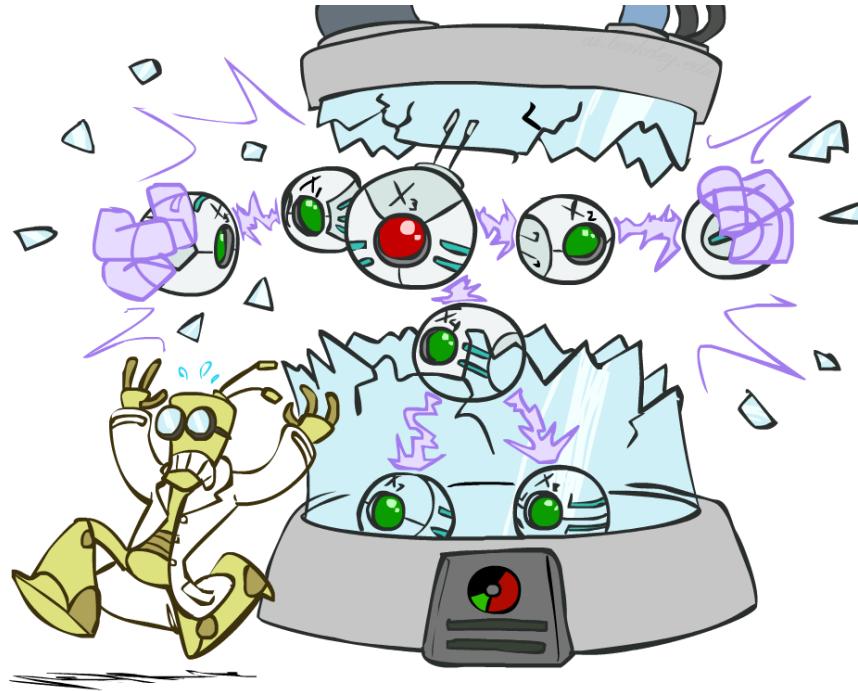


CS 161: Fundamentals of Artificial Intelligence

Bayes' Nets: Independence



Conditional Independence

- X and Y are **independent** if

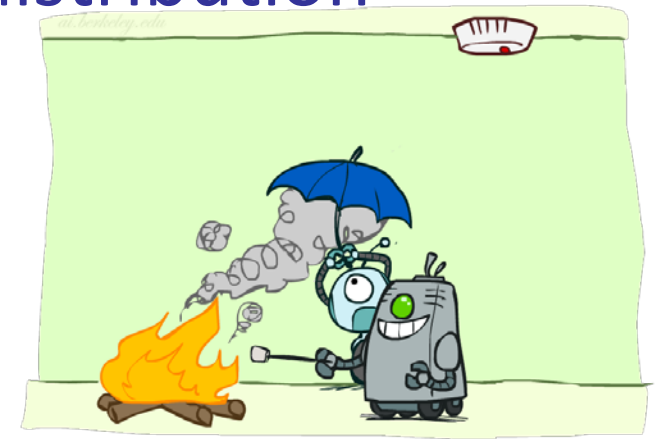
$$\forall x, y \quad P(x, y) = P(x)P(y) \quad \text{---} \rightarrow \quad X \perp Y$$

- X and Y are **conditionally independent** given Z

$$\forall x, y, z \quad P(x, y|z) = P(x|z)P(y|z) \quad \text{---} \rightarrow \quad X \perp Y|Z$$

- (Conditional) independence is a property of a distribution

- Example: $Alarm \perp Fire|Smoke$



Bayes Nets: Assumptions

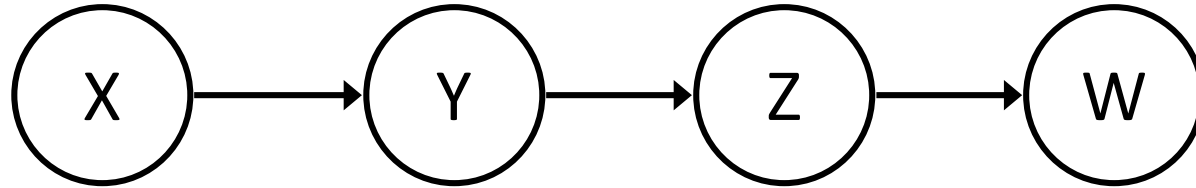
- Assumptions we are required to make to define the Bayes net when given the graph:

$$P(x_i | x_1 \cdots x_{i-1}) = P(x_i | \text{parents}(X_i))$$

- Beyond above “chain rule → Bayes net” conditional independence assumptions
 - Often additional conditional independences
 - They can be read off the graph
- Important for modeling: understand assumptions made when choosing a Bayes net graph



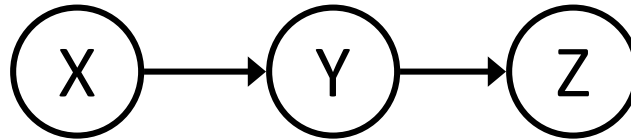
Example



- Conditional independence assumptions directly from simplifications in chain rule:
- Additional implied conditional independence assumptions?

