CHEMICAL ENGINEERING 4G03

Module 05

Linear Programming (III) – Sensitivity

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Updated February 13, 2023

Outline of Module

This module consists of the following topics. This is a continuation of module 03 and 04, so we will be using those concepts extensively here.

Suggested Readings

Rardin (1st edition): Chapter 7.1, 7.2, 7.6

Rardin (2nd edition): Chapter 6.1, 6.2, 6.5

Note that this section deals only with the qualitative and quantitative analysis of sensitivity variables, NOT how those sensitivity variables are computed. For a review of the concept of **Duality**, which we will not go into in this course, you are welcome to check out chapter 6 in Rardin (the whole thing). Otherwise, it is sufficient for you to have an idea of how sensitivity analysis are used and interpreted for this course.

Sensitivity Analysis – Why Bother?

OK. Here is an admission about optimization: **All optimization answers need to be taken with a grain of salt**. Why is this important? Well, remember that the mathematical optimum of any optimization program is simply the combination of variables that gives us the best possible value of a pre-determined objective. However, we have some questions to ask ourselves:

- Do we trust this outcome if our model is imperfect (remember, *all* models are wrong…)
- Are the relationships between parameters and variables that we defined appropriate?
- What if *any* of our assumptions about costs, demands, profits, yields, *etc*. are wrong? They are, of course, *uncertain* and thus could be completely different!
- Does changing any of our assumptions **change the structure of our solution**?

Well, if any of these questions have been bothering you, you have come to the right place (for linear programs, at least). Luckily for us, linear programs lend themselves to **sensitivity analyses** or **what-if analyses** rather conveniently!

A **sensitivity analysis** attempts to describe the behaviour of an optimization program's *solution* with respect to changes in the model's *parameters*. Consider our example from last time:

$$
\begin{aligned}
\max_{x} \phi &= x_1 + x_2 \\
-x_1 + x_2 &\geq 0 \\
x_1 &\leq 2 \\
x_2 &\leq 3 \\
x_i &\geq 0 \qquad \forall\, i
\end{aligned}
$$

What would happen if we changed the second constraint to be $x_1 \leq 3$, for example? Would this have an immediate impact on the solution? If so, how much would it be, and how can we tell?

- Well, we could always **re-solve** the model using our Simplex Search, GAMS built-in solvers, or any other method. However, this might end up being *very time costly* if the optimization program we are trying to solve is very large and complex.
- Conveniently, performing a **sensitivity analysis** requires very little computation and allows us to build insight not only on how our *solution may change*, but also [therefore] the **robustness** of our model to changes in assumed parameters.

Sweet. Sounds like fun (does it?). So, what are the parameters that we can change easily in a sensitivity analysis? Well, we can change the RHS of our constraints and the cost coefficients in our objectives (think why). Let's talk about these qualitatively/graphically first, and quantitatively second.

Qualitative Sensitivity Analyses

As usual, it is useful for us to visualize what is going on before we try to compute anything numerically. This will give us key insights into what we are doing so we can **interpret the results** with meaning.

Relaxing and Tightening Constraints

Relaxing and tightening constraints can be interpreted as changing the *size* of the feasible region by *translating* (sliding) constraints to either make the region smaller (tightening) or larger (relaxing). This is done by changing the RHS of our constraints (that is, b in $Ax = b$) to affect the constraint intercept.

