## **PROPOSITIONAL LOGIC:**

Week 5

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#### MODELS AND POSSIBLE WORLDS

- Logicians typically think in terms of models, which are formally structured worlds with respect to which truth can be evaluated.
- m is a model of a sentence a if a is true in m
- M(a) is the set of all models of a
- Possible worlds ~ models
  - Possible worlds: potentially real environments
  - Models: mathematical abstractions that establish the truth or falsity of every sentence

#### **ENTAILMENT**

- One sentence follows logically from another
- $a \models \beta$ 
  - $\circ$  a entails sentence  $\beta$  *if and only if*  $\beta$  is true in all worlds where a is true.
  - $\circ$  e.g.,  $x+y=4 \not= 4=x+y$
- Entailment is a relationship between sentences that is based on semantics.

### **PROPOSITIONAL LOGIC: SYNTAX**

- Propositional logic is the simplest logic
  - illustrates basic ideas
- Symbols of propositional logic
  - Logical constants
    - True and False
  - Propositional Variables (Symbols)
    - Atom
    - **■** E.g., P, Q, R
    - Each variable can have binary value
      - P = {true | false}
  - Logical Connectives
    - $\blacksquare$   $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\Rightarrow$ ,  $\Leftrightarrow$
  - Sentences
    - Made by putting these symbols together

### **LOGICAL CONNECTIVES**

- Logical Connectives
  - ¬: not
    - $\blacksquare$  ¬P: negation of P
  - $\circ$   $\wedge$  : and
    - $\blacksquare$  P  $\land$  Q, P  $\land$  (Q  $\lor$  R) : conjunction. Its parts are the **conjuncts**
    - $\blacksquare$  P  $\land$  (Q  $\lor$  R) is a conjunction of the conjuncts P and (Q  $\lor$  R)
  - $\circ \lor : or$ 
    - lacksquare P  $\lor$  Q, P  $\lor$  (Q  $\land$  R) : disjunction . Its parts are the **disjuncts**
    - ightharpoonup igh
  - $\circ \Rightarrow : implies$ 
    - $\blacksquare$  (P  $\land$  Q)  $\Rightarrow$  R : implication.
    - $\blacksquare$  (P  $\land$  Q): premise, antecedent, R: conclusion, consequent
  - $\bigcirc$   $\Leftrightarrow$  : equivalent
    - $\blacksquare$  (P  $\land$  Q)  $\Leftrightarrow$  (P  $\land$  Q) : equivalence, biconditional

### **LOGICAL CONNECTIVES**

$$\bullet \quad A \Rightarrow B = \neg A \lor B$$

$$\neg \neg A = A$$

#### **SENTENCES**

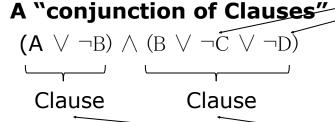
#### Sentences:

- If S is a sentence, ¬S is a sentence (negation)
- $\circ$  If S<sub>1</sub> and S<sub>2</sub> are sentences, S<sub>1</sub>  $\wedge$  S<sub>2</sub> is a sentence (conjunction)
- $\circ$  If S<sub>1</sub> and S<sub>2</sub> are sentences, S<sub>1</sub>  $\vee$  S<sub>2</sub> is a sentence (**disjunction**)
- $\circ$  If  $S_1$  and  $S_2$  are sentences,  $S_1 \rightarrow S_2$  is a sentence (implication)
- O If  $S_1$  and  $S_2$  are sentences,  $S_1 \Leftrightarrow S_2$  is a sentence (equivalence)

### **CNF and DNF**

CNF (Conjunctive Normal Form)

literals

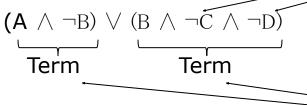


DNF (Disjunctive Normal Form)

Disjunction of literals

literals

A "disjunction of terms"



Conjunction of literals

- Horn Form (Special case of CNF)
  - Conjunction of Horn Clauses
    - Horn Clauses: a clause with at most one positive literal

### PROPOSITIONAL LOGIC: SEMANTICS

- Each model/world specifies true or false for each proposition symbol
  - E.g. P1 , P2, P3
  - With these symbols, 8 possible models, can be enumerated automatically.

