Chemical Engineering 4G03

Tutorial 5 Practice Activity

Here for your own benefit and practice (best to do it individually)
Recommended completion: End of week 06 (or before midterm exam)
Grading: 0% (Practice for assignments and tests)

Problem Adapted from Rardin (2017) Chapter 5

Formulation

This linear optimization problem can be formulated as

$$
\begin{aligned}
\max_{x}\ \phi &= 12x_1 + 9x_2 \\
\textit{s.t.} & \\
x_1 &\leq 1000 && \textit{Maximum availability of passionfruit juice} \\
x_2 &\leq 1500 && \textit{Maximum availability of mango concentrate} \\
x_1 + x_2 &\leq 1750 && \textit{Maximum availability of raspberry purée} \\
4x_1 + 2x_2 &\leq 4800 && \textit{Maximum availability of spring water} \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Standard Form

We can introduce slack variables to the problem in order to write the constraint as equalities. With this, the problem becomes

$$
\begin{aligned}
\max_{x}\ \phi &= 12x_1 + 9x_2 \\
& \textit{s.t.} \\
x_1 + x_3 &= 1000 && \textit{Maximum availability of passionfruit juice} \\
x_2 + x_4 &= 1500 && \textit{Maximum availability of mango concentrate} \\
x_1 + x_2 + x_5 &= 1750 && \textit{Maximum availability of raspberry purée} \\
4x_1 + 2x_2 + x_6 &= 4800 && \textit{Maximum availability of spring water} \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Note that no substitutions are required for any existing variables in this problem as all variables are nonnegative as defined and neither are unrestricted variables.

The optimization problem can now be written in standard form as

$$\boldsymbol{c}^T = [\,-12 \quad -9 \quad 0 \quad 0 \quad 0 \quad 0\,]$$

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$\begin{aligned} \min_{\boldsymbol{x}} \ \phi' &= \boldsymbol{c}^T\boldsymbol{x} \\ &\text{s.t.} \\ \mathrm{A}\boldsymbol{x} &= \boldsymbol{b} \\ \boldsymbol{x} &\geq \boldsymbol{0} \end{aligned}$$

Initial Basic Solution

In standard form, this problem has six variables and four equations. Therefore, there must be two nonbasic variables (i.e., 6 variables – 4 equations) variables that are fixed to zero to find each basic solution.

Basic Solution

Nonbasic Variables $\{x_1,\ x_2\} = 0$

Basic Variables $\mathcal{B} = \{x_3,\ x_4,\ x_5,\ x_6\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

