CHEMICAL ENGINEERING 4G03

Supplement to Module 02

Convex Constrained Sets

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Description of Module

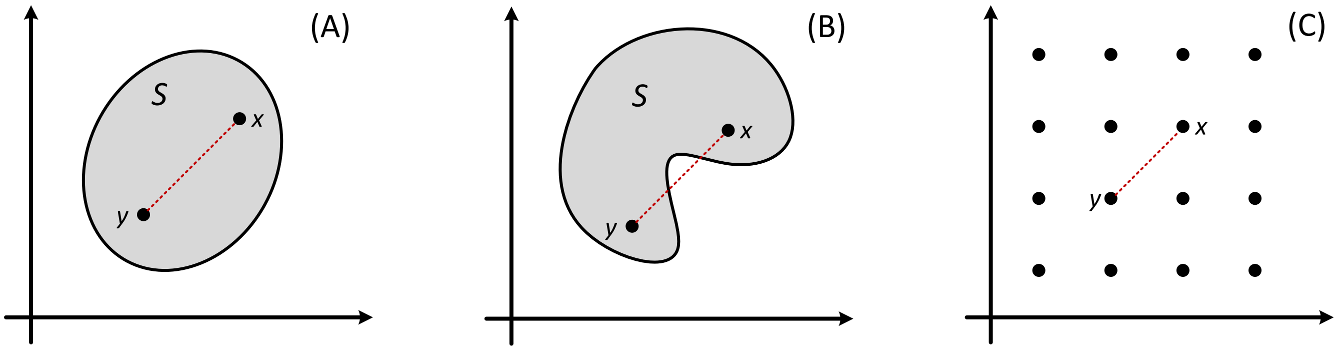
This module is an addition to section 02 in which I describe some examples of convex constraints resulting in convex sets. I want to make sure that we avoid confusion using all resources possible.

Suggested Readings

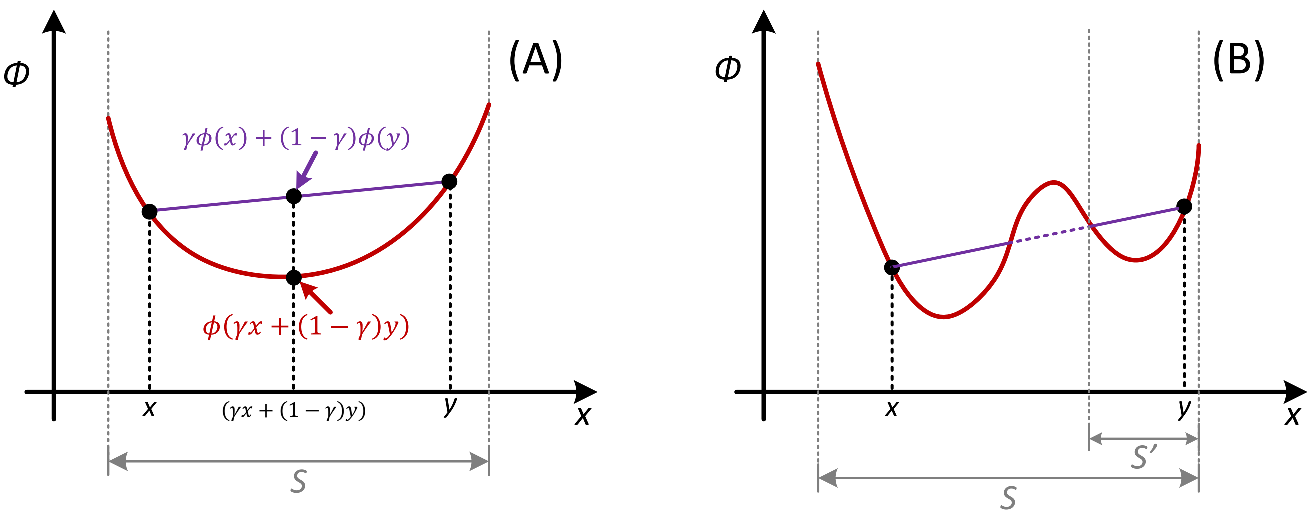
N/A

A review of Convex Sets

Recall that I define a convex set $\mathcal{S}$ to be the set of all feasible points in an optimization problem in which I am able to select *any* two values and ALL values between that selected pair will remain in $\mathcal{S}$. As a reminder, we have the following figures (recall that a discrete set is by definition non-convex).



This is **not to be confused with a convex function**. A convex function is a function for which I can select any two points on the function (can be done in infinite dimensions) and if I draw a line (2D), plane (3D) or some other multi-dimensional surface (4D+) between those two points, that line/plane/whatever will have a "height" *greater* than the function evaluated at any points along the line. As a reminder, here is the picture of convex versus non-convex *functions* again.