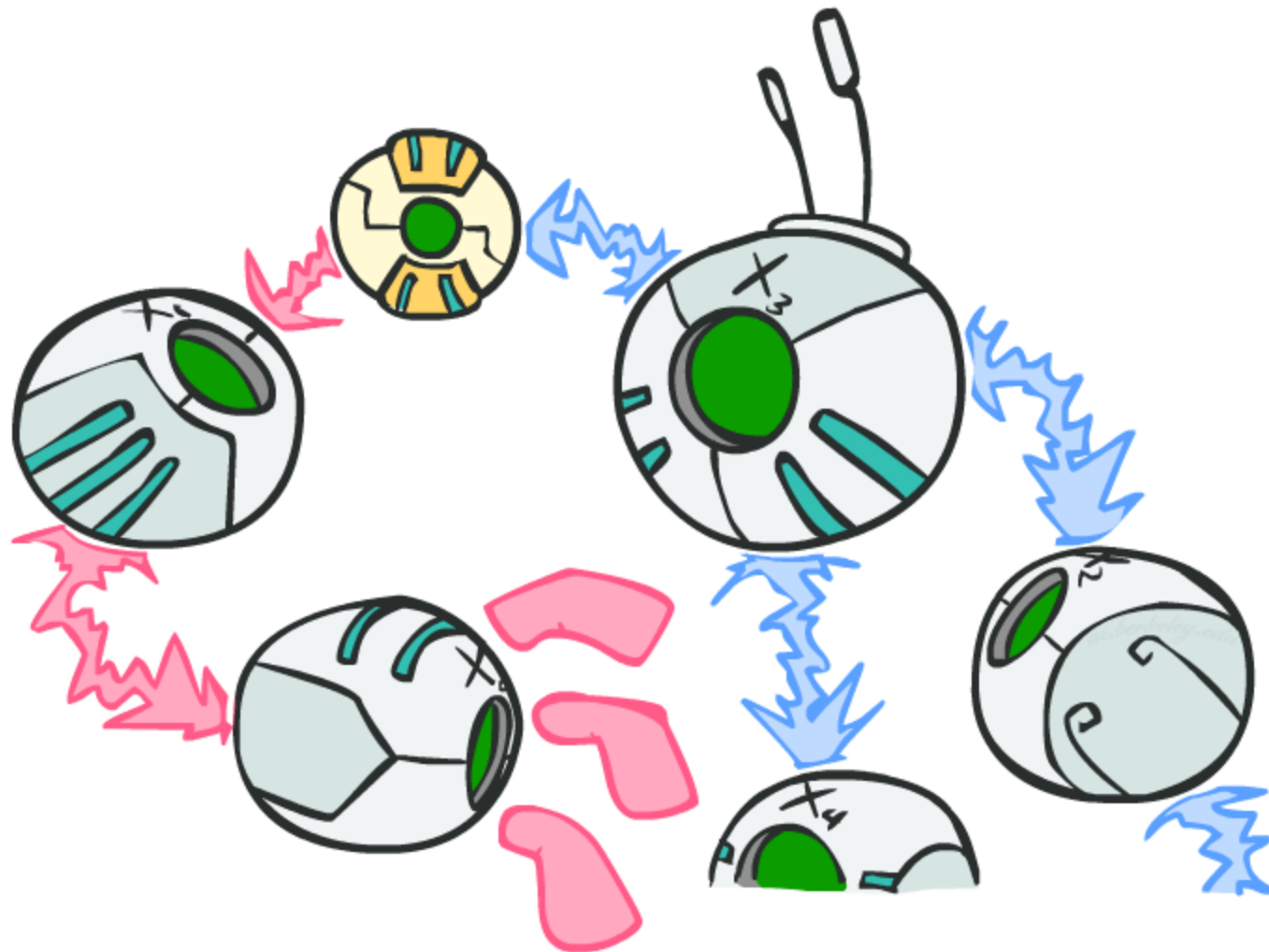
Independence in a BN

- Important question about a BN:
 - Are two nodes independent given certain evidence?
 - If yes, can prove using algebra (tedious in general)
 - If no, can prove with a counter example
 - Example:



- Question: are X and Z necessarily independent?
 - Answer: no. Example: low pressure causes rain, which causes traffic.
 - X can influence Z, Z can influence X (via Y)
 - Addendum: they *could* be independent: how?

D-separation: Outline



D-separation: Outline

- Study independence properties for triples
- Analyze complex cases in terms of member triples
- D-separation: a condition / algorithm for answering such queries

Causal Chains

- This configuration is a “causal chain”



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z ? **No!**

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.

- Example:

- Low pressure causes rain causes traffic, high pressure causes no rain causes no traffic

- In numbers:

$$P(+y \mid +x) = 1, P(-y \mid -x) = 1, \\ P(+z \mid +y) = 1, P(-z \mid -y) = 1$$

Causal Chains

- This configuration is a “causal chain”

- Guaranteed X independent of Z given Y?