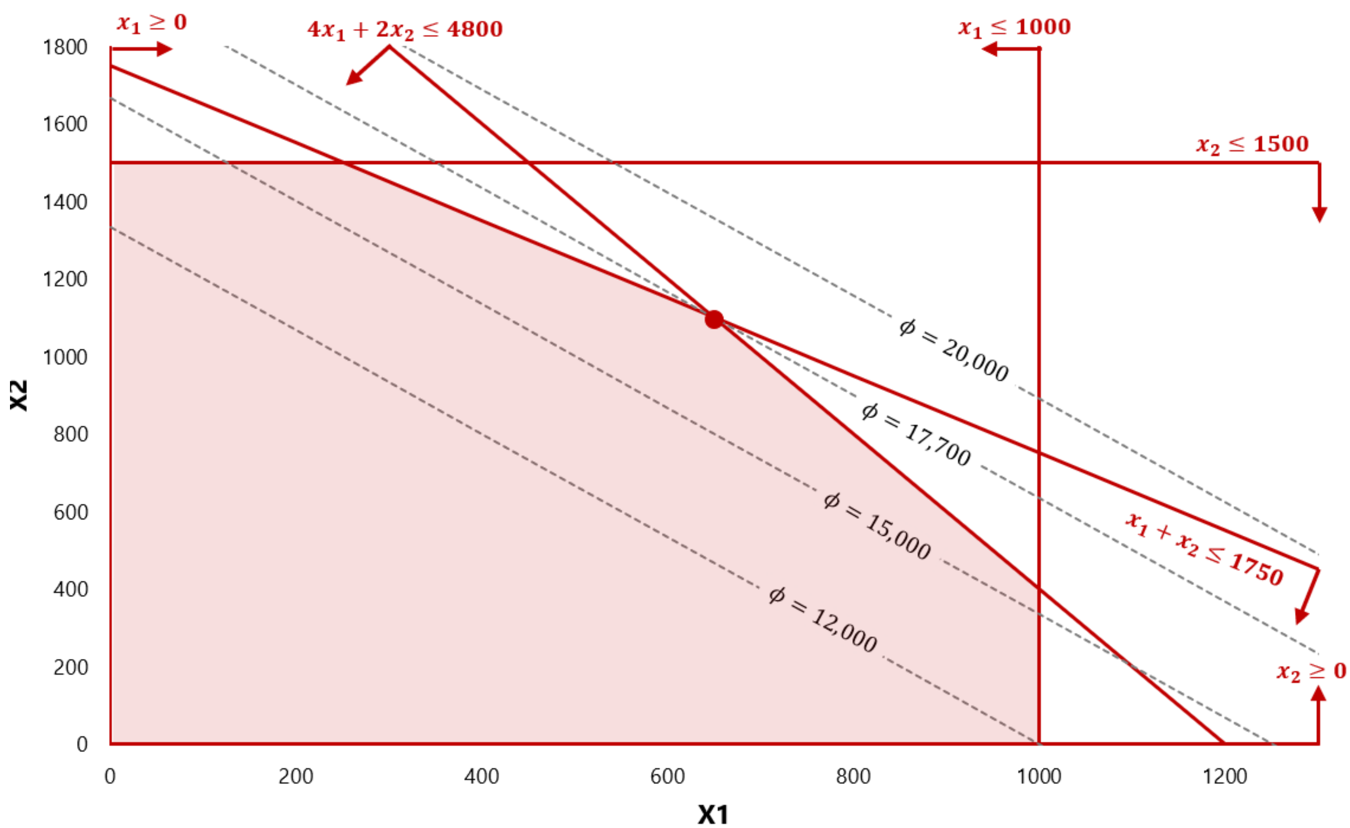
$$x_3 = 1000$$
$$x_4 = 1500$$
$$x_5 = 1750$$
$$x_6 = 4800$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

This initial basic solution is feasible (all variables are positive).

Graphical Solution

The optimization problem is plotted below, with the feasible region shaded in red as defined by the constraints. The dashed grey lines represent objective contours. The optimum is identified on the plot as a red dot and lies on the point $(x_1, x_2) = (650{,}1100)$, with a corresponding optimal objective function value of $\phi = \$17{,}700$.



