The Quantum Theory of Light

Third Edition

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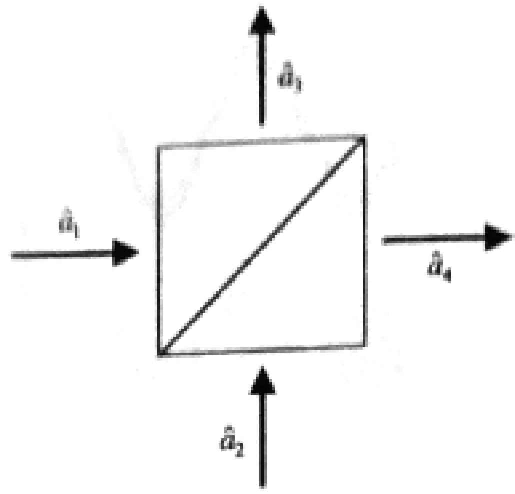

Fig. 5.15. Representation of a lossless beam-splitter showing the notation for the destruction operators associated with the input and output fields.

tized field operators. The relations between the classical input and output fields at a beam splitter are discussed in §3.2 on the basis of the conservation of the classical energy flow. These relations are ultimately determined by the boundary conditions for the electromagnetic fields at the partially reflecting and transmitting interface within the beam splitter. The boundary conditions are the same for the classical fields and for the quantum-mechanical field operators. It follows that the basic relations (3.2.12) satisfied by the reflection and transmission coefficients of a symmetric beam splitter for monochromatic incident fields, namely

$$|\mathcal{R}|^2 + |\mathcal{T}|^2 = 1 \quad \text{and} \quad \mathcal{R}\mathcal{T}^* + \mathcal{T}\mathcal{R}^* = 0, \tag{5.7.1}$$

remain the same. Similarly, the relations (3.2.1) between the classical input and output fields convert into analogous relations between the quantized field operators. For a symmetric beam splitter, these are expressed in the relations

$$\hat{a}_3 = \mathcal{R}\hat{a}_1 + \mathcal{T}\hat{a}_2 \quad \text{and} \quad \hat{a}_4 = \mathcal{T}\hat{a}_1 + \mathcal{R}\hat{a}_2 \tag{5.7.2}$$

between the destruction operators for the input and output modes. The inverse relations

$$\hat{a}_1 = \mathcal{R}^*\hat{a}_3 + \mathcal{T}^*\hat{a}_4 \quad \text{and} \quad \hat{a}_2 = \mathcal{T}^*\hat{a}_3 + \mathcal{R}^*\hat{a}_4 \tag{5.7.3}$$

are sometimes useful, and these are readily obtained from eqn (5.7.2) with the use of eqn (5.7.1). The corresponding relations between the input and output creation operators are given by the Hermitian conjugates of eqns (5.7.2) and (5.7.3).

We assume that the input fields in arms 1 and 2 are independent, with creation and destruction operators that satisfy the boson commutation relations

$$\left[\hat{a}_1, \hat{a}_1^\dagger\right] = \left[\hat{a}_2, \hat{a}_2^\dagger\right] = 1 \tag{5.7.4}$$

and

$$\left[\hat{a}_1,\hat{a}_2^\dagger\right]=\left[\hat{a}_2,\hat{a}_1^\dagger\right]=0. \tag{5.7.5}$$

Then with the use of eqns (5.7.1) and (5.7.2),

$$\left[\hat{a}_3,\hat{a}_3^\dagger\right]=\left[\mathcal{R}\hat{a}_1+\mathcal{T}\hat{a}_2,\mathcal{R}^*\hat{a}_1^\dagger+\mathcal{T}^*\hat{a}_2^\dagger\right]=|\mathcal{R}|^2+|\mathcal{T}|^2=1, \tag{5.7.6}$$

$$\left[\hat{a}_3,\hat{a}_4^\dagger\right]=\left[\mathcal{R}\hat{a}_1+\mathcal{T}\hat{a}_2,\mathcal{T}^*\hat{a}_1^\dagger+\mathcal{R}^*\hat{a}_2^\dagger\right]=\mathcal{R}\mathcal{T}^*+\mathcal{T}\mathcal{R}^*=0 \tag{5.7.7}$$

and similarly

$$\left[\hat{a}_4,\hat{a}_4^\dagger\right]=1. \tag{5.7.8}$$

Thus the output mode operators also have independent boson commutation relations. An alternative, and more fundamental, approach to the beam-splitter theory begins with the basic requirement that the output mode operators should have independent boson commutators, when the relations (5.7.1) between the reflection and transmission coefficients follow as consequences.

