Chemical Engineering 4G03

Tutorial 3 Formulation Solution

Here for your own benefit and practice (best to do it individually)
Recommended completion: week 04 (or before the midterm exam)
Grading: 0% (Practice for assignments and tests)

Problem Adapted from Rardin (2017) Chapter 4

Objective Function

The objective in this problem is simply to maximize NPV, where the NPV of allocating one acre of forest to one of three prescriptions has a certain NPV (different for each forest section). In this case, I will select $\boldsymbol{x_{i,j}}$ as my variable defining the number of acres in forest section i allocated to prescription j. **For example**, $x_{2,3}$ would be the number of acres in forest section 2 that are allocated to prescription 3. My objective is:

$$\max_{x} \phi = 503x_{1,1} + 140x_{1,2} + 203x_{1,3} + \cdots + 705x_{7,1} + 60x_{7,2} + 400x_{7,3}$$

Or, in indexed notation:

$$\boxed{\max_{\boldsymbol{x}} \boldsymbol{\phi} = \sum_{\boldsymbol{i=1}}^{\boldsymbol{7}} \sum_{\boldsymbol{j=1}}^{\boldsymbol{3}} \boldsymbol{NPV_{i,j} x_{i,j}}}$$

Allocation Constraints

I am only allowed to allocate as many acres as are available in each of the forest sections. Moreover, each acre MUST be allocated to one of the three prescriptions (I can't leave forest unallocated). Therefore, I am going to have a set of **equality constraints**. Take, for example, forest area $i = 1$. I must allocate all 75,000 acres to one of the three prescriptions, or:

$$x_{1,1} + x_{1,2} + x_{1,3} = 75{,}000$$

I can then use summation for this to denote the same thing:

$$\sum_{j=1}^{3} x_{1,j} = 75{,}000$$

AND since I have this same constraint (but different areas) for each of the forest regions (I already denoted this S_i), I can simply repeat this constraint for all i:

$$\boxed{\sum_{\boldsymbol{j=1}}^{\boldsymbol{3}} \boldsymbol{x_{i,j}} = \boldsymbol{S_i} \qquad \forall \quad \boldsymbol{i = 1 \dots 7}}$$

Timber, Grazing, and Wilderness Constraints

In this case, I just need to make sure that the sum-total of timber produced and grazing potential for the entire forest meets the required constraints. For the timber production, I need to ensure that the total timber produced (in feet) from each section under each prescription sums to *at least* 40 million:

$$\boxed{\sum_{i=1}^{7}\sum_{j=1}^{3} \boldsymbol{T_{i,j} x_{i,j}} \geq \mathbf{40{,}000{,}000}}$$

For grazing, I have the exact same thing, just a different matrix and a different lower bound:

$$\boxed{\sum_{i=1}^{7}\sum_{j=1}^{3} \boldsymbol{G_{i,j} x_{i,j}} \geq \mathbf{5000}}$$

Finally, the wilderness index is a little different. I need the *average* wilderness index of the **whole forest** to be at least 70. I can use a blending constraint for this one, saying that the sum total of "wilderness points" divided by the total area of the forest. Luckily for us, the total area of the forest is easily computed as the sum of the vector S_i. So we have:

$$\frac{\sum_{i=1}^{7}\sum_{j=1}^{3} \boldsymbol{W_{i,j} x_{i,j}}}{\sum_{i=1}^{7} \boldsymbol{S_i}} \geq \mathbf{70}$$

Or I could make sure I don't run into any division issues and elevate the sum in the denominator to the numerator of the RHS (this is optional):

$$\boxed{\sum_{i=1}^{7}\sum_{j=1}^{3} \boldsymbol{W_{i,j} x_{i,j}} \geq \mathbf{70} \sum_{i=1}^{7} \boldsymbol{S_i}}$$

When all is said and done, I have (trumpets please!):

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$$
\begin{aligned}
\max_{x} \boldsymbol{\phi} &= \sum_{i=1}^{7}\sum_{j=1}^{3} NPV_{i,j}x_{i,j} \\
\textit{Subject to} & \\
\sum_{j=1}^{3} x_{i,j} &= S_i && \forall\, i \\
\sum_{i=1}^{7}\sum_{j=1}^{3} T_{i,j}x_{i,j} &\geq 40,000,000 \\
\sum_{i=1}^{7}\sum_{j=1}^{3} G_{i,j}x_{i,j} &\geq 5000 \\
\sum_{i=1}^{7}\sum_{j=1}^{3} W_{i,j}x_{i,j} &\geq 70\sum_{i=1}^{7} S_i \\
x_{i,j} &\geq 0 && \forall\, i,j
\end{aligned}
$$