S are always real.

Let $S \in \mathbb{R}^{n\times n}$ be a symmetric matrix.

$$\max_{x \neq 0} \frac{x^T S x}{x^T x}$$

$$\frac{\partial}{\partial x_i}(x^T x) = \frac{\partial}{\partial x_i}\left(x_1^2 + \cdots + x_i^2 + \cdots + x_n^2 \right) = 2x_i = 2(x)_i$$

$$\frac{\partial}{\partial x_i}(x^T S x) = \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{n} \sum_{j=1}^{n} S_{kj}\, x_j\, x_k \right) = 2\sum_{j=1}^{n} S_{ij} x_j = 2(Sx)_i$$

$$\frac{\partial}{\partial x_i}\left(\frac{x^T S x}{x^T x}\right) = 0 \qquad \text{for} \quad i = 1, 2, \cdots, n$$

$$(x^T x)\, 2(Sx)_i - (x^T S x)\, 2(x)_i = 0 \qquad \text{for } i = 1, 2, \cdots, n$$

$$\Rightarrow \quad Sx = \left(\frac{x^T S x}{x^T x}\right) x \qquad \text{—} (*)$$

Eq^n (*) suggests that a vector $x \in \mathbb{R}^n$ which satisfies $Sx = \alpha x$ is a stationary pt.

$\Rightarrow$ (α, x) should be an eigen pair.

$\because$ S is a symmetric matrix, there are n real eigenvalues & corresponding eigenvectors.

choose the eigenvalue $\alpha_1 = \max$ of all eigenvalues. & denote by v_1 the corresponding eigenvector.

This implies that v_1 is the maximizer of the $\frac{x^T S x}{x^T x}$ and the maximum value is given by the largest eigenvalue of S (α_1).

WLOG assume that $\|v_1\|_2 = 1$. $(v_1^T v_1 = 1)$.

Step 2:

$$\max_{x \neq 0} \frac{x^T S x}{x^T x}$$

$$\text{s.t.} \quad x^T v_1 = 0$$

This step will yield second largest eigenvalue as the maximum & corresponding eigenvector (v_2) as maximizer. Clearly $v_2 \perp v_1$.

Now step III:

$$\max_{x \neq 0} \frac{x^T S x}{x^T x}$$

$$\text{s.t.} \quad x^T v_1 = 0, \quad x^T v_2 = 0$$

$$\frac{x^T S x}{x^T x} + \lambda x^T v_1$$

stationarity conditions:

$$\frac{\partial}{\partial x_i}\left(\qquad \right) = 0$$

$$\frac{\partial}{\partial \lambda}\left(\qquad \right) = 0 \leftarrow \text{constraint.}$$

Continuing these steps, we conclude that any symmetric matrix $S \in \mathbb{R}^{n \times n}$ has n real eigenvalues and orthogonal eigenvectors.

In fact $S = Q \Lambda Q^T$ where Λ is the real diagonal matrix of eigenvalues & columns of Q contain corresponding eigenvectors.