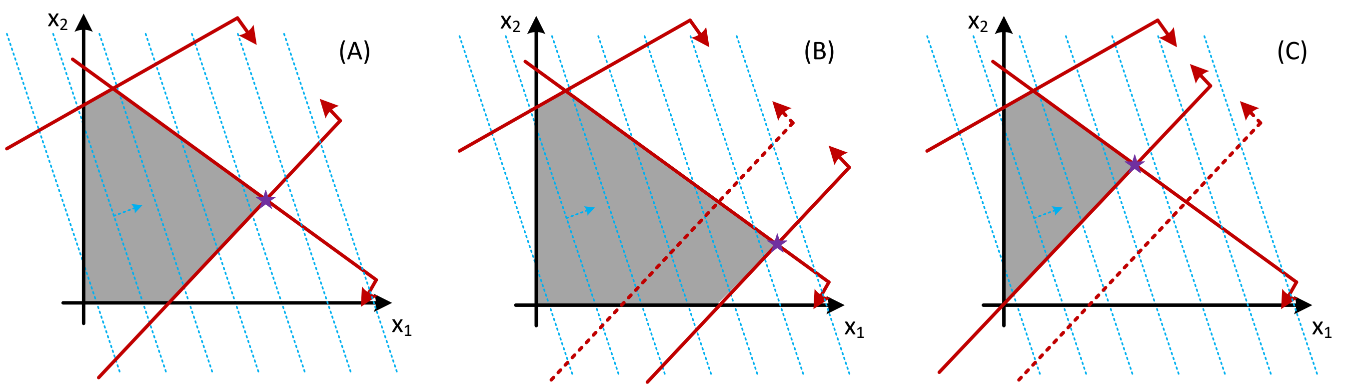
Relaxing and Tightening Constraints

A *modification* to a nominal constraint is said to be *relaxing* if it increases the size of the feasible region, and thus either **improves the optimal objective function value** (↑ max or ↓ min) OR leaves the optimal objective value **unchanged** at worst.

A *modification* to a nominal constraint is said to be *tightening* if it reduces the size of the feasible region, and thus either **makes the optimal objective function value worse** OR leaves it **unchanged** at best.

NB – tightening constraints too much may cause the problem to become *infeasible* (think why!)

Consider the figure below, in which a one of the constraints defining the nominal feasible set in (A) is relaxed (B) and tightened (C). Note how in this case the location of the optimum moves with the altered constraint, resulting in a new solution $x^\star$ and $\phi^\star$ with each change.



You can instantly notice that the size of the feasible set $\mathcal{S}$ changes as the perturbed constraint is translated left or right. Moreover, looking at the fact that this is a convex set, you should be able to convince yourself that making the feasible region *larger* can in no way make the optimum *worse*. This is why "relaxing" constraints, leading to larger feasible regions $\mathcal{S}$, will always either improve the objective at the optimum or leave it unchanged in a worst-case scenario. Vice-versa for tightening, clearly.

Class Workshop – Relaxing and Tightening Constraints

Consider the optimization program below. For each of the following *single* changes to the program, identify whether you anticipate the optimal objective function value $\phi^\star$ will increase, decrease or stay the same.

1. Change the RHS of the first constraint to -2.
2. Eliminate the second constraint.
3. Allow variable x_2 to be unrestricted (URS)
4. Change the first constraint to be an equality
5. Change the coefficient for x_1 in the first constraint to -2.
6. Add a new constraint $3x_1 + 5x_2 \leq 9$.

$$\begin{aligned} \max_{x} \phi &= x_1 + x_2 \\ -x_1 + x_2 &\geq 0 \\ x_1 &\leq 2 \\ x_2 &\leq 3 \\ x_i &\geq 0 \quad \forall i \end{aligned}$$

Workshop Solution – Relaxing and Tightening Constraints

1.

2.

3.

4.

5.