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

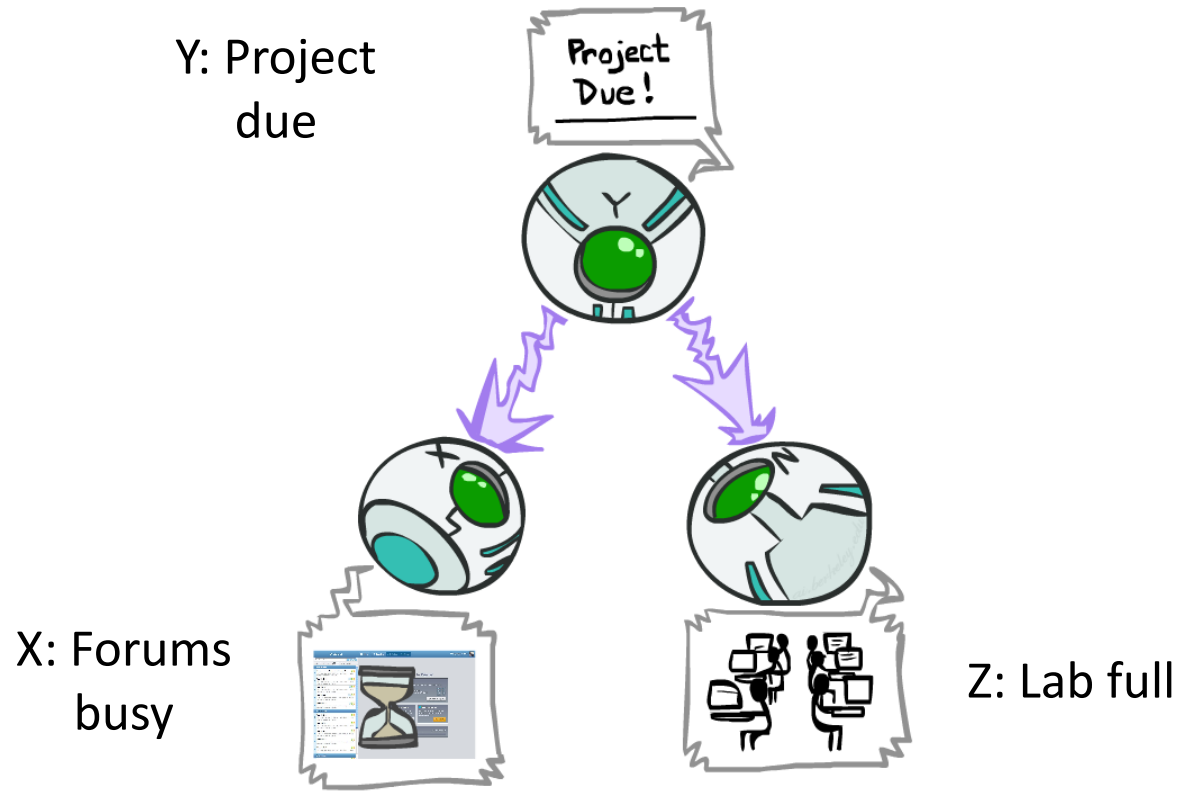
$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)} \\ &= P(z|y) \end{aligned}$$

Yes!

- Evidence along the chain “blocks” the influence

Common Cause

- This configuration is a “common cause”



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X independent of Z ? **No!**

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.

- Example:

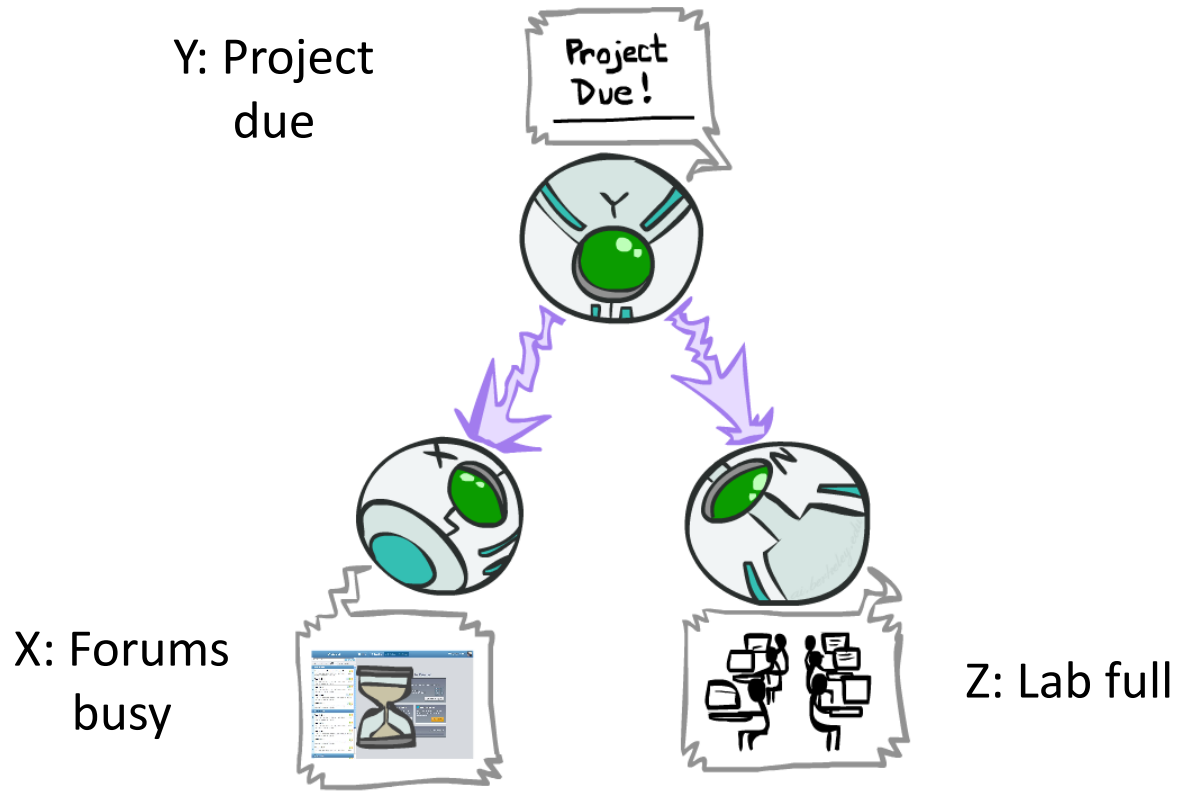
- Project due causes both forums busy and lab full