Simplex Search Solution

	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_6}$	
$\max \boldsymbol{\phi} = \boldsymbol{c}^T \rightarrow$	12	9	0	0	0	0	$\boldsymbol{b}$
A	1	0	1	0	0	0	1000
	0	1	0	1	0	0	1500
	1	1	0	0	1	0	1750
	4	2	0	0	0	1	4800
$\boldsymbol{\mathcal{B}}^{(0)}$	**N**	**N**	**B**	**B**	**B**	**B**	
$x^{(0)}$	0	0	1000	1500	1750	4800	$c^T x^{(0)} = 0$
Δx for x_1	1	0	-1	0	-1	-4	$\bar{\phi}_1 = 12$
Δx for x_2	0	1	0	-1	-1	-2	$\bar{\phi}_2 = 9$
$\dfrac{x^{(0)}}{-\Delta x_1}$	—	—	**1000**	—	1750	1200	$\alpha = 1000$
$\boldsymbol{\mathcal{B}}^{(1)}$	**B**	**N**	**N**	**B**	**B**	**B**	
$x^{(1)}$	1000	0	0	1500	750	800	$c^T x^{(1)} = 12{,}000$
Δx for x_2	0	1	0	-1	-1	-2	$\bar{\phi}_2 = 9$
Δx for x_3	-1	0	1	0	1	4	$\bar{\phi}_3 = -12$
$\dfrac{x^{(1)}}{-\Delta x_2}$	—	—	—	1500	750	**400**	$\alpha = 400$
$\boldsymbol{\mathcal{B}}^{(2)}$	**B**	**B**	**N**	**B**	**B**	**N**	
$x^{(2)}$	1000	400	0	1100	350	0	$c^T x^{(2)} = 15{,}600$
Δx for x_3	-1	2	1	-2	-1	0	$\bar{\phi}_3 = 6$
Δx for x_6	0	-0.5	0	0.5	0.5	1	$\bar{\phi}_6 = -4.5$
$\dfrac{x^{(2)}}{-\Delta x_3}$	1000	—	—	550	**350**	—	$\alpha = 350$
$\boldsymbol{\mathcal{B}}^{(3)}$	**B**	**B**	**B**	**B**	**N**	**N**	
$x^{(3)}$	650	1100	350	400	0	0	$c^T x^{(3)} = 17{,}700$
Δx for x_5	1	-2	-1	2	1	0	$\bar{\phi}_5 = -6$
Δx for x_6	-0.5	0.5	-0.5	-0.5	0	1	$\bar{\phi}_6 = -1.5$
NO IMPROVING DIRECTIONS. SIMPLEX SEARCH TERMINATED.							

Additional Basic Solutions (FYI)

Basic Solution 2

Nonbasic Variables $\{x_1,\ x_3\} = 0$

Basic Variables $\mathcal{B} = \{x_2,\ x_4,\ x_5,\ x_6\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ 0 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

The resultant A matrix is not full-rank (there is a row of all zeros) and so this system cannot be solved. Therefore, this basic solution does not exist.

Basic Solution 3

Nonbasic Variables $\{x_1,\ x_4\} = 0$

Basic Variables $\mathcal{B} = \{x_2,\ x_3,\ x_5,\ x_6\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ x_3 \\ 0 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{aligned} x_3 &= 1000 \\ x_2 &= 1500 \\ x_2 + x_5 &= 1750 \ \rightarrow\ x_5 = 250 \\ 2x_2 + x_6 &= 4800 \ \rightarrow\ x_6 = 1800 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1500 \\ 1000 \\ 0 \\ 250 \\ 1800 \end{bmatrix}$$

This is a basic solution that exists and is feasible (all variables are positive).

Basic Solution 4

Nonbasic Variables $\{x_1,\ x_5\} = 0$

Basic Variables $\mathcal{B} = \{x_2,\ x_3,\ x_4,\ x_6\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ x_3 \\ x_4 \\ 0 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{aligned} x_3 &= 1000 \\ x_2 + x_4 &= 1500 \\ x_2 &= 1750 \\ 2x_2 + x_6 &= 4800 \\ x_4 &= -250 \\ x_6 &= 1300 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1750 \\ 1000 \\ -250 \\ 0 \\ 1300 \end{bmatrix}$$

This is a basic solution that exists but is infeasible (x_4 is negative and thereby violates the positivity constraint).

Basic Solution 5

Nonbasic Variables $\{x_1,\ x_6\} = 0$

Basic Variables $\mathcal{B} = \{x_2,\ x_3,\ x_4,\ x_5\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{aligned} x_3 &= 1000 \\ x_2 + x_4 &= 1500 \\ 2x_2 &= 4800 \\ x_2 + x_5 &= 1750 \\ x_2 &= 2400 \\ x_4 &= -900 \\ x_5 &= -650 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 2400 \\ 1000 \\ -900 \\ -650 \\ 0 \end{bmatrix}$$

This is a basic solution that exists but is infeasible (x_4 and x_5 are negative and thereby violate the positivity constraint).