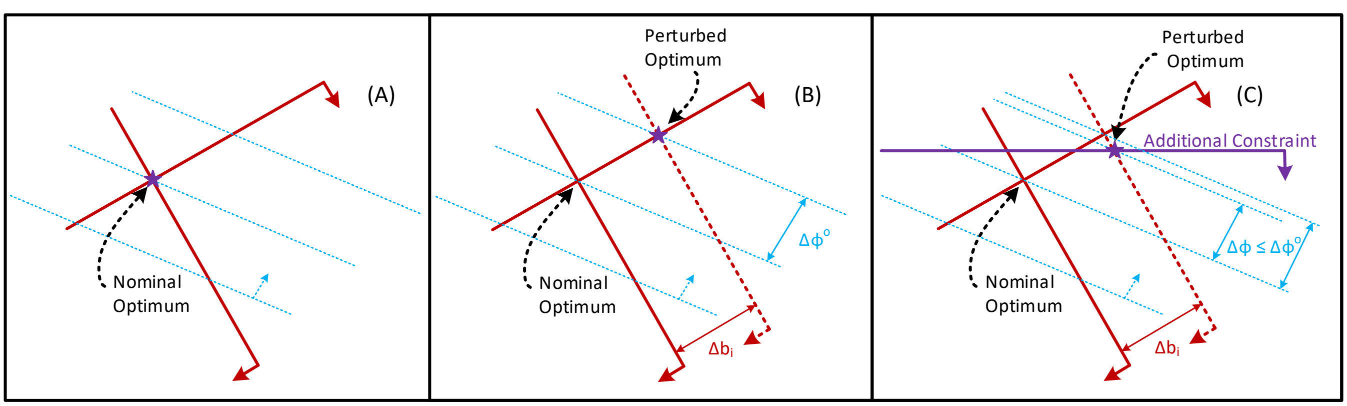
6.

If we translate a constraint left or right, we are effectively tracing the optimum **along another constraint** (see the figure above) until we stop. Thus, the optimal solution value ϕ^* changes *monotonically* as the feasible region is enlarged via relaxation or reduced via tightening. This can be seen in the figure above as ϕ^* slides along the left-most constraint at the slope of that constraint, and thus visits different contours monotonically. This is because there is no change in the **active set** (set of constraints active at the optimum).

Changes in the Active Set

Consider instead the following figure, which shows another linear program with the optimum located at the intersection of two constraints (the red lines). If I slide one of the constraints (b_i) to the right by some Δb_i, I can improve the value of the optimum by some amount $\Delta\phi^o$ (moving from one contour to another). This correlates to moving from figure (A) to (B). **HOWEVER**, what happens if I introduce a new constraint (purple line) and perform the same move to my original constraint? I end up with a value for ϕ that is still better than I started with, but **my active set has changed**, meaning that I don't get quite as much relative improvement as I did originally. In other words, $\Delta\phi < \Delta\phi^o$ for the same Δb_i. This correlates to moving from figure (A) to (C). To summarize:

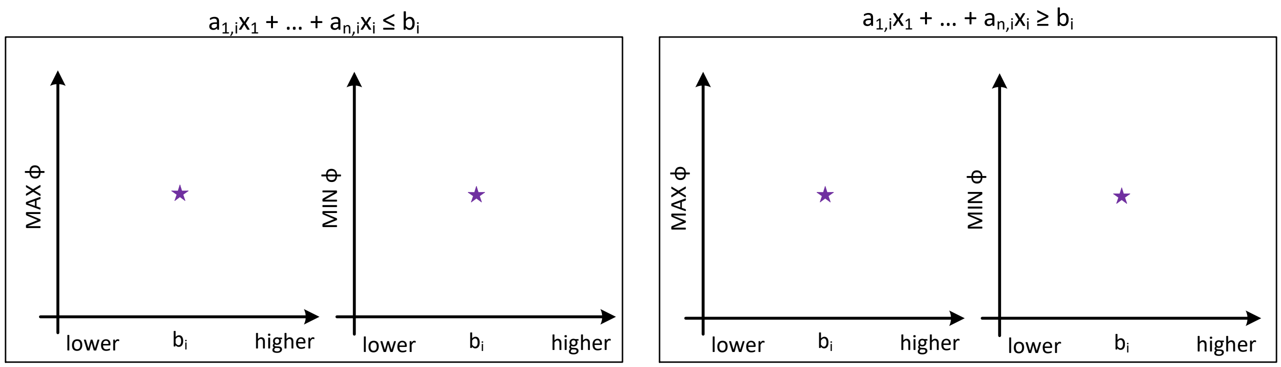
- **Without** a change in the active set: $\frac{\Delta\phi^o}{\Delta b_i} \triangleq v_i^o$ is the relative change in the objective function ϕ.
- **With** a change in the active set: $\frac{\Delta\phi}{\Delta b_i} \triangleq v_i < v_i^o$ is the relative change in the objective function ϕ.



