Correctness Proof of Shortest Distances using BFS

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Lemma 1 If u is discovered before v, then $dist[u] \leq dist[v]$.

Proof:

- By contradiction. Let v be the *first* vertex such that $dist[u] > dist[v]$ for some u that was discovered before v.
- Denote $d = dist[v]$; then $dist[u] \geq d + 1$.
- Let $\pi[v] = v_1$; then $dist[v_1] = d - 1$.
- Let $\pi[u] = u_1$; then $dist[u_1] \geq d$.
- Note that v_1 was discovered before u_1 since v is the first "out-of-order" vertex.
- So, in order of discovery, we have v_1, u_1, u, v.
- Vertex v was discovered while processing $Adj[v_1]$ and vertex u was discovered while processing $Adj[u_1]$.
- This means that v was discovered before u, a contradiction.

Lemma 2 If uv is any edge, then $|dist[u] - dist[v]| \leq 1$.

Proof: WLOG suppose u is discovered before v, so we explore uv in the direction $u \to v$. We identify three cases:

(1) Suppose v is white when we process $Adj[u]$. Then $dist[v] = dist[u] + 1$.

(2) Suppose v is grey when we process $Adj[u]$. Let $\pi[v] = v_1$; then v was discovered when $Adj[v_1]$ was being processed. So v_1 was discovered before u. By Lemma 1, $dist[v_1] \leq dist[u]$. Also, $dist[v] = dist[v_1] + 1$, so $dist[u] \geq dist[v] - 1$. Since u was discovered before v, we have $dist[u] \leq dist[v]$ by Lemma 1. Therefore, $dist[u] \leq dist[v] \leq dist[u]+1$.

(3) Suppose v is black when we process $Adj[u]$. Then $Adj[v]$ has been completely processed and we would already have discovered u from v. This is a contradiction, so this case does not occur.

Theorem For every vertex v, $dist[v]$ equals the length of the shortest path from s to v.

Proof:

- Let $\delta(v)$ denote the length of the shortest path from s to v.
- Consider the path

$$v \quad \pi[v] \quad \pi[\pi[v]] \quad \cdots \quad s.$$

- This path has length $dist[v]$, so $\delta(v) \leq dist[v]$.
- To complete the proof, we show that $\delta(v) \geq dist[v]$; we will prove this by induction on $\delta(v)$.

  Base case: $\delta(v) = 0$. Then $v = s$ and $dist[v] = 0 = \delta(v)$.

  Induction assumption: Assume $\delta(v) \geq dist[v]$ if $\delta(v) \leq d - 1$. Now suppose $\delta(v) = d$. Let

$$s \quad v_1 \quad v_2 \quad \cdots \quad v_{d-1} \quad v_d = v$$

  be a shortest path (having length d). Then $\delta(v_{d-1}) = d-1 = dist[v_{d-1}]$ by induction. Now, $dist[v] \leq dist[v_{d-1}] + 1$ (by Lemma 2). But $dist[v_{d-1}] = d - 1$, so $dist[v] \leq d = \delta(v)$ and we're done.