Basic Solution 6

Nonbasic Variables $\{x_2,\ x_3\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_4,\ x_5,\ x_6\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 1000 \\ x_4 &= 1500 \\ x_1 + x_5 &= 1750 \\ 4x_1 + x_6 &= 4800 \\ x_5 &= 750 \\ x_6 &= 800 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ 1500 \\ 750 \\ 800 \end{bmatrix}$$

This is a basic solution that exists and is feasible (all variables are positive).

Basic Solution 7

Nonbasic Variables $\{x_2,\ x_4\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_3,\ x_5,\ x_6\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

The resultant A matrix is not full-rank (there is a row of all zeros) and so this system cannot be solved. Therefore, this basic solution does not exist.

Basic Solution 8

Nonbasic Variables $\{x_2,\ x_5\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_3,\ x_4,\ x_6\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ x_4 \\ 0 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_4 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_3 &= 1000 \\ x_4 &= 1500 \\ x_1 &= 1750 \\ 4x_1 + x_6 &= 4800 \\ x_3 &= -750 \end{aligned}$$

$$x_6 = -2200$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1750 \\ 0 \\ -750 \\ 1500 \\ 0 \\ -2200 \end{bmatrix}$$

This is a basic solution that exists but is infeasible (x_3 and x_6 are negative and thereby violate the positivity constraint).

Basic Solution 9

Nonbasic Variables $\{x_2,\ x_6\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_3,\ x_4,\ x_5\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_3 &= 1000 \\ x_4 &= 1500 \\ x_1 + x_5 &= 1750 \\ 4x_1 &= 4800 \\ x_1 &= 1200 \\ x_3 &= -200 \\ x_5 &= 550 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1200 \\ 0 \\ -200 \\ 1500 \\ 550 \\ 0 \end{bmatrix}$$

This is a basic solution that exists but is infeasible (x_3 is negative and thereby violates the positivity constraint).

Basic Solution 10

Nonbasic Variables $\{x_3,\ x_4\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_2,\ x_5,\ x_6\}$

$$\begin{bmatrix}1&0&1&0&0&0\\0&1&0&1&0&0\\1&1&0&0&1&0\\4&2&0&0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\0\\0\\x_5\\x_6\end{bmatrix}=\begin{bmatrix}1000\\1500\\1750\\4800\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\4&2&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_5\\x_6\end{bmatrix}=\begin{bmatrix}1000\\1500\\1750\\4800\end{bmatrix}$$

$$\begin{aligned}x_1 &= 1000\\ x_2 &= 1500\\ x_1+x_2+x_5 &= 1750\\ 4x_1+2x_2+x_6 &= 4800\\ x_5 &= -750\\ x_6 &= -2200\end{aligned}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=\begin{bmatrix}1000\\1500\\0\\0\\-750\\-2200\end{bmatrix}$$

This is a basic solution that exists but is infeasible (x_5 and x_6 are negative and thereby violate the positivity constraint).

Basic Solution 11

Nonbasic Variables $\{x_3,\ x_5\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_2,\ x_4,\ x_6\}$

$$\begin{bmatrix}1&0&1&0&0&0\\0&1&0&1&0&0\\1&1&0&0&1&0\\4&2&0&0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\0\\x_4\\0\\x_6\end{bmatrix}=\begin{bmatrix}1000\\1500\\1750\\4800\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\0&1&1&0\\1&1&0&0\\4&2&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_4\\x_6\end{bmatrix}=\begin{bmatrix}1000\\1500\\1750\\4800\end{bmatrix}$$

$$\begin{aligned}x_1 &= 1000\\ x_2+x_4 &= 1500\\ x_1+x_2 &= 1750\\ 4x_1+2x_2+x_6 &= 4800\end{aligned}$$

$$x_2 = 750$$
$$x_4 = 750$$
$$x_6 = -700$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 750 \\ 0 \\ 750 \\ 0 \\ -700 \end{bmatrix}$$

This is a basic solution that exists but is infeasible (x_6 is negative and thereby violates the positivity constraint).