Class Workshop – Qualitative Behaviour of Constraint Movement

In the space below, sketch what you would expect the typical qualitative behaviour of changing the RHS b_i of the constraints $a_{1,i}x_1 + \cdots + a_{n,i}x_n \leq b_i$ and $a_{1,i}x_1 + \cdots + a_{n,i}x_n \geq b_i$ would have on a nominal objective function value $\phi^\star$ for both a maximization and minimization objective. Assume that the active set changes over the range of b_i.

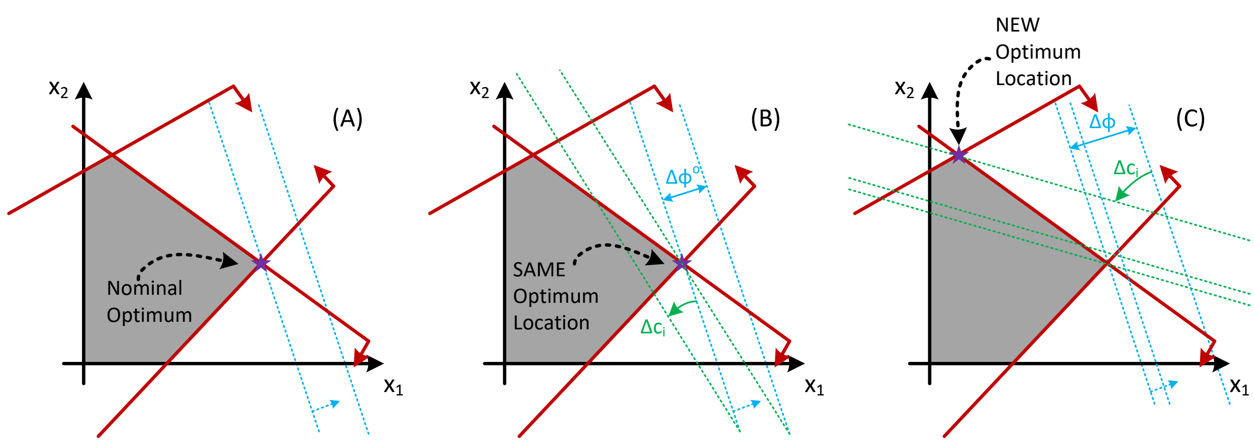
Workshop Solution – Qualitative Behaviour of Constraint Movement



Change in Cost Coefficients c_i

The other thing we can change via sensitivity are the cost coefficients c_i. In two dimensions, this is equivalent to changing ***the slope of the objective function contours***. It is important to note here that the feasible region thus does not change when there are changes in c_i. However, this does not mean the value of the objective function does not change! Furthermore, it also does not mean that the active set of constraints (and hence the solution location x^*) does not change either. Both can change, as shown in the figure below. When transitioning from figure (A) to (B), the slope of the objective function changes so that the location of the optimum does not move, but there is still an increase in the objective function value. When going from (A) to (C), the location of the optimum actually changes since the slopes of the objective contours change **SO** much that a better contour can be found elsewhere in the feasible region (at a corner point, of course).