- In numbers:

$$P(+x \mid +y) = 1, P(-x \mid -y) = 1, \\ P(+z \mid +y) = 1, P(-z \mid -y) = 1$$

Common Cause

- This configuration is a “common cause”
- Guaranteed X and Z independent given Y?



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

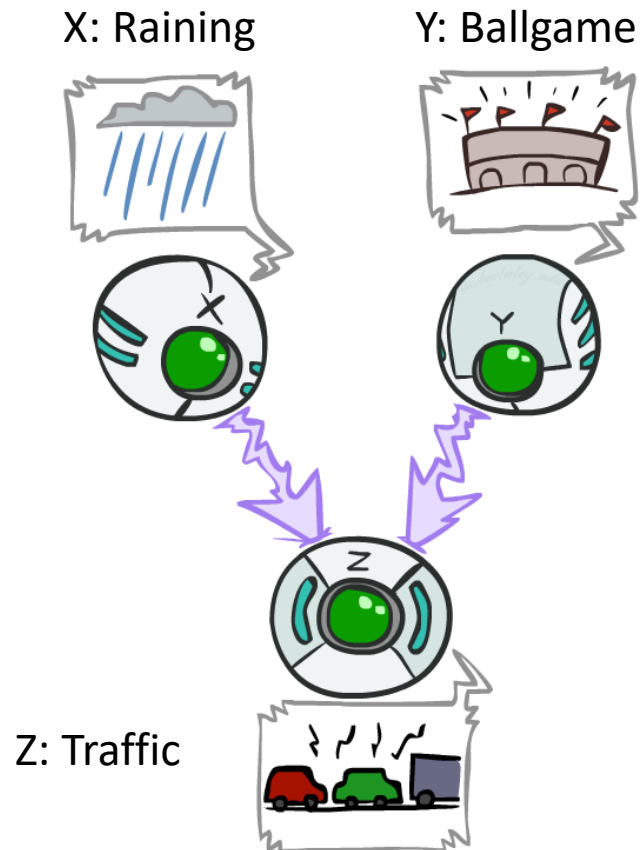
$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)} \\ &= P(z|y) \end{aligned}$$

Yes!

- Observing the cause blocks influence between effects.

Common Effect

- Last configuration: two causes of one effect (v-structures)



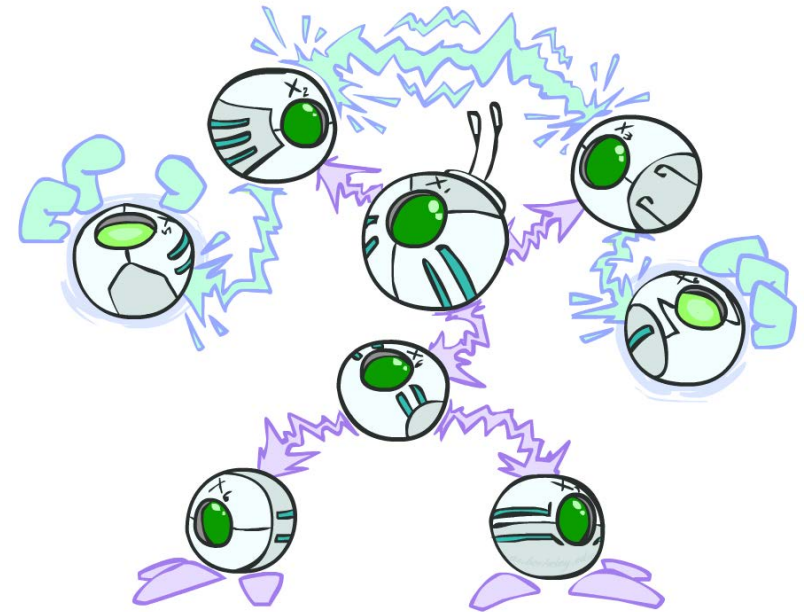
- Are X and Y independent?
 - **Yes**: the ballgame and the rain cause traffic, but they are not correlated
 - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
 - **No**: seeing traffic puts the rain and the ballgame in competition as explanation.
- **This is backwards from the other cases**
 - Observing an effect **activates** influence between possible causes.