The input–output relations (5.7.2) provide immediate connections between the single-mode electric-field operators, as defined by eqn (5.1.3), for the four arms of the beam splitter. Thus with the arm labels denoted by subscripts on the fields, it follows that

$$\begin{cases}\hat{E}_3(\chi)=|\mathcal{R}|\hat{E}_1(\chi-\phi_{\mathcal{R}})+|\mathcal{T}|\hat{E}_2(\chi-\phi_{\mathcal{T}})\\ \hat{E}_4(\chi)=|\mathcal{T}|\hat{E}_1(\chi-\phi_{\mathcal{T}})+|\mathcal{R}|\hat{E}_2(\chi-\phi_{\mathcal{R}}),\end{cases} \tag{5.7.9}$$

where $\phi_{\mathcal{R}}$ and $\phi_{\mathcal{T}}$ are the phases of the beam-splitter reflection and transmission coefficients as defined in eqn (3.2.11). The mean values of the input and output fields satisfy the same relations as in eqn (5.7.9). It is easily shown that their variances, defined as in eqn (5.1.6), are related by

$$\begin{cases}(\Delta E_3(\chi))^2=|\mathcal{R}|^2\left(\Delta E_1(\chi-\phi_{\mathcal{R}})\right)^2+|\mathcal{T}|^2\left(\Delta E_2(\chi-\phi_{\mathcal{T}})\right)^2\\ (\Delta E_4(\chi))^2=|\mathcal{T}|^2\left(\Delta E_1(\chi-\phi_{\mathcal{T}})\right)^2+|\mathcal{R}|^2\left(\Delta E_2(\chi-\phi_{\mathcal{R}})\right)^2,\end{cases} \tag{5.7.10}$$

where the input fields are assumed to be uncorrelated. The beam splitter thus transmits the field fluctuations with appropriate coefficients $|\mathcal{R}|^2$ and $|\mathcal{T}|^2$ and changes of phase $\phi_{\mathcal{R}}$ and $\phi_{\mathcal{T}}$. These phase shifts in the noise are of course only significant for input states with phase-dependent noise, for example the squeezed states treated in §§5.5 and 5.6. The relations (5.7.10) represent a kind of conservation of quadrature noise, or fluctuations, between the beam-splitter input and output arms.

The photon number operators for the beam-splitter arms are defined as

$$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i \quad (i = 1,2,3,4) \tag{5.7.11}$$

and it follows from eqn (5.7.2) that

$$\hat{n}_3 = |\mathcal{R}|^2 \hat{a}_1^\dagger \hat{a}_1 + \mathcal{R}^* \mathcal{T} \hat{a}_1^\dagger \hat{a}_2 + \mathcal{T}^* \mathcal{R} \hat{a}_2^\dagger \hat{a}_1 + |\mathcal{T}|^2 \hat{a}_2^\dagger \hat{a}_2 \tag{5.7.12}$$

and

$$\hat{n}_4 = |\mathcal{T}|^2 \hat{a}_1^\dagger \hat{a}_1 + \mathcal{T}^* \mathcal{R} \hat{a}_1^\dagger \hat{a}_2 + \mathcal{R}^* \mathcal{T} \hat{a}_2^\dagger \hat{a}_1 + |\mathcal{R}|^2 \hat{a}_2^\dagger \hat{a}_2. \tag{5.7.13}$$

Addition of eqn (5.7.12) to eqn (5.7.13) with the use of eqn (5.7.1) gives

$$\hat{n}_3 + \hat{n}_4 = \hat{n}_1 + \hat{n}_2, \tag{5.7.14}$$

which represents photon-number conservation between the input and output arms of the beam splitter. Again, the basic relations (5.7.1) for the beam splitter reflection and transmission coefficients could be derived by requiring the validity of the conservation law (5.7.14) and indeed this is the quantum analogue of the classical energy-conservation method used to derive the relations in §3.2.

