Reasoning Under Uncertainty

Agents which use logic to represent domain knowledge, probabilities to capture uncertainty, and probabilistic inference to reason about that knowledge.

Agents may need to handle uncertainty

- Partial observability
  - not complete knowledge of the world
- Nondeterminism
  - actions do not always have their intended consequences

Random Variables

- A random variable *X* has a domain of possible values $\{a_1, \ldots, a_n\}$ and an associated probability distribution
- E.g.:
  - random variable: Weather
  - domain: {Sunny, Rain, Cloudy, Snow}
  - P(Weather = Sunny) = 0.7
    P(Weather = Rain) = 0.2
    P(Weather = Cloudy) = 0.08
    P(Weather = Snow) = 0.02

Shorthand

- If X is a Boolean random variable (i.e., the domain is {true, false}) we write:
  - $P(X)$ as a shorthand for $P(X = \text{true})$
  - $P(\neg X)$ as a shorthand for $P(X = \text{false})$
- Can also ask the probability of more complex sentences:
  - $P(X \wedge \neg Y)$ as a shorthand for $P(X = \text{true}, Y = \text{false})$

Probabilities

- $P(X)$
  - *prior* or *unconditional* probability that X is true in the absence of any other information
- $P(X \mid Y)$
  - *posterior* or *conditional* probability that X is true given that all we know is that Y is true
  - $P(X \mid Y) = P(X \wedge Y) / P(Y) \qquad \text{if } P(Y) > 0$

Joint Probability Distribution

- Probabilistic model of a domain
  – set of random variables
- Atomic event
  – assignment of a value to each random variable in the model (i.e., a complete specification of a state of the domain)
- Joint probability distribution
  – assignment of a probability to each possible atomic event

Holmes scenario

Mr. Holmes lives in a high crime area and therefore has installed a burglar alarm. He relies on his neighbors to phone him when they hear the alarm sound. Mr. Holmes has two neighbors, Dr. Watson and Mrs. Gibbon.

Unfortunately, his neighbors are not entirely reliable. Dr. Watson is known to be a tasteless practical joker and Mrs. Gibbon, while more reliable in general, has occasional drinking problems.

Mr. Holmes also knows from reading the instruction manual of his alarm system that the device is sensitive to earthquakes and can be triggered by one accidentally. He realizes that if an earthquake has occurred, it would surely be on the radio news.

Holmes example

- Boolean random variables

  W means Watson calls
  A means the alarm is going off
  B means there is a burglary in progress

- Joint probability distribution

$$P(\neg W, \neg A, \neg B) = p_0$$
$$P(\neg W, \neg A, B) = p_1$$
$$P(\neg W, A, \neg B) = p_2$$
$$P(\neg W, A, B) = p_3$$
$$P(W, \neg A, \neg B) = p_4$$
$$P(W, \neg A, B) = p_5$$
$$P(W, A, \neg B) = p_6$$
$$P(W, A, B) = p_7$$

$$p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = 1$$

Determine probability of any sentence

- $P(B) =$
- $P(W \wedge B) =$
- $P(W \vee B) =$
- $P(W \vee \neg W) =$
- $P(B \mid W) = P(B \wedge W) / P(W) =$
- $P(\neg B \mid W \wedge A) = P(\neg B \wedge W \wedge A) / P(W \wedge A) =$

Some important rules (I)

- Product rule:

$$P(X, Y) = P(X \mid Y)\, P(Y) = P(Y \mid X)\, P(X)$$

- Sum rule:

$$P(X = a) = \sum_{b \in dom(Y)} P(X = a \mid Y = b)\, P(Y = b)$$

- Bayes' rule:

$$P(Y \mid X) = P(X \mid Y)\, P(Y) / P(X)$$

Examples of probabilistic reasoning

Some important rules (II)

- Chain rule:

$$
\begin{aligned}
\mathrm{P}(X_1, \ldots, X_n) = {} & \mathrm{P}(X_n \mid X_{n-1}, \ldots, X_1) \\
& \mathrm{P}(X_{n-1} \mid X_{n-2}, \ldots, X_1) \\
& \ldots \\
& \mathrm{P}(X_2 \mid X_1) \\
& \mathrm{P}(X_1)
\end{aligned}
$$

$$
= \prod_{i=1}^{n} \mathrm{P}(X_i \mid X_{i-1}, \ldots, X_1)
$$

Independence

- Defn: X is <u>independent</u> of Y if

$$P(X \mid Y) = P(X)$$

If X is indep.of Y, Y is indep. of X; i.e.,

$$P(Y \mid X) = P(Y)$$

- Shorthand for:

$$\forall x \in dom(X) \,.\, \forall y \in dom(Y) \,.$$

$$P(X{=}x \mid Y{=}y) = P(X{=}x)$$

Examples of (non) independence

- Burglary is independent of alarm? **No**

$$P(B \mid A) \stackrel{?}{=} P(B) \qquad P(A \mid B) \stackrel{?}{=} P(A)$$

$$P(B \mid \neg A) \stackrel{?}{=} P(B) \qquad P(A \mid \neg B) \stackrel{?}{=} P(A)$$

$$P(\neg B \mid A) \stackrel{?}{=} P(\neg B) \qquad P(\neg A \mid B) \stackrel{?}{=} P(\neg A)$$

$$P(\neg B \mid \neg A) \stackrel{?}{=} P(\neg B) \qquad P(\neg A \mid \neg B) \stackrel{?}{=} P(\neg A)$$

Examples of (non) independence

- Watson is independent of Gibbon? **No**

$$P(W \mid G) \stackrel{?}{=} P(W) \qquad P(G \mid W) \stackrel{?}{=} P(G)$$

$$P(W \mid \neg G) \stackrel{?}{=} P(W) \qquad P(G \mid \neg W) \stackrel{?}{=} P(G)$$

$$P(\neg W \mid G) \stackrel{?}{=} P(\neg W) \qquad P(\neg G \mid W) \stackrel{?}{=} P(\neg G)$$

$$P(\neg W \mid \neg G) \stackrel{?}{=} P(\neg W) \qquad P(\neg G \mid \neg W) \stackrel{?}{=} P(\neg G)$$

Examples of (non) independence

- Earthquake is independent of Burglary? **Yes**

$$P(E \mid B) \stackrel{?}{=} P(E) \qquad P(B \mid E) \stackrel{?}{=} P(B)$$

$$P(E \mid \neg B) \stackrel{?}{=} P(E) \qquad P(B \mid \neg E) \stackrel{?}{=} P(B)$$

$$P(\neg E \mid B) \stackrel{?}{=} P(\neg E) \qquad P(\neg B \mid E) \stackrel{?}{=} P(\neg B)$$

$$P(\neg E \mid \neg B) \stackrel{?}{=} P(\neg E) \qquad P(\neg B \mid \neg E) \stackrel{?}{=} P(\neg B)$$

Conditional Independence

- Defn: X is <u>conditionally independent</u> of Y given Z if

$$P(X \mid Y, Z) = P(X \mid Z)$$

If X is cond. indep. of Y given Z, Y is cond. indep. of X given Z; i.e.,

$$P(Y \mid X, Z) = P(Y \mid Z)$$

- Shorthand for:

$$\forall x \in dom(X) \,.\, \forall y \in dom(Y) \,.\, \forall z \in dom(Z) \,.$$

$$P(X{=}x \mid Y{=}y, Z{=}z) = P(X{=}x \mid Z{=}z)$$

Examples of (non) conditional independence

- Watson conditionally independent of Gibbon given Burglary?

$$P(W \mid G, B) \stackrel{?}{=} P(W \mid B) \qquad\qquad P(G \mid W, B) \stackrel{?}{=} P(G \mid B)$$

No

Example reasoning: The joint event of Burglary and Gibbon calling saying the alarm is going constitute stronger evidence for the occurrence of the alarm than burglary alone.

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Examples of (non) conditional independence

- Watson conditionally independent of Gibbon given Burglary?

$$P(W \mid G, B) \overset{?}{=} P(W \mid B) \qquad\qquad P(G \mid W, B) \overset{?}{=} P(G \mid B)$$

No

Example reasoning: The joint event of Burglary and Gibbon calling saying the alarm is going constitute stronger evidence for the occurrence of the alarm than burglary alone.

Examples of (non) conditional independence

- Watson conditionally independent of Gibbon given Alarm?

$$P(W \mid G, A) \stackrel{?}{=} P(W \mid A) \qquad\qquad P(G \mid W, A) \stackrel{?}{=} P(G \mid A)$$

Yes

Example reasoning: Gibbon is an unreliable reporter of a fact that we know with certainty (whether the alarm is going or not). So knowing whether Gibbon is calling does not add anything.

Examples of (non) conditional independence

- Watson conditionally independent of Burglary, Earthquake, and Gibbon, given Alarm?

$$P(W \mid B, E, G, A) \stackrel{?}{=} P(W \mid A)$$

$$P(B, E, G \mid W, A) \stackrel{?}{=} P(B, E, G \mid A)$$

Yes