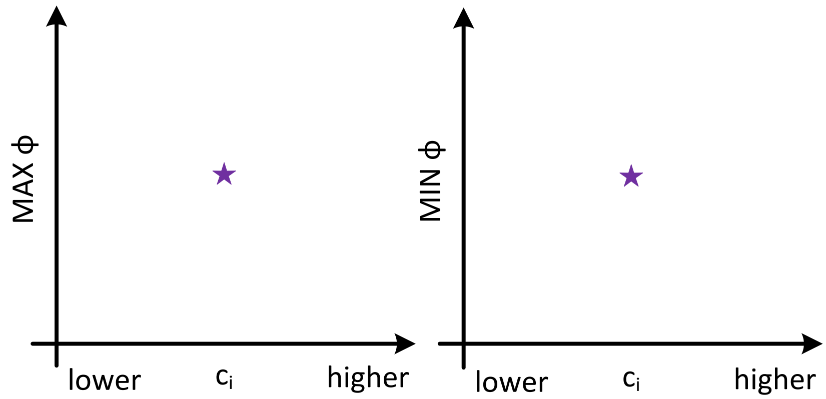
- **Without** a change in the active set: $\frac{\Delta\phi^o}{\Delta c_i} \triangleq \xi_i^o$ is the relative change in the objective function ϕ.
- **With** a change in the active set: $\frac{\Delta\phi}{\Delta c_i} \triangleq \xi_i > \xi_i^o$ is the relative change in the objective function ϕ.



Class Workshop – Qualitative Behaviour of Cost Coefficients

In the space below, sketch what you would expect the typical qualitative behaviour of changing the cost coefficient c_i of the objective function would have on the optimal value $\phi^\star$ for both a maximization and minimization objective. Assume that the active set changes over the range of c_i.

Workshop Solution – Qualitative Behaviour of Cost Coefficients



Quantitative Sensitivity Analysis

OK, it is all well and good to see things qualitatively. However, what we really need is a method to compare values quantitatively. The caveat to this is that we can only do this when the active constraint set does not change. Overall, see the following comments about the **marginal price** of a constraint:

- Quantitative analyses of marginal prices are **limited to optima with the same active constraints** as the nominal solution.
- Marginal prices obtained via quantitative sensitivity analyses provide the **exact** value of the relative change in the objective versus a change in the parameter ($\frac{\Delta\phi^\star}{\Delta b_i}$).
- Quantitative sensitivity analyses also provide the range over which a parameter can change before the active set changes (and the marginal price is thus invalid).

Wait… Marginal price?

Marginal (Shadow) Price

The rate of change in the optimal solution $\phi^\star$ to an infinitesimal *increase* in the right-hand side coefficient b_i for any constraint $\sum_{j=1}^{n} a_{i,j}x_j \; \{\leq, =, \geq\} \; b_i$ is known as the **marginal price** $\boldsymbol{\nu_i}$ of constraint i.

NB – The marginal price is also referred to as the *shadow price* or *dual variable* of constraint i. We do not cover duality theory in this class (it is too advanced), but the course text has some nice duality discussions if you are interested. Or talk to me. Or go to grad school.

Computing Marginal Prices

As I mentioned in the definition box above, we are not going to go into Duality theory in 4G03. If you are interested, the computation of the value of ν_i is derived from partitioning the basic and nonbasic variables at the optimal solution and computing the changes in objective for adjusting the i^{th} basic variable. This will correspond to changing the location of the solution, but NOT the active constraint set.

This procedure of computing adjusted costs for only the basic variables at a given solution has another name: it is known (well, a loose interpretation of this anyway) as the **formation of the optimization program DUAL**. There is some nifty theory in here, but the point I want you to take away is this:

> *For the purposes of this course, it is sufficient to assume that you will never need to compute the numerical value of the marginal prices for any constraints. Rather, it is our objective to have the computer software we are using (such as GAMS) to output the marginal prices for our program so that we may interpret the results.*

The one other thing that is important is:

> *The marginal price for ANY constraint that is inactive is identically equal to zero*.

This being said, let's look at the validity range for marginal prices and how to interpret/use them!

Range of Validity of Marginal Prices

The question that needs answering is: "How much can the RHS coefficient b_i be changed *without modifying the set of active constraints*?" In other words, for how long is the (constant) value of the marginal price ν_i valid? Recall our class workshop a few pages above... Once we change the active set, the rate of change of ϕ^* changes (and thus the marginal price changes) because the optimum exists at a different set of basic variables.