OK, so that is all well and good. However, people seemed to be getting confused relating the intersection of *constraints* as being convex *sets*. When I am forming an optimization problem, my region $\mathcal{S}$ is **defined as the set of points that satisfy ALL constraints**. If my optimization program involves inequality constraints, the feasible set $\mathcal{S}$ is the collection of points that satisfy each of those constraints. In a two-dimensional sense, this means that it is the region *bounded* on all sides by my constraints. Some examples of feasible regions defined by constraints are given again for your reference below (from Module 01).

What we need to understand is how we can claim if the grey-shaded regions bounded by the *constraints* (see below) form a set of possible points that is also *convex* (as in the first figure). This means that we are not defining $\mathcal{S}$ arbitrarily as we have in the first figure, but by using the bounds of variables given in the figure below. However, if the shape of the region $\mathcal{S}$ bounded by my inequality results in a region that obeys my convex set laws (I can draw a line between two points and never leave $\mathcal{S}$), then my *constraints are said to yield a convex set*.

Constraint set (A)

$x_1 + x_2 \leq 2$

$3x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$

Constraint set (B)

$x_1 + x_2 \leq 2$

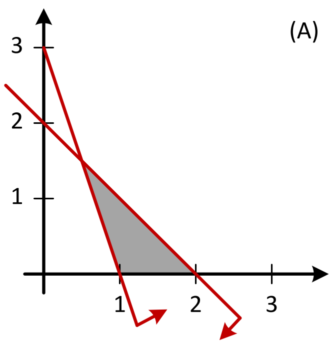
$3x_1 + x_2 = 3$

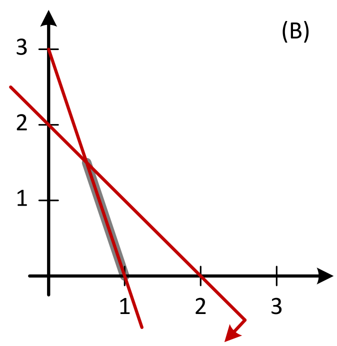
$x_1, x_2 \geq 0$

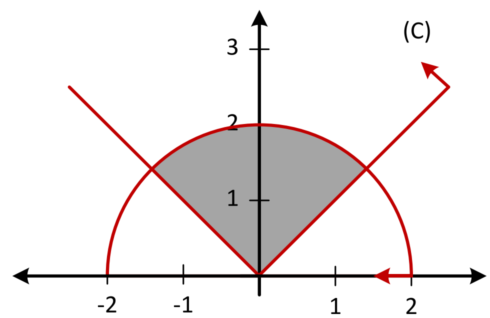
Constraint set (C)

$x_1^2 + x_2^2 \leq 4$

$|x_1| - x_2 \leq 0$







WHEW! OK. So, here is the kicker:

> ***IF THE EQUATIONS THAT DEFINE MY INEQUALITY CONSTRAINTS g(x) ≤ 0 ARE <u>CONVEX FUNCTIONS</u>, AND THE EQUATIONS THAT DEFINE MY INEQUALITY CONSTRAINTS g(x) ≥ 0 ARE <u>CONCAVE FUNCTIONS</u>, THEN THE INTERSECTION OF THOSE CONSTRAINTS THAT FORM THE FEASIBLE REGION IS CONVEX.***

So, consider the first constraint set above. Since both of the constraints are *linear*, I know that they are convex, and hence the intersection of those constraints forms a convex set, as we expect.

Consider instead constraint set (c). WHAT exactly is going on here. Well, let's look at each of those constraints one at a time. Remember that the *function* defining the constraints, which we call $g(x)$, exists on the LHS of the inequality. If I re-arrange the first constraint in set (c) so I have $g(x) \leq 0$ I get:

$$x_1^2 + x_2^2 - 4 \leq 0$$

Therefore the equation of $g(x)$ is:

$$\boldsymbol{g(x) = x_1^2 + x_2^2 - 4}$$

What does this look like? Well, there is a plot of it below. Now ask yourself, is the **function** $g(x)$ convex? I think you will find that it is! I can draw any line/plane between any two points on the surface, and the line/plane will remain above the surface for all points in between. Nifty. OK, so now that we know $g(x)$ as a **FUNCTION** is convex, we need only now re-arrange our variables to plot the **constraint** that it represents. If I re-arrange:

