PROPOSITIONAL LOGIC:

Week 5

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Models and possible worlds

- Logicians typically think in terms of models, which are formally structured worlds with respect to which truth can be evaluated.
- m is a model of a sentence a if a is true in m
- $M(a)$ is the set of all models of a
- Possible worlds ~ models
  - Possible worlds: potentially real environments
  - Models: mathematical abstractions that establish the truth or falsity of every sentence

ENTAILMENT

- One sentence follows logically from another
- $\alpha \models \beta$
  - α entails sentence β *if and only if* β is true in all worlds where α is true.
  - e.g., $x+y=4 \models 4=x+y$
- Entailment is a relationship between sentences that is based on semantics.

Propositional logic: Syntax

- Propositional logic is the simplest logic
  - illustrates basic ideas
- Symbols of propositional logic
  - Logical constants
    - True and False
  - Propositional Variables (Symbols)
    - Atom
    - E.g., P, Q, R
    - Each variable can have binary value
      - P = {true | false}
  - Logical Connectives
    - $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$
  - Sentences
    - Made by putting these symbols together

Logical Connectives

- Logical Connectives
  - ¬ : not
    - ¬P : negation of P
  - $\wedge$: and
    - P $\wedge$ Q, P $\wedge$ (Q $\vee$ R) : conjunction. Its parts are the **conjuncts**
    - P $\wedge$ (Q $\vee$ R) is a conjunction of the conjuncts P and (Q $\vee$ R)
  - $\vee$: or
    - P $\vee$ Q, P $\vee$ (Q $\wedge$ R) : disjunction . Its parts are the **disjuncts**
    - P $\vee$ (Q $\wedge$ R) is a disjunction of the disjuncts P and (Q $\wedge$ R)
  - $\Rightarrow$: implies
    - (P $\wedge$ Q) $\Rightarrow$ R : implication.
    - (P $\wedge$ Q) : premise, antecedent, R : conclusion, consequent
  - $\Leftrightarrow$: equivalent
    - (P $\wedge$ Q) $\Leftrightarrow$ (P $\wedge$ Q) : equivalence, biconditional

Logical Connectives

- $A \Rightarrow B = \neg A \vee B$
- $\neg(A \wedge B) = \neg A \vee \neg B$
- $\neg(A \vee B) = \neg A \wedge \neg B$
- $\neg\neg A = A$
- $A \Rightarrow B = \neg B \Rightarrow \neg A$

SENTENCES

- Sentences:
  - If S is a sentence, $\neg$S is a sentence (**negation**)
  - If S_1 and S_2 are sentences, $S_1 \wedge S_2$ is a sentence (**conjunction**)
  - If S_1 and S_2 are sentences, $S_1 \vee S_2$ is a sentence (**disjunction**)
  - If S_1 and S_2 are sentences, $S_1 \Rightarrow S_2$ is a sentence (**implication**)
  - If S_1 and S_2 are sentences, $S_1 \Leftrightarrow S_2$ is a sentence (**equivalence**)

CNF and DNF

- CNF (Conjunctive Normal Form)

A "conjunction of Clauses"

literals

$$\underbrace{(A \vee \neg B)}_{\text{Clause}} \wedge \underbrace{(B \vee \neg C \vee \neg D)}_{\text{Clause}}$$

Disjunction of literals

- DNF (Disjunctive Normal Form)

A "disjunction of terms"

literals

$$\underbrace{(A \wedge \neg B)}_{\text{Term}} \vee \underbrace{(B \wedge \neg C \wedge \neg D)}_{\text{Term}}$$

Conjunction of literals

- Horn Form (Special case of CNF)
  - Conjunction of Horn Clauses
    - Horn Clauses : a clause with at most one positive literal

Propositional logic: Semantics

- Each model/world specifies true or false for each proposition symbol
  - E.g. P1 , P2, P3
  - With these symbols, 8 possible models, can be enumerated automatically.

P1	P2	P3
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Propositional logic: Semantics

- Rules for evaluating truth with respect to a model *m*:
  - $\neg S$ is true iff S is false
  - $S_1 \wedge S_2$ is true iff S_1 is true and S_2 is true
  - $S_1 \vee S_2$ is true iff S_1 is true or S_2 is true
  - $S_1 \Rightarrow S_2$ is true iff S_1 is false or S_2 is true
  - $S_1 \Leftrightarrow S_2$ is true iff $S_1 \Rightarrow S_2$ is true and $S_2 \Rightarrow S_1$ is true
- Simple evaluates an arbitrary sentence, e.g., $\neg P_1 \wedge (P_2 \vee P_3)$

P1	P2	P3	¬P1	(P2 ∨ P3)	¬P1 ∧ (P2 ∨ P3)
T	T	T	F	T	F
T	T	F	F	T	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	F	F

TRUTH TABLES FOR CONNECTIVES

- True tables for the five logical connectives

P	Q	$\neg P$	$P \land Q$	$P \lor Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
$false$	$false$	$true$	$false$	$false$	$true$	$true$
$false$	$true$	$true$	$false$	$true$	$true$	$false$
$true$	$false$	$false$	$false$	$true$	$false$	$false$
$true$	$true$	$false$	$true$	$true$	$true$	$true$

VALIDITY

- A sentence is valid if it is true in all models
  - e.g $((P \lor H) \land \neg H) \Rightarrow P$

P	H	$P \lor H$	$(P \lor H) \land \neg H$	$((P \lor H) \land \neg H) \Rightarrow P$
F	F	F	F	T
F	T	T	F	T
T	F	T	T	T
T	T	T	F	T

- Sentence $((P \lor H) \land \neg H) \Rightarrow P$ is Valid

Semantics properties

- Deduction Theorem:
  - $KB \models \alpha$ if and only if $(KB \Rightarrow \alpha)$ is valid
  - $M(\alpha) = \{ w : w \models \alpha \}$
- Consistency, satisfiability
  - $M(\alpha) \neq \emptyset$, α is consistent, satisfiable if it is true in some model
  - $M(\alpha) = \emptyset$, α inconsistent, unsatisfiable if it is false in all models
- Validity
  - $M(\alpha)$ = Whole world, α is valid
- Equivalence
  - $M(\alpha) = M(\beta)$, α is equivalent β
- Mutually Exclusive
  - $M(\alpha \wedge \beta) = \emptyset$
  - $M(\alpha) \wedge M(\beta) = \emptyset$

Propositional Logic Problems

Q. Is the Propositional Logic (PL) sentence $(A \Leftrightarrow B) \wedge (\neg A \vee B)$ valid, unsatisfiable, or satisfiable?

A	B	$A \Leftrightarrow B$	$\neg A \vee B$	$(A \Leftrightarrow B) \wedge (\neg A \vee B)$
T	T	T	T	T
T	F	F	F	F
F	T	F	T	F
F	F	T	T	T

A. Not Valid and Satisfiable
Since the last column contains both T and F, the sentence is satisfiable.

Propositional Logic Problems

Q. Prove $(A \wedge B) \models (A \Leftrightarrow B)$ using a truth table.

A	B	$A \wedge B$	$A \Leftrightarrow B$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	T

A. Since for each row where the next to last column is T, the last column is also T (this only occurs here for the first row), entailment is proved.