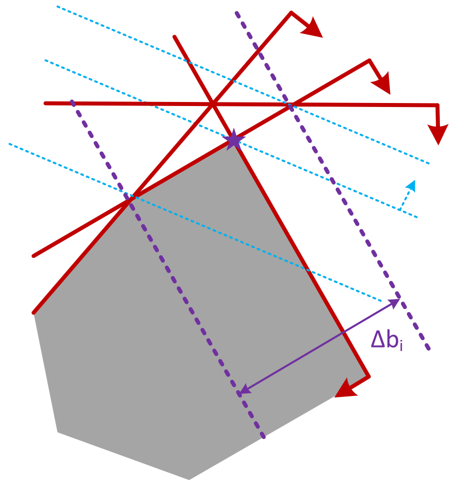
Valid Range of Marginal Prices

The valid range for a marginal price ν_i is the range in which a change in the objective function value for a proportional change in the RHS of the constraint b_i is **constant**.

Equivalently, a marginal price is valid for the region in which it is *constant*.

Equivalently, a marginal price is valid for the region in which the active set of constraints at the solution **is not changed** for any reason.

The range of a marginal price is typically reported as a maximum change in the positive and negative directions, and is demonstrated in the figure below. Note that in the figure below, any change to the constraint b_i more than Δb_i would result in a different feasible set.



Sensitivity to Changes in Cost Coefficients

The change in objective function value due to a change in cost coefficient i (Δc_i) can be computed as the product of the optimal solution at variable i $x_i^{\star}$ and the change in cost coefficient.

Change in Objective via Cost Coefficients

A change Δc_i in the cost coefficient for variable x_i results in a change in the optimum **when the active constraints remain identical** as:

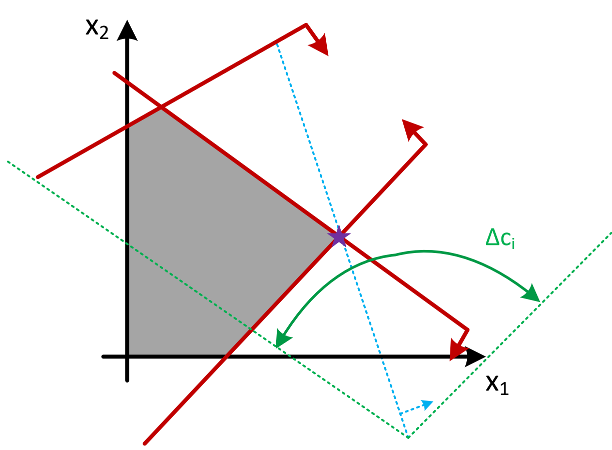
$\Delta\phi = 0$ IF x_i is nonbasic (why?).

$\Delta\phi = \Delta c_i x_i$ IF x_i is basic (why?).

Moreover, the solution point remains the same, or $\Delta x^{\star} = 0$ and the feasible set is unaffected.

Should the change in cost coefficient Δc_i cause the *active set* to change, the problem must be re-solved from scratch.

The figure below gives an example of the maximum allowable changes to the cost coefficients before the active set changes (and hence the location of the solution changes). Note, however, that changing the cost coefficients c_i do NOT affect the shape of the feasible set $\mathcal{S}$. ALSO – ask yourself why, in the figure below, the perturbed objective contours are at the angles they are. It has to do with the constraints!



Interpreting Computer Outputs of Sensitivity Analyses

Software products (such as GAMS) can report a variety of marginal price information. When using the correct options, GAMS is able to output:

- The **marginal price** ν_i and the associated **range of validity** Δb_i for every constraint.
- The **sensitivity to a change in cost coefficient** ξ_i and associated range of validity Δc_i.