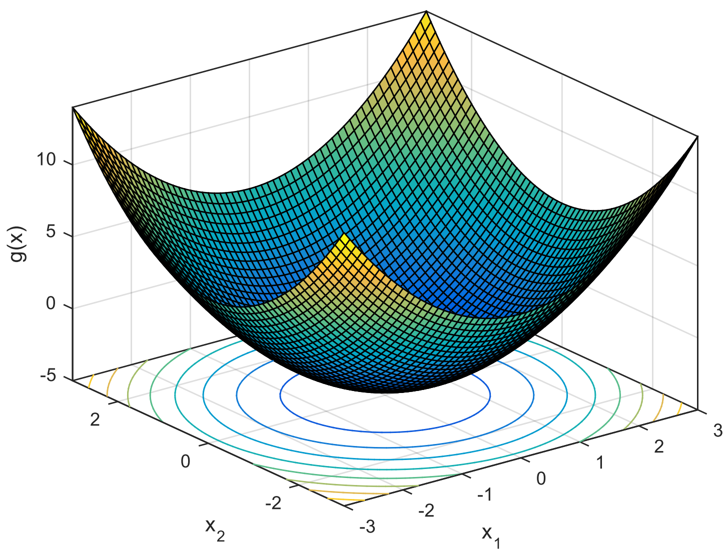
$$g(x) \leq 0 \quad \Rightarrow \quad x_1^2 + x_2^2 - 4 \leq 0$$

$$\boldsymbol{x_2 \leq \pm\sqrt{4 - x_1^2}} \ : \ -2 \leq x_1 \leq 2$$

I get x_2 being described as the top-and bottom-halves of a circle of radius 2. Since my variables can only be positive, it is defined only for the positive half, and so I plot the function:

$$x_2 \leq \sqrt{4 - x_1^2} \;\; : \;\; -2 \leq x_1 \leq 2$$

What do I get?? Well, I get all of the area **under the semicircle in constraint set (c)!!** Hence, since $g(x)$ is a convex function, the constraint $g(x) \leq 0$ results in a convex set.



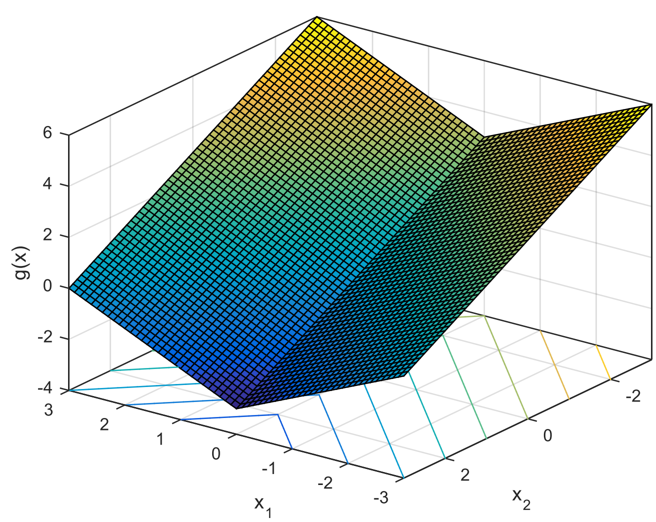
NOW consider the second constraint in set (c) above:

$$|x_1| - x_2 \leq 0$$

In this case, I get:

$$g(x) = |x_1| - x_2$$

Is this a convex function? Let's plot it! When I do, I get this chevron-looking thing:



Now, is **THIS** function convex? Well, of course it is! Just like in the bowl-shaped function, I can draw that line or plane and always stay above the function! So now when I re-arrange $g(x)$ to plot the constraint:

$$-x_2 \leq -|x_1|$$

And then multiply by -1 (flip the inequality):

$$x_2 \geq |x_1|$$

I get the kinked-line looking thing in constraint set (c), which is ALSO a convex collection of points! Now, If I take all of the points *between* to form my region $\mathcal{S}$, I get a pie slice that is convex. Therefore, the intersection of regions defined by constraints $g(x) \leq 0$ (the inequality is important) where all functions $g(x)$ are convex functions is a convex set. BAM.

Now… Why Do I need CONCAVE Functions for ≥ Inequalities???