The General Case



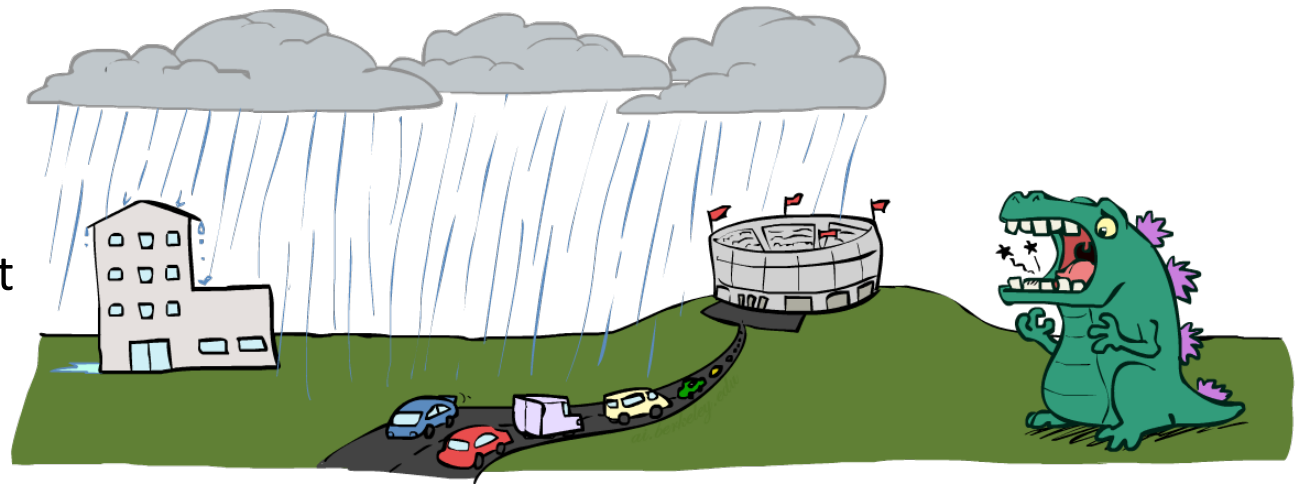
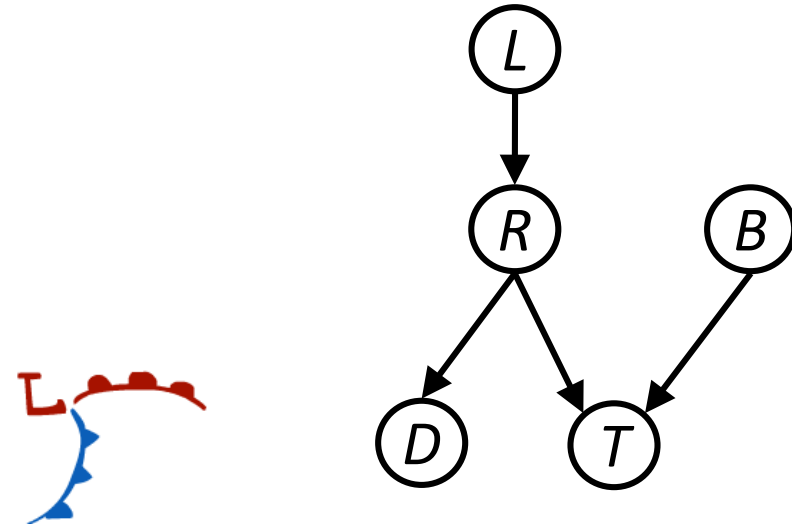
The General Case

- General question: in a given BN, are two variables independent (given evidence)?
- Solution: analyze the graph
- Any complex example can be broken into repetitions of the three canonical cases



Reachability

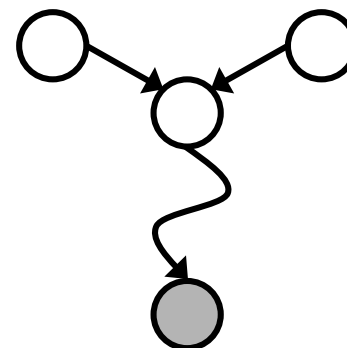
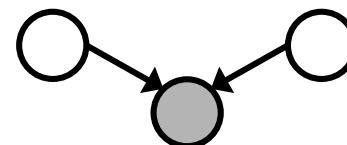
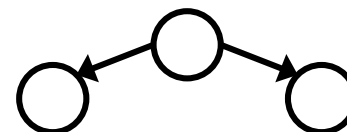
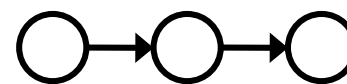
- Recipe: shade evidence nodes, look for paths in the resulting graph
- Attempt 1: if two nodes are connected by an undirected path not blocked by a shaded node, they are conditionally independent
- Almost works, but not quite
 - Where does it break?
 - Answer: the v-structure at T doesn't count as a link in a path unless "active"



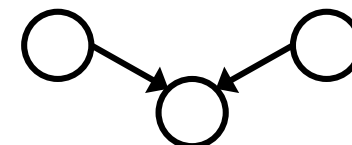
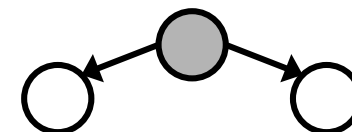
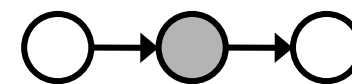
Active / Inactive Paths

- Question: Are X and Y conditionally independent given evidence variables {Z}?
 - Yes, if X and Y “d-separated” by Z
 - Consider all (undirected) paths from X to Y
 - No active paths = independence!
- A path is active if each triple is active:
 - Causal chain $A \rightarrow B \rightarrow C$ where B is unobserved (either direction)
 - Common cause $A \leftarrow B \rightarrow C$ where B is unobserved
 - Common effect (aka v-structure)
 $A \rightarrow B \leftarrow C$ where B or one of its descendants is observed
- All it takes to block a path is a single inactive segment