Basic Solution 12

Nonbasic Variables $\{x_3,\ x_6\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_2,\ x_4,\ x_5\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 4 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 1000 \\ x_2 + x_4 &= 1500 \\ x_1 + x_2 + x_5 &= 1750 \\ 4x_1 + 2x_2 &= 4800 \\ x_2 &= 400 \\ x_4 &= 1100 \\ x_5 &= 350 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 400 \\ 0 \\ 1100 \\ 350 \\ 0 \end{bmatrix}$$

This is a basic solution that exists and is feasible (all variables are positive).

Basic Solution 13

Nonbasic Variables $\{x_4,\ x_5\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_2,\ x_3,\ x_6\}$

$$\begin{bmatrix}1 & 0 & 1 & 0 & 0 & 0\\0 & 1 & 0 & 1 & 0 & 0\\1 & 1 & 0 & 0 & 1 & 0\\4 & 2 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\0\\0\\x_6\end{bmatrix}=\begin{bmatrix}1000\\1500\\1750\\4800\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 0\\1 & 1 & 0 & 0\\4 & 2 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_6\end{bmatrix}=\begin{bmatrix}1000\\1500\\1750\\4800\end{bmatrix}$$

$$\begin{aligned}x_1 + x_3 &= 1000\\x_2 &= 1500\\x_1 + x_2 &= 1750\\4x_1 + 2x_2 + x_6 &= 4800\\x_1 &= 250\\x_6 &= 800\\x_3 &= 750\end{aligned}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=\begin{bmatrix}250\\1500\\750\\0\\0\\800\end{bmatrix}$$

This is a basic solution that exists and is feasible (all variables are positive).

Basic Solution 14

Nonbasic Variables $\{x_4,\ x_6\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_2,\ x_3,\ x_5\}$

$$\begin{bmatrix}1 & 0 & 1 & 0 & 0 & 0\\0 & 1 & 0 & 1 & 0 & 0\\1 & 1 & 0 & 0 & 1 & 0\\4 & 2 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\0\\x_5\\0\end{bmatrix}=\begin{bmatrix}1000\\1500\\1750\\4800\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 0\\1 & 1 & 0 & 1\\4 & 2 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_6\end{bmatrix}=\begin{bmatrix}1000\\1500\\1750\\4800\end{bmatrix}$$

$$\begin{aligned}x_1 + x_3 &= 1000\\x_2 &= 1500\\x_1 + x_2 + x_5 &= 1750\\4x_1 + 2x_2 &= 4800\end{aligned}$$

$$x_1 = 450$$
$$x_3 = 550$$
$$x_5 = -200$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 450 \\ 1500 \\ 550 \\ 0 \\ -200 \\ 0 \end{bmatrix}$$

This is a basic solution that exists but is infeasible (x_5 is negative and thereby violates the positivity constraint).

Basic Solution 15

Nonbasic Variables $\{x_5,\ x_6\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_2,\ x_3,\ x_4\}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 4 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_3 &= 1000 \\ x_2 + x_4 &= 1500 \\ x_1 + x_2 &= 1750 \\ 4x_1 + 2x_2 &= 4800 \\ 4(1750 - x_2) + 2x_2 &= 4800 \\ 7000 - 2x_2 &= 4800 \\ x_2 &= 1100 \\ x_1 &= 650 \\ x_4 &= 400 \\ x_3 &= 350 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 650 \\ 1100 \\ 350 \\ 400 \\ 0 \\ 0 \end{bmatrix}$$

This is a basic solution that exists and is feasible (all variables are positive).