Good question! Let's start by considering constraint set (c) again, but let's flip the inequality in the first one, so I get:

$$x_1^2 + x_2^2 - 4 \geq 0$$

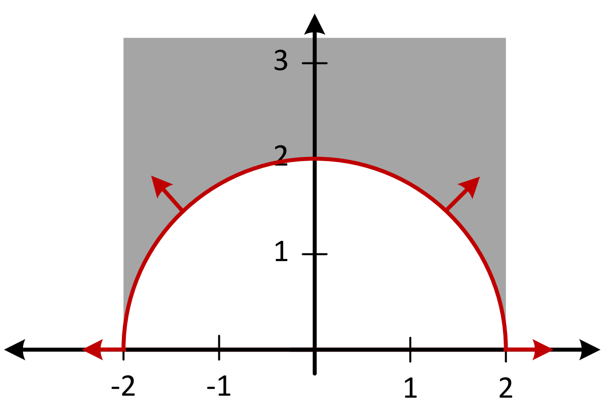
OK. Here is the thing. I actually did NOT change my function $g(x)$. It is still:

$$g(x) = x_1^2 + x_2^2 - 4$$

And it STILL looks like the bowl-shaped region a couple of figures above, and it is STILL CONVEX (as opposed to concave, or pointing downwards). Nothing is different. All that is different is the inequality. OK, now let's go ahead and plot our constraint with the new inequality. After the same re-arrangement, we arrive at:

$$x_2 \geq \sqrt{4 - x_1^2} \;\; : \;\; -2 \leq x_1 \leq 2$$

WHOA WHOA now. You might say "Jake, you dummy, that is still the equation of a circle, and circles are convex sets". To which I would say "You are right about one thing, but in fact there is a DISTINCT difference in what we are looking at here". You need only plot the *constraint* to see what I mean. Let's plot it and IGNORE the linear constraint for now. I get:

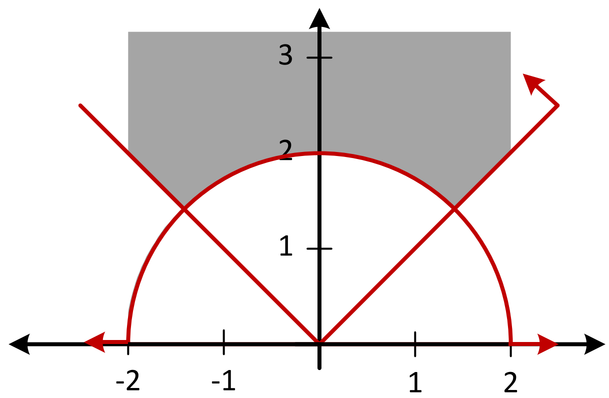


WAIT!! The set $\mathcal{S}$ is now no longer convex! Now that we have flipped the inequality, I have to stay *above* the red line (between -2 and 2 in x_1). I can easily choose two points, say $(-2,1)$ and $(2,1)$ and draw a line between them that EXITS $\mathcal{S}$ and then returns to it, which violates our definition of a convex set.

Now if I keep the other inequality constraint the same:

$$|x_1| - x_2 \leq 0$$

And add that back to the figure, I end up with (I hope you like this picture, it took me like 10 minutes to draw it):



NOW you can see that, even though the second constraint (≤) is still convex, since the first constraint is also convex (but is a ≥) our feasible region $\mathcal{S}$ is NON-CONVEX!!

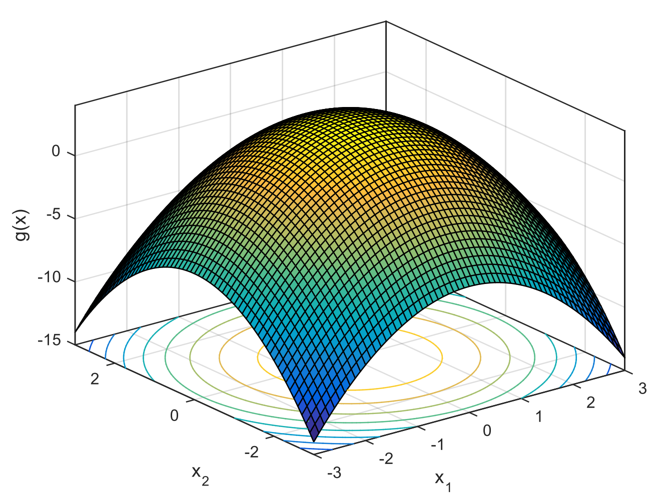
This leads us to our fundamental results:

- ANY constraint $g(x) \leq 0$ results in a convex region if the function $g(x)$ is **convex (bowl pointing upward)**.
- ANY constraint $g(x) \geq 0$ results in a convex region if the function $g(x)$ is **concave (bowl pointing downward)**. This is because $g(x) \geq 0$ with $g(x)$ concave is the exact same as $-g(x) \leq 0$ with $(-g(x))$ convex.