Outside of the **ranges** reported by the software, there is very little we can do. We will have to resolve the problem! However, we CAN use our qualitative knowledge to form "best case" and "worst case" scenarios for when we stray from the bounds of our values of ν_i and ξ_i.

A *typical* sensitivity output from an optimization solver might be something along the lines of:

Constraint ID	Status	Slack	Marginal Price	Lower Range	Upper Range
Max. reflux flow	Active	0	3.74	123	47
Max. Pump 7	Inactive	321	0	321	1.0E+30

In the above example, `Constraint ID` is the name of the constraint we are changing the RHS of, `Status` is whether or not the constraint is active, `Slack` is how much we can change that constraint before it either becomes active or the `Marginal Price` (the impact on the objective) is no longer valid, and the `Lower` and `Upper Range` are how much we can increase or decrease the RHS (Δb_i in each direction) before the marginal price changes. We can now answer the following questions:

- What is the effect on profit (objective ϕ) of increasing the reflux flow by 2.0?

  2.0 is in the upper range of 47, so the marginal price is valid. The increase in profit will be $\Delta b \times \nu = 2.0 \times 3.74 = 7.48$ *increase!*

- Are the slack and marginal price values consistent for Pump 7?

  YES. The marginal price is 0, and thus the constraint is INACTIVE. We would need to reduce the RHS by $\Delta b = -321$ *before the constraint is activated and that variable has any impact on the solution.*

- What does a maximum allowable increase of `1.0E30` mean?

  It means that we can increase the maximum flow of pump 7 infinitely and it will STILL have no impact on the solution (clearly we are not using the maximum available flow of pump 7 so increasing it will not help us).

- Your co-worker suggests increasing the capacity for pump 7. What is your opinion?

  Since we are already not using the full capacity of pump 7 (negative slack available), increasing its capacity will not improve our objective function. We should avoid it.

- What is the effect of reducing the reflux flow by 125?

*Our maximum negative move according to our lower range is 123. We can therefore conclude that the cost of reducing the flow rate by **123** will be $\Delta\phi = \Delta b v = 3.74 \times (-123) = $ $\mathbf{-460.}$ The cost for the final 2 units is not known PRECISELY, but we know it will be GREATER than 3.74 per unit. We will need to re-solve the model to know for sure, but the **best-case scenario** is that we lose $125 \times 3.74 = 467.5$ if the marginal price becomes only very slightly worse. ALSO note that the problem may become infeasible if we reduce the reflux flow by more than 123.*

Multiple Parameter Changes

In the event that we want to analyze the effects of multiple changes to RHS coefficients b_i and/or cost coefficients c_j **at the same time**, we can use the provided marginal prices **as long as we abide by the rule of 100**:

Rule of 100

Consider the *percentage* of allowable changes to parameters being made, and **add all percentages**.

- If the sum is **no more than 100%**, the current active constraints will not change (all marginal prices and cost coefficient prices are valid)
- If the sum is **more than 100%**, the current set of active constraints at the optimum *might change* (and thus the marginal values may not be valid)

NB – The Rule of 100 does NOT apply when mixing RHS Δb_i and cost coefficient Δc_j moves simultaneously, only for a collection of one or the other.

Class Workshop – Rule of 100

Consider the following output from an optimization solver. What is the effect of increasing the reflux flow by 25 and reducing the capacity of pump 5 by 19 at the same time?

Constraint ID	Status	Slack	Marginal Price	Lower Range	Upper Range
Max. reflux flow	Active	0	3.74	123	47
Max. Pump 7	Inactive	321	0	321	1.0E+30
Max. Pump 5	Active	0	1.27	55	23

Workshop Solution – Rule of 100

The CPLEX Sensitivity Report in GAMS

The CPLEX sensitivity report in GAMS will look like the following figure. This is for our **Crude Oil Allocation Problem** from the first lecture module.