Evaluation and Solution

Of the 15 above combinations of potential basic variables, 13 produce a basic solution. Of those 13 basic solutions, 6 are feasible and 7 are infeasible. The feasible basic solutions are numbered above as solutions 1, 3, 6, 12, 13 and 15. These basic feasible solutions will now be evaluated based on the objective function.

Basic Solution 1

Nonbasic Variables $\{x_1,\ x_2\} = 0$

Basic Variables $\mathcal{B} = \{x_3,\ x_4,\ x_5,\ x_6\}$

$$\phi' = \boldsymbol{c}^T\boldsymbol{x} = \begin{bmatrix} -12 \\ -9 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \times \begin{bmatrix} 0 \\ 0 \\ 1000 \\ 1500 \\ 1750 \\ 4800 \end{bmatrix}$$

$$\phi' = 0$$

$$\phi = 0$$

The income for this basic feasible solution is \$0.

Basic Solution 3

Nonbasic Variables $\{x_1,\ x_4\} = 0$

Basic Variables $\mathcal{B} = \{x_2,\ x_3,\ x_5,\ x_6\}$

$$\phi' = \boldsymbol{c}^T\boldsymbol{x} = \begin{bmatrix} -12 \\ -9 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \times \begin{bmatrix} 0 \\ 1500 \\ 1000 \\ 0 \\ 250 \\ 1800 \end{bmatrix}$$

$$\phi' = -13{,}500$$

$$\phi = 13{,}500$$

The income for this basic feasible solution is \$13,500.

Basic Solution 6

Nonbasic Variables $\{x_2,\ x_3\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_4,\ x_5,\ x_6\}$

$$\phi' = \boldsymbol{c}^T\boldsymbol{x} = \begin{bmatrix} -12 \\ -9 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \times \begin{bmatrix} 1000 \\ 0 \\ 0 \\ 1500 \\ 750 \\ 800 \end{bmatrix}$$

$$\phi' = -12{,}000$$

$$\phi = 12{,}000$$

The income for this basic feasible solution is $12,000.

Basic Solution 12

Nonbasic Variables $\{x_3,\ x_6\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_2,\ x_4,\ x_5\}$

$$\phi' = \boldsymbol{c}^T\boldsymbol{x} = \begin{bmatrix} -12 \\ -9 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \times \begin{bmatrix} 1000 \\ 400 \\ 0 \\ 1100 \\ 350 \\ 0 \end{bmatrix}$$

$$\phi' = -15{,}600$$

$$\phi = 15{,}600$$

The income for this basic feasible solution is $15,600.

Basic Solution 13

Nonbasic Variables $\{x_4,\ x_5\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_2,\ x_3,\ x_6\}$

$$\phi' = \boldsymbol{c}^T\boldsymbol{x} = \begin{bmatrix} -12 \\ -9 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \times \begin{bmatrix} 250 \\ 1500 \\ 750 \\ 0 \\ 0 \\ 800 \end{bmatrix}$$

$$\phi' = -16{,}500$$

$$\phi = 16{,}500$$

The income for this basic feasible solution is $16,500.

Basic Solution 15

Nonbasic Variables $\{x_5,\ x_6\} = 0$

Basic Variables $\mathcal{B} = \{x_1,\ x_2,\ x_3,\ x_4\}$

$$\phi' = \boldsymbol{c}^T\boldsymbol{x} = \begin{bmatrix} -12 \\ -9 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \times \begin{bmatrix} 650 \\ 1100 \\ 350 \\ 400 \\ 0 \\ 0 \end{bmatrix}$$

$$\phi' = -17{,}700$$

$$\phi = 17{,}700$$

The income for this basic feasible solution is $17,700.

Therefore, the optimal solution for this problem is $x_1 = 650$ and $x_2 = 1100$. This corresponds to making 650 Jamie Juices and 1100 Lloyd Libations. The income produced by this solution is $17,700.