P1	P2	Р3
Т	Т	Т
Т	Т	F
Т	F	Т
Т	F	F
F	Т	Т
F	Т	F
F	F	Т
F	F	F

### PROPOSITIONAL LOGIC: SEMANTICS

- $\bullet$  Rules for evaluating truth with respect to a model m:
  - ¬S is true iff S is false
  - $\circ$   $S_1 \wedge S_2$  is true iff  $S_1$  is true and  $S_2$  is true
  - $\circ$   $S_1 \vee S_2$  is true iff  $S_1$  is true or  $S_2$  is true
  - $\circ$   $S_1 \Rightarrow S_2$  is true iff  $S_1$  is false or  $S_2$  is true
  - $\circ$   $S_1 \Leftrightarrow S_2$  is true iff  $S_1 \Rightarrow S_2$  is true and  $S_2 \Rightarrow S_1$  is true
- Simple evaluates an arbitrary sentence, e.g.,  $\neg P_1 \land (P_2 \lor P_3)$

P1	P2	Р3	¬P1	(P2∨ P3)	$\neg$ P1 $\wedge$ (P2 $\vee$ P3)
Т	Т	Т	F	Т	F
Т	Т	F	F	Т	F
Т	F	Т	F	Т	F
Т	F	F	F	F	F
F	Т	Т	Т	Т	Т
F	Т	F	Т	Т	Т
F	F	Т	Т	Т	Т
F	F	F	Т	F	F

## **TRUTH TABLES FOR CONNECTIVES**

True tables for the five logical connectives

P	Q	$\neg P$	$P \wedge Q$	$P \lor Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
false	false	true	false	false	true	true
false	true	true	false	true	true	false
true	false	false	false	true	false	false
true	true	false	true	true	true	true

## **VALIDITY**

A sentence is valid if it is true in all models

O e.g ((P 
$$\vee$$
 H)  $\wedge \neg$ H)  $\Rightarrow$  P

Р	Н	P V H	<b>(</b> P ∨ H ) ∧ ¬H	$((P \lor H) \land \neg H) \Rightarrow P$
F	F	F	F	Т
F	Т	Т	F	Т
Т	F	Т	Т	Т
Т	Т	Т	F	Т

• Sentence ((P  $\vee$  H)  $\wedge \neg$ H)  $\Rightarrow$  P is Valid

## **Semantics properties**

- Deduction Theorem:
  - $\bigcirc$  KB  $\models \alpha$  if and only if (KB  $\Rightarrow \alpha$ ) is valid
  - $\bigcirc M(a) = \{ w : w \models a \}$
- Consistency, satisfiability
  - $\circ$  M(a)  $\neq$  Ø, a is consistent, satisfiable if it is true in some model
  - $\circ$  M(a) =  $\emptyset$ , a inconsistent, unsatisfiable if it is false in all models
- Validity
  - $\circ$  M(a) = Whole world, a is valid
- Equivalence
  - $\circ$  M(a) = M( $\beta$ ), a is equivalent  $\beta$
- Mutually Exclusive
  - $\bigcirc \mathsf{M}(\mathsf{a} \land \beta) = \emptyset$
  - $\circ M(a) \wedge M(\beta) = \emptyset$

# **Propositional Logic Problems**

Q. Is the Propositional Logic (PL) sentence (A  $\Leftrightarrow$  B)  $\land$  (  $\neg$ A  $\lor$  B) valid, unsatisfiable, or satisfiable?

Α	В	$A \Leftrightarrow B$	$\neg A \lor B$	$(A \Leftrightarrow B) \land (\neg A \lor B)$
T	T	T	T	T
T	F	F	F	F
F	T	F	T	F
F	F	T	T	T

A. Not Valid and Satisfiable Since the last column contains both T and F, the sentence is satisfiable.

# **Propositional Logic Problems**

Q. Prove (A  $\wedge$  B) |= (A  $\Leftrightarrow$  B) using a truth table.

Α	В	$A \wedge B$	$A \Leftrightarrow B$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	T

A. Since for each row where the next to last column is T, the last column is also T (this only occurs here for the first row), entailment is proved.