Active Triples



Inactive Triples



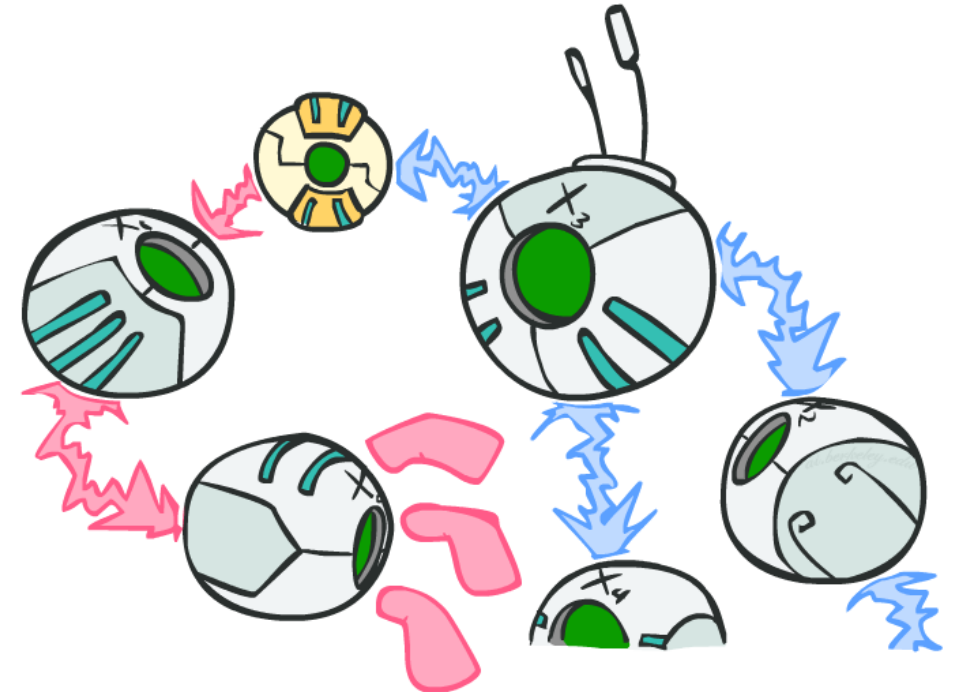
D-Separation

- Query: $X_i \perp\!\!\!\perp X_j \mid \{X_{k_1}, \dots, X_{k_n}\} ?$
- Check all (undirected!) paths between X_i and X_j
 - If one or more active, then independence not guaranteed