The first section of the output, given in **red**, shows the current RHS values of the constraints b_i (including the objective) and the current cost coefficients c_i (under the `VARIABLE NAME` header). This is where we get our information for the upper and lower bounds for Δb_i and Δc_i.

Under this, you will see the usual output that GAMS has always put out. You now know what the MARGINAL column means! It is the marginal price ν_i (for the equations) and cost dependence ξ_i (for the variables) in your sensitivity analysis! If you combine the table of allowable ranges with the marginal prices in the typical output, you can perform all relevant sensitivity calculations.

```
EQUATION NAME                                   LOWER      CURRENT       UPPER
-------------                                   -----      -------       -----
TOTAL_COST                                       -INF            0        +INF
DEMAND_GAS                                       1386         2000        2850
DEMAND_JET                                        750         1500        2667
DEMAND_LBRCT                                     -INF         1000        1587
AVAILABLE_SAUDI                                  2609         9000        +INF
AVAILABLE_VEN                                    3043         6000        +INF


VARIABLE NAME                                   LOWER      CURRENT       UPPER
-------------                                   -----      -------       -----
X1                                              28.12           50         100
X2                                              18.75         37.5       66.67
TC                                               -INF            1        +INF



                      LOWER     LEVEL     UPPER    MARGINAL

---- EQU TOTAL_COST     .         .         .        1.000
---- EQU DEMAND_GAS  2000.000  2000.000     +INF     65.217
---- EQU DEMAND_JET  1500.000  1500.000     +INF     76.087
---- EQU DEMAND_LB~  1000.000  1586.957     +INF       .
---- EQU AVAILABLE~     -INF   2608.696  9000.000      .
---- EQU AVAILABLE~     -INF   3043.478  6000.000      .

  TOTAL_COST  Objective function
  DEMAND_GAS  Demand for gasoline must be met
  DEMAND_JET  Demand for jet fuel must be met
  DEMAND_LBRCT  Demand for lubricants must be met
  AVAILABLE_SAUDI  Total availability from Saudi Arabia
  AVAILABLE_VEN  Total availability from Venezuela

                      LOWER     LEVEL     UPPER    MARGINAL

---- VAR X1             .      2608.696     +INF       .
---- VAR X2             .      3043.478     +INF       .
---- VAR TC           -INF   2.4457E+5     +INF       .
```

Class Workshop – Interpreting GAMS Output

Consider the GAMS CPLEX output for the two-crude distillation study and answer the following:

1. The demand for gasoline is anticipated to increase to 2,600. How much more per period is this anticipated to cost us?
2. Due to new import tariffs (among other things…) the cost of Saudi crude is increasing to $95 per barrel. Are we going to use less Saudi crude?
3. The demand for jet fuel in this period is anticipated to be only 700. What will be the anticipated impact on our costs? Are we going to use the same combination of oils?
4. The availability of Venezuelan crude has been cut to 3500 barrels due to a shipping incident. Do we change the blend? Will it cost us more?

Workshop Solution – Interpreting GAMS Output

Conclusions

Well, after all of that we now hopefully understand what a sensitivity analysis is and why it might be useful. Moreover, we know how to interpret the output of solvers to answer some questions:

- What are the effects of single changes to RHS or cost coefficients Δb_i or Δc_i?
- What happens when we change multiple RHS coefficients concurrently (but independent of c_i)?
- What happens when we change multiple cost coefficients concurrently (but independent of b_i)?

If our sensitivity analysis does NOT apply (rule of 100 violated or we are mixing changes), we must **re-solve the model**. We also DID NOT look at what happens when changing the LHS coefficients (A in $Ax = b$). We also did not examine precise computation of the marginal prices ν_i or cost dependencies ξ_i. Such topics are out of the scope of this course and are instead a graduate-level topic in linear duality.

Interpreting results and analyzing sensitivities are crucial skills for a professional engineer. We DO NOT want to just "take the numbers for what they are." It is key for us to take all results with a grain of salt, and look at the big picture!

~~ *END OF MODULE* ~~

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