Constrained least squares.

Given A, b, c, d of appropriate sizes

$$\min_x \|Ax - b\|_2^2$$

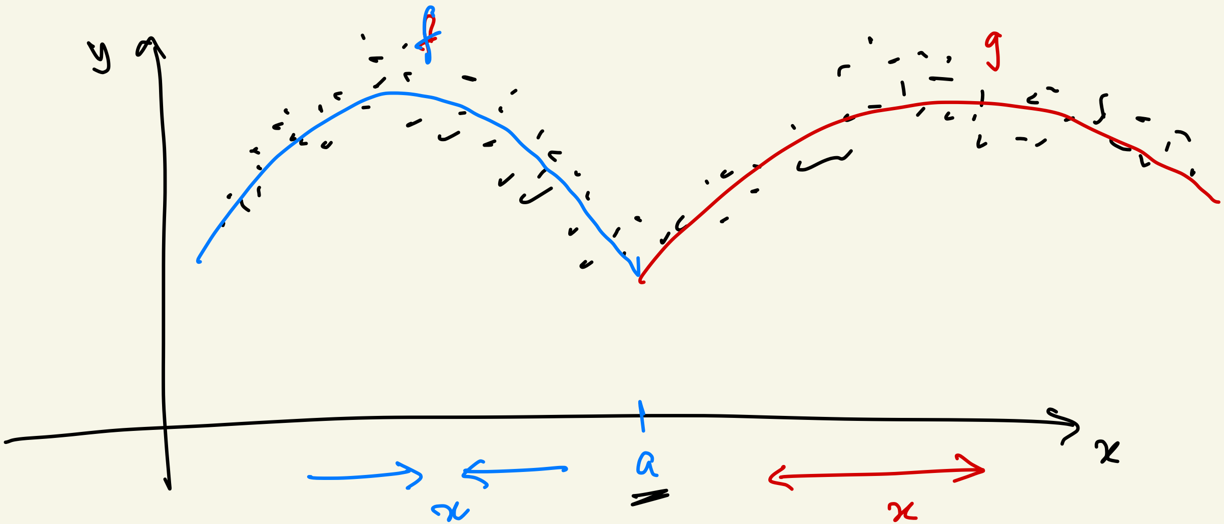
s.t. $\boxed{Cx = d}$

$\mathcal{D} = \{(x_i, y_i)\}_{i=1}^{N}$

$x_i \in \mathbb{R}$

$y_i \in \mathbb{R}$

(x_i, y_i)



M no. obs. $< a$

$N - M$ obs. $> a$

Let $x_1, x_2, \ldots, x_N$ be such that

$$x_1 < x_2 < \cdots < x_N.$$

Let $M > 0$, $M \in \mathbb{N}$ be such that

$$x_1 < x_2 < \cdots x_M \leq a \leq x_{M+1} < x_{M+2} < \cdots < x_N$$

Let degree of f be p & $\deg(g(x)) = q$.

$$f(x) = \theta_0 + \theta_1 x + \cdots + \theta_p x^p \quad ; \quad g(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_q x^q$$

$$\min_{\substack{\theta_0, \ldots, \theta_p \\ \alpha_0, \ldots, \alpha_q}} \left[\sum_{i=1}^{M} \left(y_i - \left(\theta_0 + \theta_1 x_i + \cdots + \theta_p x_i^p\right)\right)^2 + \sum_{i=M+1}^{N} \left(y_i - \left(\alpha_0 + \alpha_1 x_i + \cdots + \alpha_q x_i^q\right)\right)^2 \right]$$

s.t. $f(a) = g(a)$, $f'(a) = g'(a)$ $\longrightarrow (*)$

$$\theta_0 + \theta_1 a + \cdots + \theta_p a^p = \alpha_0 + \alpha_1 a + \cdots + \alpha_q a^q$$

$$\theta_1 + 2\theta_2 a + \cdots + p\theta_p a^{p-1} = \alpha_1 + 2\alpha_2 a + \cdots + q\alpha_q a^{q-1}$$

$$A = \begin{bmatrix} 1 & x_1 & \cdots & x_1^p & & & & \\ 1 & x_2 & \cdots & x_2^p & & 0_{M\times(q+1)} & & \\ \vdots & & & & & & & \\ 1 & x_M & \cdots & x_M^p & & & & \\ & & & & 1 & x_{M+1} & \cdots & x_{M+1}^q \\ & 0_{(N-M)\times(p+1)} & & & 1 & x_{M+2} & \cdots & x_{M+2}^q \\ & & & & \vdots & & & \\ & & & & 1 & x_N & \cdots & x_N^q \end{bmatrix} \in \mathbb{R}^{N\times(p+q+2)} \; ; \; x = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_p \\ \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_q \end{bmatrix} \in \mathbb{R}^{(p+q+2)\times 1}$$

$$b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \\ y_{M+1} \\ \vdots \\ y_N \end{bmatrix} \in \mathbb{R}^{N\times 1} \; , \quad C = \begin{bmatrix} 1 & a & a^2 & \cdots & a^p & -1 & -a & \cdots & -a^q \\ 0 & 1 & 2a & & pa^{p-1} & 0 & -1 & \cdots & -qa^{q-1} \end{bmatrix} \in \mathbb{R}^{2\times(p+q+2)} \; ,$$

$$d = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{2\times 1}$$

Piecewise polynomial fitting problem:

$$\min_x \; \|Ax - b\|_2^2$$
$$\text{s.t.} \quad Cx = d$$

Ex: Least norm solution:

Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$

$$\min \|x\|_2^2$$
$$\text{s.t.} \quad Ax = b$$

Consider the constrained least square problem as follows:

$$\min_x \|Ax - b\|_2^2$$
$$\text{s.t.} \quad Cx = d$$

The feasible set is the set $\{x : Cx = d\}$

Let $\hat{x} \in$ feasible set is a minimizer if

$$\|A\hat{x} - b\|_2^2 \leq \|Ax - b\|_2^2 \qquad \forall x \in \text{feasible set.}$$