As another example, if we return to the ≥ case for the circular constraint, but have the function $g(x)$ as concave, or:

$$-x_1^2 - x_2^2 + 4 \geq 0$$

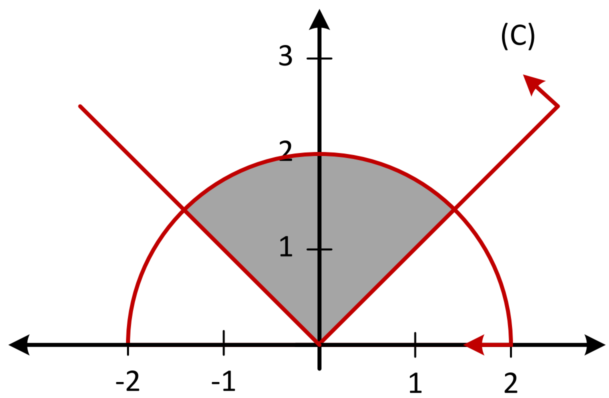
Which looks like an upside-down bowl, and is a **concave** function (a line drawn between any two points is *below* the function, or the opposite of convex):



I can re-arrange my constraint for plotting as:

$$x_2 \leq \sqrt{4 - x_1^2} \;:\; -2 \leq x_1 \leq 2$$

Which is the exact same constraint that I had before! And when I combine it with the second constraint, I get my original set as defined by constraint set (c):

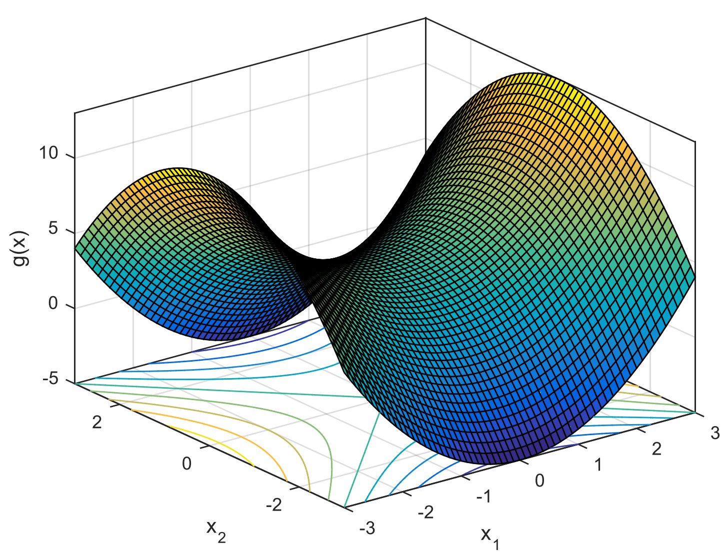


This is possible because, even though I have a ≥ constraint, the function $g(x)$ in $g(x) \geq 0$ is a concave function, and therefore results in a convex set when the constraint is plotted.

And… FINALLY, when I choose $g(x)$ to be **neither convex nor concave** I can't claim anything (even though it might work). Consider:

$$g(x) = x_1^2 - x_2^2 + 4$$

Which is neither convex nor concave. It looks like a magic carpet:



If my inequality is a ≤, I get **lucky** and end up with a convex set. If the inequality is a ≥, I get **unlucky** and don't have a convex set (try it yourself to see why!). So, if I end up with a constraint that is non-convex (or concave) I cannot conclude that my feasible set $\mathcal{S}$ is (or is not) convex.

Conclusions

Hopefully now we are a little more confident about what I mean as a convex set and how inequalities can form convex sets. One thing you should remember is that **there are concave and convex functions, but there is no such thing as a "concave set." However, convex sets can be formed by concave functions**.

The key things to ask to determine whether your program is convex are:

1. Are my inequality (and equality, but they were not discussed here) LINEAR?
2. If I have a nonlinear inequality constraint function, is it convex (≤ 0) or concave (≥ 0)?
3. Is my objective function $\phi(x)$ convex (minimization) or concave (maximization)?

If you answered YES to all of the above, your feasible region is convex (by [1] and [2]) and your objective is convex (by [3]) and thus you have a convex program.

Some things that will immediately **DENY** the convexity of your program:

1. Do I have an *equality* constraint that is nonlinear (non-affine as we described it?)
2. Is my objective function non-convex?

And, finally, the one thing that really does not help you in this case is:

1. Are any of my inequality constraints non-convex (or non-concave)?

In the case of the statement above, you really can't conclude anything, you only know that your feasible region $\mathcal{S}$ *might* be convex and *might* not be… But you can't tell (and likely can't draw it either).

I hope this has helped clear up the issue of forming convex feasible regions (sets) $\mathcal{S}$ using inequalities! Please let me know if it is still unclear at any time.

~~ END OF MODULE SUPPLEMENT~~

(You need a break):

https://www.youtube.com/watch?v=CTAud5O7Qqk

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