The Modified General Version of the Master Theorem

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Our goal is to prove the following generalization of the Master Theorem stated on slide 60. This is similar to the version given in [CLRS], except that the condition in the third case has been replaced by a simpler condition[1].

Theorem 1. *Suppose that $a \geq 1$ and $b > 1$. Denote $x = \log_b a$. Suppose $T(n)$ is defined by the recurrence*

$$T(n) = \begin{cases} a\,T\left(\frac{n}{b}\right) + f(n) & \text{if } n > 1 \\ d & \text{if } n = 1. \end{cases}$$

Then, for $n = b^j$, it holds that

$$T(n) \in \begin{cases} \Theta(n^x) & \text{if } f(n) \in O(n^{x-\epsilon}) \text{ for some } \epsilon > 0 \\ \Theta(n^x \log n) & \text{if } f(n) \in \Theta(n^x) \\ \Theta(f(n)) & \text{if } f(n)/n^{x+\epsilon} \text{ is an increasing function of } n \\ & \text{for some } \epsilon > 0. \end{cases}$$

We begin by using the recursion tree method to obtain a formula for $T(n)$. This is very similar to what was done in class. The resulting formula is:

$$T(n) = a^j f(1) + \sum_{i=0}^{j-1} a^i f(n/b^i) = d\,n^x + \sum_{i=0}^{j-1} a^i f(n/b^i), \tag{1}$$

where we use the fact that $a^j = (b^x)^j = (b^j)^x = n^x$.

[1] This condition was pointed out to me by Timothy Chan.

case 1

Here we have $f(n) \in O(n^{x-\epsilon})$ for some $\epsilon > 0$. Denote $y = x - \epsilon$; then $f(n) \in O(n^y)$. Note that $x - y = \epsilon > 0$, so $b^{x-y} > 1$.

First, from (1), we have $T(n) \geq d\,n^x$, so $T(n) \in \Omega(n^x)$.

For the upper bound, we use the fact that $f(n) \in O(n^y)$, which implies that there is a constant $c > 0$ such that $f(n) \leq c\,n^y$ for all $n > 1$. Now, from (1), we have

$$\begin{aligned}
T(n) &\leq d\,n^x + \sum_{i=0}^{j-1} c\,a^i (n/b^i)^y \\
&= d\,n^x + c\,n^y \sum_{i=0}^{j-1} (a/b^y)^i \\
&= d\,n^x + c\,n^y \sum_{i=0}^{j-1} (b^{x-y})^i \\
&\in O(n^x + n^y (b^{x-y})^j), \text{ since } \textstyle\sum_{i=0}^{j-1} (b^{x-y})^i \in O((b^{x-y})^j) \text{ (slide 42)} \\
&= O(n^x + n^y (b^j)^{x-y}) \\
&= O(n^x + n^y n^{x-y}) \\
&= O(n^x).
\end{aligned}$$

Because $T(n) \in \Omega(n^x)$ and $T(n) \in O(n^x)$, we have that $T(n) \in \Theta(n^x)$.

case 2

In this case, we have that $f(n) \in \Theta(n^x)$. The proof of this case is the same as the proof of the corresponding case of the simplified version of the Master Theorem.

case 3

Here, $f(n)/n^{x+\epsilon}$ is an increasing function of n. Denote $y = x + \epsilon$. Then $x - y = \epsilon < 0$, so $b^{x-y} < 1$. Note also that $n^y \in O(f(n))$.

First, from (1), we have $T(n) \geq f(n)$, so $T(n) \in \Omega(f(n))$.

For the upper bound, we use the fact that $f(n)/n^y$ is an increasing function of n. Therefore,

$$\frac{f(n)}{n^y} > \frac{f\left(\frac{n}{b}\right)}{\left(\frac{n}{b}\right)^y} > \frac{f\left(\frac{n}{b^2}\right)}{\left(\frac{n}{b^2}\right)^y} > \cdots,$$

which implies that

$$f(n) > b^y f\left(\frac{n}{b}\right) > b^{2y} f\left(\frac{n}{b^2}\right) > \cdots > b^{(j-1)y} f\left(\frac{n}{b^{j-1}}\right). \qquad (2)$$

Now, from (1) and (2), we obtain

$$
\begin{aligned}
T(n) &= d\,n^x + \sum_{i=0}^{j-1} a^i f(n/b^i) \\
&< d\,n^x + \sum_{i=0}^{j-1} (a/b^y)^i f(n) \\
&= d\,n^x + f(n) \sum_{i=0}^{j-1} (a/b^y)^i \\
&\in O(n^x + f(n)), \text{ since } \textstyle\sum_{i=0}^{j-1} (a/b^y)^i \in O(1) \text{ (slide 42)} \\
&= O(f(n)),
\end{aligned}
$$

since $x < y$ and $n^y \in O(f(n))$.