T-Shirt Cannon Solution

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Part 1: Find $x(\theta)$

Denote $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$. We will assume:

1. No air resistance, so $\ddot{x} = 0$ and $\ddot{y} = -g \approx -9.81\text{m/s}^2$;
2. The T-shirts are launched from the ground, so $y_0 = y(0) = 0$;
3. Without loss of generality, the rows are 1m deep (we are free to choose any depth, since the problem is invariant in the x direction).

$\ddot{x} = 0$ implies $\dot{x}$ is constant. Then:

$$x(t) = \dot{x}t \tag{1}$$

and

$$y(t) = \dot{y}_0 t - \frac{1}{2}gt^2 = 0 \tag{2}$$

for some time $t \in \mathbb{R}^+$. Introduce angle θ and define $\dot{x}$ and $\dot{y}_0$ in terms of θ:

$$\dot{x} = v_0 \cos\theta \tag{3a}$$
$$\dot{y}_0 = v_0 \sin\theta \tag{3b}$$

for some initial speed v_0. Our goal is to find x in terms of θ that minimizes v_0.

First, we combine equations (1) and (3a):

$$\begin{aligned} t &= \frac{x}{\dot{x}} \\ &= \frac{x}{v_0 \cos\theta} \end{aligned} \tag{4}$$

Next we combine (2) and (3b):

$$\begin{aligned} y(x) &= v_0 \sin\theta \cdot \frac{x}{v_0 \cos\theta} - \frac{g}{2} \cdot \left(\frac{x}{v_0 \cos\theta}\right)^2 \\ &= x\left[\tan\theta - \frac{gx}{2} \cdot \left(\frac{1}{v_0 \cos\theta}\right)^2\right] \end{aligned} \tag{5}$$

Equation (5) has two roots: the trivial root $x = 0$, and a non-trivial root corresponding to the point when the launched T-shirt returns to the ground. Solving it for this non-trivial root yields

$$x = \frac{v_0^2}{g} \sin 2\theta \tag{6}$$

From assumption 3, $x_{\text{max}} = 100$, so we normalize (6) to

$$x = 100 \sin 2\theta \tag{7}$$

This is the equation that we are looking for.

Part 2: Finding the best seat

If we take $\varphi = \theta - \frac{\pi}{4}$, we get

$$\begin{aligned} x &= 100 \sin\left[2\left(\varphi + \frac{\pi}{4}\right)\right] \\ &= 100 \cos 2\varphi \end{aligned} \tag{8}$$

which is symmetric about $\varphi = 0$. Thus the solution is the same whether $\theta \in [0, \frac{\pi}{2}]$ or $\theta \in [0, \frac{\pi}{4}]$. We assume that $\theta \sim U(0, \frac{\pi}{4})$, the continuous uniform distribution. Consider equation (7)'s derivative

$$\frac{dx}{d\theta} = 200 \cos 2\theta \tag{9}$$

$dx/d\theta > 0$ everywhere on $\theta \in [0, \frac{\pi}{4})$, and $x = 100$ has the unique solution on $[0, \frac{\pi}{4}]$ of $\theta = \frac{\pi}{4}$. Thus (7) is invertible on $[0, \frac{\pi}{4}]$, and there we may define $\theta(x)$ as the inverse of $x(\theta)$:

$$\theta = \frac{1}{2} \sin^{-1} \frac{x}{100} \tag{10}$$

which maps $[0, 100]$ onto $[0, \frac{\pi}{4}]$. The goal is to find $x^* \in [0, \frac{\pi}{4}]$ that gives

$$\max F(x^*) = \int_{x^*-1}^{x^*} \theta(x)\, dx \tag{11}$$

the probability that the T-shirt lands on row x^*. $d\theta/dx = 1/(dx/d\theta)$, so $F'(x)$ is strictly increasing over $(0, 100)$. Thus we may change the optimization problem to

$$\begin{aligned} \max F'(x^*) &= \frac{d}{dx} \int_0^{x^*} \theta(x)\, dx - \frac{d}{dx} \int_0^{x^*-1} \theta(x)\, dx \\ &= \theta(x^*) - \theta(x^* - 1) \end{aligned} \tag{12}$$

Expression (12) is also the slope of secant lines of $\theta(x)$ with $\Delta x = 1$. Since $\theta(x)$ is strictly increasing, $F'(x)$ is maximized when x is maximized, or $x^* = 100$.

Therefore, the optimal place to sit is on the very back row.