The effects of transmission through the beam splitter on the photon number fluctuations are calculated with the use of eqns (5.7.12) and (5.7.13). The expressions for the photon-number variances of the two output arms are quite complicated for general input states and we restrict attention to the common experimental arrangement where only one of the inputs is illuminated. Thus with the arm 2 input in its vacuum state, the output variances are

$$\begin{cases} (\Delta n_3)^2 = |\mathcal{R}|^4 (\Delta n_1)^2 + |\mathcal{R}|^2 |\mathcal{T}|^2 \langle n_1 \rangle \\ (\Delta n_4)^2 = |\mathcal{T}|^4 (\Delta n_1)^2 + |\mathcal{T}|^2 |\mathcal{R}|^2 \langle n_1 \rangle . \end{cases} \tag{5.7.15}$$

Each output photon-number variance has a contribution from the input variance, with appropriate scaling, plus an additional contribution proportional to the mean input photon number. This second contribution is regarded as generated by a beating of the input field in arm 1 with the vacuum field fluctuation in arm 2, or as a *partition noise* caused by the random division of the input photon stream at the beam splitter with probabilities $|\mathcal{R}|^2$ and $|\mathcal{T}|^2$ for the two output arms.

5.8 Single-photon input

The simplest application of the beam-splitter input–output relations occurs when a single photon is incident in arm 1, with arm 2 in its vacuum state. The input state is denoted

$$|1\rangle_1|0\rangle_2 = \hat{a}_1^\dagger|0\rangle, \tag{5.8.1}$$

where we have made use of the standard ladder property (4.3.33) of the harmonic oscillator states and $|0\rangle$ denotes the joint vacuum states of the beam-splitter arms. The input state is converted to the corresponding output state by use of the Hermitian conjugate of eqn (5.7.3)

$$|1\rangle_1|0\rangle_2 = \left(\mathcal{R}\hat{a}_3^\dagger + \mathcal{T}\hat{a}_4^\dagger\right)|0\rangle = \mathcal{R}|1\rangle_3|0\rangle_4 + \mathcal{T}|0\rangle_3|1\rangle_4. \tag{5.8.2}$$

This conversion of the input state to a linear superposition of the two possible output states is the basic quantum-mechanical process performed by the beam splitter. A state of the form shown on the right-hand side of eqn (5.8.2), with the property that each contribution to the superposition is a product of states for different systems (output arms), is said to be *entangled*. Note that although the state as a whole is pure, the individual output arms are not in pure states. Both product states in the superposition represent a photon in one arm and none in the other. Such superpositions occur only in quantum mechanics and they have no analogues in the classical theory. The entanglement is responsible for the importance of the beam splitter in an extensive range of experiments in quantum optics.

Consider first the effects on the beam-splitter output state of observations made on one of the output arms; for example, arm 3. According to von Neumann measurement theory [14], the state of the system *after* the observation is given by the projection of the state *before* the observation on to the state determined by the measurement. Suppose that the observation finds the output in arm 3 to be in its vacuum state $|0\rangle_3$. The state of the system before the observation is given by eqn (5.8.2) and the state after the observation is

$$N_3\langle 0|\left\{\mathcal{R}|1\rangle_3|0\rangle_4 + \mathcal{T}|0\rangle_3|1\rangle_4\right\} = N\mathcal{T}|1\rangle_4 = |1\rangle_4, \tag{5.8.3}$$

where N is a normalization constant, here equal to $1/\mathcal{T}$. In words, the state of the beam-splitter output *conditioned* on the observation of the vacuum state in arm 3 is a single photon in arm 4. This description of the effects of a measurement extends to arbitrary beam-splitter input states [15].

Now consider the electric-field expectation values for the output state on the right-hand side of eqn (5.8.2). It is easily shown, with use of the definition of the single-mode field operator in eqn (5.1.3), that the mean fields vanish in all arms of the beam splitter. The field variances are also calculated without difficulty.