$$X_i \not\perp\!\!\!\perp X_j \mid \{X_{k_1}, \dots, X_{k_n}\}$$

- Otherwise (i.e. if all paths are inactive), then independence is guaranteed

$$X_i \perp\!\!\!\perp X_j \mid \{X_{k_1}, \dots, X_{k_n}\}$$



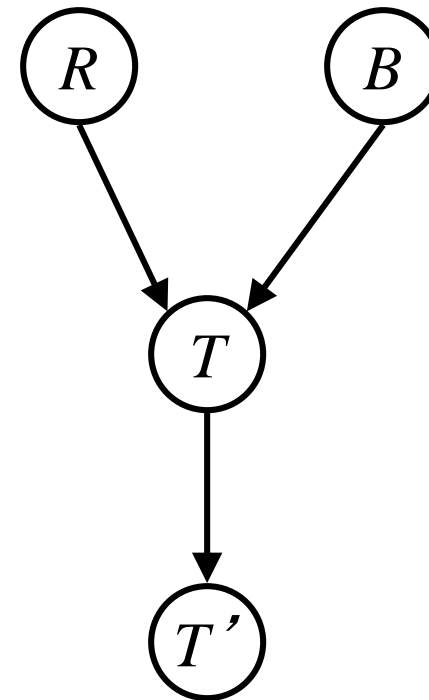
Example

$$R \perp\!\!\!\perp B$$

Yes

$$R \perp\!\!\!\perp B | T$$

$$R \perp\!\!\!\perp B | T'$$



Example

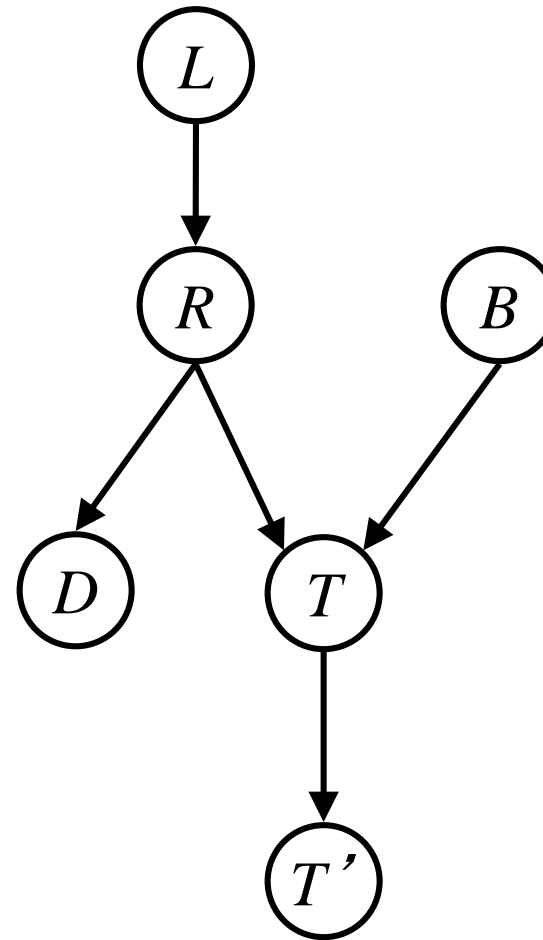
$L \perp\!\!\!\perp T' | T$ *Yes*

$L \perp\!\!\!\perp B$ *Yes*

$L \perp\!\!\!\perp B | T$

$L \perp\!\!\!\perp B | T'$

$L \perp\!\!\!\perp B | T, R$ *Yes*



Example

- Variables:

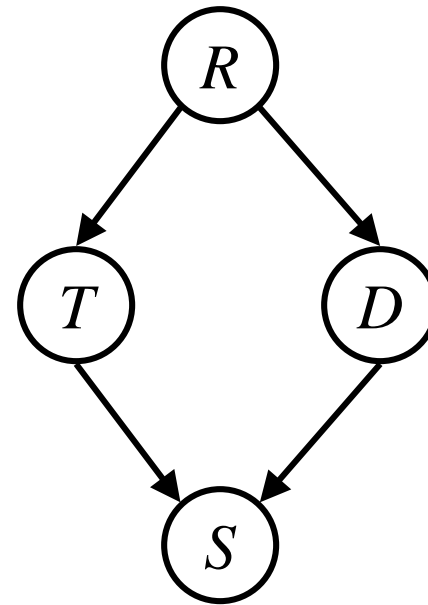
- R: Raining
- T: Traffic
- D: Roof drips
- S: I'm sad

- Questions:

$$T \perp\!\!\!\perp D$$

$$T \perp\!\!\!\perp D | R \quad \text{Yes}$$

$$T \perp\!\!\!\perp D | R, S$$

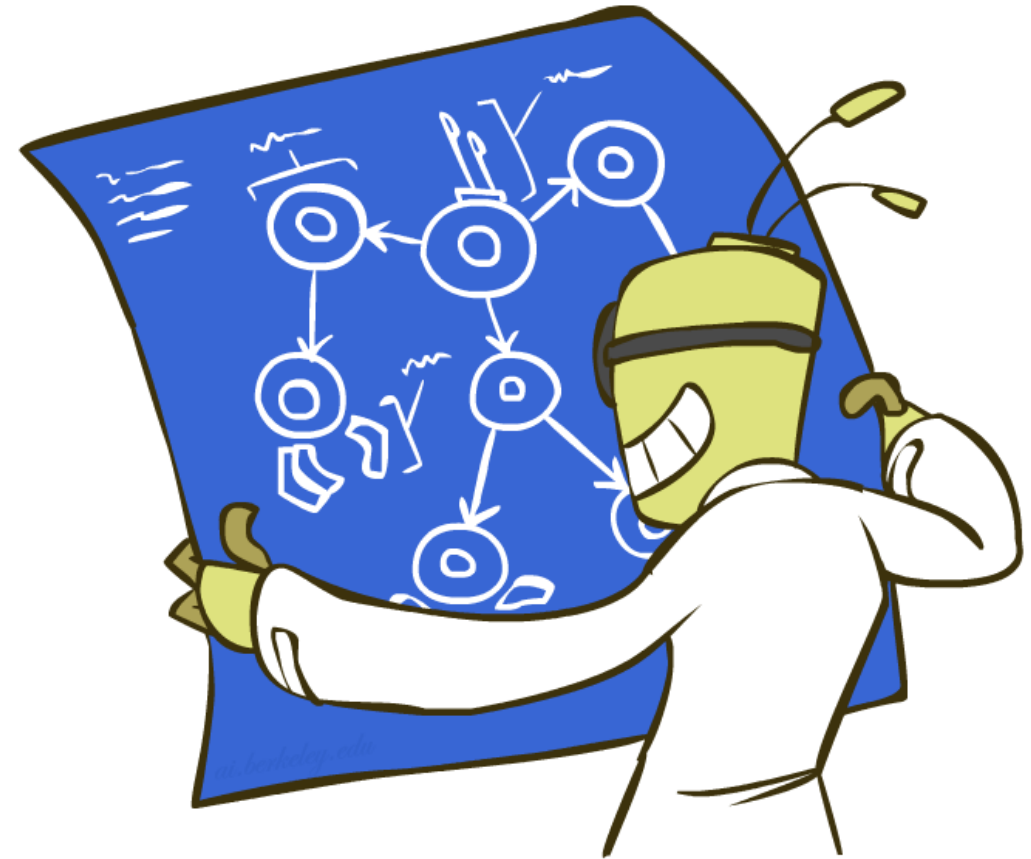


Structure Implications

- Given a Bayes net structure, can run d-separation algorithm to build a complete list of conditional independences that are necessarily true of the form

