1 Solution

You win with probability $\frac{4}{7}$.

2 Proof

This problem can be modeled as a discrete-time Markov chain with left stochastic matrix

$$P = \begin{bmatrix} 1 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 1 \end{bmatrix} \tag{1}$$

and initial state vector

$$\mathbf{s}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T \tag{2}$$

where the first row or column denotes a win state for us, the second denotes that it is our turn, the third that it is our opponent's turn, and the fourth that our opponent wins. $\mathbf{s}_n = P^n \mathbf{s}_0$ gives us the state probabilities after n turns. Denote $\mathbf{s}_n^{(j)}$ as the jth element of $\mathbf{s}_n$. The problem asks us to find $\lim_{n\to\infty} \mathbf{s}_n^{(1)}$.

After iterating the first few instances of $\mathbf{s}_i$, we note that $\mathbf{s}_n^{(1)}$ only changes when i is odd:

$$\begin{aligned} \mathbf{s}_1^{(1)} &= \mathbf{s}_2^{(1)} = \frac{1}{4} \\ \mathbf{s}_3^{(1)} &= \mathbf{s}_4^{(1)} = \frac{25}{64} \\ \mathbf{s}_5^{(1)} &= \mathbf{s}_6^{(1)} = \frac{481}{1024} \\ \mathbf{s}_7^{(1)} &= \mathbf{s}_8^{(1)} = \frac{8425}{16384} \\ \mathbf{s}_9^{(1)} &= \mathbf{s}_{10}^{(1)} = \frac{141361}{262144} \end{aligned} \tag{3}$$

etc. This matches our intuition that the probability of having won increases if and only if it is our turn. So we can think of $\mathbf{s}_1$ as the true base case and left multiply by P^2 to get subsequent relevant cases.

Observation. *For all positive odd values of i, $\mathbf{s}_i^{(2)} = 0$.*

Proof. We proceed by induction. The base case, $i = 1$, is trivial. Suppose that k is odd and $\mathbf{s}_k = \begin{bmatrix} a & b & 0 & c \end{bmatrix}^T$ for arbitrary values a, b, and c. Then

$$\mathbf{s}_{k+2} = P^2 \mathbf{s}_k = \begin{bmatrix} 1 & \frac{1}{4} & \frac{3}{16} & 0 \\ 0 & \frac{9}{16} & 0 & 0 \\ 0 & 0 & \frac{9}{16} & 0 \\ 0 & \frac{3}{16} & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} a + \frac{1}{4}b \\ \frac{9}{16}b \\ 0 \\ \frac{3}{16}b + c \end{bmatrix} \tag{4}$$

□

Since $\mathbf{s}_{k+2}^{(1)}$ and $\mathbf{s}_{k+2}^{(2)}$ only depend on a combination of a and b, we may reduce the first two rows of (4) to the set of difference equations

$$\begin{aligned} x_{n+1} &= x_i + \frac{1}{4} y_n, \qquad & x_0 = 0 \\ y_{n+1} &= \frac{9}{16} y_n, & y_0 = 1 \end{aligned} \tag{5}$$

which has the solution

$$\begin{aligned} x_n &= \frac{4}{7} \left(1 - \left(\frac{9}{16} \right)^n \right) \\ y_n &= \left(\frac{9}{16} \right)^n \end{aligned} \tag{6}$$

Finally, $\lim_{n \to \infty} \mathbf{s}_n^{(1)} = \lim_{n \to \infty} x_n = \frac{4}{7}$, our chances of winning the game.