Problem 5.15 Prove that

$$\begin{cases} \left(\Delta E_3(\chi_3)\right)^2 = \frac{1}{2}|\mathcal{R}|^2 + \frac{1}{4} \\ \left(\Delta E_4(\chi_4)\right)^2 = \frac{1}{2}|\mathcal{T}|^2 + \frac{1}{4}, \end{cases} \tag{5.8.4}$$

in agreement with eqn (5.7.10) when the input variances are inserted from eqns (5.1.8) and (5.2.9). Show also that the correlation between output fields is

$$\left\langle \hat{E}_3(\chi_3)\hat{E}_4(\chi_4)\right\rangle = \tfrac{1}{2}|\mathcal{R}||\mathcal{T}|\cos\left(\phi_{\mathcal{R}} - \phi_{\mathcal{T}} - \chi_3 + \chi_4\right). \tag{5.8.5}$$

This nonzero correlation is important for the formation of fringes in the Mach–Zehnder interferometer.

The mean photon numbers in the two output arms are determined by replacement of output operators by input operators with the use of eqn (5.7.3). Thus,

$$\langle n_3\rangle = {}_2\langle 0|_1\langle 1|\hat{n}_3|1\rangle_1|0\rangle_2 = {}_2\langle 0|_1\langle 1|\left(\mathcal{R}^*\hat{a}_1^\dagger + \mathcal{T}^*\hat{a}_2^\dagger\right)\left(\mathcal{R}\hat{a}_1 + \mathcal{T}\hat{a}_2\right)|1\rangle_1|0\rangle_2 = |\mathcal{R}|^2, \tag{5.8.6}$$

where the ground state property of eqn (4.3.23) is used for arm 2, and similarly

$$\langle n_4\rangle = |\mathcal{T}|^2. \tag{5.8.7}$$

Alternatively, the same results are obtained by evaluating the number operators expressed in terms of creation and destruction operators for arms 3 and 4 for the output state on the right of eqn (5.8.2).

These expressions for the mean output photon numbers resemble the corresponding classical results for the division of the input electromagnetic energy in accordance with the intensity reflection and transmission coefficients $|\mathcal{R}|^2$ and $|\mathcal{T}|^2$ respectively. However, a result very different from the classical theory is found for the correlation between output photon numbers, as measured in the quantum interpretation of the Brown–Twiss interferometer. Consider a series of identical experiments for the single input photon, in which the numbers of photons observed in arms 3 and 4 are multiplied together. The quantum-mechanical average for this product is

$$\begin{aligned}\langle n_3 n_4\rangle &= {}_2\langle 0|_1\langle 1|\hat{n}_3\hat{n}_4|1\rangle_1|0\rangle_2\\ &= {}_2\langle 0|_1\langle 1|\left(\mathcal{R}^*\hat{a}_1^\dagger + \mathcal{T}^*\hat{a}_2^\dagger\right)\left(\mathcal{R}\hat{a}_1 + \mathcal{T}\hat{a}_2\right)\left(\mathcal{T}^*\hat{a}_1^\dagger + \mathcal{R}^*\hat{a}_2^\dagger\right)\left(\mathcal{T}\hat{a}_1 + \mathcal{R}\hat{a}_2\right)|1\rangle_1|0\rangle_2\\ &= \mathcal{R}^*\mathcal{R}\mathcal{T}^*\mathcal{T}\,{}_1\langle 1|\hat{a}_1^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_1|1\rangle_1 + \mathcal{R}^*\mathcal{T}\mathcal{R}^*\mathcal{T}\,{}_2\langle 0|\hat{a}_2\hat{a}_2^\dagger|0\rangle_2\,{}_1\langle 1|\hat{a}_1^\dagger\hat{a}_1|1\rangle_1\\ &= \mathcal{R}^*\left(\mathcal{R}\mathcal{T}^* + \mathcal{T}\mathcal{R}^*\right)\mathcal{T} = 0,\end{aligned} \tag{5.8.8}$$

where eqn (5.7.1) is used. The zero average is a clear consequence of the particle-like aspect of the single photon, in that its presence in one output arm requires its absence in the other and each experimental run produces a correlation 1×0 or 0×1. By contrast, no input field that can be described by the classical theory is capable of producing a zero correlation of output intensities.

Now consider the more complicated example of a Mach–Zehnder interfero-