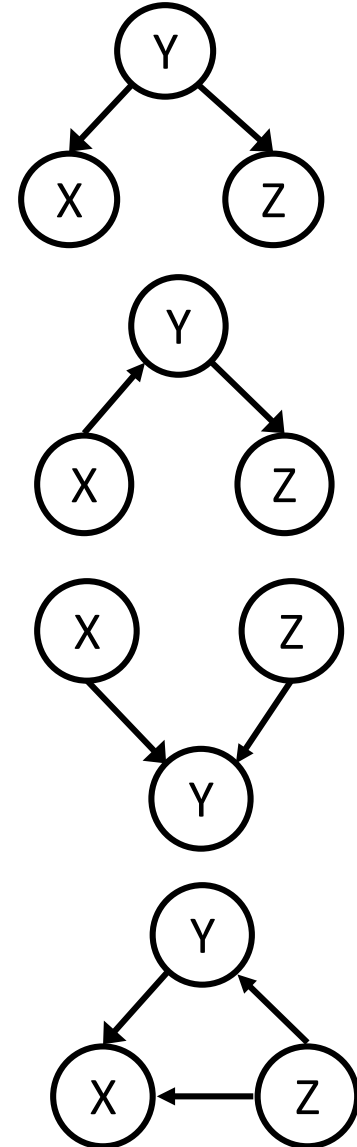
$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$$

- This list determines the set of probability distributions that can be represented



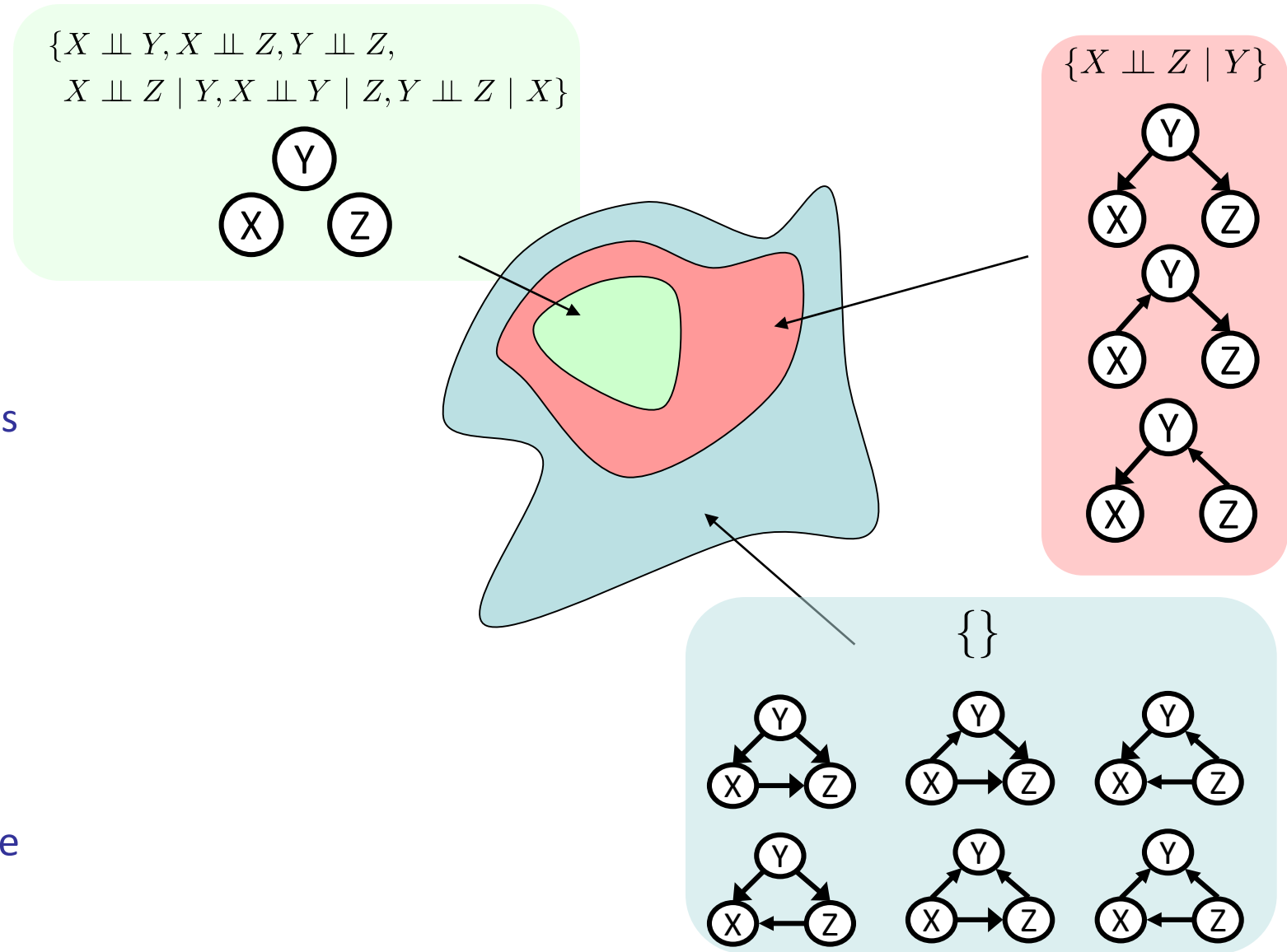
Computing All Independences

COMPUTE ALL THE
INDEPENDENCES!



Topology Limits Distributions

- Given some graph topology G , only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution



Bayes Nets Representation Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution