Quantum Field Theory
Part II: Gauge theories

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Chapter 1 Functional methods

Invented for QED by Feynman and developed for and applied to

- non-Abelian gauge theories by Faddeev and Popov, etc.,
- spontaneously broken gauge theories by 't Hooft, Lee, etc.

Merits of functional methods:

- Easy derivation of Feynman rules, especially for theories with derivative interactions: scalar QED, non-Abelian gauge theories.
- All symmetries of a theory are more manifestly preserved since the method works with $\mathcal{L}$ instead of $\mathcal{H}$.

1.1 Path integrals in QM

1.2 Functional quantization of scalar fields

1.3 Quantization of the electromagnetic field

1.4 Functional quantization of spinor fields

1.5 Symmetries in the functional formalism

1.1 Path integrals in Quantum Mechanics

1. Motivating the functional method

Consider an NRQM particle moving in 1-dim:

$$H = \frac{p^2}{2m} + V(x) \tag{1}$$

Amplitude to travel from point x_a to point x_b after a duration of time T:

$$U(x_a, x_b, T) = \langle x_b | e^{-\frac{i}{\hbar}HT} | x_a \rangle \tag{2}$$

On the other hand, *Superposition Principle* implies:
total amplitude = coherent sum of amplitudes for alternative ways for the process to occur

$$\Rightarrow U(x_a, x_b; T) \sim \sum_{\text{all paths}} e^{i\cdot\text{phase}} \quad \equiv \quad \int \mathcal{D}x(t)\ e^{i\cdot\text{phase}} \tag{3}$$

equal weights of all paths

integral over a continuous space of functions;
functional integral, path integral, $[x(t)]$;
mapping functions to a complex number

Guess of "phase":
In *classical limit*, only one path contributes to amplitude, which is determined by the stationary condition:

$$\frac{\delta}{\delta x(t)}\Big(\text{phase}[x(t)]\Big)\Big|_{x_{\text{cl.}}} = 0$$

Comparing it to *Principle of Least Action*, $\frac{\delta}{\delta x(t)} S[x(t)]\Big|_{x_{\text{cl.}}} = 0$, we guess

$$\text{phase}[x(t)] = \hbar^{-1} S[x(t)] \tag{4}$$

$$\Rightarrow \quad \langle x_b | e^{-\frac{i}{\hbar}HT} | x_a \rangle = U(x_a, x_b; T) = \int \mathcal{D}x(t)\ e^{\frac{i}{\hbar}S[x(t)]} \tag{5}$$

Justification of (4) for double-slit experiment:

$$\text{action for path } 1 = \frac{1}{2}mv_1^2 t, \quad v_1 = \frac{D}{t} \implies (\text{phase } 1) = \frac{1}{2\hbar t}mD^2$$

$$\text{action for path } 2 = \frac{1}{2}mv_2^2 t, \quad v_2 = \frac{D+d}{t} \implies (\text{phase } 2) = \frac{1}{2\hbar t}m(D+d)^2$$

For monochromatic electron, $v_1 \approx v_2$, i.e., $D \gg d$, then (phase 2) − (phase 1) $\approx \frac{mDd}{\hbar t} \approx \frac{pd}{\hbar} = \frac{d}{\lambda}$, as expected for interference phenomenon

2. Equivalence of LH & RH in eqn.(5)

(1) Discretization of a path (a curve in (x,t) plane) to a zigzagged line

$(x_a, 0) \to (x_1, \epsilon) \to \cdots \to (x_{N-1}, T-\epsilon) \to (x_b, T)$:

$$T = \overbrace{\epsilon + \epsilon + \cdots + \epsilon}^{N}$$

$$S = \int_0^T dt \left[\frac{1}{2} m\dot{x}^2 - V(x)\right] \to \sum_{k=0}^{N-1} \left[\frac{1}{2} m \frac{(x_{k+1} - x_k)^2}{\epsilon} - \epsilon V\left(\frac{x_{k+1} + x_k}{2}\right)\right] \quad \begin{matrix} x_0 = x_a \\ x_N = x_b \end{matrix}$$

Define

$\int \mathcal{D}x(t)$ path integral

$$= \lim_{\epsilon \to 0} \frac{1}{C(\epsilon)} \int \frac{dx_1}{C(\epsilon)} \int \frac{dx_2}{C(\epsilon)} \cdots \int \frac{dx_{N-1}}{C(\epsilon)} = \lim_{\epsilon \to 0} \frac{1}{C(\epsilon)} \prod_{k=1}^{N-1} \left[\int_{-\infty}^{\infty} \frac{dx_k}{C(\epsilon)}\right] \text{ suppressed below} \quad (6)$$

(2) Differential equation fulfilled by RH of eqn.(5)

$$U(x_a, x_b; T) = \int \frac{dx'}{C(\epsilon)} \exp\left\{ \frac{i}{\hbar} \left[\frac{m(x_b - x')^2}{2\epsilon} - \epsilon V\left(\frac{x_b + x'}{2}\right)\right]\right\} \cdot U(x_a, x'; T - \epsilon) \quad (7)$$

$(x_a, 0) \to (x_b, T)$

contribution from the N-th slice
$(x', T - \epsilon) \to (x_b, T), x' = x_{N-1}$

$(x_a, 0) \to (x', T - \epsilon)$

$\epsilon \to 0$:

$$\begin{aligned} & U(x_a, x_b; T) \\ =& \int \frac{dx'}{C(\epsilon)} \exp\left[\frac{i}{\hbar}\frac{m(x_b - x')^2}{2\epsilon}\right] \left[1 - \frac{i}{\hbar}\epsilon V(x_b) + \cdots\right] \\ & \times \left[1 + (x' - x_b)\frac{\partial}{\partial x_b} + \frac{1}{2}(x' - x_b)^2 \frac{\partial^2}{\partial x_b^2} + \cdots\right] U(x_a, x_b; T - \epsilon) \\ =& \int \frac{dx'}{C(\epsilon)} \exp\left[\frac{i}{\hbar}\frac{m(x_b - x')^2}{2\epsilon}\right] \left[1 - \frac{i}{\hbar}\epsilon V(x_b) + \frac{1}{2}(x' - x_b)^2 \frac{\partial^2}{\partial x_b^2} + \cdots\right] U(x_a, x_b; T - \epsilon) \end{aligned}$$

will not contri. to the desired $O(\epsilon)$ term below

odd, vanishing

Using $\int d\xi \; e^{-b\xi^2} = \sqrt{\frac{\pi}{b}}, \quad \int d\xi \; \xi^2 e^{-b\xi^2} = \frac{1}{2b}\sqrt{\frac{\pi}{b}}, \quad b = \frac{m}{i2\hbar\epsilon}$

the above becomes

$$U(x_a, x_b; T) = \frac{1}{C(\epsilon)}\sqrt{\frac{i2\pi\hbar\epsilon}{m}}\left\{1 - \frac{i\epsilon}{\hbar}V(x_b) + \frac{i\hbar\epsilon}{2m}\frac{\partial^2}{\partial x_b^2} + O(\epsilon^2)\right\}U(x_a, x_b; T-\epsilon)$$

$$\therefore \quad C(\epsilon) = \sqrt{\frac{i2\pi\hbar\epsilon}{m}} \tag{8}$$

$O(\epsilon)$ terms give, as desired:

$$i\hbar\frac{\partial}{\partial T}U(x_a, x_b; T) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2} + V(x_b)\right)U(x_a, x_b; T) \equiv HU(x_a, x_b; T) \tag{9}$$

(3) Initial condition for RH of eqn. (5)

$T = \epsilon \to 0$: one slice, no integration at all:

$$\frac{1}{C(\epsilon)}\exp\frac{i}{\hbar}\left[\frac{m(x_b - x_a)^2}{2\epsilon} - \epsilon V\left(\frac{x_a + x_b}{2}\right)\right] \xrightarrow{\epsilon\to 0} \delta(x_a - x_b), \quad \text{as desired}$$

3. General formula

$H(q,p)$, conjugate (q^i, p^i)

$$U(q_a, q_b; T) = \langle q_b | e^{-iHT} | q_a \rangle \quad \text{(superscript } i \text{ often suppressed below, } \hbar = 1\text{)}$$

$$T = \underbrace{\epsilon + \epsilon + \cdots + \epsilon}_{N} \;\; , \quad e^{-iHT} = \underbrace{e^{-iH\epsilon} e^{-iH\epsilon} \cdots e^{-iH\epsilon}}_{N}$$

Inserting the completeness condition

$$\mathbf{1} = \int \Big(\prod_i dq_k^i \Big) |q_k\rangle\langle q_k| \quad (k = 1, \cdots, N-1) \tag{10}$$

as follows: $U(q_a, q_b; T) = \langle q_b| \int dq_{N-1}\, e^{-iH\epsilon} |q_{N-1}\rangle\langle q_{N-1}| \cdots \int dq_1 |q_1\rangle\langle q_1| e^{-iH\epsilon} |q_a\rangle$

Now work out the N factors thereof:

$$\langle q_{k+1}| e^{-iH\epsilon} |q_k\rangle = \langle q_{k+1}| \Big(1 - iH\epsilon + O(\epsilon^2)\Big) |q_k\rangle, \;\; k = 0, \cdots, N-1; \;\; q_0 \equiv q_a, \;\; q_N \equiv q_b \tag{11}$$

(1) Terms $f(q)$ in H

$$\begin{aligned} \langle q_{k+1}| f(q) |q_k\rangle &= f(q_k) \prod_i \delta(q_{k+1}^i - q_k^i) \\ &= f\Big(\frac{1}{2}(q_{k+1} + q_k)\Big) \int \prod_i \left(\frac{dp_k^i}{2\pi}\right) \exp\Big[i \sum_i p_k^i (q_{k+1}^i - q_k^i) \Big] \end{aligned} \tag{12}$$

(2) Terms $f(p)$ in H

$$\langle q_{k+1}|f(p)|q_k\rangle = \int \prod_i \left(\frac{dp_k^i}{2\pi}\right) f(p_k) \exp\left[i \sum_i p_k^i (q_{k+1}^i - q_k^i)\right] \quad \text{by inserting completeness condition in } p\text{-space} \tag{13}$$

(3) Mixed q and p terms in H

When Weyl-ordered, we can substitute simply: $q \to \frac{1}{2}(q_{k+1} + q_k)$, $p \to p_k$

Example: $\text{Weyl}(q^2p^2) = \frac{1}{4}(q^2p^2 + 2qp^2q + p^2q^2)$

Assume $H(q,p)$ is always Weyl-ordered. Then we have generally:

$$\langle q_{k+1}|H(q,p)|q_k\rangle = \int \prod_i \left(\frac{dp_k^i}{2\pi}\right) H\left(\frac{1}{2}(q_{k+1} + q_k), p_k\right) \exp\left[i \sum_i p_k^i (q_{k+1}^i - q_k^i)\right] \tag{14}$$

Exponentiating it back:

$$\langle q_{k+1}|e^{-i\epsilon H}|q_k\rangle = \int \prod_i \left(\frac{dp_k^i}{2\pi}\right) \exp\left\{i\Big[\sum_i p_k^i (q_{k+1}^i - q_k^i) - \epsilon H\left((q_{k+1} + q_k)/2, p_k\right)\Big]\right\}$$

Putting N slices together ($q_k: \ k = 1, \cdots, N-1$; $p_k: \ k = 1, \cdots, N$):

$$U(q_a, q_b; T) = \int \prod_{i,k} \left(\frac{dq_k^i dp_k^i}{2\pi} \right) \exp \left\{ i \sum_k \left[\sum_i p_k^i (q_{k+1}^i - q_k^i) - \epsilon H\big((q_{k+1} + q_k)/2, p_k\big) \right] \right\}$$

$$\equiv \int \prod_i \left(\mathcal{D}q^i(t) \mathcal{D}p^i(t) \right) \exp \left[i \int_0^T dt \left(\sum_i p^i \dot{q}^i - H(q^j, p^j) \right) \right] \qquad (15)$$

where the endpoints of $q(t)$ are constrained while those of $p(t)$ not.

Recovery of NRQM

$$H = \frac{p^2}{2m} + V(q)$$

$$\int \frac{dp_k}{2\pi} \exp \left[i \left(p_k (q_{k+1} - q_k) - \epsilon \frac{p_k^2}{2m} \right) \right] = \frac{1}{C(\epsilon)} \exp \left[\frac{im}{2\epsilon} (q_{k+1} - q_k)^2 \right]$$

N factors of $C(\epsilon)$ as needed

$\Rightarrow$ eqns $(5, 6)$

1.2 Functional quantization of scalar fields

Goal: deriving Feynman rules from functional integral expressions

Eqn. (15) holds for any quantum system, esp. for a real scalar field:

$$q^i \to \varphi(\mathbf{x})$$

$$\Rightarrow \quad H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + V(\varphi)\right) \tag{16}$$

$$\langle\varphi_b(\mathbf{x})|e^{-iHT}|\varphi_a(\mathbf{x})\rangle$$

$$= \int \mathcal{D}\varphi\mathcal{D}\pi \exp\left[i\int_0^T d^4x \left(\pi\dot{\varphi} - \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\varphi)^2 - V(\varphi)\right)\right] \quad \begin{array}{l}\varphi(x)|_{t=0} = \varphi_a(\mathbf{x}) \\ \varphi(x)|_{t=T} = \varphi_b(\mathbf{x})\end{array}$$

$$= \int \mathcal{D}\varphi \exp\left[i\int_0^T d^4x\ \mathcal{L}\right] \quad \text{(normalization constant ignored)} \tag{17}$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - V(\varphi) \tag{18}$$

(17) is better in manifesting symmetries.

Assumption: take eqns. (17,18) to define the quantum dynamical system.

1. Correlation functions

What's the time-ordered Green's function? Consider:

$$\int \mathcal{D}\varphi(x)\ \varphi(x_1)\varphi(x_2)\exp\left[i\int_{-T}^{T} d^4x\ \mathcal{L}(\varphi)\right] \quad \left(\begin{array}{l} \varphi(x)|_{t=-T} = \varphi_a(\mathbf{x}) \\ \varphi(x)|_{t=T} = \varphi_b(\mathbf{x}) \end{array}\right. \tag{19}$$

Split the measure as follows:

$$\int \mathcal{D}\varphi(x) = \int \mathcal{D}\varphi_1(\mathbf{x})\mathcal{D}\varphi_2(\mathbf{x}) \int_{\text{constraints}} \mathcal{D}\varphi(x) \quad \begin{array}{l} \varphi(x)|_{t=-T} = \varphi_a(\mathbf{x}), \varphi(x)|_{t=T} = \varphi_b(\mathbf{x}) \\ \varphi(x)|_{t=x_1^0} = \varphi_1(\mathbf{x}), \varphi(x)|_{t=x_2^0} = \varphi_2(\mathbf{x}) \end{array} \tag{20}$$

Then:

$$(19) = \int \mathcal{D}\varphi_1(\mathbf{x})\mathcal{D}\varphi_2(\mathbf{x})\ \varphi_1(\mathbf{x}_1)\varphi_2(\mathbf{x}_2) \int_{\text{constraints}} \mathcal{D}\varphi(x)\exp\left[i\int_{-T}^{T} d^4x\ \mathcal{L}(\varphi)\right]$$

For T sufficiently large, we have either $-T < x_1^0 < x_2^0 < T$ or $-T < x_2^0 < x_1^0 < T$.

For $-T < x_1^0 < x_2^0 < T$, the above factorizes using eq. (17) into

$$(19) \quad \overset{x_1^0<x_2^0}{=} \int \mathcal{D}\varphi_1(\mathbf{x})\mathcal{D}\varphi_2(\mathbf{x})\ \varphi_1(\mathbf{x}_1)\varphi_2(\mathbf{x}_2) \int_{(\cdots)} \mathcal{D}\varphi(x) \exp\left[i\int_{-T}^{x_1^0} d^4x\ \mathcal{L}(\varphi)\right]$$

$$\times \int_{(\cdots)} \mathcal{D}\varphi(x) \exp\left[i\int_{x_1^0}^{x_2^0} d^4x\ \mathcal{L}\right] \int_{(\cdots)} \mathcal{D}\varphi(x) \exp\left[i\int_{x_2^0}^{T} d^4x\ \mathcal{L}\right]$$

$$= \int \mathcal{D}\varphi_1(\mathbf{x})\mathcal{D}\varphi_2(\mathbf{x})\ \varphi_1(\mathbf{x}_1)\varphi_2(\mathbf{x}_2)$$

$$\times \langle\varphi_b|e^{-iH(T-x_2^0)}|\varphi_2\rangle\langle\varphi_2|e^{-iH(x_2^0-x_1^0)}|\varphi_1\rangle\langle\varphi_1|e^{-iH(x_1^0+T)}|\varphi_a\rangle$$

$$= \int \mathcal{D}\varphi_1(\mathbf{x})\mathcal{D}\varphi_2(\mathbf{x})\langle\varphi_b|e^{-iH(T-x_2^0)}\varphi_S(\mathbf{x}_2)|\varphi_2\rangle \quad \text{field operator in S. picture}$$

$$\times \langle\varphi_2|e^{-iH(x_2^0-x_1^0)}\varphi_S(\mathbf{x}_1)|\varphi_1\rangle\langle\varphi_1|e^{-iH(x_1^0+T)}|\varphi_a\rangle \quad \text{completeness relation}$$

$$= \langle\varphi_b|e^{-iH(T-x_2^0)}\varphi_S(\mathbf{x}_2)e^{-iH(x_2^0-x_1^0)}\varphi_S(\mathbf{x}_1)e^{-iH(x_1^0+T)}|\varphi_a\rangle$$

$$= \langle\varphi_b|e^{-iHT}\varphi_H(x_2)\varphi_H(x_1)e^{-iHT}|\varphi_a\rangle \quad \text{field operator in H. picture}$$

Similarly,

$$(19) \quad \overset{x_1^0 > x_2^0}{=} \langle \varphi_b | e^{-iHT} \varphi_H(x_1) \varphi_H(x_2) e^{-iHT} | \varphi_a \rangle$$

$$\therefore \quad (19) = \langle \varphi_b | e^{-iHT} T \left(\varphi_H(x_1) \varphi_H(x_2) \right) e^{-iHT} | \varphi_a \rangle \tag{21}$$

Use the same trick as in §4.2-QFT, to isolate the vacuum:

$$e^{-iHT} |\varphi_a\rangle = \sum_n e^{-iE_n T} |n\rangle \langle n | \varphi_a \rangle \xrightarrow{T \to \infty(1-i\epsilon)} \langle \Omega | \varphi_a \rangle \cdot e^{-iE_0 \cdot \infty (1-i\epsilon)} |\Omega\rangle$$

The strange factors appear independently of the two factors of φ. Thus,

$$\begin{aligned} &\langle \Omega | T \left(\varphi_H(x_1) \varphi_H(x_2) \right) |\Omega\rangle \\ = \quad &\lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}\varphi \; \varphi(x_1)\varphi(x_2) \exp\left[i \int d^4x \; \mathcal{L} \right]}{\int \mathcal{D}\varphi \; \exp\left[i \int d^4x \; \mathcal{L} \right]} \quad \text{key formula} \end{aligned} \tag{22}$$

Easily generalizable to higher-point functions.

[finished in 3 units on Sept 7, 2012.]

2. Feynman rules

(1) Real, free scalar: 2-point function

$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 \Rightarrow$ path integral in Gaussian form, thus computable

Discretization:

$\varphi(x) \to \varphi(x_i)$ at lattice site x_i; lattice spacing ϵ, space-time volume $L^4 \equiv V$

$$\varphi(x_i) = V^{-1}\sum_n e^{-ik_n\cdot x_i}\varphi(k_n) \quad , \quad k_n^\mu = \frac{2\pi}{L}n^\mu \ , \ n^\mu : \text{integer with } |n^\mu| < \frac{L}{\epsilon} \tag{23}$$

$\varphi^*(x_i) = \varphi(x_i) \Rightarrow \varphi^*(k_n) = \varphi(-k_n),$ can restrict ourselves to $\varphi(k_n)$ with $k_n^0 > 0$

$$D\varphi \to \prod_i d\varphi(x_i) \to \prod_{k_n^0>0}[d\mathrm{Re}\varphi_n \ d\mathrm{Im}\varphi_n] \qquad \varphi_n \equiv \varphi(k_n) \tag{24}$$

$$S_0 = \int d^4x \ \mathcal{L}_0 \to -V^{-1}\sum_n \frac{1}{2}(m^2 - k_n^2)|\varphi(k_n)|^2$$

$$= -V^{-1}\sum_{k_n^0>0}(m^2 - k_n^2)\Big[(\mathrm{Re}\varphi_n)^2 + (\mathrm{Im}\varphi_n)^2\Big].$$

Continuum limit:

$$L \to \infty \quad , \quad \epsilon \to 0 \quad , \quad V^{-1} \sum_n \leftrightarrow \int \frac{d^4 k}{(2\pi)^4} \tag{25}$$

Denominator of eqn.(22) ($> \equiv (k_n^0 > 0)$):

$$\begin{aligned}
& \int \mathcal{D}\varphi \; e^{iS_0} \\
\to \quad & \int \prod_{>} \Big(d\mathrm{Re}\varphi_n \; d\mathrm{Im}\varphi_n \Big) \exp \left\{ -iV^{-1} \sum_{>} (m^2 - k_n^2) \Big[(\mathrm{Re}\varphi_n)^2 + (\mathrm{Im}\varphi_n)^2 \Big] \right\} \\
= \quad & \int \prod_{>} \Big(d\mathrm{Re}\varphi_n \Big) \exp \left\{ -i \frac{1}{V} \sum_{>} (m^2 - k_n^2)(\mathrm{Re}\varphi_n)^2 \right\} \\
& \times \int \prod_{>} \Big(d\mathrm{Im}\varphi_n \Big) \exp \left\{ -i \frac{1}{V} \sum_{>} (m^2 - k_n^2)(\mathrm{Im}\varphi_n)^2 \right\} \\
= \quad & \prod_{>} \left(\sqrt{\frac{\pi V}{i(m^2 - k_n^2)}} \cdot \sqrt{\frac{\pi V}{i(m^2 - k_n^2)}} \right) = \prod_{k_n} \sqrt{\frac{\pi V}{i(m^2 - k_n^2)}}
\end{aligned} \tag{26}$$

Comments on (26)

- $T \to \infty(1 - i\epsilon) \Leftrightarrow t \to t(1 - i\epsilon)$

$\Leftrightarrow k^0 \to k^0(1 + i\epsilon)$ to keep Fourier expansion intact to $O(\epsilon^1)$

$\therefore m^2 - k_n^2 \to m^2 - k_n^2 - i\epsilon \leftrightarrow$ convergence factor $\exp\left[-\epsilon V^{-1}|\varphi_n|^2\right]$ in PI

- Generalization of ordinary Gaussian integral:

$$\int \Big(\prod_k d\xi_k\Big) \exp\left(-\xi_i B_{ij} \xi_j\right) = \prod_i \sqrt{\frac{\pi}{b_i}} = \text{const.} \times (\det B)^{-\frac{1}{2}} \tag{27}$$

where B_{ij}: symmetric, with eigenvalues b_i and $\mathrm{Re}\, b_i > 0$

In our case:

$$S_0 = \int d^4x \, \frac{1}{2} \varphi(x)(-\partial^2 - m^2)\varphi(x) + \text{surface term}$$

$$\int \mathcal{D}\varphi \; e^{iS_0} \to \text{const.} \times \left[\det(\partial^2 + m^2)\right]^{-\frac{1}{2}} \qquad \text{functional determinant} \tag{28}$$

whose equivalent form in momentum space is given in eq. (26).

Numerator of eqn. (22)

$$
\begin{aligned}
\varphi(x_1)\varphi(x_2) \quad &\to \quad V^{-1}\sum_m e^{-ik_m\cdot x_1}\varphi_m \cdot V^{-1}\sum_l e^{-ik_l\cdot x_2}\varphi_l \\
\text{numerator} \quad &\to \quad V^{-2}\sum_{m,l} e^{-i(k_m\cdot x_1+k_l\cdot x_2)}\int\prod_{>}(d\mathrm{Re}\varphi_n\; d\mathrm{Im}\varphi_n) \\
&\quad \times(\mathrm{Re}\varphi_m+i\mathrm{Im}\varphi_m)(\mathrm{Re}\varphi_l+i\mathrm{Im}\varphi_l) \\
&\quad \times\exp\left[-iV^{-1}\sum_{>}(m^2-k_n^2)\left((\mathrm{Re}\varphi_n)^2+(\mathrm{Im}\varphi_n)^2\right)\right] \qquad (29)
\end{aligned}
$$

Most terms vanish either because the integrand is odd or because cancelation occurs between Re and Im terms ($k_m = k_l$).

Remaining term: $k_m = -k_l$. Re and Im contribute the same

$$
\text{numerator} \quad \to \quad V^{-2}\sum_l e^{-ik_l\cdot(x_1-x_2)}\underbrace{\left(\prod_{>}\frac{\pi V}{i(m^2-k_n^2)}\right)}_{\text{canceled by denominator}}\cdot\frac{V}{i(m^2-k_l^2-i\epsilon)}
$$

Result of eqn. (22)

$$\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle \quad \to \quad V^{-1}\sum_l e^{-ik_l\cdot(x_1-x_2)}\frac{1}{i(m^2-k_l^2-i\epsilon)}$$
$$\to \quad \int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik\cdot(x_1-x_2)} = D_F(x_1-x_2) \qquad (30)$$

(2) Real, free scalar: 4-point function

Numerator contains $(\mathrm{Re}\varphi_m+i\mathrm{Im}\varphi_m)(\mathrm{Re}\varphi_l+i\mathrm{Im}\varphi_l)(\mathrm{Re}\varphi_p+i\mathrm{Im}\varphi_p)(\mathrm{Re}\varphi_q+i\mathrm{Im}\varphi_q)$.

Non-vanishing terms are only from pairings like $k_l=-k_m, k_q=-k_p$, where Im parts double the Re terms' contributions:

$$V^{-4}\sum_{m,p} e^{-ik_m\cdot(x_1-x_2)}e^{-ik_p\cdot(x_3-x_4)}\left(\prod_{>}\frac{\pi V}{i(m^2-k_n^2)}\right)\frac{V}{i(m^2-k_m^2-i\epsilon)}\frac{V}{i(m^2-k_p^2-i\epsilon)}$$

$$\xrightarrow{V\to\infty}\left(\prod_{>}\frac{\pi V}{i(m^2-k_n^2)}\right)D_F(x_1-x_2)D_F(x_3-x_4) \quad \text{canceled by denominator}$$

Similarly for the other two pairings.

$$\therefore \quad \langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle \quad = \quad \frac{\displaystyle\int \mathcal{D}\varphi \; \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) \; e^{iS_0}}{\displaystyle\int \mathcal{D}\varphi \; e^{iS_0}}$$

$$= \quad \text{sum of all full contractions} \tag{31}$$

Easily extendable to higher functions. Wick's theorem always reproduced.

(3) Real scalar φ^4

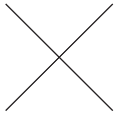
$$\mathcal{L} = \mathcal{L}_0 - \frac{\lambda}{4!}\varphi^4$$

We don't know how to integrate PI *exactly*.
But for λ small, we can Taylor expand

$$\exp\left(i\int d^4x \; \mathcal{L}\right) = \exp(iS_0)\left[1 - \frac{i\lambda}{4!}\int d^4y \; \varphi^4(y) + \cdots\right]$$

Both numerator and denominator can be worked out order by order in λ

using Gaussian integrals as for a free field. The result is identical to §4.2-QFT.



$$= -i\lambda(2\pi)^4\delta^4(\Sigma p) \tag{32}$$

3. Functional derivatives and generating functional

Want a more *systematic method* to compute Green's functions!

Functional derivative

$$\frac{\delta}{\delta J(x)} J(y) = \delta^4(x-y) \longleftrightarrow \frac{\delta}{\delta J(x)} \int d^4y\; J(y)\varphi(y) = \varphi(x) \tag{33}$$

Examples:

$$\frac{\delta}{\delta J(x)} \exp\left[i\int d^4y\; J(y)\varphi(y)\right] = i\varphi(x)\exp\left[i\int d^4y\; J(y)\varphi(y)\right]$$

$$\frac{\delta}{\delta J(x)} \int d^4y\; \left(\partial_\mu J(y)\right) V^\mu(y) = -\partial_\mu V^\mu(x)$$

Generating functional

$$Z[J] = \int \mathcal{D}\varphi \ \exp\left[iS + i\int d^4x \ J(x)\varphi(x)\right] \qquad \text{source} \tag{34}$$

e.g.,
$$\langle\Omega|T\varphi(x_1)\varphi(x_2)|\Omega\rangle = \frac{1}{Z_0}\left(\frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)}Z[J]\right)_{J=0} \quad , \quad Z_0 \equiv Z[0] \tag{35}$$

Similarly for higher functions.

Example 1: free theory

$$\int d^4x \ [\mathcal{L}_0(\varphi) + J\varphi] = \int d^4x \ \left[\frac{1}{2}\varphi(-\partial^2 - m^2 + i\epsilon)\varphi + J\varphi\right] \quad \text{convergence and boundary condition}$$

$$= \quad \int d^4x \ \frac{1}{2}\varphi'(-\partial^2 - m^2 + i\epsilon)\varphi' - \int d^4x d^4y \ \frac{1}{2}J(x)\left(-iD_F(x-y)\right)J(y)$$

where the following formulas are used:

$$\begin{cases} \varphi'(x) \equiv \varphi(x) - i\int d^4y \ D_F(x-y)J(y) \\ (-\partial^2 - m^2 + i\epsilon)D_F(x-y) = i\delta^4(x-y) \end{cases}$$

More formally, we write

$$\varphi' = \varphi + (-\partial^2 - m^2 + i\epsilon)^{-1} J \tag{36}$$

$$\int d^4x [\mathcal{L}_0(\varphi) + J\varphi] = \int d^4x \Big[\frac{1}{2}\varphi'(-\partial^2 - m^2 + i\epsilon)\varphi' - \frac{1}{2}J(-\partial^2 - m^2 + i\epsilon)^{-1}J\Big] \tag{37}$$

$$\Rightarrow \quad Z[J] = \int \mathcal{D}\varphi \; e^{iS_0[\varphi] + i\int J\varphi}$$

$$= \left(\int \mathcal{D}\varphi' \; e^{iS_0[\varphi']}\right) \exp\left(+\frac{i^2}{2}\int d^4x d^4y J(x) D_F(x-y) J(y)\right) \quad \text{Jacobian} \; = 1$$

$$= Z_0 \exp\left[-\frac{1}{2}\int d^4x d^4y \; J(x) D_F(x-y) J(y)\right] \tag{38}$$

Use of (35) and (38) yields:

$$\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle = -\left(\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)} \exp\Big[-\frac{1}{2}\int d^4x d^4y J(x) D_F(x-y) J(y)\Big]\right)_{J=0}$$

$$= \quad -\left(\frac{\delta}{\delta J(x_1)}\left\{-\frac{1}{2}\int d^4y D_F(x_2 - y) J(y) - \frac{1}{2}\int d^4x J(x) D_F(x - x_2)\right\} \exp[\cdots]\right)_{J=0}$$

$$= \quad D_F(x_1 - x_2)$$

and in brief notations

$$\langle 0|T\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle = i^{-4}\left[\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_4}\exp\Big(-\frac{1}{2}J_x D_{xy} J_y\Big)\right]_{J=0}$$

$$= \left[\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}(-J_x D_{x4})\exp\Big(-\frac{1}{2}J_x D_{xy} J_y\Big)\right]_{J=0} = \cdots$$

$$= D_{34}D_{12} + D_{24}D_{13} + D_{14}D_{23}$$

Example 2: φ^4

(35) is useful for interaction theory because $Z[J]$ can be worked out order by order in terms of $Z_{\text{free}}[J]$:

$$\begin{aligned} Z[J] &= \int\mathcal{D}\varphi\; e^{iS_0[\varphi]+i\int J\varphi}\;\exp\left[-\frac{i\lambda}{4!}\int\varphi^4\right] \\ &= \int\mathcal{D}\varphi\;\exp\left[-\frac{i\lambda}{4!}\int d^4z\frac{\delta^4}{i^4\delta J^4(z)}\right]e^{iS_0[\varphi]+i\int J\varphi} \\ &= \exp\left[-\frac{i\lambda}{4!}\int d^4z\frac{\delta^4}{i^4\delta J^4(z)}\right]Z_{\text{free}}[J], \end{aligned}$$

where $Z_{\text{free}}[J]$ is known from (38).

1.3 Quantization of the electromagnetic field

We wrote down the photon propagator without proof in §4.8-QFT.
Now we derive it!

1. Why difficult to quantize: gauge invariance!

$$\mathcal{L} = -\frac{1}{4}\left(F_{\mu\nu}\right)^2$$

$$\Rightarrow \quad S = \frac{1}{2}\int d^4x \; A_\mu(x)(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu(x) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\tilde{A}_\mu(k)(-k^2 g^{\mu\nu} + k^\mu k^\nu)\tilde{A}_\nu(k)$$

$S = 0$ for $\tilde{A}_\mu(k) = k_\mu C(k)$, i.e., for all field configurations that are gauge equivalent to $A_\mu(x) = 0$!

$\Rightarrow \int \mathcal{D}A \; e^{iS[A]}$ contains a divergent factor due to gauge invariance!

Or, try to find the photon propagator:

$$(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)D_F^{\nu\rho}(x-y) = i g_\mu^\rho \delta^4(x-y) \leftrightarrow (-k^2 g_{\mu\nu} + k_\mu k_\nu)\tilde{D}_F^{\nu\rho}(k) = i g_\mu^\rho$$

from gauge invariance — not invertible!

2. Solution via the Faddeev-Popov method

Strip the factor due to gauge invar and count each *physically independent* configuration only once

(1) Isolating gauge equivalent factor

Gauge condition: $G(A) = 0$; e.g., Lorentz gauge: $G(A) = \partial_\mu A^\mu$

Inserting in PI the identity:

$$\mathbf{1} = \int \mathcal{D}\alpha(x)\ \delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) \quad \text{generalization of ordinary formula} \tag{39}$$

$$A^\alpha_\mu \equiv A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x), \quad \text{e.g., Lorentz gauge: } \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \det(e^{-1}\partial^2)$$

Assume $\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$ is indept. of A:

$$\int \mathcal{D}A\ e^{iS[A]} = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) \int \mathcal{D}\alpha \int \mathcal{D}A\ \delta(G(A^\alpha)) e^{iS[A]} \tag{40}$$

Using $\mathcal{D}A = \mathcal{D}A^\alpha,\ S[A] = S[A^\alpha]$,

$$\int \mathcal{D}A\, e^{iS[A]} = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\int\mathcal{D}\alpha\int\mathcal{D}A^\alpha\,\delta(G(A^\alpha))e^{iS[A^\alpha]}$$

$$= \det(\cdots)\underbrace{\left(\int\mathcal{D}\alpha\right)}_{\text{volume of gauge equivalent parameter space}}\underbrace{\left(\int\mathcal{D}A\,\delta(G(A))e^{iS[A]}\right)}_{\text{independent of }\alpha}\quad(\text{renaming}) \tag{41}$$

(2) Gauge-fixed $\mathcal{L}$

$\delta(G(A))$ inconvenient for computation. Try to trade it for a new term in $\mathcal{L}$. Consider

$$G(A) = \partial^\mu A_\mu(x) - \omega(x),\quad \omega : \text{any scalar function} \tag{42}$$

$$\int\mathcal{D}A\, e^{iS[A]} = \det(e^{-1}\partial^2)\left(\int\mathcal{D}\alpha\right)\left(\int\mathcal{D}A\,\delta(\partial A-\omega)e^{iS[A]}\right) \tag{43}$$

The above holds for any $\omega(x)$. Thus we can form a linear combination of different ω's and do a Gaussian weighting centered on $\omega = 0$

$$\begin{aligned}
\int \mathcal{D}A\, e^{iS[A]} &= N(\xi) \int \mathcal{D}\omega \, \exp\left[-\frac{i}{2\xi}\int d^4x\, \omega^2(x)\right] \det(e^{-1}\partial^2) \quad \text{normalization factor} \\
&\times \left(\int \mathcal{D}\alpha\right)\left(\int \mathcal{D}A\, \delta(\partial A - \omega) e^{iS[A]}\right) \quad \xi: \text{ any real number} \\
&= N(\xi)\det(e^{-1}\partial^2)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A\, \exp\left[iS[A] - \frac{i}{2\xi}\int d^4x (\partial A)^2\right] \qquad (44)
\end{aligned}$$

$\Rightarrow$ Except for A-indept. factors, $\delta(G(A))$ is equivalent to the change:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 \Rightarrow \mathcal{L}_{\text{eff}} = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \qquad (45)$$

Since the only property used in the derivation is gauge invar, those factors are cancelled in Green's functions of gauge invariant operators:

$$\langle\Omega|T\mathcal{O}(A)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}A\, \mathcal{O}(A) \exp\left[i\int_{-T}^{T} d^4x\, \mathcal{L}_{\text{eff}}\right]}{\int \mathcal{D}A\, \exp\left[i\int_{-T}^{T} d^4x\, \mathcal{L}_{\text{eff}}\right]} \qquad (46)$$

(3) Photon propagator

Differential equation for Green's function of $\mathcal{L}_{\text{eff}}$ gives

$$\left(-k^2 g_{\mu\nu} + k_\mu k_\nu (1 - \xi^{-1})\right) \tilde{D}_F^{\nu\rho}(k) = i g_\mu^\rho$$

which is invertible: by assuming $\tilde{D}_F^{\nu\rho}(k) \equiv -i[A(k^2) g^{\nu\rho} + k^\nu k^\rho B(k^2)]$, one gets

$$\tilde{D}_F^{\nu\rho}(k) = \frac{-i}{k^2 + i\epsilon}\left[g^{\nu\rho} - (1-\xi)\frac{k^\nu k^\rho}{k^2}\right], \quad \xi = \begin{cases} 0, & \text{Landau gauge} \\ 1, & \text{Feynman gauge} \end{cases} \tag{47}$$

Ward identity guarantees that the S matrix computed with the ξ-dependent photon propagator is ξ-indept and unitary.

[finished in 3 units on Sept 14, 2012.]

1.4 Functional quantization of spinor fields

boson fields	fermion fields
$\updownarrow$	$\updownarrow$
commuting ordinary functions in PI	anticommuting (Grassmannian) functions in PI
$\updownarrow$	$\updownarrow$
quantum canonical commutation relations $[,]$	quantum canonical anticommutation relations $\{,\}$

1. Anticommuting (Grassmann) numbers

Algebra: Grassmann numbers $\eta, \theta, \cdots$; c-numbers $c_i, \cdots$:

$$\eta\theta = -\theta\eta, \quad \eta^2 = \theta^2 = 0; \quad c_1\eta + c_2\theta;$$

$$(c_1\eta + c_2\theta)(c_3\eta + c_4\theta) = (c_1c_4 - c_2c_3)\eta\theta; \quad (\eta\theta)^* = \theta^*\eta^* = -\eta^*\theta^*$$

Differentiation:

$$\frac{d}{d\eta}\eta = 1, \quad \frac{d}{d\eta}(\theta\eta) = -\frac{d}{d\eta}(\eta\theta) = -\theta, \quad \frac{d}{d\eta}(c_1\eta) = c_1$$

Integration: Any function $f(\theta)$ of θ terminates in the linear term: $f(\theta) = A + B\theta$

- Linear: $\int d\theta (A + B\theta) = \int d\theta A + \int d\theta B\theta = \begin{cases} -A \int d\theta + B \int d\theta\theta \ (f : \text{Grassmann}) \\ A \int d\theta - B \int d\theta\theta \ (f : \text{c-number}) \end{cases}$
- Invariant under const. shift of integration variable: $\theta \to \theta + \eta$

$$\int d\theta (A + B\theta) = \int d\theta ((A + B\eta) + B\theta) = \int d\theta (A + B\eta) + \int d\theta B\theta$$

$\therefore$ can only depend on B. Define:

$$\int d\theta \ 1 = 0, \quad \int d\theta \ \theta = 1; \quad \text{convention} : \int d\theta \int d\eta \ \eta \ \theta = \int d\theta \ \theta = 1 \quad \text{inner first}$$

Gaussian integrals:

θ and θ^* can be treated as 2 indept. Grassmann numbers.

$$\int d\theta^* d\theta \ e^{-\theta^* b\theta} = \int d\theta^* d\theta \ (1 - \theta^* b\theta) = 0 + b \int d\theta^* d\theta \ \theta\theta^* = b \ \text{ c-number} \tag{48}$$

$$\int d\theta^* d\theta \ \theta\theta^* e^{-\theta^* b\theta} = \int d\theta^* d\theta \ \theta\theta^* \cdot 1 = 1 = b \cdot \frac{1}{b} \tag{49}$$

Compare to the ordinary case:

$$\int dz^* dz\ e^{-b|z|^2} = \left(\int dz_1\ e^{-bz_1^2}\right)^2 = \frac{\pi}{b}; \quad \int dz^* dz\ |z|^2 e^{-b|z|^2} = \frac{\pi}{b}\cdot\frac{1}{b}$$

Under a unitary transformation, $\theta_i' = \sum_j U_{ij}\theta_j,\ U^\dagger = U^{-1}$:

$$\prod_i \theta_i' = \cdots = (\det U)\prod_i \theta_i \quad , \quad \prod_i d\theta_i' = \cdots = \prod_i d\theta_i (\det U)$$

$$\Rightarrow \quad \Big(\prod_i \theta_i'\Big)\Big(\prod_j \theta_j'^*\Big) = \Big(\prod_i \theta_i\Big)\Big(\prod_j \theta_j^*\Big) \quad , \quad \Big(\prod_i d\theta_i'\Big)\Big(\prod_j d\theta_j'^*\Big) = \Big(\prod_i d\theta_i\Big)\Big(\prod_j d\theta_j^*\Big)$$

Then, for a Hermitian matrix B with eigenvalues b_i $(U^\dagger BU = b)$, upon $\theta \to U\theta$:

$$\int \prod_i (d\theta_i^* d\theta_i) \exp\Big[-\sum_{ij}\theta_i^* B_{ij}\theta_j\Big] = \int \prod_i (d\theta_i^* d\theta_i) \exp\Big[-\sum_j b_j \theta_j^* \theta_j\Big]$$

$$= \quad \prod_i \left(\int d\theta_i^* d\theta_i\ \exp\left[-b_i\theta_i^*\theta_i\right]\right) = \prod_i b_i = \det B \qquad (50)$$

$$\int\prod_i(d\theta_i^* d\theta_i)\exp\Big[-\sum_{ij}\theta_i^* B_{ij}\theta_j\Big]\theta_k\theta_l^* = \int\prod_i(d\theta_i^* d\theta_i)\sum_{k'l'}U_{kk'}U_{ll'}^*\theta_{k'}\theta_{l'}^*\exp\Big[-\sum_j b_j\theta_j^*\theta_j\Big]$$

$$= \int\prod_i(d\theta_i^* d\theta_i)\sum_{k'}U_{kk'}U_{lk'}^*\theta_{k'}\theta_{k'}^*\exp\Big[-\sum_j b_j\theta_j^*\theta_j\Big] = \sum_{k'}U_{kk'}U_{lk'}^*\frac{\prod_j b_j}{b_{k'}}$$

$$= \det B\sum_{k'}U_{kk'}b_{k'}^{-1}U_{lk'}^* = (\det B)(Ub^{-1}U^\dagger)_{kl} = (\det B)(B^{-1})_{kl} \tag{51}$$

Compare to the ordinary case:

$$\int\prod_i\Big(dz_i^* dz_i\Big)\exp\Big[-\sum_{ij}B_{ij}z_i^* z_j\Big] = \int\prod_i\Big(dz_i^* dz_i\Big)\exp\Big[-\sum_j b_j|z_j|^2\Big] = \Big[\det(B/\pi)\Big]^{-1},$$

$$\int\prod_i\Big(dz_i^* dz_i\Big)z_k z_l^*\exp\Big[-\sum_{ij}B_{ij}z_i^* z_j\Big] = \int\prod_i\Big(dz_i^* dz_i\Big)\sum_{k'l'}U_{kk'}U_{ll'}^* z_{k'}z_{l'}^* e^{-\sum_j b_j|z_j|^2}$$

$$= \sum_{k'}U_{kk'}U_{lk'}^*\int\prod_i\Big(dz_i^* dz_i\Big)|z_{k'}|^2\exp\Big[-\sum_j b_j|z_j|^2\Big] = \sum_{k'}U_{kk'}U_{lk'}^*\cdot b_{k'}^{-1}\Big[\det(B/\pi)\Big]^{-1}$$

$$= (Ub^{-1}U^\dagger)_{kl}\Big[\det(B/\pi)\Big]^{-1} = (B^{-1})_{kl}\Big[\det(B/\pi)\Big]^{-1}.$$

2. Dirac propagator

Dirac field is a Grassmann field:

$$\underset{\text{Dirac field}}{\psi(x)} = \sum \underset{\text{Grass. number}}{\psi_i} \cdot \underset{\text{basis of orthogonal c-number functions}}{\varphi_i(x)} \tag{52}$$

$$\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ \psi(x_1)\bar{\psi}(x_2) \exp\left[i\int d^4x\ \bar{\psi}(i\partial\!\!\!/ - m)\psi\right]}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[i\int d^4x\ \bar{\psi}(i\partial\!\!\!/ - m)\psi\right]} \tag{53}$$

Using formulas (50, 51):

$$\text{denominator} \sim \det(i\partial\!\!\!/ - m), \quad \text{numerator} \sim [-i(i\partial\!\!\!/ - m)]^{-1}\det(i\partial\!\!\!/ - m)$$

As for the scalar field, the inverse may be found by Fourier transf.:

$$\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle = S_F(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k\!\!\!/ - m + i\epsilon}e^{-ik\cdot(x_1-x_2)} \tag{54}$$

Easily generalizable to higher-point functions.

3. Generating functional for Dirac field

More systematic and generalizable to interaction theory.

$$Z[\bar{\eta}, \eta, \cdots] \quad = \quad \int (\mathcal{D}\bar{\psi}\mathcal{D}\psi \cdots) \exp\left[iS + i\int d^4x (\bar{\eta}\psi + \bar{\psi}\eta) + \cdots \right] \tag{55}$$

other fields

Grassmannian external sources for ψ & $\bar{\psi}$

other source terms

e.g., free Dirac theory:

$$Z[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[i\int d^4x \left(\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta \right) \right]$$

It may be evaluated as in the scalar case by change of variables:

$$\begin{aligned} \psi(x) \quad &\rightarrow \quad \psi'(x) = \psi(x) - \int d^4y \; iS_F(x-y)\eta(y), \\ \bar{\psi}(x) \quad &\rightarrow \quad \bar{\psi}'(x) = \bar{\psi}(x) - \int d^4y \; \bar{\eta}(y) iS_F(y-x), \end{aligned}$$

with the result (check!):

$$Z[\bar{\eta},\eta] \quad = \quad \underbrace{Z_0}_{\substack{Z[0,0],\\ \text{free Gaussian integration}}} \exp\underbrace{\left[-\int d^4x d^4y\; \bar{\eta}(x) S_F(x-y)\eta(y)\right]}_{\substack{\text{remaining term from}\\ \text{completing the square}}} \tag{56}$$

Then,

$$\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle = Z_0^{-1}\left(\frac{\delta}{i\delta\bar{\eta}(x_1)}\frac{\delta}{-i\eta(x_2)}Z[\bar{\eta},\eta]\right)_{\bar{\eta}=\eta=0} \tag{57}$$

4. QED

Results in §1.3, §1.4 can be combined to work out Feynman rules in QED:

Propagators:	known
Vertex:	can be directly worked out from PI or using generating functional defined with gauge-fixed QED Lagrangian: $\mathcal{L}_{\text{eff}}^{\text{QED}}$
Minus signs:	present due to Grassmann field

1.5 Symmetries in the functional formalism

PI uses $\mathcal{L}$, instead of $\mathcal{H}$; thus symmetries more manifest

We saw:
symmetries $\leftrightarrow$ conservation laws
$\rightarrow$ relations among Green's functions (e.g. Ward-Takahashi identities)

How to formulate symmetries in PI?

1. Equations of motion

Example: free, real scalar φ
Consider:

$$\langle\Omega|T\varphi(x_1)\varphi(x_2)\varphi(x_3)|\Omega\rangle = Z_0^{-1}\int\mathcal{D}\varphi\ \exp\left(i\int d^4x\ \mathcal{L}[\varphi]\right)\varphi(x_1)\varphi(x_2)\varphi(x_3) \quad (58)$$

$$\mathcal{L}[\varphi] = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2$$

It is invariant under a change of integration variables, in particular under the shift:

$$\varphi(x) \longrightarrow \varphi'(x) = \varphi(x) + \epsilon(x) \tag{59}$$

Noting $\mathcal{D}\varphi = \mathcal{D}\varphi'$, we have

$$\int \mathcal{D}\varphi \; e^{i\int d^4x \; \mathcal{L}[\varphi]} \varphi(x_1)\varphi(x_2)\varphi(x_3) = \int \mathcal{D}\varphi \; e^{i\int d^4x \; \mathcal{L}[\varphi']} \varphi'(x_1)\varphi'(x_2)\varphi'(x_3)$$

Expansion to $\mathcal{O}(\epsilon^1)$ and integration by parts:

$$\begin{aligned}
0 \quad &= \quad \int \mathcal{D}\varphi \; e^{i\int \mathcal{L}[\varphi]} \Big\{ i \int d^4x \; \epsilon(x)[-\partial_x^2 - m^2]\varphi(x) \cdot \varphi(x_1)\varphi(x_2)\varphi(x_3) \\
&\qquad\qquad + \epsilon(x_1)\varphi(x_2)\varphi(x_3) + \varphi(x_1)\epsilon(x_2)\varphi(x_3) + \varphi(x_1)\varphi(x_2)\epsilon(x_3) \Big\} \\
&= \quad \int \mathcal{D}\varphi \; e^{i\int \mathcal{L}[\varphi]} (-i) \int d^4x \; \epsilon(x) \Big\{ (\partial_x^2 + m^2)\varphi_x\varphi_1\varphi_2\varphi_3 \qquad \varphi_x \equiv \varphi(x), \;\; \varphi_j \equiv \varphi(x_j) \\
&\qquad\qquad + i\delta(x - x_1)\varphi_2\varphi_3 + \varphi_1 i\delta(x - x_2)\varphi_3 + \varphi_1\varphi_2 i\delta(x - x_3) \Big\}
\end{aligned}$$

Since it holds for any $\epsilon(x)$, we must have

$$\int \mathcal{D}\varphi \; e^{i\int \mathcal{L}[\varphi]} \Big\{ (\partial_x^2 + m^2)\varphi_x\varphi_1\varphi_2\varphi_3 + \cdots \Big\} = 0 \tag{60}$$

$$\begin{aligned} \Leftrightarrow \quad (\partial_x^2 + m^2)\langle\Omega|T\varphi_x\varphi_1\varphi_2\varphi_3|\Omega\rangle &= -i\delta(x-x_1)\langle\Omega|T\varphi_2\varphi_3|\Omega\rangle \\ &\quad -i\delta(x-x_2)\langle\Omega|T\varphi_1\varphi_3|\Omega\rangle \\ \text{K.-G. operator} \qquad\qquad &\quad -i\delta(x-x_3)\langle\Omega|T\varphi_1\varphi_2|\Omega\rangle \end{aligned} \tag{61}$$

Schwinger-Dyson eqs
quantum analogue of classical EoM

"contact" terms (non-vanishing only at $x = x_1$ etc.)
(arising from $\partial_0\theta$ in Hamiltonian formalism)

Comments:

- generalizable to n-point functions
- generalizable to any field; inva. of Grass. integral under variable shift!
- change of $S[\varphi]$ under the shift is generally:

$$\int d^4x \; \epsilon(x) \frac{\delta}{\delta\varphi(x)} \left(\int d^4y \; \mathcal{L}[\varphi(y)] \right) \quad = \quad \int d^4x \; \epsilon(x) \left[\frac{\partial \mathcal{L}}{\partial\varphi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi(x))} \right]$$

would vanish according to E.-L. equation

2. Conservation laws

Classical: symmetries $\rightarrow$ conserved currents (Noether theorem)

What's quantum analogue?

Example: free, complex scalar φ

$$\mathcal{L} = |\partial_\mu \varphi|^2 - m^2 |\varphi|^2$$

inva. under global phase transf. $\varphi \rightarrow e^{i\alpha}\varphi$. Consider its local form:

$$\varphi(x) \rightarrow \varphi'(x) \equiv e^{i\alpha(x)}\varphi(x) \doteq (1 + i\alpha(x))\varphi(x) \quad \text{infinitesimal} \tag{62}$$

Consider the PI for 2-point function:

$$\int \mathcal{D}\varphi \mathcal{D}\varphi^* \exp\left[i \int d^4x \mathcal{L}[\varphi]\right] \varphi(x_1)\varphi^*(x_2), \quad \text{often suppressed below}$$

which is invariant under (62) since it is just a change of integration variables:

$$\mathcal{D}\varphi = \mathcal{D}\varphi' \text{ (or more explicitly, } \mathcal{D}\varphi\mathcal{D}\varphi^* = \mathcal{D}\varphi'\mathcal{D}\varphi'^*)$$

$$\varphi(x_1) \to \varphi'(x_1) \approx (1 + i\alpha(x_1))\varphi(x_1), \;\; \varphi^*(x_2) \to \varphi^{*\prime}(x_2) \approx (1 - i\alpha(x_2))\varphi^*(x_2)$$

$$\mathcal{L}[\varphi] \to \mathcal{L}[\varphi'] \approx \mathcal{L}[\varphi] + i\partial^\mu\alpha((\partial_\mu\varphi^*)\varphi - \varphi^*\partial_\mu\varphi)$$

$$\Rightarrow \quad \int\mathcal{D}\varphi \;\exp\left[i\int d^4x \;\mathcal{L}[\varphi]\right]\varphi(x_1)\varphi^*(x_2) = \int\mathcal{D}\varphi \;\exp\left[i\int d^4x \;\mathcal{L}[\varphi']\right]\varphi'(x_1)\varphi'^*(x_2)$$

$\mathcal{O}(\alpha)$ terms must vanish:

$$0 = \int\mathcal{D}\varphi \;\exp\left[i\int d^4x\mathcal{L}[\varphi]\right]\Big\{i\int d^4y \;\partial^\mu\alpha \; i\big((\partial_\mu\varphi^*)\varphi - \varphi^*\partial_\mu\varphi\big)\varphi(x_1)\varphi^*(x_2)$$

$$+i\alpha(x_1)\varphi(x_1)\varphi^*(x_2) + \varphi(x_1)(-i\alpha(x_2))\varphi^*(x_2)\Big\}$$

integration by parts: $-\,i\int d^4y \;\alpha\partial^\mu j_\mu, \;\; j_\mu = i\big((\partial_\mu\varphi^*)\varphi - \varphi^*\partial_\mu\varphi\big)$ Noether current

$$= \int\mathcal{D}\varphi \;\exp\left[i\int d^4x\mathcal{L}\right](+i)\int d^4y \;\alpha(y)\Big\{-\partial^\mu j_\mu(y)\varphi(x_1)\varphi^*(x_2)$$

$$+\delta(x_1 - y)\varphi(x_1)\varphi^*(x_2) - \varphi(x_1)\delta(x_2 - y)\varphi^*(x_2)\Big\}$$

$$\Rightarrow \quad \int\mathcal{D}\varphi \;\exp\left[i\int d^4x\mathcal{L}\right]\Big\{\cdots\Big\} = 0$$

Dividing by Z_0, its T-ordered form is

$$\langle \partial^\mu j_\mu(y)\varphi(x_1)\varphi^*(x_2)\rangle = (-i)\Big\langle i\varphi(x_1)\delta(x_1-y)\varphi^*(x_2) + \varphi(x_1)(-i\varphi^*(x_2))\delta(x_2-y)\Big\rangle$$

understood to be acting from outside of $\langle\Omega|T\cdots|\Omega\rangle$

contact terms (63)

Schwinger-Dyson equations for current conservation

Comments:

- Generalizable to general sym. transf.

Assume $\int \mathcal{L}[\varphi]$ is inva. under an x-indept, infinitesimal transf.:

$$\varphi_a(x) \to \varphi'_a(x) = \varphi_a(x) + \epsilon\Delta\varphi_a(x), \text{ i.e., } \mathcal{L}[\varphi] \to \mathcal{L}[\varphi] + \epsilon\partial_\mu\mathcal{J}^\mu$$

Then, for x-dept $\epsilon(x)$:

$$\begin{aligned}
\mathcal{L}[\varphi] &\to \mathcal{L}[\varphi] + (\partial_\mu\epsilon(x))\Delta\varphi_a\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)} + \epsilon(x)\partial_\mu\mathcal{J}^\mu \\
\text{i.e., } \int\mathcal{L}[\varphi] &\to \int\mathcal{L}'[\varphi'] = \int\mathcal{L}[\varphi] - \int\epsilon(x)\partial_\mu j^\mu(x)
\end{aligned} \tag{64}$$

where j is the Noether current corresponding to the symm:

$$j^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi_a)}\Delta\varphi_a - \mathcal{J}^{\mu}$$

Similar derivation as for (63) gives

$$\langle \partial_{\mu} j^{\mu}(x)\varphi_a(x_1)\varphi_b(x_2)\rangle = (-i)\Big\langle \Delta\varphi_a(x_1)\delta(x_1 - x)\varphi_b(x_2) + \varphi_a(x_1)\Delta\varphi_b(x_2)\delta(x_2 - x)\Big\rangle \quad (65)$$

- Generalizable to n-point functions

3. Ward-Takahashi identity

An application of the above discussion. Consider PI in QED:

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\ \psi(x_1)\bar{\psi}(x_2)\ \exp\left[i\int\mathcal{L}\right]$$

Make the change of integration variables:

$$\begin{aligned} \psi(x) &\to \psi'(x) \approx (1 + ie\alpha(x))\psi(x) \\ \bar{\psi}(x) &\to \bar{\psi}'(x) \approx (1 - ie\alpha(x))\bar{\psi}(x) \end{aligned} \qquad (A_{\mu} \text{ not changed – not charged}) \quad (66)$$

In PI, measure is not changed and

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} - (e\partial_\mu \alpha)\bar{\psi}\gamma^\mu\psi = \mathcal{L} - (\partial_\mu\alpha)j^\mu, \quad j^\mu = e\bar{\psi}\gamma^\mu\psi \quad \text{Noether current} \tag{67}$$

$$\psi(x_1)\bar{\psi}(x_2) \to \psi(x_1)\bar{\psi}(x_2) + ie\big(\alpha(x_1) - \alpha(x_2)\big)\psi(x_1)\bar{\psi}(x_2)$$

PI is not changed, thus $\mathcal{O}(\alpha)$ terms vanish:

$$0 = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\; e^{i\int\mathcal{L}}\Big\{ -i\int(\partial_\mu\alpha\; j^\mu)\psi(x_1)\bar{\psi}(x_2) + ie\big(\alpha(x_1) - \alpha(x_2)\big)\psi(x_1)\bar{\psi}(x_2)\Big\}$$

integration by parts

$$\xrightarrow{\times Z_0^{-1}} i\partial_\mu\langle\Omega|Tj^\mu(x)\psi(x_1)\bar{\psi}(x_2)|\Omega\rangle = -ie\delta(x - x_1)\langle\Omega|T\psi(x_1)\bar{\psi}(x_2)|\Omega\rangle \quad \text{contact}$$
$$+ie\delta(x - x_2)\langle\Omega|T\psi(x_1)\bar{\psi}(x_2)|\Omega\rangle \quad \text{terms} \tag{68}$$

Schwinger-Dyson equation for global $U(1)$ symmetry of QED

The above can be Fourier-transformed to momentum space to yield the Ward-Takahashi identity for 2 external fermions derived in §7.4-QFT.

Hwk: Problem 9.1

[finished in 3 units on Sept 21, 2012.]

Chapter 2 Systematics of renormalization

Goals

- Determine which diagrams are ultraviolet (UV) divergent (div); classification of QFT
- How UV divs are removed in renormalizable theories

2.1 Counting of ultraviolet divergences

2.2 Renormalized perturbation theory

2.3 Renormalization of QED

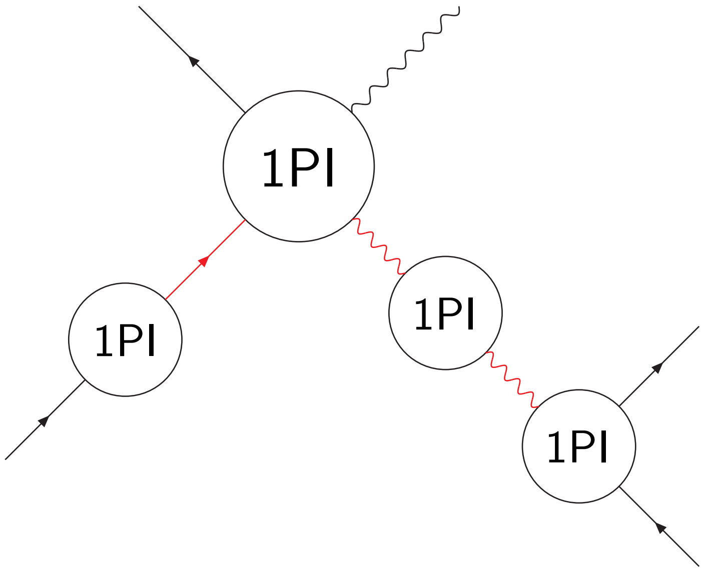
2.4 Renormalization beyond the leading order

2.1 Counting of UV div

1. Primitively divergent diagrams

any connected diagrams $=$ multiplication of 1PI diagrams by propagators whose momenta are not integrated over

e.g.:



propagators whose momenta are not integrated over connect 1PI pieces of a diagram

$\Rightarrow$ *It's sufficient to study 1PI diagrams.*

Take QED as our example. Notations:

$$
\begin{aligned}
N_e &= \text{No. of external } e \text{ lines} \\
N_\gamma &= \text{No. of external } \gamma \text{ lines} \\
P_e &= \text{No. of internal } e \text{ propagators} \\
P_\gamma &= \text{No. of internal } \gamma \text{ propagators} \\
V &= \text{No. of vertices} \\
L &= \text{No. of loops}
\end{aligned}
$$

Superficial degree of div for a 1PI diagram:

$$
D = \overbrace{4L}^{\text{each loop has an } \int d^4k} - P_e - 2P_\gamma \tag{1}
$$

$(\not{k} - m + i\epsilon)^{-1}$ counts as -1 in the UV

$(k^2 + i\epsilon)^{-1}$ counts as -2 in the UV

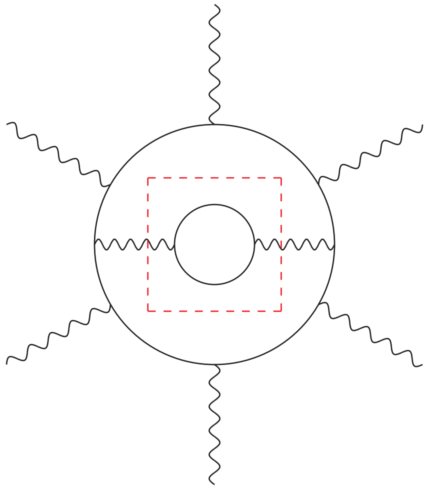
$D > 0:$ UV divergent as a power

$D < 0:$ UV finite

$D = 0:$ UV logarithmically div

Comments:

• Applies to 1PI diagram *as a whole* which may contain UV div. subdiagram. e.g.:



$P_e = 10, P_\gamma = 2, L = 3$

$\Longrightarrow D = 4 \cdot 3 - 10 - 2 \cdot 2 = -2 < 0$

but actually UV div. due to the div. subdiagram

$\Longrightarrow$ "UV finite for $D < 0$" only when no subdivergences appear

• *Actual* degree of div. may be lower than D because of symmetries: Lorentz, gauge, C, etc. Examples later.

- Better form of D:

$$\begin{cases} L = P_e + P_\gamma - (V-1) & \left(\begin{array}{l} \text{Each internal propagator has an } \displaystyle\int d^4k; \\ \text{each vertex has a } (2\pi)^4\delta^4(\Sigma k); \\ \text{a global } \delta^4(\Sigma p) \text{ remains for external momenta.} \end{array} \right. \\ \begin{aligned} V &= 2P_\gamma + N_\gamma \\ 2V &= 2P_e + N_e \end{aligned} & \left(\begin{array}{l} \text{Each internal propagator connects 2 vertices;} \\ \text{each external line connects 1 vertex;} \\ \text{each vertex has 2 } e \text{ and 1 } \gamma. \end{array} \right. \end{cases}$$

$$\Longrightarrow D = 4 - N_\gamma - \frac{3}{2}N_e \tag{2}$$

Superficial degree of div. depends only on No. of external lines!

3 primitively div. diagrams

There are only a few cases with $D \geqslant 0$: $N_\gamma \leqslant 4$, $N_e(\text{always even}) \leqslant 2$:

	$N_\gamma = 0$	1	2	3	4
$N_e = 0$	4	3	2	1	0
2	1	0	-1	-2	-3

7 cases with $D \geqslant 0$ can be further reduced:

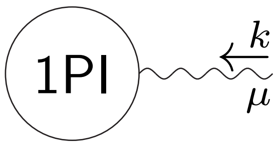
(a) $N_\gamma = N_e = 0$: vacuum bubble, not interesting to us

(b) $N_e = 0, N_\gamma$ odd $(1,3)$: vanishing duo to C conservation; e.g.

$$
\begin{aligned}
& \langle\Omega|T \uparrow A_\alpha(x) \uparrow A_\beta(y) \uparrow A_\gamma(z) \uparrow |\Omega\rangle \qquad \uparrow = C^{-1}C = 1 \\
= \quad & \langle\Omega|C^{-1}T(-1)A_\alpha(x)(-1)A_\beta(y)(-1)A_\gamma(z)C|\Omega\rangle = (-1)\cdot \text{LHS} = 0
\end{aligned}
$$

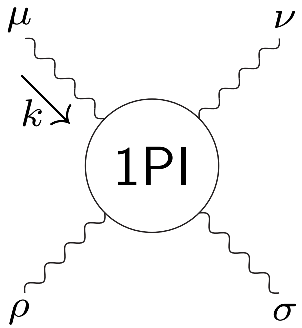
applicable to all odd-point pure photon Green's functions (Furry's theorem)
Equivalent form for 1PI diagrams: $\langle\Omega|Tj_{\mu_1}(x_1)\cdots j_{\mu_n}(x_n)|\Omega\rangle = 0$ for all odd n.

$(N_e = 0, N_\gamma = 1)$ vanishes also due to Lorentz covariance; most manifest in momentum space:



$\propto k_\mu$ by Lorentz covariance; but $k_\mu = 0$ due to momentum conservation

(c) $N_e = 0, N_\gamma = 4$: finite due to symmetries



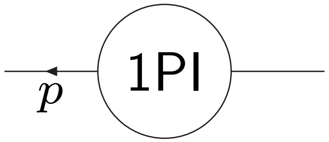
$\times k_\mu = 0$ by W.-T. identity and Furry's theorem

$\therefore \quad \propto (g_{\mu\alpha} k_\beta - g_{\mu\beta} k_\alpha)$

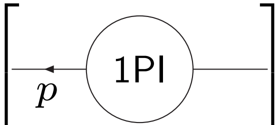
4 such factors reduce effectively D by 4, i.e., $D = 0 \to -4$: finite!

$\Longrightarrow$ Only 3 primitively div. diagrams

(i) electron self-energy



$= A_0 + A_1 \not{p} + A_2 p^2 + \cdots$, Taylor expansion at $\not{p} = 0$ OK for UV counting
(Caution: IR div.)

$$A_n = \frac{1}{n!} \frac{d^n}{d\not{p}^n} \Big[$$  $$\Big]_{\not{p}=0} \qquad (3)$$

(show an example!)

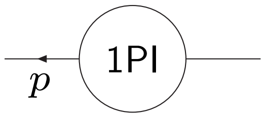
Since $\not{p}$ appears in internal propagators, each derivative reduces D by 1.

(Caution: subdiv. not counted)

$$\therefore \qquad A_0 \leftrightarrow D_{\text{eff}} = 1, \quad A_1 \leftrightarrow D_{\text{eff}} = 0, \quad A_{n\geqslant 2} \leftrightarrow D_{\text{eff}} < 0$$

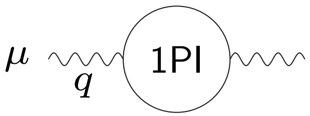
linear UV div. log UV div.

Further, chiral sym. in QED:
$m = 0$ exactly if $m = 0$ in $\mathcal{L}$, i.e., mass shift $\propto m \Longrightarrow A_0$ actually log div.

$\Longrightarrow$ 

$$= a_0 m \ln \Lambda + a_1 \not{p} \ln \Lambda + (\text{finite}), \ \Lambda : \ \text{UV cutoff} \tag{4}$$

(ii) photon self-energy:

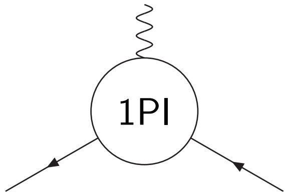


$$= (g_{\mu\nu} q^2 - q_\mu q_\nu)\Pi(q^2) \quad \text{due to W.-T. identity \& Furry's theorem}$$

i.e., $(q)^0$ $[D_{\text{eff}} = 2]$, $(q)^1$ $[D_{\text{eff}} = 1]$ terms of Taylor expansion vanish!

$$\therefore \ \text{UV div. only in } \Pi(0) \sim \ln \Lambda \tag{5}$$

(iii) γee vertex:



$$= (-ie\gamma_\mu) v_0 \ln \Lambda + (\text{finite}) \tag{6}$$

Summary of QED

- 3 primitively div. diagrams (all known at one loop in QFT-chapt 7)
- 4 UV div. coefficients, to be treated by renormalization.

2. Classification of QFT

Classify QFT according to its D.

Example 1: QED in d dimensions

$$\left.\begin{aligned} D &= dL - P_e - 2P_\gamma \\ L &= P_e + P_\gamma - V + 1 \\ V &= 2P_\gamma + N_\gamma \\ 2V &= 2P_e + N_e \end{aligned}\right\} \Longrightarrow D = d + \frac{1}{2}(d-4)V - \frac{1}{2}(d-2)N_\gamma - \frac{1}{2}(d-1)N_e \tag{7}$$

$d = 4$:

$D \geqslant 0$ only for a finite set of $(N_\gamma,\ N_e)$, i.e., only a finite number of amplitudes are superficially div.; but div. occurs at all orders

"renormalizable theory"

(for 1PI: more external lines, less div.)

$d < 4$:

$D \geqslant 0$ only for a finite set of $(N_\gamma,\ N_e,\ V)$; i.e., only a finite number of diagrams are div.

"super-renormalizable theory"

(for 1PI: more vertices/external lines, more convergent)

$d > 4$:

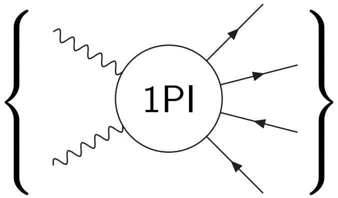
$D \geqslant 0$ for large enough V and for any N_γ and N_e; i.e., all amplitudes are div. at high enough orders

"non-renormalizable theory"

(for 1PI: higher orders, more div.)

Further understanding of (7)

For dimension analysis, any 1PI diagram can be considered as an effective interaction term in $\mathcal{L}$

N_γ 

$N_e \sim \mathcal{A} A^{N_\gamma} \psi^{N_e}$ Feyn amplitude excluding external factors u, v, etc
$\mathcal{A}(p, e, m, \Lambda, \mu)$ Λ : UV cutoff, μ : IR cutoff

$[\mathcal{L}] = [\mathrm{mass}]^d \Rightarrow$ "dimension d", denoted as $[\mathcal{L}] = d$:

$$[\bar{\psi}\partial\!\!\!/\psi] = d \to [\psi] = \frac{1}{2}(d-1); \ [F_{\mu\nu}^2] = d \to [A] = \frac{1}{2}(d-2);$$

$$[e\bar{\psi}A\!\!\!/\psi] = d \to [e] = \frac{1}{2}(4-d)$$

$$[\mathcal{A} A^{N_\gamma} \psi^{N_e}] = d \to [\mathcal{A}] = d - N_\gamma(d-2)/2 - N_e(d-1)/2$$

superficially most div term in $\mathcal{A}$: $\Lambda^D e^V$, $[\Lambda^D e^V] = D + V(4-d)/2$

Identification $[\mathcal{A}] = [\Lambda^D e^V]$ gives (7), which can be recast in the form

$$D = d - N_\gamma[A] - N_e[\psi] - V[e] \tag{8}$$

Therefore,

- 'super-renormalizable': $[e] > 0$, i.e., coupling of positive dim in mass
- 'renormalizable': $[e] = 0$, i.e., dimensionless coupling
- 'non-renormalizable': $[e] < 0$, i.e., coupling of negative dim in mass

Example 2. Real scalar $\lambda\varphi^n$ in d dimensions

Work out its classification by yourselves !

2.2. Renormalized perturbation theory

We only treat renormalizable QFTs, i.e., in which all UV div. at any given order in perturbation theory can be consistently removed from observables.

Compute observables at the next-to-leading order in perturbation theory:

- Compute corresponding amplitudes at next-to-leading (NL) order.
(regularization if UV div appears)

- Redefine physical input parameters (masses, couplings) at NL order.
(renormalization)

- Reexpress amplitudes in terms of redefined physical input parameters.

Important: Renormalization is *not just* to remove UV div, but is a procedure that is required by the consistent implementation of perturbation theory.

Two conventional procedures to do perturbation theory

(1) Working with bare quantities

- Regularize and compute amplitude in terms of bare quantities (e_0, m_0, etc) and regularization parameters (cutoff Λ, or $(4-d)^{-1}$ in DR, etc)

- Compute physical input parameters (e, m, etc) in terms of bare quantities and regularization parameters to the same order consistent with the order of the required amplitude

- Reexpress bare quantities in terms of physical parameters

- Use eqn.(7.36) in QFT to finish computing the amplitude and express it in terms of physical parameters

(2) Renormalized perturbation theory

- Original $\mathcal{L}_0$ is expressed in terms of bare quantities (parameters and fields).

- Each bare quantity is split into a renormalized piece and a counter-term piece, so is $\mathcal{L}_0 = \mathcal{L}_{\rm r} + \mathcal{L}_{\rm c.t.}$. Treat $\mathcal{L}_{\rm c.t.}$ as new interaction terms.

- Define physical parameters (and fields) to the order consistent with the desired order of the amplitude to be computed. This fixes $\mathcal{L}_{\rm r}$ and $\mathcal{L}_{\rm c.t.}$.

- Compute the required amplitude to the desired order by including all contributions of the same order, in particular those from insertions of c.t..

We work with the renormalized perturbation theory.

Example Real φ^4 theory

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \varphi_0)^2 - \frac{1}{2} m_0^2 \varphi_0^2 - \frac{1}{4!}\lambda_0 \varphi_0^4$$

$_0$ bare parameters & fields

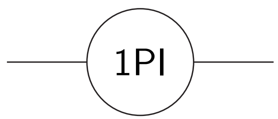
UV div amplitudes – For amplitudes with N external legs :

$D = 4 - N$ (similar analysis leading to (7) or (8))

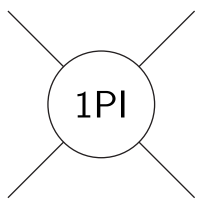
Reflection sym. : $\varphi \to -\varphi \implies$ All amplitudes with N odd vanish identically.

$\therefore$ UV div. restricted to :

$N = 0$	2	4
$D = 4$	2	0
vacuum bubble ignored	2-point function	4-point function



$\sim$ "c_1" Λ^2 + "c_2" $\ln \Lambda$ + finite

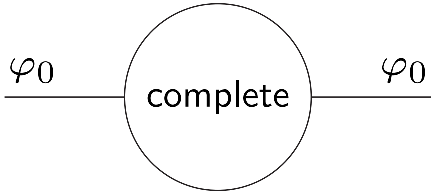


$\sim$ "c_3" $\ln \Lambda$ + finite

3 UV div. constants

[finished in 3 units on Sep 28, 2012.]

Introduction of renormalized quantities and counterterms



$\xrightarrow{p^2 \to m^2} \quad \dfrac{iZ}{p^2 - m^2}$ $\qquad m$: physical mass

suggesting

$$\varphi_0 = Z^{\frac{1}{2}}\varphi, \;\; Z = \text{wavefunc renormalization const (real but UV div.)} \tag{9}$$

$$\Longrightarrow \quad \mathcal{L}_0 = \frac{1}{2}Z(\partial_\mu\varphi)^2 - \frac{1}{2}m_0^2 Z\varphi^2 - \frac{1}{4!}\lambda_0 Z^2\varphi^4 \tag{10}$$

Further splitting:

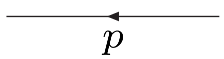
$$Z = 1 + \delta_Z, \quad m_0^2 Z = m^2 + \delta_m, \quad \lambda_0 Z^2 = \lambda + \delta_\lambda \quad \text{renormalization constants} \tag{11}$$

$$\Longrightarrow \quad \mathcal{L}_0 = \mathcal{L}_{\mathrm{r}} + \mathcal{L}_{\mathrm{c.t.}}$$

$$\mathcal{L}_{\mathrm{r}} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4 \quad \text{renormalized parameters and field} \tag{12}$$

$$\mathcal{L}_{\mathrm{c.t.}} = \frac{1}{2}\delta_Z(\partial_\mu\varphi)^2 - \frac{1}{2}\delta_m\varphi^2 - \frac{1}{4!}\delta_\lambda\varphi^4$$

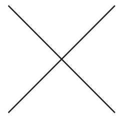
Feynman rules



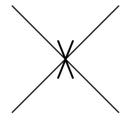
$$= \frac{i}{p^2 - m^2 + i\epsilon}$$

$$= i(p^2\delta_Z - \delta_m)$$

$\Leftarrow \mathcal{L}_{\mathrm{r}}$ $\qquad$ $\mathcal{L}_{\text{c.t.}} \Rightarrow$ both as new interactions



$$= -i\lambda$$



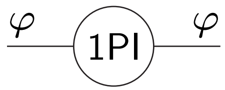
$$= -i\delta_\lambda$$

Renormalization conditions

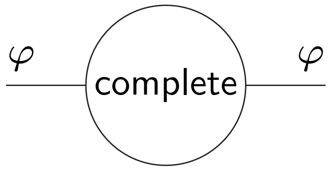
Redefine parameters and fields to the relevant order. This also fixes c.t..

Need 3 renormalization conditions to fix uniquely $\delta_Z, \delta_m,\ \delta_\lambda$.

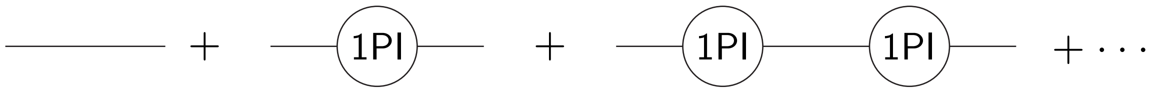
(1) 2-point function $\Rightarrow\ \delta_m,\ \delta_Z$



$\equiv \quad -iM(p^2)$ $\quad$ (excluding tree) $\qquad$ both renormalized!



$=$



$$= \frac{i}{p^2 - m^2 - M(p^2)}$$

$*$ The pole locates at the physical m^2:

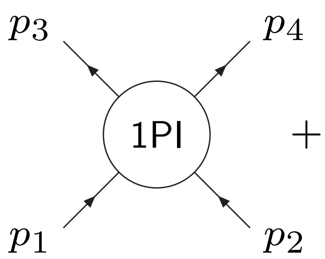
$$\left[p^2 - m^2 - M(p^2)\right]_{p^2=m^2} = 0 \Rightarrow M(m^2) = 0 \tag{13}$$

$*$ The residue of the propagator at the pole is $1\ (\times i)$:

$$\frac{d}{dp^2}\left[p^2 - m^2 - M(p^2)\right]_{p^2=m^2} = 1 \Rightarrow \left.\frac{dM(p^2)}{dp^2}\right|_{p^2=m^2} = 0 \tag{14}$$

(2) 4-point function $\Rightarrow \delta_\lambda$

Scattering amplitude in renormalized pert. theory *happens* to be 1PI at one loop because renormalized field has $Z = 1$ at one loop ((QFT-7.36)):



$$+(-i\lambda) \equiv \ i\mathcal{M}(p_1p_2 \to p_3p_4), \text{ as a function of } \begin{cases} s = (p_1+p_2)^2 \\ t = (p_1-p_3)^2 \\ u = (p_1-p_4)^2 \end{cases}$$

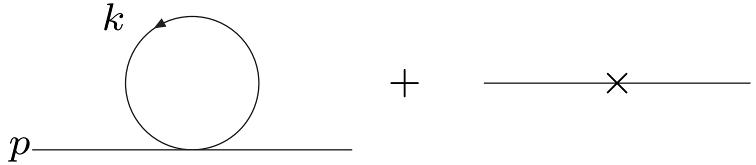
renormalized, including c.t. and tree-level term

λ specifies the strength of scattering. May use any convenient momentum configuration to define λ. We define

$$\mathcal{M}\Big|_{s=4m^2,t=u=0} = -\lambda\ , \text{ i.e., at the kinematic point where all } \mathbf{p} = 0 \tag{15}$$

Compute δ_m & δ_Z at one loop

We regularize UV div. by dim. reg.



both of $\mathcal{O}(\lambda^1)$

(a) (b)

$$
\begin{aligned}
(a) \quad &= \quad \frac{-i\lambda}{2!}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2+i\epsilon} = \frac{-i\lambda}{2!}\int\frac{id^dk_E}{(2\pi)^d}\frac{i}{-k_E^2-m^2+i\epsilon} \quad \text{Wick r.} \\
&= \quad -\frac{i\lambda}{2}\frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}\int_0^\infty\frac{x^{d-1}dx}{(2\pi)^d}\frac{1}{x^2+m^2} \quad \text{angular integration} \\
&= \quad -\frac{i\lambda}{2}\frac{(m^2)^{(d-2)/2}}{(4\pi)^{d/2}}\Gamma\Big(1-\frac{d}{2}\Big), \quad \text{indept. of } p^2! \\
(b) \quad &= \quad i(p^2\delta_Z-\delta_m)
\end{aligned}
$$

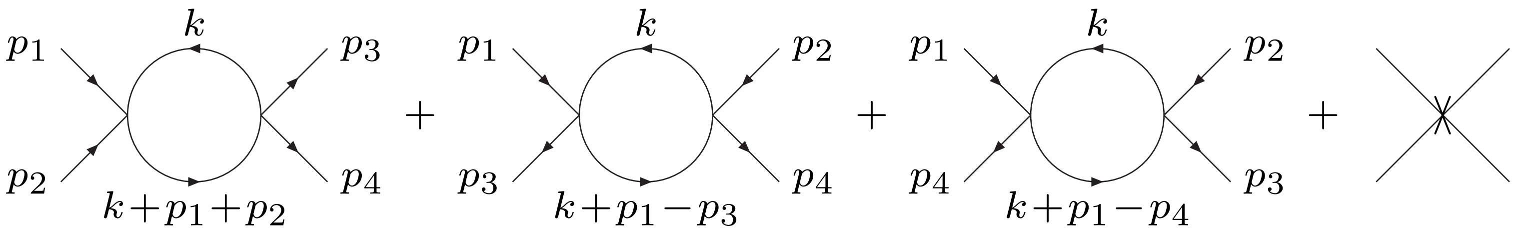
$$\therefore \qquad M(p^2) = \frac{\lambda}{2}\frac{(m^2)^{(d-2)/2}}{(4\pi)^{d/2}}\Gamma\Big(1-\frac{d}{2}\Big) - (p^2\delta_Z - \delta_m) \quad \text{at one loop}$$

$$\xrightarrow[(14)]{(13)} \quad \begin{cases} m^2\delta_Z - \delta_m = \frac{\lambda}{2}\frac{(m^2)^{(d-2)/2}}{(4\pi)^{d/2}}\Gamma\Big(1-\frac{d}{2}\Big) \\ \\ \delta_Z = 0 \end{cases} \tag{16}$$

Comments

- δ_m is UV div. because $\Gamma\Big(1-\frac{d}{2}\Big) = \frac{\Gamma(2-\frac{d}{2})}{1-\frac{d}{2}} \xrightarrow{d\to 4} -\frac{1}{2-\frac{d}{2}} + \text{regular}$
- $\delta_Z = 0$ is a special feature of $\lambda\varphi^4$ theory. $\delta_Z \neq 0$ at higher loops.

Compute δ_λ at one loop



(a) (b) (c) (d)

all of $O(\lambda^2)$

$$
\begin{aligned}
(a) \quad &= \quad \frac{(-i\lambda)^2}{2!}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(k+p_1+p_2)^2-m^2+i\epsilon} \\
&= \quad \frac{\lambda^2}{2!}\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\Big\{k^2+2k\cdot(p_1+p_2)x+(p_1+p_2)^2x-m^2+i\epsilon\Big\}^{-2} \\
&= \quad \frac{\lambda^2}{2!}\int dx\int\frac{d^d\ell}{(2\pi)^d}\Big\{\ell^2-m^2+(p_1+p_2)^2x(1-x)+i\epsilon\Big\}^{-2} \quad [\ell\equiv k+(p_1+p_2)x] \\
&= \quad \frac{\lambda^2}{2!}\int dx\, i\int\frac{d^d\ell_E}{(2\pi)^d}\Big\{\ell_E^2+m^2-sx(1-x)-i\epsilon\Big\}^{-2} \quad [s=(p_1+p_2)^2] \\
&= \quad \frac{\lambda^2}{2!}\int dx\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\big[m^2-sx(1-x)-i\epsilon\big]^{\frac{d}{2}-2} \quad \text{using QFT-(7.64)} \\
&= \quad \frac{i\lambda^2}{2}\frac{1}{(4\pi)^2}\int_0^1 dx\Big\{\frac{1}{2-\frac{d}{2}}-\gamma_E+\ln 4\pi-\ln[m^2-sx(1-x)-i\epsilon]\Big\}+\mathcal{O}\Big((4-d)^1\Big)
\end{aligned}
$$

where in the last step the following expansions are used:

$$
\Gamma(z)\xrightarrow{z\to 0}\frac{1}{z}-\gamma_E+\mathcal{O}(z^1), \quad A^z\xrightarrow{z\to 0}1+z\ln A+\mathcal{O}(z^2)
$$

The other graphs give

$$(b) = (a)|_{s\to t}, \quad (c) = (a)|_{s\to u}, \quad (d) = -i\delta_\lambda$$

Including the lowest order term, the *amplitude up to one loop order* is

$$i\mathcal{M} = -i\lambda + (a) + (b) + (c) + (d) \tag{17}$$

(15) demands

$$\Big[(a) + (b) + (c) + (d)\Big]_{s=4m^2,t=u=0} = 0$$

which fixes uniquely

$$\begin{aligned}
\delta_\lambda \quad = \quad & \frac{\lambda^2}{2(4\pi)^2}\int_0^1 dx \Big\{ \frac{2}{4-d} - \gamma_E + \ln 4\pi - \ln\left[m^2 - 4m^2 x(1-x)\right] && \leftarrow (a) \\
& + \frac{2}{4-d} - \gamma_E + \ln 4\pi - \ln m^2 && \leftarrow (b) \\
& + \frac{2}{4-d} - \gamma_E + \ln 4\pi - \ln m^2 \Big\} && \leftarrow (c)
\end{aligned} \tag{18}$$

Final result for 2 to 2 scattering at one loop

All physical parameters have been defined at one-loop precision.

Although $\mathcal{M} = -\lambda$ at one loop at the kinematic point $s = 4m^2, t = u = 0$, we generally have $\mathcal{M} \neq -\lambda$ at any other kinematic point. This serves as a test of the theory:

$$\left.\begin{matrix}(17)\\(18)\end{matrix}\right\} \implies \mathcal{M} = -\lambda + \frac{\lambda^2}{32\pi^2}\int_0^1 dx \Bigg\{ \ln\frac{m^2 - 4m^2x(1-x)}{m^2 - sx(1-x) - i\epsilon} + \ln\frac{m^2}{m^2 - tx(1-x) - i\epsilon} + \ln\frac{m^2}{m^2 - ux(1-x) - i\epsilon}\Bigg\} \tag{19}$$

2.3. Renormalization of QED

Summary of renormalization program in pert. theory

(1) Rescale fields and separate bare parameters into renormalized and c.t. pieces.

(2) $\mathcal{L}_0$ is split into a renormalized piece $\mathcal{L}_{\mathrm{r}}$, which contains only renormalized fields and parameters, and a c.t. piece $\mathcal{L}_{\mathrm{c.t.}}$, which is expressed in terms of renormalized fields and parameters, and c.t..

(3) $\mathcal{L}_{\mathrm{c.t.}}$ is treated as interactions. Work out its Feynman rules.

(4) Specify renormalization conditions that define the renormalized fields and parameters to the order considered. This fixes the c.t..

(5) Compute amplitudes with the new Feynman rules. All contributions of the same order, including those from $\mathcal{L}_{\mathrm{c.t.}}$, must be included.

Renormalization constants and c.t.

$$\mathcal{L}_0 = -\frac{1}{4}(F^0_{\mu\nu})^2 + \overline{\psi_0}(i\partial\!\!\!/ - m_0)\psi_0 - e_0\overline{\psi_0}\gamma_\mu\psi_0 A^\mu_0, \quad F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu$$

$_0$ or ^{0} : bare

Field rescaling: $A^0_\mu = Z_3^{\frac{1}{2}} A_\mu, \ \ \psi^0 = Z_2^{\frac{1}{2}}\psi$

$$\Rightarrow \mathcal{L}_0 = -\frac{1}{4}Z_3(F_{\mu\nu})^2 + Z_2\overline{\psi}(i\partial\!\!\!/ - m_0)\psi - e_0 Z_2 Z_3^{\frac{1}{2}}\overline{\psi}\gamma_\mu\psi A^\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Parameter c.t.: $Z_2 m_0 = m + \delta_m, \ \ e_0 Z_2 Z_3^{\frac{1}{2}} = eZ_1$

renormalized

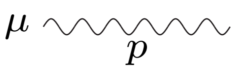
Define:

$$\delta_i = Z_i - 1, \ \ i = 1, 2, 3$$

Then

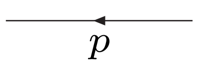
$$\begin{aligned} \mathcal{L}_0 &= \mathcal{L}_{\rm r} + \mathcal{L}_{\rm c.t.} \\ \mathcal{L}_{\rm r} &= -\frac{1}{4}(F_{\mu\nu})^2 + \overline{\psi}(i\partial\!\!\!/ - m)\psi - e\overline{\psi}A\!\!\!/\psi \qquad (20) \\ \mathcal{L}_{\rm c.t.} &= -\frac{1}{4}\delta_3(F_{\mu\nu})^2 + \overline{\psi}(\delta_2 i\partial\!\!\!/ - \delta_m)\psi - e\delta_1\overline{\psi}A\!\!\!/\psi \end{aligned}$$

Feynman rules

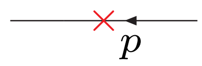


$$= \frac{-ig_{\mu\nu}}{p^2 + i\epsilon} \quad \text{in Feynman gauge}$$

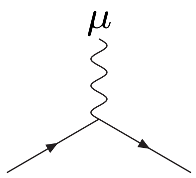
$$= -i(p^2 g_{\mu\nu} - p_\mu p_\nu)\delta_3$$



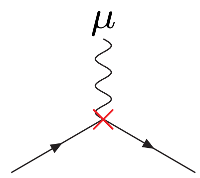
$$= \frac{i}{\not{p} - m + i\epsilon}$$



$$= i(\not{p}\delta_2 - \delta_m)$$



$$= -ie\gamma_\mu$$



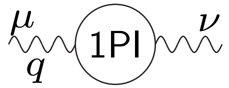
$$= -ie\delta_1\gamma_\mu$$

$\Uparrow$ $\mathcal{L}_\mathrm{r}$ $\qquad$ $\Uparrow$ $\mathcal{L}_\mathrm{c.t.}$

Renormalization conditions

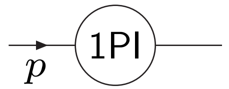
From §2.1., 3 primitively div. amplitudes with 4 div constants:
2 field renormalizations δ_2, δ_3
1 mass renormalization δ_m
1 coupling renormalization δ_1

Notations (for renormalized quantities):



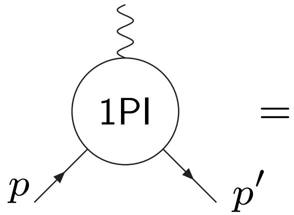
$$= i\Pi_{\mu\nu}(q) = i(g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi(q^2)$$

$$\Rightarrow \text{prop.} = \frac{-i}{q^2[1-\Pi(q^2)]}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - \frac{i}{q^2}\frac{q_\mu q_\nu}{q^2}$$



$$= -i\Sigma(\not{p})$$

$$\Rightarrow \text{prop.} = \frac{i}{\not{p} - m - \Sigma(\not{p})}$$



$$= -ie\delta\Gamma_\mu(p', p), \quad \Gamma_\mu(p', p) = \gamma_\mu + \delta\Gamma_\mu(p', p)$$

(1) 2-point functions $\Rightarrow \delta_2,\ \delta_3,\ \delta_m$

$*$ The electron propagator has the pole at $\not{p} = m$:

$$\left[\not{p} - m - \Sigma(\not{p})\right]_{\not{p}\to m} = 0 \Rightarrow \Sigma(m) = 0 \tag{21}$$

∗ The electron propagator has residue 1 at the pole:

$$\frac{d}{d\not{p}}\Big[\not{p}-m-\Sigma(\not{p})\Big]_{\not{p}\to m}=1\Rightarrow\Big[\frac{d}{d\not{p}}\Sigma(\not{p})\Big]_{\not{p}\to m}=0 \tag{22}$$

∗ The photon propagator has residue 1 at the pole:

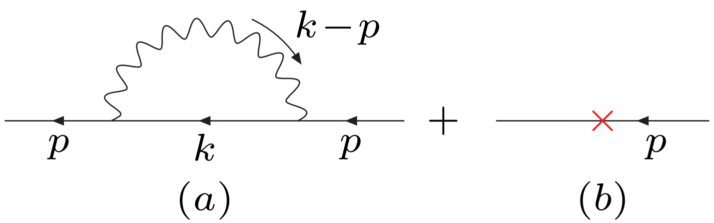
$$\frac{d}{dq^2}\Big[q^2(1-\Pi(q^2))\Big]_{q^2\to 0}=1\Rightarrow\Pi(0)=0 \tag{23}$$

(2) 3-point function $\Rightarrow \delta_1$

∗ The coupling constant e is the one defined in low energy (Thomson) limit:

$$\Big[-ie\Gamma_\mu(p',p)\Big]_{\not{p},\not{p}'\to m}=-ie\gamma_\mu \tag{24}$$

$\delta_2,\ \delta_m$ at one loop



UV div. reg. : dim. reg.
IR div. reg. : photon mass μ

$$
\begin{aligned}
(a) \quad &= \quad \int \frac{d^d k}{(2\pi)^d} (-ie\gamma^\mu) \frac{i}{\not{k} - m + i\epsilon} (-ie\gamma_\mu) \frac{-i}{(k-p)^2 - \mu^2 + i\epsilon} \\
&= \quad -e^2 \int \frac{d^d k}{(2\pi)^d} \frac{(2-d)\not{k} + dm}{(k^2 - m^2 + i\epsilon)[(k-p)^2 - \mu^2 + i\epsilon]} \qquad \leftarrow \gamma^\mu (\not{k} + m) \gamma_\mu = (2-d)\not{k} + md
\end{aligned}
$$

Using

$$
\frac{1}{(k^2 - m^2 + i\epsilon)[(k-p)^2 - \mu^2 + i\epsilon]} = \int_0^1 dx \left\{ (k - px)^2 + p^2 x(1-x) - \mu^2 x - m^2 (1-x) + i\epsilon \right\}^{-2}
$$

we have

$$
\begin{aligned}
(a) &= -e^2 \int \frac{d^d \ell}{(2\pi)^d} \int_0^1 dx \frac{(2-d)x\not{p} + dm}{[\ell^2 + p^2 x(1-x) - \mu^2 x - m^2(1-x) + i\epsilon]^2} \qquad \leftarrow \ell \equiv k - px \\
&= -\frac{ie^2}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) \int dx \frac{(2-d)x\not{p} + dm}{\left[\mu^2 x + m^2(1-x) - p^2 x(1-x) - i\epsilon\right]^{2-d/2}} \qquad \leftarrow \text{QFT-(7.63)} \\
(b) &= i(\not{p}\delta_2 - \delta_m)
\end{aligned}
$$

Then,

$$-i\Sigma_2(p) = (a) + (b), \qquad\qquad e^2$$

$*$ $\Sigma_2(m) = 0$ gives

$$m\delta_2 - \delta_m = \frac{e^2 m}{(4\pi)^{d/2}}\Gamma\Big(2-\frac{d}{2}\Big)\int dx \frac{(2-d)x+d}{[\mu^2 x + m^2(1-x)^2 - i\epsilon]^{2-d/2}} \tag{25}$$

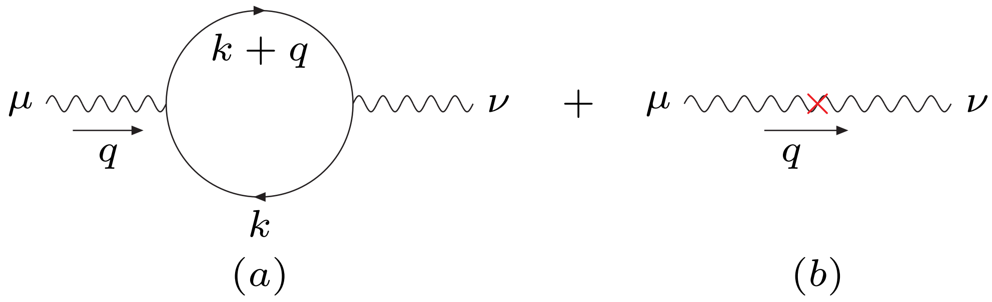
$*$ $\dfrac{d}{d\not{p}}\left[\Sigma_2(\not{p})\right]_{\not{p}\to m} = 0$ gives

$$\begin{aligned}\delta_2 \quad = \quad & \frac{e^2}{(4\pi)^{d/2}}\Gamma\Big(2-\frac{d}{2}\Big)\int dx\,[\mu^2 x + m^2(1-x)^2 - i\epsilon]^{\frac{d}{2}-2} \\ & \times\left\{(2-d)x + \Big(2-\frac{d}{2}\Big)\frac{2m^2 x(1-x)[(2-d)x+d]}{\mu^2 x + m^2(1-x)^2 - i\epsilon}\right\},\end{aligned} \tag{26}$$

where the 2nd term is IR div.

[almost finished in 2.5 units on Oct 12, 2012.]

δ_3 at one loop



$$(a) = (-1)\int\frac{d^dk}{(2\pi)^d}\mathrm{Tr}\left[(-ie\gamma_\mu)\frac{i}{\not k - m + i\epsilon}(-ie\gamma_\nu)\frac{i}{\not k + \not q - m + i\epsilon}\right]$$

$$= -4e^2\int\frac{d^dk}{(2\pi)^d}\frac{k_\mu(k+q)_\nu + k_\nu(k+q)_\mu - g_{\mu\nu}[k\cdot(k+q) - m^2]}{[k^2 - m^2 + i\epsilon][(k+q)^2 - m^2 + i\epsilon]} \quad \text{QFT-§7.5}$$

$$= -4e^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{2\ell_\mu\ell_\nu - g_{\mu\nu}\ell^2 - 2x(1-x)q_\mu q_\nu + g_{\mu\nu}[m^2 + x(1-x)q^2] + (\text{linear in } \ell)}{[\ell^2 + x(1-x)q^2 - m^2 + i\epsilon]^2} \quad \ell \equiv k + xq$$

Symmetric integration:

$\ell_\mu\ell_\nu \to g_{\mu\nu}\ell^2/d$

terms linear in ℓ vanish upon integration

We have

$$(a) = -4e^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2\left[\frac{2}{d}-1\right]g_{\mu\nu} - 2x(1-x)q_\mu q_\nu + g_{\mu\nu}\left[m^2 + x(1-x)q^2\right]}{[\ell^2 + x(1-x)q^2 - m^2 + i\epsilon]^2}$$

$$= -\frac{i4e^2}{(4\pi)^{d/2}} \int_0^1 dx \left\{ \left(\frac{2}{d}-1\right) g_{\mu\nu} \cdot (-1) \frac{d}{2} \frac{\Gamma(1-\frac{d}{2})}{\Gamma(2)} \left[m^2 - x(1-x)q^2 - i\epsilon\right]^{\frac{d}{2}-1} \right.$$ QFT-(7.64) (7.65)

$$\left. + \left[-2x(1-x)q_\mu q_\nu + g_{\mu\nu}\left(m^2 + x(1-x)q^2\right)\right] \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left[m^2 - x(1-x)q^2 - i\epsilon\right]^{\frac{d}{2}-2} \right\}$$

$$= -\frac{i4e^2}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) (g_{\mu\nu}q^2 - q_\mu q_\nu) \int_0^1 dx\ 2x(1-x) \left[m^2 - x(1-x)q^2 - i\epsilon\right]^{\frac{d}{2}-2},$$

$$(b) = -i(q^2 g_{\mu\nu} - q_\mu q_\nu)\delta_3.$$ $a\Gamma(a) = \Gamma(a+1)$

Then,

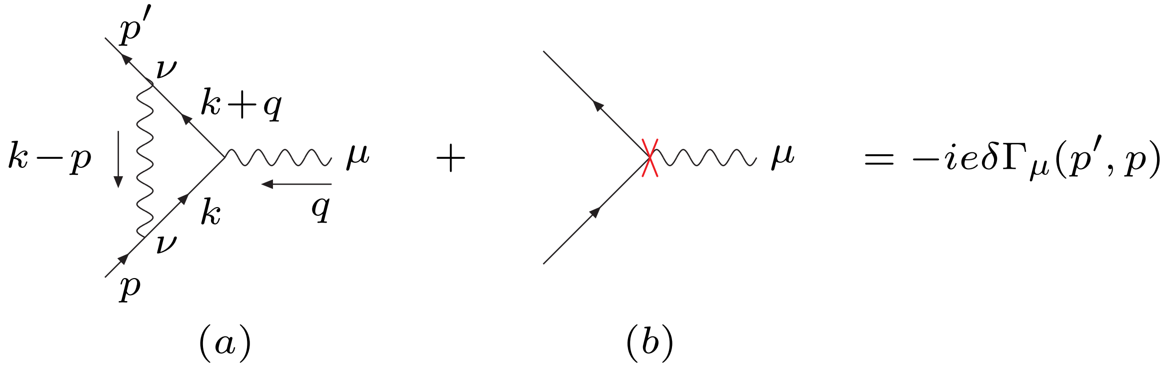
$$i\Pi_{\mu\nu}(q) = (a) + (b)$$

$$\Rightarrow \quad \Pi_2(q^2) = -\frac{4e^2}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \int dx\ 2x(1-x) \left[\cdots\right]^{\frac{d}{2}-2} - \delta_3.$$

$*$ $\Pi_2(0) = 0$ gives

$$\delta_3 = -\frac{4e^2}{(4\pi)^{d/2}}\Gamma\Big(2-\frac{d}{2}\Big)\int dx\ 2x(1-x)(m^2)^{\frac{d}{2}-2}. \tag{27}$$

δ_1 at one loop



$$(a) = (-ie)^3\int\frac{d^dk}{(2\pi)^d}\ \gamma_\nu\frac{i}{\not k+\not q-m+i\epsilon}\gamma_\mu\frac{i}{\not k-m+i\epsilon}\gamma^\nu\frac{-i}{(k-p)^2-\mu^2+i\epsilon} = -e^3\int\frac{d^dk}{(2\pi)^d}\frac{N_\mu}{D},$$

where

$$\begin{aligned} N_\mu = & -2\not k\gamma_\mu(\not k+\not q) + (4-d)(\not k+\not q)\gamma_\mu\not k + m^2(2-d)\gamma_\mu \leftarrow \text{QFT-}(7.68) \\ & +m\big[4(2k+q)_\mu - (4-d)(\not k+\not q)\gamma_\mu - (4-d)\gamma_\mu\not k\big] \end{aligned}$$

and

$$\frac{1}{D} = \left\{[k^2 - m^2 + i\epsilon][(k+q)^2 - m^2 + i\epsilon][(k-p)^2 - \mu^2 + i\epsilon]\right\}^{-1} \leftarrow \text{QFT-§6.3}$$

$$= \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2}{[\ell^2 + xyq^2 - (1-z)^2 m^2 - z\mu^2 + i\epsilon]^3}$$

$$\ell \equiv k + yq - zp,\ p'^2 = p^2 = m^2,\ 2p \cdot q = -q^2$$

Upon shifting the loop momentum, symm. integration, and $(4-d) \cdot \text{finite} \to 0$,

$$N_\mu \to \frac{(2-d)^2}{d}\ell^2\gamma_\mu - 2(z\not{p} - y\not{q})\gamma_\mu[z\not{p} + \not{q}(1-y)] + m^2(2-d)\gamma_\mu$$
$$+4m[2zp_\mu + (1-2y)q_\mu]$$
$$\to \frac{(2-d)^2}{d}\ell^2\gamma_\mu - 2[mz - (z+y)\not{q}]\gamma_\mu[mz + (1-y)\not{q}] + m^2(2-d)\gamma_\mu$$
$$+4m[z(2m\gamma_\mu - \not{q}\gamma_\mu) + (1-2y)q_\mu],$$

where $\bar{u}(p')\not{p}' = m\bar{u}(p'),\ \not{p}u(p) = mu(p)$ are applied. Using further:

$$\not{q}\gamma_\mu\not{q} = -q^2\gamma_\mu + 2q_\mu\not{q},\ \not{q}\gamma_\mu = \frac{1}{2}[\not{q}, \gamma_\mu] + q_\mu,\ \gamma_\mu\not{q} = -\frac{1}{2}[\not{q}, \gamma_\mu] + q_\mu,$$

N_μ simplifies into

$$N_\mu \to \frac{(2-d)^2}{d}\ell^2\gamma_\mu - 2m^2z^2\gamma_\mu + m^2(2-d)\gamma_\mu + 8m^2z\gamma_\mu$$
$$+2(y+z)(1-y)(-q^2\gamma_\mu + 2q_\mu \not{q}) + 2mz(y+z)\Big(\frac{1}{2}[\not{q},\gamma_\mu] + q_\mu\Big)$$
$$-2mz(1-y)\Big(-\frac{1}{2}[\not{q},\gamma_\mu] + q_\mu\Big) - 4mz\Big(\frac{1}{2}[\not{q},\gamma_\mu] + q_\mu\Big) + 4m(1-2y)q_\mu$$

Further manipulations:

$$q_\mu: \;\; 2m[z(y+z) - z(1-y) - 2z + 2(1-2y)] = 2m(z-2)(y-x), \;\; \text{vanishing upon} \int dxdy$$

$$\frac{1}{2}[\not{q},\gamma_\mu]: \;\; -2mz(1-z), \qquad q_\mu\not{q} = q_\mu[\not{p} - (\not{p} - \not{q})] \to 0$$

Namely,

$$N_\mu \quad \to \quad \frac{(2-d)^2}{d}\ell^2\gamma_\mu + \gamma_\mu[(-2z^2 + 8z + (2-d))m^2 - 2(1-x)(1-y)q^2]$$
$$-2mz(1-z)i\sigma_{\mu\nu}q^\nu$$

Therefore, $-ie\delta\Gamma_\mu(p',p) = (a) + (b)$ with

$$(a) = -2e^3 \int dxdydz\ \delta(x+y+z-1) \int \frac{d^d\ell}{(2\pi)^d}$$

$$\times \frac{\frac{(2-d)^2}{d}\ell^2\gamma_\mu + \left(\gamma_\mu[-2m^2(z^2-4z+1) - 2q^2(1-x)(1-y)] - 2mz(1-z)i\sigma_{\mu\nu}q^\nu\right)}{[\ell^2 + xyq^2 - (1-z)^2m^2 - z\mu^2 + i\epsilon]^3}$$

$$= -2e^3 \int dxdydz\ \delta(x+y+z-1) \cdot \frac{i}{(4\pi)^{d/2}} \qquad \triangle \equiv (1-z)^2m^2 + z\mu^2 - xyq^2 - i\epsilon$$

$$\times \left\{ \frac{(2-d)^2}{d}\gamma_\mu \cdot \frac{d}{2}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(3)}\triangle^{d/2-2} - (\cdots) \cdot \frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)}\triangle^{d/2-3} \right\}$$

$$(b) = -ie\delta_1\gamma_\mu$$

$*$ $[\Gamma_\mu(p',p)]_{\not p',\not p\to m} = \gamma_\mu$, i.e., $[\delta\Gamma_\mu(p',p)]_{\not p',\not p\to m} = 0$, gives

$$\delta_1 = -\frac{2e^2}{(4\pi)^{d/2}} \int dxdydz\ \delta(x+y+z-1) \qquad \Delta(0) \equiv \Delta\Big|_{q^2=0}$$

$$\times \left[\frac{1}{4}(2-d)^2\Gamma\left(2-\frac{d}{2}\right)\Delta(0)^{d/2-2} + \frac{m^2(z^2-4z+1)}{\Delta(0)}\right] \tag{28}$$

Ward identity and its physical implication

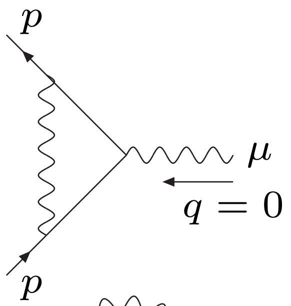
It can be checked that $\delta_1 = \delta_2$ at one loop. This is a result of Ward identity

$$\text{QFT-}(7.54): \; q_\mu \Gamma^\mu(p+q,p) = \Sigma(p) - \Sigma(p+q) \quad \text{for renormalized quantities}$$

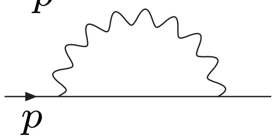
$$\frac{\partial}{\partial q_\nu}\Big|_{q=0}: \; \Gamma_\nu(p,p) = -\frac{\partial}{\partial p_\nu}\Sigma(p) \quad \text{for renormalized quantities} \tag{29}$$

Eqn. (29) holds separately for unrenormalized one-loop diagrams and c.t.

Check for unrenor. one-loop diagrams:



$$= \quad -e^3 \int \frac{d^dk}{(2\pi)^d} \gamma_\nu \frac{1}{\not{k} - m + i\epsilon} \gamma_\mu \frac{1}{\not{k} - m + i\epsilon} \gamma^\nu \frac{1}{(k-p)^2 - \mu^2 + i\epsilon}$$

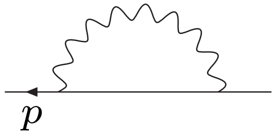


$$= \quad -e^2 \int \frac{d^dk}{(2\pi)^d} \gamma_\nu \frac{1}{\not{k} - m + i\epsilon} \gamma^\nu \frac{1}{(k-p)^2 - \mu^2 + i\epsilon}$$

$$= \quad -e^2 \int \frac{d^dk}{(2\pi)^d} \gamma_\nu \frac{1}{(\not{k} + \not{p}) - m + i\epsilon} \gamma^\nu \frac{1}{k^2 - \mu^2 + i\epsilon}$$

Thus,

$$\frac{\partial}{\partial p^\mu}$$

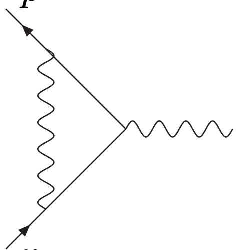


$$= e^2 \int \frac{d^dk}{(2\pi)^d} \gamma_\nu \frac{1}{(\not{k}+\not{p}) - m + i\epsilon} \gamma_\mu \frac{1}{(\not{k}+\not{p}) - m + i\epsilon} \gamma^\nu \frac{1}{k^2 - \mu^2 + i\epsilon}$$

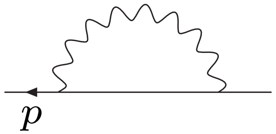
$$= e^2 \int \frac{d^dk}{(2\pi)^d} \gamma_\nu \frac{1}{\not{k} - m + i\epsilon} \gamma_\mu \frac{1}{\not{k} - m + i\epsilon} \gamma^\nu \frac{1}{(k-p)^2 - \mu^2 + i\epsilon}$$

demonstrate the derivation on blackboard

i.e.,

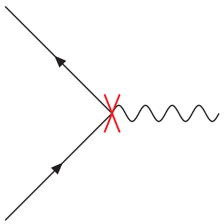


$$= (-e) \frac{\partial}{\partial p^\mu}$$

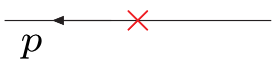


as expected

Then the relation must hold also for c.t. diagrams:



$$= (-e) \frac{\partial}{\partial p^\mu}$$



$$\Rightarrow -ie\delta_1\gamma_\mu = (-e)i\gamma_\mu\delta_2 \Rightarrow \quad \delta_1 = \delta_2 \qquad (30)$$

The relation $\delta_1 = \delta_2$ (or $Z_1 = Z_2$) holds order by order in pert. theory in a similar manner. Then

$$\left.\begin{array}{l} e_0 Z_2 Z_3^{\frac{1}{2}} = e Z_1 \\ Z_1 = Z_2 \end{array}\right\} \Rightarrow \quad e_0 Z_3^{\frac{1}{2}} = e \tag{31}$$

Renormalization of charge does not depend on the reference charged field that is chosen to do renormalization, but is uniquely determined by the photon field renormalization.

With $\delta_1 = \delta_2, \ \delta_3$ and δ_m known, all QED processes to one-loop order are well-defined and free of UV div.

[finished in 3 units on Oct 19, 2012.]

2.4. Renormalization beyond the leading order

In the last sections,

* We classified QFT according to D.
* In renormalizable QFT, $D \geqslant 0$ only for a finite number of amplitudes.
* UV divs are polynomials in external momenta at one-loop order.

We also warned that the actual UV behavior can be

* better due to sym., or
* worse at higher orders due to subdiv..

$\mathcal{L}$ is local in the sense that it is a function of fields and derivatives of fields of finite order.

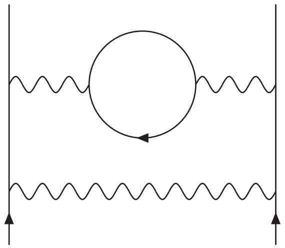
$\Rightarrow \mathcal{L}_{\mathrm{c.t.}}$ is a polynomial in external momenta in momentum space.

Can all UV div. in renormalizable QFT be removed by renormalization at higher orders?

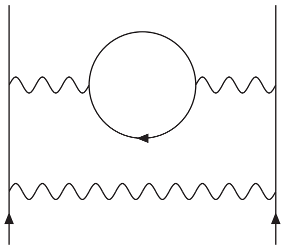
Bogliubov, Parasiuk, Hepp & Zimmermann: Yes!

BPHZ theorem: All UV divs. in a renormalizable QFT can be removed order by order in pert. theory by c.t. of superficially div. amplitudes.

Example 1: QED



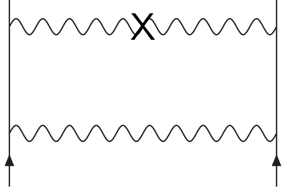
$$D = 4 - 4 \cdot \frac{3}{2} = -2 < 0,\ \text{but UV div. due to subgragh div.!}$$



$+$

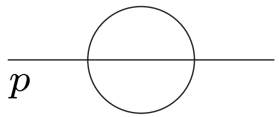
one-loop c.t.

$\downarrow$

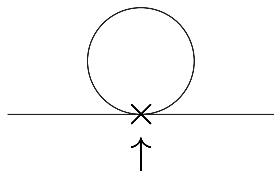


$=$ finite

Example 2: φ^4

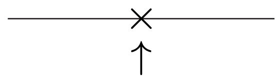


UV div: both subdiv and overlap div



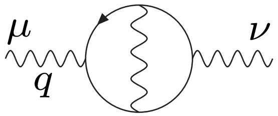
one-loop c.t.

Sum of them : subdiv removed but still UV div., cancelled by

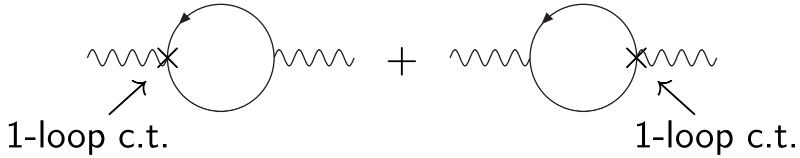


2-loop c.t.

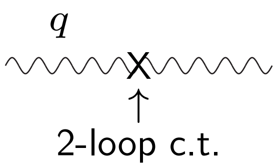
Example 3: QED



UV div., not a polynomial in q

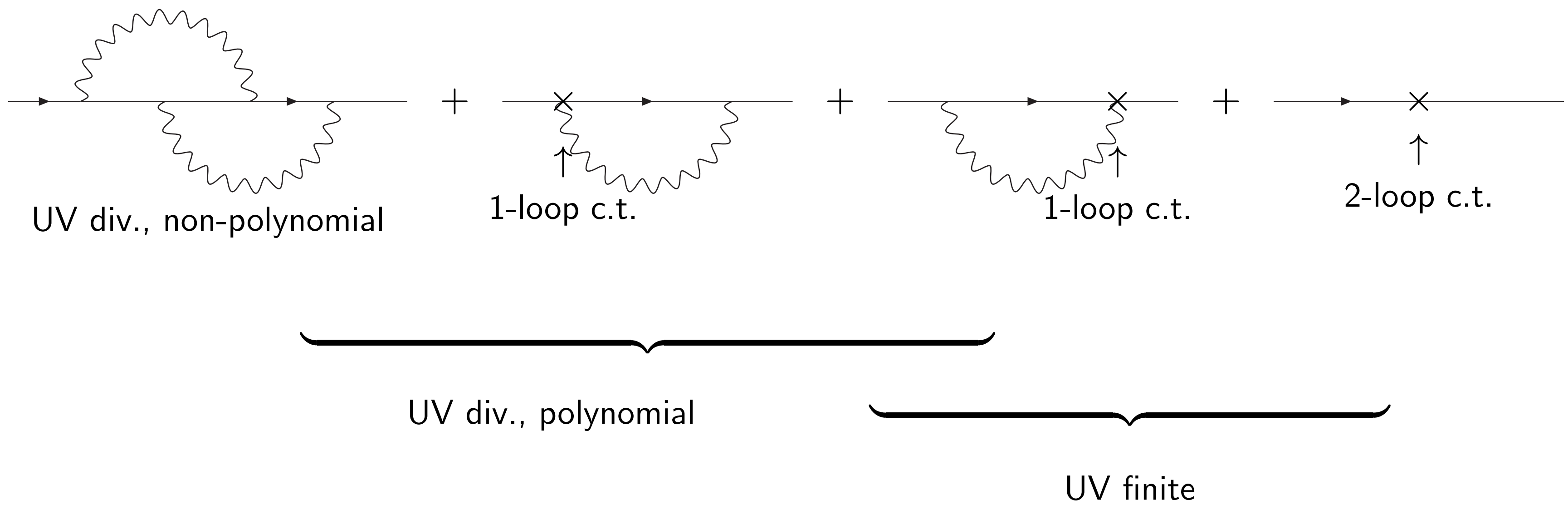


Sum of them : still UV div., but a polynomial in q which is cancelled by



$\Rightarrow$ total: finite and $\propto (q^2 g_{\mu\nu} - q_\mu q_\nu)$

Example 4: QED



Homework : Problem 10.1, 10.2

Chapter 3 Renormalization group

Follow S. Pokorski, Gauge field theories (Cambridge Univ. Press 1987)

3.1. Renormalization group equetion

1. Renormalization ‘group’ (RG)

$*$ Renormalized parameters (masses, couplings) depend on renormalization conditions (or prescriptions) used to fix them.

$*$ Physical quantities (physical masses, S matrix etc) are indept. of the prescriptions.

$*$ Different prescriptions yield different relations between renormalized parameters and physical quantities.

In perturbation theory, a new prescription results in a re-organization of the perturbative expansion compared to an old one; thus, perturbative results to the apparently same order using different prescriptions can differ by a higher order term.

Renormalization ‘group’ describes transformations of different prescriptions.

renormalization conditions $\longleftrightarrow$ renor. prescriptions $\longleftrightarrow$ renor. schemes

1PI Green's function: $\Gamma \begin{cases} \text{bare: } \Gamma_B \\ \text{renormalized in } R \text{ scheme: } \Gamma_R \end{cases}$

$\Gamma_R = Z(R)\Gamma_B$ $\quad$ ($Z(R)$ is a product of $Z_R^{1/2}$ for each field; compare with full or connected Green's functions!

For a different scheme R',

$$\Gamma_{R'} = Z(R')\Gamma_B \Rightarrow \Gamma_{R'} = Z(R', R)\Gamma_R,\ Z(R', R) = Z(R')/Z(R)$$

Properties of $Z(R', R)$:

$*$ $\quad Z(R_3, R_1) = Z(R_3, R_2)Z(R_2, R_1)$

$*$ $\quad Z^{-1}(R_2, R_1) = Z(R_1, R_2)$

$*$ $\quad Z(R, R) = 1$

But $\quad Z(R_1, R_2)Z(R_3, R_4) \neq Z(R_5, R_6)$ for $R_2 \neq R_3$

$\Rightarrow$ The set of all $Z(R_1, R_2)$ does not form a group.

2. Derivation of Renormalization Group Equation (RGE)

Restrict ourselves to mass-indept. renormalization schemes.
In such schemes, all renor. const. Z are mass-indept. $\to$ simplest RGE.

Example: 2 schemes in QED

<u>on-shell renor. scheme</u>

$$\Sigma(\not{p})\Big|_{\not{p}\to m} = 0$$

renormalized mass m is the physical mass, i.e., the location of the pole of the propagator:

$$m\delta_2 - \delta_m = \frac{\alpha}{4\pi}\frac{2}{\varepsilon}3m + \text{fin.} \quad (d = 4 - \varepsilon)$$

$$\text{Similarly, } \delta_2 = \frac{\alpha}{4\pi}\left(-\frac{2}{\varepsilon}\right) + \text{fin.}$$

<u>minimal subtraction</u> (MS) <u>scheme</u>
include only the singular term ($1/\varepsilon$ at one loop) in renor. consts:

$$m\delta_2 - \delta_m = \frac{\alpha}{4\pi}\frac{2}{\varepsilon}3m$$

$$\text{Similarly, } \delta_2 = \frac{\alpha}{4\pi}\left(-\frac{2}{\varepsilon}\right)$$

$$\begin{aligned} \text{renormalized mass} \quad &\neq \quad \text{physical mass} \\ \text{residue at pole} \quad &\neq \quad 1 \end{aligned}$$

Derivation for $\lambda\varphi^4$

dimensional regularization ($d = 4 - \varepsilon$), minimal subtraction

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \varphi_0)^2 - \frac{1}{2} m_0^2 \varphi_0^2 - \frac{1}{4!}\lambda_0 \varphi_0^4 \quad \text{in } d\text{-dim} \begin{cases} [\varphi_0] = [M]^{\frac{d}{2}-1} = [M]^{1-\frac{\varepsilon}{2}} \\ [\lambda_0] = [M]^{4-d} = [M]^{\varepsilon} \end{cases}$$

Introduce renor. consts:

$$\varphi_0 = Z_3^{1/2}\varphi_R,\ m_0^2 = Z_3^{-1} Z_0 m_R^2,\ \lambda_0 = Z_3^{-2} Z_1 \lambda_R \mu^{\epsilon} \quad \mu : \text{renor. scale} \tag{1}$$

All Z_i & λ_R are dimensionless! $Z_i = Z_i(\lambda_R, \varepsilon)$ in mass-indept. schemes.

Consider n-point 1PI:

$$\Gamma_R^{(n)}(p_i, \lambda_R, m_R, \mu, \varepsilon) = Z_3^{n/2}\Gamma_0^{(n)}(p_i, \lambda_0, m_0, \varepsilon) \tag{2}$$

$$\Gamma_0^{(n)} \text{ is indept. of } \mu : \ \mu \frac{d\Gamma_0^{(n)}}{d\mu} = 0 \tag{3}$$

$$\Rightarrow \left(\mu\frac{\partial}{\partial\mu} + \mu\frac{d\lambda_R}{d\mu}\frac{\partial}{\partial\lambda_R} + \mu\frac{dm_R}{d\mu}\frac{\partial}{\partial m_R}\right)\Gamma_R^{(n)} = \left(\frac{1}{2} n Z_3^{\frac{n}{2}-1}\mu\frac{dZ_3}{d\mu}\right)\Gamma_0^{(n)} = \left(\frac{n}{2}\frac{1}{Z_3}\mu\frac{dZ_3}{d\mu}\right)\Gamma_R^{(n)} \tag{4}$$

Define

$$\begin{cases} \beta(\lambda_R, \varepsilon) = \mu \frac{d\lambda_R}{d\mu} \\ = -\varepsilon \mu^{-\varepsilon} \lambda_0 Z_3^2 Z_1^{-1} + \mu^{-\varepsilon} \lambda_0 \mu \frac{d}{d\mu}(Z_3^2 Z_1^{-1}) \\ \gamma(\lambda_R, \varepsilon) = \frac{1}{2}\frac{1}{Z_3}\mu \frac{dZ_3}{d\mu} = \frac{1}{2}\mu \frac{d\ln Z_3}{d\mu} \\ \gamma_m(\lambda_R, \varepsilon) = \frac{1}{m_R}\mu \frac{dm_R}{d\mu} = \frac{1}{2}\mu \frac{d\ln(Z_3 Z_0^{-1})}{d\mu} \end{cases} \Rightarrow \begin{cases} \lim\limits_{\varepsilon\to 0} \beta(\lambda_R, \varepsilon) = \beta(\lambda_R) \\ \lim\limits_{\varepsilon\to 0} \gamma(\lambda_R, \varepsilon) = \gamma(\lambda_R) \\ \lim\limits_{\varepsilon\to 0} \gamma_m(\lambda_R, \varepsilon) = \gamma_m(\lambda_R) \end{cases} \qquad (5)$$

$\lim\limits_{\varepsilon\to 0}$ exists for eqn. (4) in a renormalizable theory
(Reasoning: $\beta(\lambda_R, \varepsilon)$, $\gamma_m(\lambda_R, \varepsilon)$ well-defined by definition;
lhs of (4) well-defined $\Rightarrow$ rhs and thus $\gamma(\lambda_R, \varepsilon)$ too.)

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda_R)\frac{\partial}{\partial \lambda_R} + m_R \gamma_m(\lambda_R)\frac{\partial}{\partial m_R} - n\gamma(\lambda_R)\right] \Gamma_R^{(n)}(p_i, \lambda_R, m_R, \mu) = 0 \quad \text{RGE} \qquad (6)$$

Remarks:

- β, γ, γ_m can depend on m_R/μ in mass-dept. schemes!
- Change of μ is compensated by changes of renor. consts and renormalized parameters.

3. Solving RGE

Denote $\mu = e^{-t}\mu_0$, μ_0 fixed; then $\mu\dfrac{\partial}{\partial\mu} = -\dfrac{\partial}{\partial t}$.

Define functions $\bar{\lambda}(t,\lambda), \bar{m}(t,\lambda,m)$ by ($_R$ suppressed from now on):

$$\begin{cases} \displaystyle\int_{\lambda}^{\bar{\lambda}(t,\lambda)} \beta^{-1}(x)dx = t & (a) \\ \bar{m}(t,\lambda,m) = m\exp\displaystyle\int_{\lambda}^{\bar{\lambda}(t,\lambda)} dx\frac{\gamma_m(x)}{\beta(x)} = m\exp\int_0^t dt'\gamma_m(\bar{\lambda}(t',\lambda)) & (b) \end{cases} \tag{7}$$

Eqs. (7) are equivalent to the following upon $\dfrac{\partial}{\partial t}$:

$$\begin{cases} \dfrac{\partial}{\partial t}\bar{\lambda}(t,\lambda) = \beta(\bar{\lambda}(t,\lambda)) \equiv \beta(\bar{\lambda}), \quad \bar{\lambda}(0,\lambda) = \lambda & (a) \\ \dfrac{\partial}{\partial t}\bar{m}(t,\lambda,m) = \gamma_m(\bar{\lambda}(t,\lambda))\bar{m}(t,\lambda,m) \equiv \gamma_m(\bar{\lambda})\bar{m}(t,\lambda,m), \;\; \bar{m}(0,\lambda,m) = m & (b) \end{cases} \tag{8}$$

Furthermore,

$$\frac{\partial}{\partial\lambda}(7a):\frac{1}{\beta(\overline{\lambda})}\frac{\partial\overline{\lambda}}{\partial\lambda}-\frac{1}{\beta(\lambda)}=0;\ \&(8a)\Rightarrow\left[-\frac{\partial}{\partial t}+\beta(\lambda)\frac{\partial}{\partial\lambda}\right]\overline{\lambda}(t,\lambda)=0 \qquad (a)$$

$$\frac{\partial}{\partial m}(7b):\frac{\partial\overline{m}}{\partial m}=\frac{\overline{m}}{m} \qquad (b) \quad (9)$$

$$\frac{\partial}{\partial\lambda}(7b):\frac{\partial\overline{m}}{\partial\lambda}=\overline{m}\left[\frac{\gamma_m(\overline{\lambda})}{\beta(\overline{\lambda})}\frac{\partial\overline{\lambda}}{\partial\lambda}-\frac{\gamma_m(\lambda)}{\beta(\lambda)}\right] \qquad (9a),\ (8b) \qquad (9b) \quad \text{see Appendix}$$

$$\Rightarrow\left[-\frac{\partial}{\partial t}+\beta(\lambda)\frac{\partial}{\partial\lambda}+\gamma_m(\lambda)m\frac{\partial}{\partial m}\right]\overline{m}(t,\lambda,m)=0 \qquad (c)$$

Any function of t, λ and m, $f(t,\lambda,m)$, satisfies the following differential eqn. if it depends on t, λ, m via combinations $\bar\lambda$ and $\bar m$, i.e., $f(t,\lambda,m)=F(\overline{\lambda},\overline{m})$:

$$\left[-\frac{\partial}{\partial t}+\beta(\lambda)\frac{\partial}{\partial\lambda}+\gamma_m(\lambda)m\frac{\partial}{\partial m}\right]f(t,\lambda,m)=0$$

Check using $(9a)$, $(9c)$:

$$\mathrm{lhs}=\frac{\partial F(\bar\lambda,\bar m)}{\partial\bar\lambda}\left[-\frac{\partial}{\partial t}+\beta(\lambda)\frac{\partial}{\partial\lambda}\right]\bar\lambda+\frac{\partial F(\bar\lambda,\bar m)}{\partial\bar m}\left[-\frac{\partial}{\partial t}+\beta(\lambda)\frac{\partial}{\partial\lambda}+\gamma_m(\lambda)m\frac{\partial}{\partial m}\right]\bar m=0$$

$\Rightarrow$ Solution of (6):

$$\Gamma_R^{(n)}(p_i, \lambda, m, \mu_0 e^{-t}) \quad = \quad \Gamma_R^{(n)}(p_i, \bar{\lambda}, \bar{m}, \mu_0) \exp\left[-n \int_0^t dt' \gamma(\bar{\lambda}(t', \lambda))\right] \qquad (10)$$

Green's function with renormalized para. at renor. scale $\mu_0 e^{-t}$	Green's function with renormalized para. at renor. scale μ_0	matching at $t = 0$ cancels non-∂ term in (6) using (9a): see App.

Discussion: physical quantity $P(m, \lambda, \mu)$ is μ-indept:

$$\mathcal{R} P(m, \lambda, \mu) \equiv \left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + \gamma_m(\lambda) m \frac{\partial}{\partial m}\right] P(m, \lambda, \mu) = 0 \qquad (11)$$

Example 1: physical mass and residue at the pole of propagator

RGE for complete scalar propagator $D(p^2, m, \lambda, \mu)$:

$$\left[\mathcal{R} + 2\gamma(\lambda)\right] D(p^2, m, \lambda, \mu) = 0 \qquad (12)$$

For $p^2 \to m_{\text{phys}}^2$,

$$D(p^2, m, \lambda, \mu) = \frac{Z}{p^2 - m_{\text{phys}}^2} + \tilde{D}(p^2, m, \lambda, \mu) \text{ regular in } p^2$$

Therefore,

$$
\begin{aligned}
&\Big[\mathcal{R} + 2\gamma(\lambda)\Big] D(\cdots) \\
= \quad & \frac{1}{p^2 - m^2_{\text{phys}}}\mathcal{R}Z + \frac{Z}{(p^2 - m^2_{\text{phys}})^2}\mathcal{R}m^2_{\text{phys}} + \frac{Z}{p^2 - m^2_{\text{phys}}}2\gamma(\lambda) + \text{regular} \\
\Rightarrow \quad & \begin{cases} \mathcal{R}m^2_{\text{phys}} = 0 & (a) \\ \Big[\mathcal{R} + 2\gamma(\lambda)\Big] Z = 0 & (b) \end{cases}
\end{aligned}
\tag{13}
$$

Example 2: S-matrix of n scalar particles, $Z^{n/2}\Gamma^{(n)}$ $\quad$ $\Gamma^{(n)}$: 1PI

$$
\mathcal{R}\Big(Z^{n/2}\Gamma^{(n)}\Big) = \Big(\mathcal{R}Z^{n/2}\Big)\Gamma^{(n)} + Z^{n/2}\mathcal{R}\Gamma^{(n)} = \frac{n}{2}\cdot(\underset{\substack{\uparrow \\ (13b)}}{-2\gamma(\lambda)})Z^{n/2}\Gamma^{(n)} + Z^{n/2}\underset{\substack{\uparrow \\ (6)}}{n\gamma(\lambda)}\Gamma^{(n)} = 0
$$

Warning:

Physical quantities calculated to $O(\lambda^n)$ in perturbation theory can have a residual scheme dependence of $O(\lambda^{n+1})$

[finished in 3 units on Oct 26, 2012.]

4. Green's functions for rescaled momenta

$$\Gamma_R^{(n)}(p_i, m, \lambda, \mu), \text{ of mass dim. } D$$

$$\Rightarrow \quad \left[\rho\frac{\partial}{\partial\rho} + m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial\mu} - D\right]\Gamma_R^{(n)}(\rho p_i, m, \lambda, \mu) = 0 \tag{14}$$

Denote $\rho = e^t$ (forgetting about $\mu = e^{-t}\mu_0$!). Eqs. (6) and (14) give

$$\left[-\frac{\partial}{\partial t} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \Big(\gamma_m(\lambda) - 1\Big)m\frac{\partial}{\partial m} - n\gamma(\lambda) + D\right]\Gamma_R^{(n)}(e^t p_i, m, \lambda, \mu) = 0 \tag{15}$$

If $m = 0$ (massless theory) and $\beta = \gamma = 0$ (non-interacting theory), we have

$$\Gamma_R^{(n)}(e^t p_i, 0, 0, \mu) \quad = \quad \underset{\uparrow}{e^{tD}}\,\Gamma_R^{(n)}(p_i, 0, 0, \mu) \tag{16}$$

canonical scaling as expected from dimensional analysis
μ cannot enter $\leftarrow$ free theory

Otherwise, (15) can be solved as for (6):

$$\Gamma_R^{(n)}(e^t p_i, \lambda, m, \mu) \quad = \quad \Gamma_R^{(n)}(p_i, \bar\lambda(t), e^{-t}\bar m(t), \mu)e^{tD}\exp\left[-n\int_0^t dt'\gamma(\bar\lambda(t', \lambda))\right] \tag{17}$$

def. as before

or, more directly:

$$\begin{aligned}\Gamma_R^{(n)}(e^t p_i, \lambda, m, \mu) &= \Gamma_R^{(n)}(e^t p_i, \bar\lambda(t), \bar m(t), e^t \mu) \exp\left[-n \int_0^t dt' \gamma(\bar\lambda(t', \lambda))\right] \quad \leftarrow (10) \\ &= e^{tD} \Gamma_R^{(n)}(p_i, \bar\lambda(t), e^{-t}\bar m(t), \mu) \exp\left[-n \int_0^t dt' \gamma(\bar\lambda(t', \lambda))\right] \leftarrow e^t e^{-t} \bar m(t)\end{aligned}$$

as a homogeneous func. of mass dim. D

Discussions

- (17) relates Green's functions with rescaled momenta, same renor. scale but different renormalized parameters
- Comparison of (16, 17) shows that interactions introduce an anomalous scaling via renormalization effects. γ: anomalous dimension of the field.
- Rescaling of momenta can only be done consistently in the Euclidean region, to avoid physical thresholds for time-like momenta. (Otherwise, analyticity properties would not match between the two sides!)

5. RGE in QED

Adding gauge fixing term in $\mathcal{L}$: $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$
then all quantities in RGE can depend on gauge parameter ξ:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\alpha,\xi)\frac{\partial}{\partial\alpha} + \gamma_m(\alpha,\xi)m\frac{\partial}{\partial m} + \delta(\alpha,\xi)\frac{\partial}{\partial\xi} - n_\gamma\gamma_\gamma(\alpha,\xi) - n_f\gamma_f(\alpha,\xi)\right]$$
$$\times\Gamma_R^{(n_\gamma,n_f)}(p_i,\alpha,m,\mu,\xi) = 0 \tag{18}$$

where

$$\beta(\alpha,\xi) = \lim_{\varepsilon\to 0}\mu\frac{d\alpha}{d\mu},\ \gamma_m(\alpha,\xi) = \lim_{\varepsilon\to 0}\mu\frac{d\ln Z_m}{d\mu},$$
$$\gamma_\gamma(\alpha,\xi) = \lim_{\varepsilon\to 0}\frac{1}{2}\mu\frac{d\ln Z_3}{d\mu},\ \gamma_f(\alpha,\xi) = \lim_{\varepsilon\to 0}\frac{1}{2}\mu\frac{d\ln Z_2}{d\mu} \tag{19}$$

with renor. consts. defined as follows:

$$m_0 \underset{\substack{\uparrow\\ \text{diff from §2.3}}}{=} mZ_m^{-1},\ A_0^\mu = Z_3^{1/2}A^\mu,\ \psi_0 = Z_2^{1/2}\psi,\ \alpha_0 \underset{\substack{\uparrow\\ Z_1=Z_2}}{=} \alpha\mu^\varepsilon Z_3^{-1},\ \xi_0 \underset{\substack{\uparrow\\ \text{new}}}{=} \xi Z_3$$

For gauge parameter, define

$$\delta(\alpha,\xi) = \lim_{\varepsilon\to 0} \mu\frac{d\xi}{d\mu} = -\lim_{\varepsilon\to 0} \xi\mu\frac{d\ln Z_3}{d\mu} = -2\xi\gamma_\gamma(\alpha,\xi) \tag{20}$$

The solution to (18) is similar to the solution (10) of (6), with the new element:

$$\frac{\partial\bar{\xi}(t,\alpha,\xi)}{\partial t} = \delta(\alpha,\xi) \quad , \quad \bar{\xi}(0,\alpha,\xi) = \xi \tag{21}$$

Appendix

1. Derivation of (9c)

1st term on rhs becomes, upon using $(9a)$, $(8a)$, and then $(8b)$

$$\bar{m}\frac{\gamma_m(\bar{\lambda})}{\beta(\bar{\lambda})}\frac{1}{\beta(\lambda)}\frac{\partial\bar{\lambda}}{\partial t} = \frac{\bar{m}\gamma_m(\bar{\lambda})}{\beta(\lambda)} = \frac{1}{\beta(\lambda)}\frac{\partial\bar{m}}{\partial t}$$

2nd term on rhs becomes, upon using $(9b)$,

$$-\frac{m}{\beta(\lambda)}\frac{\partial\bar{m}}{\partial m}$$

2. Derivation for last factor in (10): similar to (7b)

$$Y = \int_0^t dt'\ \gamma(\bar{\lambda}(t', \lambda)) = \int_\lambda^{\bar{\lambda}(t,\lambda)} dx \frac{\gamma(x)}{\beta(x)},$$

which is a function of $\bar{\lambda}(t, \lambda)$ and λ. Acting on it with $-\frac{\partial}{\partial t} + \beta(\lambda)\frac{\partial}{\partial \lambda}$ yields two terms:

$$\Big[-\frac{\partial}{\partial t} + \beta(\lambda)\frac{\partial}{\partial \lambda}\Big]\bar{\lambda} \cdot \frac{\partial Y}{\partial \bar{\lambda}} + \beta(\lambda)\frac{\partial Y}{\partial \lambda} = 0 + \beta(\lambda) \cdot (-1)\frac{\gamma(\lambda)}{\beta(\lambda)} = -\gamma(\lambda)$$

The differential operations acting on $\exp[-nY]$ in (10) give

$$\exp[-nY] \cdot (-n)\big(-\gamma(\lambda)\big) = \exp[-nY] \cdot n\gamma(\lambda)$$

which cancels the last term in (6).

3.2. Calculation of RG functions $\beta,\ \gamma,\ \gamma_m$

MS scheme, dim. regularization (DR) in $(4-\varepsilon)$

1. φ^4 theory

$*\ \beta$

$$(5):\beta(\lambda,\varepsilon)=-\varepsilon\lambda+\lambda Z_\lambda\mu\frac{dZ_\lambda^{-1}}{d\mu}=-\varepsilon\lambda-\lambda Z_\lambda^{-1}\mu\frac{dZ_\lambda}{d\mu}\quad\left(Z_\lambda\equiv Z_3^{-2}Z_1\right)\tag{22}$$

MS & DR:

$$Z_\lambda = 1+\sum_{\nu=1}\frac{a_\nu(\lambda)}{\varepsilon^\nu}\tag{23}$$

$$a_\nu(\lambda) = a_{\nu,\nu}\lambda^\nu+a_{\nu,\nu+1}\lambda^{\nu+1}+\cdots,\ \text{pole }\frac{1}{\varepsilon^\nu}\text{ from at least }\nu\text{-loop, i.e., }\mathcal{O}(\lambda^\nu)$$

$$\mu\frac{dZ_\lambda}{d\mu} = \mu\frac{d\lambda}{d\mu}\frac{dZ_\lambda}{d\lambda}=\beta(\lambda,\varepsilon)\frac{dZ_\lambda}{d\lambda}\tag{24}$$

$$(24) \to (22): \;\; \beta(\lambda,\varepsilon) Z_\lambda + \varepsilon\lambda Z_\lambda + \beta(\lambda,\varepsilon)\lambda \frac{dZ_\lambda}{d\lambda} = 0$$

together with $\beta(\lambda,\varepsilon) = \sum_{\nu=0} \beta_\nu \varepsilon^\nu$ and (23) we have

$$\Big(\sum_{\nu=0} \beta_\nu \varepsilon^\nu\Big) \frac{d}{d\lambda}\Big[\lambda\Big(1 + \sum_{r=1} \frac{a_r}{\varepsilon^r}\Big)\Big] + \varepsilon\lambda\Big(1 + \sum_{r=1} \frac{a_r}{\varepsilon^r}\Big) = 0 \tag{25}$$

Balance of ε^n yields

$$n \geqslant 2 \quad : \quad \beta_n + \sum_{\nu=1} \beta_{\nu+n} \frac{d}{d\lambda}(\lambda a_\nu) = 0$$

$$n = 1 \quad : \quad \beta_1 + \sum_{\nu=1} \beta_{\nu+1} \frac{d}{d\lambda}(\lambda a_\nu) + \lambda = 0$$

$$n = 0 \quad : \quad \beta_0 + \sum_{\nu=1} \beta_{\nu} \frac{d}{d\lambda}(\lambda a_\nu) + \lambda a_1 = 0$$

$$n < 0 \quad : \quad \sum_{\nu=0} \beta_{\nu} \frac{d}{d\lambda}(\lambda a_{\nu+|n|}) + \lambda a_{1+|n|} = 0$$

Denoting $y_\nu = \dfrac{d}{d\lambda}(\lambda a_\nu)$, '$n \geqslant 2$' becomes

$$\begin{pmatrix} 1 & y_1 & y_2 & \cdots & \cdots \\ & 1 & y_1 & \cdots & \cdots \\ & & 1 & y_1 & \cdots \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \vdots \\ \vdots \end{pmatrix} = 0, \quad \det = 1, \text{ unique solution: } \beta_{n \geqslant 2} = 0$$

Then, the others simplify to

$$n = 1: \;\; \beta_1 + \lambda = 0$$

$$n = 0: \;\; \beta_0 + \beta_1 \frac{d}{d\lambda}(\lambda a_1) + \lambda a_1 = 0$$

$$n < 0: \;\; \beta_0 \frac{d}{d\lambda}(\lambda a_{|n|}) + \beta_1 \frac{d}{d\lambda}(\lambda a_{1+|n|}) + \lambda a_{1+|n|} = 0$$

$$\Rightarrow \quad \begin{cases} \beta(\lambda, \varepsilon) = -\varepsilon\lambda + \beta(\lambda) \\ \beta(\lambda) = \lambda^2 \dfrac{da_1}{d\lambda} \\ \lambda^2 \dfrac{da_{\nu+1}}{d\lambda} = \beta(\lambda) \dfrac{d}{d\lambda}(\lambda a_\nu) \end{cases} \tag{26}$$

Remarks:

- $\beta(\lambda)$ is completely determined by the simple pole term in Z_λ. But all loops can contribute to the simple pole term.
- All higher order poles of Z_λ are determined by the simple pole term: a reflection of renormalizability.
- Using the last eqn. in (26), we can verify by induction that $a_{\nu,\nu'} = 0$ for $\nu > \nu'$.

Meaning: lower orders in λ cannot produce a higher order pole in ε.

Proof:

$$a_\nu \equiv \sum_{r=1}^{\infty} a_{\nu,r}\lambda^r,\ \ \nu \geqslant 1;\ \ \beta(\lambda) = \lambda^2 \frac{da_1}{d\lambda} = \sum_{r=0}^{\infty} a_{1,r+1}(r+1)\lambda^{r+2}$$

$$\Rightarrow \quad \text{rhs of } (26) = \beta(\lambda)\frac{d}{d\lambda}(\lambda a_\nu) = \cdots = \sum_{t=0}^{\infty} \lambda^{t+3} \sum_{s=0}^{t} a_{1,t-s+1}\ a_{\nu,s+1}(t-s+1)(s+2)$$

$$\text{lhs of } (26) = \lambda^2 \frac{d}{d\lambda} a_{\nu+1} = \cdots = \sum_{t=0}^{\infty} \lambda^{t+2} a_{\nu+1,t+1}(t+1)$$

$$\lambda^2 :\ a_{\nu+1,1} = 0,\ \text{ i.e., } a_{\nu,1} = 0 \text{ for all } \nu > 1$$

Assuming $a_{\nu,t}=0$ for all $\nu>t$ with $t\leqslant t_0$, verify $a_{\nu+1,t_0+1}=0$:

$$\lambda^{t_0+2} \quad : \quad a_{\nu+1,t_0+1}(t_0+1)=\sum_{s=0}^{t_0-1} a_{1,t_0-s}\, a_{\nu,s+1}(t_0-s)(s+2)=0$$

$$a_{\nu,1}=a_{\nu,2}=\cdots=a_{\nu,t_0}=0$$

$*\ \gamma$ and γ_m

Can be similarly worked out in MS scheme with DR:

$$Z_3=1+\sum_{\nu=1}\frac{Z_3^{(\nu)}(\lambda)}{\varepsilon^{\nu}},\quad Z_m\equiv Z_0Z_3^{-1}=1+\sum_{\nu=1}\frac{Z_m^{(\nu)}(\lambda)}{\varepsilon^{\nu}} \tag{27}$$

$$(5)\quad\Rightarrow\quad \gamma(\lambda,\varepsilon)=\frac{1}{2}\frac{1}{Z_3}\mu\frac{dZ_3}{d\mu}=\frac{1}{2}\frac{1}{Z_3}\mu\frac{d\lambda}{d\mu}\frac{dZ_3}{d\lambda}=\frac{1}{2}\beta(\lambda,\varepsilon)Z_3^{-1}\frac{dZ_3}{d\lambda}$$

$$\gamma_m(\lambda,\varepsilon)=\frac{1}{2}Z_m\mu\frac{dZ_m^{-1}}{d\mu}=-\frac{1}{2}\beta(\lambda,\varepsilon)Z_m^{-1}\frac{dZ_m}{d\lambda}$$

Their limits as $\varepsilon\to 0$ must exist (see note after eq (5)). This gives relations among $Z_3^{(\nu)}$ and $Z_m^{(\nu)}$ respectively, and the $\mathcal{O}(\varepsilon^0)$ term:

$$\begin{aligned}
\gamma(\lambda) &= \lim_{\varepsilon\to 0}\gamma(\lambda,\varepsilon) = -\frac{1}{2}\lambda\frac{dZ_3^{(1)}(\lambda)}{d\lambda} \\
\gamma_m(\lambda) &= \lim_{\varepsilon\to 0}\gamma_m(\lambda,\varepsilon) = \frac{1}{2}\lambda\frac{dZ_m^{(1)}(\lambda)}{d\lambda}
\end{aligned} \tag{28}$$

$*$ Results at one loop

§ 2.2 gave

$$\delta_Z = \mathcal{O}(\lambda^2),\quad \delta_m = m^2\frac{\lambda}{(4\pi)^2}\frac{1}{4-d} + \text{fin.},\quad \delta_\lambda = \frac{\lambda^2}{(4\pi)^2}\frac{3}{4-d} + \text{fin.}$$

translated to our notations here:

$$Z_3 - 1 = \delta_Z,\quad Z_0 - 1 = \frac{\delta_m}{m^2},\quad Z_1 - 1 = \frac{\delta_\lambda}{\lambda}$$

and

$$Z_\lambda = Z_3^{-2}Z_1 = (1+\delta_Z)^{-2}\left(1+\frac{\delta_\lambda}{\lambda}\right),\quad Z_m = Z_0 Z_3^{-1} = \left(1+\frac{\delta_m}{m^2}\right)(1+\delta_Z)^{-1}$$

In MS scheme, this gives

$$Z_3^{(1)}(\lambda) = 0 + \mathcal{O}(\lambda^2), \quad a_1(\lambda) = \frac{3\lambda}{(4\pi)^2} + \mathcal{O}(\lambda^2), \quad Z_m^{(1)}(\lambda) = \frac{\lambda}{(4\pi)^2} + \mathcal{O}(\lambda^2)$$

$$\therefore \quad \beta = \frac{3\lambda^2}{(4\pi)^2} + \mathcal{O}(\lambda^3), \quad \gamma = \mathcal{O}(\lambda^2), \quad \gamma_m = \frac{1}{2}\frac{\lambda}{(4\pi)^2} + \mathcal{O}(\lambda^2) \tag{29}$$

2. QED

§ 2.3 gave $\delta_3 = \dfrac{\alpha}{4\pi}\Big(-\dfrac{8}{3}\Big)\dfrac{1}{4-d} + \text{fin.}$;

i.e., in MS scheme, $Z_3 = 1 - \dfrac{\alpha}{4\pi}\dfrac{8}{3}\dfrac{1}{4-d} + \mathcal{O}(\alpha^2)$

Then (be careful here, $d = 4 - \varepsilon,\ e_0 = e\mu^{\varepsilon/2} Z_3^{-1/2}$),

$$Z_\alpha = Z_3^{-1} = 1 + \frac{\alpha}{4\pi}\frac{8}{3}\frac{1}{4-d} + \mathcal{O}(\alpha^2) \quad (\alpha_0 = \alpha\mu^\varepsilon Z_\alpha, \ Z_\alpha = Z_3^{-1} \text{ due to } Z_1 = Z_2)$$

$$\left.\begin{array}{l} \Rightarrow \beta(\alpha) = \alpha^2 \dfrac{da_1(\alpha)}{d\alpha} = \dfrac{2\alpha^2}{3\pi} + \mathcal{O}(\alpha^3) \\ \text{Define } \beta(e) = \mu \dfrac{de}{d\mu} \end{array}\right\} \beta(e) = \frac{2\pi}{e}\beta(\alpha) = \frac{e^3}{12\pi^2} + \mathcal{O}(e^5) \tag{30}$$

3.3. Fixed points; effective coupling constant

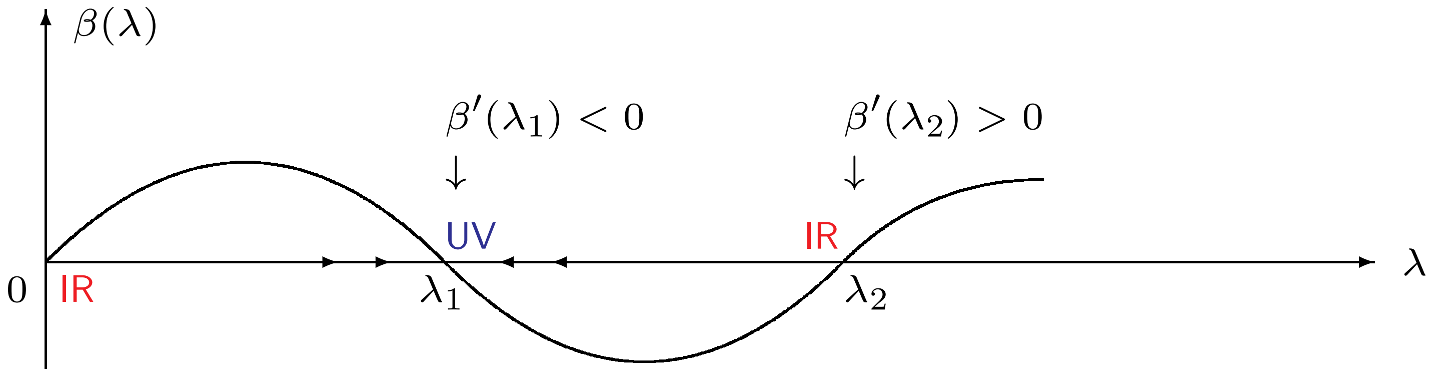
1. Fixed points

Eqn. (17) relates a Green's function to its large ($t \to +\infty$) or small momentum ($t \to -\infty$) limit. The behaviour is controlled by effective coupling $\bar{\lambda}(t, \lambda)$:

$$\frac{\partial \bar{\lambda}(t, \lambda)}{\partial t} = \beta(\bar{\lambda})$$

fixed points = zeros of $\beta(\bar{\lambda})$

Consider the example (λ denotes any coupling):



3 fixed points at $0, \lambda_1, \lambda_2$

$$\left.\begin{array}{l} 0 < \lambda < \lambda_1,\ \beta(\lambda) > 0 : \bar{\lambda} \nearrow \text{ as } t \nearrow \Rightarrow \bar{\lambda} \to \lambda_1 \text{ as } t \to +\infty \\ \lambda_1 < \lambda < \lambda_2,\ \beta(\lambda) < 0 : \bar{\lambda} \searrow \text{ as } t \nearrow \Rightarrow \bar{\lambda} \to \lambda_1 \text{ as } t \to +\infty \end{array}\right\} \text{UV stable fixed point at } \lambda_1$$

$$0 < \lambda < \lambda_1,\ \beta(\lambda) > 0 : \bar{\lambda} \searrow \text{ as } t \searrow \Rightarrow \bar{\lambda} \to 0 \text{ as } t \to -\infty$$

IR stable fixed point at $\lambda = 0$, λ_2 similar

Since $\beta(0) = 0$, $\lambda = 0$ is always a fixed point; e.g.:

$\lambda\varphi^4$, QED: "λ" $= 0$ is an IR stable fixed point $\leftarrow$ "IR stable theory"

QCD: "λ" $= 0$ is a UV stable fixed point $\leftarrow$ "asymptotically free theory"
(almost free theory in the large momentum region)

2. Effective coupling constant

$$\beta(\lambda) = \sum_{n=1}^{\infty} \lambda \cdot \left(\frac{\lambda}{\pi}\right)^n b_n \tag{31}$$

Solve eqn.(8a) for $\bar{\lambda}(t,\lambda)$ by series expansion in λ:

$$\bar{\lambda}(t,\lambda) = \sum_{n=1}^{\infty} a_n(t)\lambda^n; \quad a_1(0) = 1,\ a_{n\geqslant 2}(0) = 0 \ \Big(\leftarrow \bar{\lambda}(0,\lambda) = \lambda \Big) \tag{32}$$

$$\frac{\partial \bar{\lambda}}{\partial t} = \beta(\bar{\lambda}) \Rightarrow \sum_{n=1} \frac{da_n(t)}{dt}\lambda^n = \sum_{n=1} \frac{b_n}{\pi^n}\left(\sum_{\ell=1}^{\infty} a_\ell(t)\lambda^\ell \right)^{n+1}$$

Balance of λ^n:

$$\lambda^1 \quad : \quad \frac{da_1(t)}{dt} = 0, \quad \text{i.e., } a_1(t) = 1$$

$$\lambda^2 \quad : \quad \frac{da_2(t)}{dt} = \frac{b_1}{\pi}a_1(t) \ \Rightarrow a_2(t) = \frac{b_1 t}{\pi}$$

$$\lambda^3 \quad : \quad \frac{da_3(t)}{dt} = \frac{b_1}{\pi}2a_1(t)a_2(t) + \frac{b_2}{\pi^2}a_1^3(t) = 2t\left(\frac{b_1}{\pi}\right)^2 + \frac{b_2}{\pi} \Rightarrow a_3(t) = \left(\frac{b_1 t}{\pi}\right)^2 + \frac{b_2 t}{\pi}$$

$$\lambda^4 \quad : \quad \frac{da_4(t)}{dt} = \frac{b_1}{\pi}\Big[a_2^2(t) + 2a_1(t)a_3(t)\Big] + \frac{b_2}{\pi^2}\Big[3a_1^2(t)a_2(t)\Big] + \frac{b_3}{\pi^3}a_1^4(t)$$

We will be interested in large t behavior, thus we concentrate on the highest power of t for each $a_n(t)$:

$$
\begin{aligned}
& a_1(t) = 1, \\
& a_2(t) = \frac{b_1 t}{\pi}, \\
& a_3(t) = [a_2(t)]^2 + \cdots \\
& a_4(t) = [a_2(t)]^3 + \cdots \\
& \vdots \\
& a_n(t) = [a_2(t)]^{n-1} + \cdots \\
\Rightarrow \quad \bar{\lambda}(t, \lambda) \quad & = \lambda + \lambda^2 a_2(t) + \lambda^3 \Big[(a_2(t))^2 + \cdots \Big] + \lambda^4 \Big[(a_2(t))^3 + \cdots \Big] + \cdots \\
& = \lambda \Big[1 + a_2(t)\lambda + \big(a_2(t)\lambda\big)^2 \Big(1 + \mathcal{O}(t^{-1}) \Big) + \big(a_2(t)\lambda\big)^3 \Big(1 + \mathcal{O}(t^{-1}) \Big) + \cdots \Big] \\
& = \frac{\lambda}{1 - a_2(t)\lambda} + \cdots
\end{aligned}
$$

For large $t = \frac{1}{2} \ln \frac{\mu_0^2}{\mu^2}$ (eqn.(10)) or $t = \frac{1}{2} \ln \frac{p^2}{p_0^2}$ (eqn.(16)), the above is called the leading-logarithm approximation to effective (running) coupling const.:

$$\bar{\lambda}(t,\lambda) = \frac{\lambda}{1-\frac{b_1\lambda}{\pi}t}, \quad \bar{\lambda}(0,\lambda) = \lambda \tag{33}$$

e.g.,

$$\lambda\varphi^4 \quad : \quad \bar{\lambda}(t,\lambda) = \frac{\lambda}{1-\frac{3\lambda}{16\pi^2}t}$$

$$\text{QED} \quad : \quad \bar{\alpha}(t,\alpha) = \frac{\alpha}{1-\frac{2\alpha}{3\pi}t}$$

3.4. Renormalization scheme- and gauge-dependence of the RGE parameters

1. Renormalization scheme dependence

mass-indept. renor. schemes : primed & unprimed

$$\lambda', \beta'(\lambda'), \gamma'(\lambda'), \gamma'_m(\lambda') \qquad \lambda, \beta(\lambda), \gamma(\lambda), \gamma_m(\lambda)$$

$$\lambda' = F(\lambda) = \begin{cases} \lambda + \mathcal{O}(\lambda^2), & \text{for } \lambda\varphi^4 \text{ (or QED with } \lambda = \alpha) \\ \lambda + \mathcal{O}(\lambda^3), & \text{for QED with } \lambda = e \end{cases}$$

[The actual expansion parameter in QED is α: $F(\alpha) = \alpha(1 + O(\alpha))$.]

Renor. consts.:

$$Z'_m(\lambda') = Z_m(\lambda)F_m(\lambda), \quad Z'_3(\lambda') = Z_3(\lambda)F_3(\lambda), \quad F_i(\lambda) = \begin{cases} 1 + \mathcal{O}(\lambda) \\ 1 + \mathcal{O}(\lambda^2) \end{cases}$$

Relations of $\beta,\ \gamma,\ \gamma_m$:

$$\begin{aligned}
\beta'(\lambda') &= \mu\frac{d\lambda'}{d\mu} = \beta(\lambda)\frac{dF(\lambda)}{d\lambda} \\
\gamma'(\lambda') &= \frac{1}{2}\mu\frac{d\ln Z_3'(\lambda')}{d\mu} = \gamma(\lambda) + \frac{1}{2}\beta(\lambda)\frac{d\ln F_3(\lambda)}{d\lambda} \\
\gamma_m'(\lambda') &= \frac{1}{2}\mu\frac{d\ln Z_m'(\lambda')}{d\mu} = \gamma_m(\lambda) + \frac{1}{2}\beta(\lambda)\frac{d\ln F_m(\lambda)}{d\lambda}
\end{aligned}$$

Properties:

(1) Fixed points λ'_* of $\beta'(\lambda') \leftrightarrow$ fixed points λ_* of $\beta(\lambda)$, with $\lambda'_* = F(\lambda_*)$

(2) $\left.\frac{d\beta'}{d\lambda'}\right|_{\lambda'_*} = \left.\frac{d\beta}{d\lambda}\right|_{\lambda_*} \rightarrow$ nature of fixed point is scheme-indept. Proof:

$$\frac{d\beta'}{d\lambda'} = \left(\frac{d\lambda'}{d\lambda}\right)^{-1}\frac{d}{d\lambda}\left(\beta\frac{dF}{d\lambda}\right) = \left(\frac{dF}{d\lambda}\right)^{-1}\left[\frac{d\beta}{d\lambda}\frac{dF}{d\lambda} + \beta\frac{d^2F}{d\lambda^2}\right] = \frac{d\beta}{d\lambda} + \beta\frac{d^2F}{d\lambda^2}\left(\frac{dF}{d\lambda}\right)^{-1}$$

Evaluate the above at $\lambda_* \leftrightarrow \lambda'_*$

(3) $\gamma'(\lambda'_*) = \gamma(\lambda_*),\ \gamma'_m(\lambda'_*) = \gamma_m(\lambda_*)$

(4) First expansion coefficients of $\beta,\ \gamma,\ \gamma_m$ are scheme-indept.

$\Rightarrow$ leading-log approximation for effective coupling is scheme-indept.

leading behavior in $t \to \pm\infty$ of eqn.(10) & (17) is scheme-indept.

2. Gauge dependence of the β function

QED : $\beta(\alpha)$ is ξ-indept. $\leftarrow Z_3$ is ξ-indept. $\leftarrow Z_3$ renormalizes the transverse part of photon propagator.

QCD or other non-Abelian gauge theories: $\beta(\alpha)$ generally ξ-dept., but

$*$ first expansion coefficient of $\beta(\alpha)$ in α is ξ-indept.

$*$ MS scheme: $\beta(\alpha)$ is ξ-indept.

Homework : Problem 12.1

[finished in 3 units on Nov 2, 2012.]

Chapter 4 Non-Abelian gauge invariance

4.1. Non-Abelian gauge theories

4.2. Basic facts about Lie algebra

Renormalizability is very restrictive.

One possible extension is to consider interactions of vector bosons.

But theories of vector bosons are plagued with negative-norm problems, with the exception of gauge theories:

gauge invariance $\rightarrow$ identities $\rightarrow$ removal of negative-norm states from physical states.

$\Longrightarrow$ extension of gauge symmetries & gauge theories

Suggestion: learn this chapter by deriving all formulae

4.1. Non-Abelian gauge theories

Viewpoint: promotion of gauge invariance as a principle
instead of an 'accidental' case of QED

1. Abelian gauge theory: QED

Start from free theory of electrons:

$$\mathcal{L}_0 = \overline{\psi}(x) i\partial\!\!\!/ \psi(x) - m\overline{\psi}(x)\psi(x) \tag{1}$$

It has the symmetry under the global phase transformation:

$$\psi(x) \longrightarrow e^{i\alpha}\psi(x) \tag{2}$$

Promote it to a local symmetry by allowing α to be x-dependent:

$$\psi(x) \longrightarrow e^{i\alpha(x)}\psi(x) \tag{3}$$

$\mathcal{L}_0$ is not inva. under (3) because of the derivative term:

$$\overline{\psi}(x)\partial_\mu\psi(x) \longrightarrow \overline{\psi}(x)\left[\partial_\mu\psi(x) + i\left(\partial_\mu\alpha(x)\right)\psi(x)\right]$$

To cure this, construct a gauge-covariant derivative that transforms as $\psi(x)$:

$$D_\mu \psi(x) \longrightarrow e^{i\alpha(x)} D_\mu \psi(x) \tag{4}$$

Then, $\overline{\psi} D_\mu \psi$ is inva. under (3).

For this, we need to introduce a new vector field, called the gauge field:

$$D_\mu \psi(x) = \partial_\mu \psi(x) + ieA_\mu(x)\psi(x), \quad \text{free parameter, electron charge later} \tag{5}$$

To fulfil (4), $A_\mu(x)$ must transform under the gauge transformation as

$$A_\mu(x) \longrightarrow A_\mu(x) - e^{-1}\partial_\mu \alpha(x) \tag{6}$$

For A_μ to become a dynamic field, we need a kinetic energy term for it.

Consider the following:

$$\begin{aligned}
[D_\mu, D_\nu]\psi &= \partial_\mu \Big(\partial_\nu \psi(x) + ieA_\nu(x)\psi(x)\Big) + ieA_\mu \Big(\partial_\nu \psi(x) + ieA_\nu(x)\psi(x)\Big) - (\mu \leftrightarrow \nu) \\
&= \Big(\partial_\mu \partial_\nu \psi - e^2 A_\mu A_\nu + ie(\partial_\mu A_\nu \psi + A_\mu \partial_\nu \psi + A_\nu \partial_\mu \psi)\Big) - (\mu \leftrightarrow \nu) \\
&= ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi = ieF_{\mu\nu}\psi
\end{aligned} \tag{7}$$

(3) and (4) imply that under the gauge transformation,

$$F_{\mu\nu}(x) \longrightarrow F_{\mu\nu}(x) \tag{8}$$

as expected from (6). We thus arrive at the QED Lagrangian:

$$\mathcal{L} = \overline{\psi} i \not{D} \psi - m \overline{\psi} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad \text{2 free parameters}: \; m, \; e \tag{9}$$

Comments:

(1) If a Lagrangian constructed from ψ, $D_\mu \psi$, $F_{\mu\nu}$ and derivatives of $F_{\mu\nu}$, is inva. to global phase transf., it is also inva. under local gauge transf.

(2) Renormalizability highly restricts the terms that can appear in $\mathcal{L}$; e.g.,

$F_{\mu\nu} \overline{\psi} \sigma^{\mu\nu} \psi$ dim $= 5$, not allowed in $\mathcal{L}$ (magnetic dipole moment)

$\varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}$ dim $= 4$, allowed by renormalizability, but not allowed by P, T inva.

also a total derivative which can be ignored in pert. theory

(3) $A_\mu A^\mu$ is not gau. inva. $\Rightarrow$ gau. bosons must be massless!

(4) The $A_\mu \bar{\psi} \psi$ coupling is dictated by gau. inva. and renormalizability.

(5) Photon is electrically neutral $\Rightarrow$ no photon self-interactions in $\mathcal{L}$.

2. Non-Abelian gau. symmetry: Yang-Mills fields

Principle of gau. sym. works nicely with QED. Try larger symmetries.

Next simplest: $SU(2)$ or $SO(3)$

Origin of $SU(2)$: isotopic spin (isospin for short) in nuclear physics
$SO(3)$: rotations in 3-space (QM)

Consider a doublet of Dirac fields:

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{10}$$

Under an $SU(2)$ transf.,

$$\psi \longrightarrow \exp\left(i\alpha^i \frac{\sigma^i}{2}\right)\psi \begin{cases} \sigma^i \text{ : Pauli matrices} \\ \alpha^i \text{ : constant group parameters, real} \\ \text{summation over repeated indices implied } (i = 1, 2, 3) \end{cases} \tag{11}$$

Say: ψ constitutes the fundamental (doublet) representation (rep) of $SU(2)$.

Warning: $SU(2)$ rotates between ψ_1 and ψ_2, and has nothing to do with 4 components of a Dirac field!

Example: $\begin{pmatrix} \text{proton} \\ \text{neutron} \end{pmatrix}$ forms the isospin-$\frac{1}{2}$ rep. of $SU(2)_{\text{isospin}}$.

Promote the symmetry to a local one:

$$\psi(x) \to \exp\left[i\alpha^i(x)\frac{\sigma^i}{2}\right]\psi(x) \equiv V(x)\psi(x) \tag{12}$$

$\mathcal{L}_0$ is not inva. under (12) because the kinetic-energy term is not:

$$\begin{aligned} \partial_\mu\psi(x) &\to V(x)\partial_\mu\psi(x) + (\partial_\mu V(x))\psi(x), \\ \overline{\psi}(x)\partial_\mu\psi(x) &\to \overline{\psi}(x)\partial_\mu\psi(x) + \overline{\psi}(x)\Big(V^\dagger(x)\partial_\mu V(x)\Big)\psi(x), \ V^\dagger(x) = \exp\left[-i\alpha^i(x)\frac{\sigma^i}{2}\right]. \end{aligned}$$

Following the procedure in QED, we construct gau.-covariant derivative by introducing a vector boson field A^i_μ:

$$D_\mu\psi(x) = \partial_\mu\psi(x) - igA^i_\mu(x)\frac{\sigma^i}{2}\psi(x), \quad \text{free para.; gau. interaction coupling later} \tag{13}$$

so that

$$D_\mu \psi(x) \longrightarrow V(x) D_\mu \psi(x) \tag{14}$$

(12), (14) then require that $A^i_\mu \frac{\sigma^i}{2}$ transform as follows:

$$A^i_\mu(x)\frac{\sigma^i}{2} \to V(x) A^i_\mu(x)\frac{\sigma^i}{2} V^\dagger(x) + \frac{i}{g} V(x)\partial_\mu V^\dagger(x) \quad \text{Check it!} \tag{15}$$

For A^i_μ to be a dynamic field, we need a kinetic-energy term for it. Consider:

$$\begin{aligned}
[D_\mu, D_\nu]\psi &= \left(\partial_\mu\left(\partial_\nu\psi - igA^i_\nu\frac{\sigma^i}{2}\psi\right) - igA^j_\mu\frac{\sigma^j}{2}\left(\partial_\nu\psi - igA^i_\nu\frac{\sigma^i}{2}\psi\right)\right) - (\mu \leftrightarrow \nu) \\
&= \left\{\left[\partial_\mu\partial_\nu\psi - ig\left(A^i_\nu\frac{\sigma^i}{2}\partial_\mu\psi + A^i_\mu\frac{\sigma^i}{2}\partial_\nu\psi\right)\right] - ig\left((\partial_\mu A^i_\nu)\frac{\sigma^i}{2} - igA^j_\mu\frac{\sigma^j}{2}A^i_\nu\frac{\sigma^i}{2}\right)\psi\right\} - (\mu \leftrightarrow \nu) \\
&= -ig\left\{\left(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu\right)\frac{\sigma^i}{2} - igA^j_\mu A^i_\nu\left[\frac{\sigma^j}{2}, \frac{\sigma^i}{2}\right]\right\}\psi \leftarrow \left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2}\right] = i\epsilon^{ijk}\frac{\sigma^i}{2} \\
&= -ig\left\{\left(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu\right)\frac{\sigma^i}{2} + g\epsilon^{ijk}A^j_\mu A^k_\nu\frac{\sigma^i}{2}\right\}\psi \equiv -igF^i_{\mu\nu}\frac{\sigma^i}{2}\psi \quad \text{Check it!}
\end{aligned} \tag{16}$$

where

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk} A^j_\mu A^k_\nu \quad \text{field strength} \tag{17}$$

(14) and (16) imply transf. law for $F^i_{\mu\nu}$:

$$F^i_{\mu\nu}(x)\frac{\sigma^i}{2} \longrightarrow V(x)F^i_{\mu\nu}(x)\frac{\sigma^i}{2}V^\dagger(x) \tag{18}$$

The kinetic-energy term for A^i_μ is a natural generalization of QED:

$$-\frac{1}{4}F^i_{\mu\nu}F^{i,\mu\nu} \qquad \text{i.e., one such term for each } i$$

It is gau. inva. because, using $\mathrm{Tr}\left(\frac{\sigma^i}{2}\frac{\sigma^j}{2}\right) = \frac{1}{2}\delta^{ij}$

$$F^i_{\mu\nu}F^{i,\mu\nu} = 2\mathrm{Tr}\left(F^i_{\mu\nu}\frac{\sigma^i}{2} \cdot F^{j,\mu\nu}\frac{\sigma^j}{2}\right) \to 2\mathrm{Tr}\left(VF^i_{\mu\nu}\frac{\sigma^i}{2}V^\dagger \cdot VF^{j,\mu\nu}\frac{\sigma^j}{2}V^\dagger\right) = F^i_{\mu\nu}F^{i,\mu\nu}$$

In summary, Yang-Mills theory with a doublet fermion:

$$\mathcal{L} = -\frac{1}{4}F^i_{\mu\nu}F^{i,\mu\nu} + \overline{\psi}i\not{D}\psi - \overline{\psi}m\psi, \quad \text{2 free para.}: \ m, \ g \tag{19}$$

Comments

(1) $A^i_\mu A^{i,\mu}$ not gau. inva. $\to$ gau. bosons must be massless!

(2) A^i_μ carries a group index, i.e., carries $SU(2)$ charge, thus self-interacts.
$\leftrightarrow$ automatically contained in $(F^i_{\mu\nu})^2$

This can be viewed in classical EoM for A^i_μ

$$\partial^\mu F^i_{\mu\nu} + g\epsilon^{ijk} A^{j,\mu} F^k_{\mu\nu} = -g\overline{\psi}\gamma_\nu \frac{\sigma^i}{2}\psi \qquad \begin{array}{l}\text{absent in QED}\\ \text{similar to QED}\end{array} \tag{20}$$

(3) Concise form in matrix notation:

$$\begin{aligned} A_\mu &\equiv A^i_\mu \frac{\sigma^i}{2}, \quad F_{\mu\nu} \equiv F^i_{\mu\nu}\frac{\sigma^i}{2} \\ D_\mu\psi &= (\partial_\mu - igA_\mu)\psi \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \\ \mathcal{L} &= -\frac{1}{2}\text{Tr } F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi \end{aligned}$$

tranf. laws :

$$\left\{ \begin{array}{l} \psi(x) \longrightarrow V(x)\psi(x) \\ A_\mu \longrightarrow VA_\mu V^\dagger + \dfrac{i}{g}V\partial_\mu V^\dagger \\ F_{\mu\nu} \longrightarrow VF_{\mu\nu}V^\dagger \end{array} \right.$$

(4) Infinitesimal tranf.: $|\alpha^i(x)| \ll 1, \; \alpha(x) \equiv \frac{1}{2}\alpha^i(x)\sigma^i$

$$V(x) = 1 + i\alpha(x) + \mathcal{O}(\alpha^2(x)) \Rightarrow \begin{cases} \psi \longrightarrow (1 + i\alpha)\psi + \mathcal{O}(\alpha^2) \\ A_\mu \longrightarrow A_\mu + i[\alpha, A_\mu] + \frac{1}{g}\partial_\mu\alpha + \mathcal{O}(\alpha^2) \\ F_{\mu\nu} \longrightarrow F_{\mu\nu} + i[\alpha, F_{\mu\nu}] + \mathcal{O}(\alpha^2) \end{cases}$$

3. Non-Abelian gau. symmetry: general case

Generalization to higher groups and arbitrary rep. of ψ is straightforward.

Gauge sym. group G with generators satisfying

$$[T^a, T^b] = if^{abc}T^c, \quad \text{abstract operators, totally anti-symmetric structure const.} \tag{21}$$

Assume ψ constitutes a rep. of G with matrix generators t^a satisfying

$$[t^a, t^b] = if^{abc}t^c \tag{22}$$

Covariant derivative:

$$D_\mu\psi = (\partial_\mu - igA^a_\mu t^a)\psi \equiv (\partial_\mu - igA_\mu)\psi, \; A_\mu \equiv A^a_\mu t^a \quad \text{coupling const., gau. field} \tag{23}$$

Field tensor:

$$[D_\mu, D_\nu]\psi = \cdots = -igF^a_{\mu\nu}t^a\psi \equiv -igF_{\mu\nu}\psi \tag{24}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{25}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

Gauge transf.:

$$\begin{aligned} \psi(x) &\to V(x)\psi(x),\ V(x) = \exp\left(i\alpha^a(x)t^a\right) \\ D_\mu\psi(x) &\to V(x)D_\mu\psi(x) \\ A_\mu(x) &\to V(x)A_\mu(x)V^\dagger(x) + ig^{-1}V(x)\partial_\mu V^\dagger(x) \end{aligned} \tag{26}$$

Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \overline{\psi}(i\not{D} - m)\psi,\ \mathrm{Tr}\,t^a t^b = \frac{1}{2}\delta^{ab} \tag{27}$$

EoM for gauge fields:

$$\partial^\mu F^a_{\mu\nu} + gf^{abc}A^{b\mu}F^c_{\mu\nu} = -gj^a_\nu,\ j^a_\nu = \overline{\psi}\gamma_\nu t^a\psi \tag{28}$$

Comments

(1) Gauge bosons massless.

(2) $\#$ of gau. bosons $=$ $\#$ of generators
$=$ $\#$ of indept 'directions' of sym. transf.

(3) Gauge fields transform (even under a global transf.)
$\rightarrow$ carry 'charges' $\rightarrow$ self-interact.

(4) Universality of non-Abelian gauge interactions for a simple gau. group:

f^{abc} fixed $\rightarrow$ rescaling of t^a in (22) not allowed $\rightarrow$ g fixed in (23) for ψ
$\rightarrow$ g also fixed for A^a_μ by gau. transf. of A^a_μ in (26)

Same g for any rep. Difference in rep. is reflected in different t^a.

Comparison to $U(1)$ case: any charge ce is allowed $\leftarrow$ no analog of (22)

Product of simple/Abelian gau. groups: each simple/Abelian factor can have an indept. coupling. (e.g., SM: $g_3,\ g_2,\ g_1$ for $SU(3)_c \times SU(2)_L \times U(1)_Y$)

4.2. Basic facts about Lie algebra

1. Lie groups and Lie algebras

We are interested in groups of symmetry transformations that act on vector space of quantum states $\to$ groups of unitary transfs.

For constructing $\mathcal{L}$ of gauge theories, only infinitesimal transformations close to identity are important:

$$\underbrace{g(\alpha)}_{\text{group element}} = \underbrace{1}_{\text{identity}} + i\underbrace{\alpha^a}_{\text{group parameter}}\overbrace{T^a}^{\text{group generator}} + \mathcal{O}(\alpha^2), \quad \text{Lie group} \tag{29}$$

T^a spans the space of infinitesimal transf. $\to$ must satisfy

$$[T^a, T^b] = i\underbrace{f^{abc}}_{\text{structure constants}}T^c \qquad \text{Lie algebra} \tag{30}$$

(Summation over repeated indices implied unless otherwise stated.)

Jacobi identity:

$$\left[T^a, [T^b, T^c]\right] + \left[T^b, [T^c, T^a]\right] + \left[T^c, [T^a, T^b]\right] = 0 \tag{31}$$

$$\leftrightarrow \quad f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0 \tag{32}$$

2. Classification of Lie algebras

groups of unitary transformations;

compact groups, finite-dim. unitary rep.;

semi-simple groups, simple groups;

A general Lie algebra is a direct sum of non-Abelian simple components and Abelian generators.

Killing & Cartan: all possible compact simple Lie algebras are classified into

$$\begin{cases} \text{classical groups} \begin{cases} \text{SU(N) \& U(1)} \\ \text{SO(N) (\& O(N))} \\ \text{Sp(N)} \end{cases} \\ \text{exceptional groups: } G_2,\ F_4,\ E_6,\ E_7,\ E_8 \quad \text{employed in grand unification} \end{cases}$$

(1) Unitary transformations of N-dim. vectors

Complex N-component column vectors

$$\xi = (\xi_1, \xi_2, \cdots, \xi_N)^T, \ \eta = (\eta_1, \eta_2, \cdots, \eta_N)^T$$

transform as $\xi \to U\xi, \ \eta \to U\eta$ with

$$\xi^\dagger \eta \to \xi^\dagger U^\dagger U \eta = \xi^\dagger \eta, \ \text{i.e.,} \ U^\dagger U = 1_{N\times N} \leftrightarrow U^\dagger = U^{-1}$$

We say they constitute a rep. of $U(N)$ group.
Subgroups:

$$U(1) : \xi \to e^{i\alpha}\xi, \ \text{i.e., same } \alpha \text{ for all components, commute with all other elements of U(N)}$$

Removing the above $U(1)$ from $U(N)$ gives the simple group $SU(N)$.

For $U \in SU(N), \ \det U = 1$
$\leftrightarrow \mathrm{Tr}\, t^a = 0 \ (a = 1, 2, \cdots, N^2 - 1)$ for generators of $SU(N)$

(2) Orthogonal transformations of N-dim. vectors

Generalization of 3-dim. rotational group $SO(3)$ or $O(3)$ to N-dim. rotational group $SO(N)$ or $O(N)$. For real vectors η, ξ and $R \in O(N)$:

$$\eta \to R\eta, \ \xi \to R\xi \Rightarrow \eta^T\xi \to \eta^T R^T R\xi = \eta^T\xi,$$

$$\text{i.e.,} \ \ R^T R = 1_{N\times N}, \ \ \det R = \begin{cases} 1, & \text{without space inversion} \\ -1, & \text{with space inversion} \end{cases}$$

$SO(N)$ has $N(N-1)/2$ generators.

(3) Symplectic transformations of N-dim. vectors (for N even)

Consist of $N \times N$ matrices $U^\dagger = U^{-1}$ that preserve the inner product

$$\eta^T E\xi, \ E = \begin{pmatrix} 0 & 1_{\frac{N}{2}\times\frac{N}{2}} \\ -1_{\frac{N}{2}\times\frac{N}{2}} & 0 \end{pmatrix}$$

$Sp(N)$ has $N(N+1)/2$ generators.

[finished in 3 units on Nov 9, 2012.]

3. Representations

Fields in gau. theories constitute naturally reps. of gau. group.

Any rep. is a direct sum of irreducible reps, denoted as r

We are interested in finite-dim. unitary irre. reps.: Lie algebra generators of groups are represented by $d \times d$ Hermitian matrices t_r^a
(d: dim. of rep.; a: enumerating generators)

Normalization convention: always possible to choose a basis of t_r^a so that

$$\mathrm{Tr}\, t_r^a t_r^b = C(r)\delta^{ab} \tag{33}$$

$$\rightarrow \quad f^{abc} = -\frac{i}{C(r)}\mathrm{Tr}\left([t_r^a, t_r^b]t_r^c\right), \text{ totally anti-sym.} \tag{34}$$

Conjugate rep.: If φ belongs to rep. r of G with generators t_r^a,

$$\varphi \rightarrow \exp(i\alpha^a t_r^a)\varphi \tag{35}$$

then φ^* belongs to its conjugate rep. $\bar{r}$

$$\varphi^* \longrightarrow \exp(i\alpha^a t_{\bar{r}}^a)\varphi^*, \text{ with } t_{\bar{r}}^a = -(t_r^a)^* = -(t_r^a)^T. \tag{36}$$

Then, $(\varphi^*)^T\varphi = \varphi^\dagger\varphi$ is inva. to transf. under G.

real rep. (strictly real, pseudo-real)

Commonly used reps.:

(1) Fundamental or defining rep.

$SU(N)$: fundamental rep. N spanned by N-dim complex vectors
complex for $N > 2$ (pseudo-real for $N = 2$)

$SO(N)$: fundamental rep. N spanned by N-dim real vectors, strictly real

$Sp(N)$: fundamental rep. N, pseudo-real

(2) Adjoint rep. G: rep. to which the generators of the algebra belong.

$$(t^b_G)_{ac} = if^{abc} \tag{37}$$

They satisfy the Lie algebra (using Jacobi identity (32)):

$$\left([t^b_G, t^c_G]\right)_{ae} = if^{bcd}(t^d_G)_{ae}$$

$(t^a_G)^* = -(t^a_G)$: adj. rep is real.

Dimension of adj. rep.:

$$d(G) = \begin{cases} N^2 - 1, & \text{for SU(N)} \\ N(N-1)/2, & \text{for SO(N)} \\ N(N+1)/2, & \text{for Sp(N)} \end{cases} \tag{38}$$

For field φ in adj. rep. G,

$$(D_\mu \varphi)_a \overset{(23)}{=} \partial_\mu \varphi_a - igA_\mu^b (t_G^b)_{ac} \varphi_c = \partial_\mu \varphi_a + gf^{abc} A_\mu^b \varphi_c \tag{39}$$

Under infinitesimal gau. transf.,

$$\varphi_a \to \left(\delta_{ab} + i\alpha^c (t_G^c)_{ab}\right) \varphi_b + \mathcal{O}(\alpha^2) = \varphi_a - \alpha^c f^{cba} \varphi_b + \mathcal{O}(\alpha^2) \tag{40}$$

Introduce matrix notations,

$$\varphi \equiv \varphi_a t_G^a,\ A_\mu \equiv A_\mu^a t_G^a,\ \alpha \equiv \alpha^a t_G^a \left(\Rightarrow D_\mu \varphi = \partial_\mu \varphi - ig[A_\mu, \varphi] \right) \tag{41}$$

Then,

$$\begin{aligned} \varphi &\rightarrow \varphi - \alpha^c f^{cba}\varphi_b t_G^a + \mathcal{O}(\alpha^2) = \varphi + i[\alpha, \varphi] + \mathcal{O}(\alpha^2) \\ A_\mu &\rightarrow A_\mu + i[\alpha, A_\mu] + \frac{1}{g}\partial_\mu \alpha + \mathcal{O}(\alpha^2) \end{aligned}$$

For finite transf. $V(\alpha) = e^{i\alpha}$,

$$\varphi \rightarrow V(\alpha)\varphi V^\dagger(\alpha),\ A_\mu \rightarrow V(\alpha)A_\mu V^\dagger(\alpha) + \frac{i}{g}V(\alpha)\partial_\mu V^\dagger(\alpha),\ F_{\mu\nu} \rightarrow V(\alpha)F_{\mu\nu}V^\dagger(\alpha) \tag{42}$$

As far as rep. is concerned, only the global transf. is relevant. We see that gau. bosons belong to adj. rep. as their index suggests.

Comparison of (28) and (39) gives

$$(D^\mu F_{\mu\nu})^a = -g j_\nu^a \quad \text{gau. cova. derivative for fields in adj. rep.} \tag{43}$$

or in matrix notations,

$$D^\mu F_{\mu\nu} = \partial^\mu F_{\mu\nu} - ig[A^\mu, F_{\mu\nu}] = -g j_\nu,\ j_\nu \equiv t_G^a \overline{\psi}\gamma_\nu t_r^a \psi \tag{44}$$

Bianchi identity (analog of $\epsilon^{\mu\nu\alpha\beta}\partial_\nu F_{\alpha\beta} = 0$ in electrodynamics):

$$\text{by antisym.} \quad 0 = \epsilon^{\mu\nu\alpha\beta}[D_\nu, [D_\alpha, D_\beta]] = \cdots \Rightarrow \epsilon^{\mu\nu\alpha\beta}(D_\nu F_{\alpha\beta})^a = 0 \tag{45}$$

4. Casimir operator

Quadratic Casimir operator for simple Lie algebra:

$$T^2 \equiv T^a T^a \qquad \text{(repeated indices summed)} \tag{46}$$

Claim: T^2 commutes with all T^a. Check:

$$[T^b T^b, T^a] = T^b [T^b, T^a] + [T^b, T^a] T^b = i f^{bac} T^b T^c + i f^{bac} T^c T^b = 0 \text{ by antisym.}$$

$\Rightarrow$ T^2 is a unit matrix in any rep. r:

$$t_r^a t_r^a = C_2(r) \mathbf{1} \tag{47}$$

In particular, for $r = G$, the above gives

$$f^{acd} f^{bcd} = C_2(G) \delta^{ab} \tag{48}$$

Some useful results

$$C_2(r)d(r) \overset{(47)}{=} \delta^{ab} \times \mathrm{Tr}\, t_r^a t_r^b \overset{(33)}{=} C(r)\delta^{ab}\delta^{ab} = C(r)d(G)$$

$$\Rightarrow \quad C_2(r)d(r) = C(r)d(G) \quad \begin{array}{l}\text{Casimir of } r, \ \text{dim of } r \\ \text{normalization of } t_r^a, \ \text{dim. of adj. rep.}\end{array} \tag{49}$$

$SU(N)$: Choose

$$\mathrm{Tr}\, t_N^a t_N^b = \delta^{ab}/2 \leftrightarrow C(N) = 1/2, \tag{50}$$

then (49) gives

$$C_2(N) = \frac{N^2 - 1}{2N}. \tag{51}$$

Some computation yields

$$C_2(G) = C(G) = N \tag{52}$$

Homework: 15.1, 15.2; learn group theory in chap 4, Gauge theory of elementary particle physics, T.P. Cheng and L.F. Li

Chapter 5 Quantization of non-Abelian theories

Difficulties in quantization of gau. theories:

Manifest Lorentz invariance requires introducing *unphysical* degrees of freedom, i.e., those which are changed by gau. transformations.

Solution in Abelian case: Ward identity

More delicate in non-Abelian case since gau. bosons are also charged:

Slavnov-Taylor identity (generalized Ward identity) $\longrightarrow$ BRS(T) invariance

We quantize using path integral.

5.1. Interactions of non-Abelian gauge bosons

5.2. The Faddeev-Popov Lagrangian

5.3. Ghosts and unitarity

5.4. Outline of one-loop divergences of non-Abelian gauge theory

5.1. Interactions of non-Abelian gauge bosons

A sketch of naive derivation of Feynman rules using methods in $\S 1.2$, $\S 1.4$. More technical treatment in the next section using $\S 1.3$.

1. Feynman rules for fermions and gauge bosons

gauge group G; fermions ψ in irreducible rep. with generator matrices t^a

$$D_\mu \psi = (\partial_\mu - igA_\mu^a t^a)\psi \quad \text{gau. coupling} \qquad \text{summation over repeated indices} \tag{1}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c \quad \text{structure consts.} \tag{2}$$

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \overline{\psi}(i\not{D} - m)\psi \quad \text{relative sign fixed by gau. covar.} \tag{3}$$

Propagators

Fermions: almost nothing new but that there are a number of them (=dim. of their rep.) and that the propagator is diagonal in their indices

$$\langle \psi_{i\alpha}(x)\overline{\psi}_{j\beta}(y)\rangle \equiv \langle 0|T\psi_{i\alpha}(x)\overline{\psi}_{j\beta}(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\left(\frac{i}{\not{k} - m}\right)_{\alpha\beta}\delta_{ij} e^{-ik\cdot(x-y)} \tag{4}$$

i, j : enumerating fermions in group space $\quad \alpha, \beta$: spinor index $\quad$ diagonal in group indices

Gauge bosons: by analogy with photon,

$$\langle A^a_\mu(x) A^b_\nu(y)\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\mu\nu}}{k^2} \delta^{ab} e^{-ik\cdot(x-y)} \qquad a, b : \text{enumerating gau. bosons in adj. rep. of } G \tag{5}$$

to be checked in §5.2.

Vertices

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\rm int}, \\
\mathcal{L}_0 &= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \overline{\psi}(i\partial\!\!\!/ - m)\psi, \\
\mathcal{L}_{\rm int} &= -\frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) g f^{abc} A^{b,\mu} A^{c,\nu} - \frac{1}{4} g^2 f^{abc} f^{ab'c'} A^b_\mu A^c_\nu A^{b',\mu} A^{c',\nu} + g\overline{\psi}\gamma^\mu t^a \psi A^a_\mu \\
&= -g f^{abc}(\partial_\mu A^a_\nu) A^{b,\mu} A^{c,\nu} - \frac{1}{4} g^2 f^{abc} f^{ab'c'} A^b_\mu A^c_\nu A^{b',\mu} A^{c',\nu} + g\overline{\psi}\gamma^\mu t^a \psi A^a_\mu
\end{aligned} \tag{6}$$

(1) $\bar{\psi}\psi A^a_\mu$: easy to figure out by analogy with QED

$$+ig\gamma_\mu t^a \qquad \begin{matrix}\text{spinor space} \\ \text{group space}\end{matrix} \qquad \text{acting in different spaces, thus not interfering} \tag{7}$$

(2) $A^a_\mu A^b_\nu A^c_\rho$: using definition in terms of operators for 3-point function

$$i(-g)\int d^4z\Big\langle A^a_\mu(x_1)A^b_\nu(x_2)A^c_\rho(x_3)f^{a'b'c'}\Big(\partial_\alpha A^{a'}_\beta(z)\Big)A^{b',\alpha}(z)A^{c',\beta}(z)\Big\rangle \qquad \text{from } S_{\rm int}$$

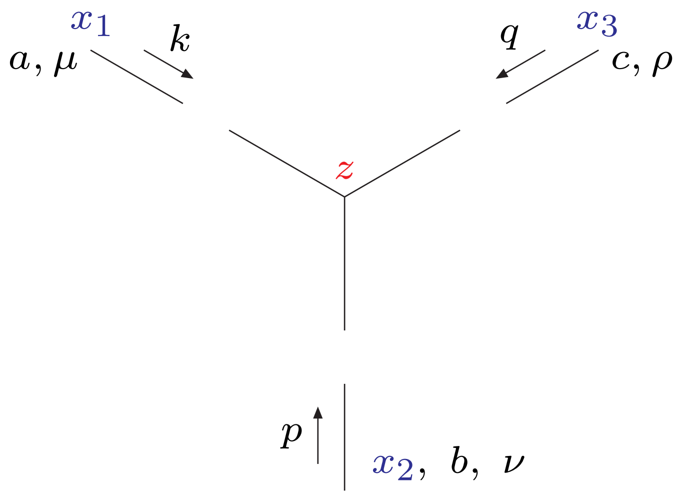
Exhaust all contractions between external fields at x_i & internal ones at z.
Each contraction yields a propagator. Each ∂ gives a momentum.
More explicitly, consider the contraction

$$A^a_\mu(x_1)A^b_\nu(x_2)A^c_\rho(x_3)\Big(\partial_\alpha A^{a'}_\beta(z)\Big)A^{b',\alpha}(z)A^{c',\beta}(z)f^{a'b'c'}$$

In momentum space, choose all momenta to be incoming into vertex at z.
Above contraction gives

$$g_{\mu\beta}g^\alpha_\nu g^\beta_\rho\delta^{aa'}\delta^{bb'}\delta^{cc'}f^{a'b'c'}(-ik_\alpha) = -ik_\nu g_{\mu\rho}f^{abc}, \quad k: \text{pointing from } x_1 \text{ to } z\text{: } e^{-ik\cdot(z-x_1)}$$

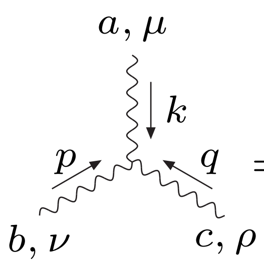
with 3 propagators of gau. bosons separated out from the vertex.



Interchange of roles played by $A^{b',\alpha}(z)$ and $A^{c',\beta}(z)$ gives another contraction:

$$-ik_\rho g_{\mu\nu} f^{acb} = +ik_\rho g_{\mu\nu} f^{abc}$$

Permutation of above two yields the complete vertex in momentum space



$$= g f^{abc} \Big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu \Big]$$

(3) $A^a_\mu A^b_\nu A^c_\rho A^d_\sigma$:

$$i(-g^2)\frac{1}{4}\int d^4z \left\langle A^a_\mu(x_1)A^b_\nu(x_2)A^c_\rho(x_3)A^d_\sigma(x_4)A^e_\alpha(z)A^f_\beta(z)A^{e',\alpha}(z)A^{f',\beta}(z)f^{hef}f^{he'f'}\right\rangle$$

sample contractions:

$$(a,\mu)(b,\nu)(c,\rho)(d,\sigma)\ \ (e,\alpha)(f,\beta)(e',\alpha)(f',\beta)\ \ f^{hef}f^{he'f'}$$

$$\Rightarrow \quad g_{\mu\alpha}\delta^{ae}g_{\nu\beta}\delta^{bf}g^\alpha_\rho\delta^{ce'}g^\beta_\sigma\delta^{df'}f^{hef}f^{he'f'} = g_{\mu\rho}g_{\nu\sigma}f^{hab}f^{hcd}$$

There are 4 sets of contractions yielding the same result, thus removing $\frac{1}{4}$:

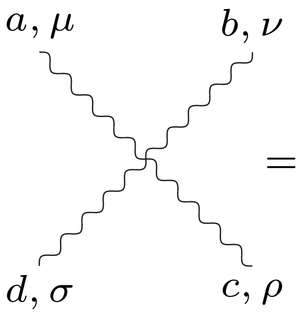
$$(1) \quad (a,\mu)(b,\nu)(c,\rho)(d,\sigma)\ \ (e,\alpha)(f,\beta)(e',\alpha)(f',\beta)$$

$$(2) \quad (e',\alpha) \leftrightarrow (f',\beta) \text{ of } (1)$$

$$(3) \quad (a,\mu)(b,\nu)(c,\rho)(d,\sigma)\ \ (e,\alpha)(f,\beta)(e',\alpha)(f',\beta)$$

$$(4) \quad (e',\alpha) \leftrightarrow (f',\beta) \text{ of } (3)$$

Other types of contractions are obtained by permutations.



$$= \quad -ig^2\Big\{ f^{abe}f^{cde}(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\nu\rho}) + f^{ace}f^{dbe}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma}) + f^{ade}f^{bce}(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\sigma\nu})\Big\}$$

Features: fixing (a, μ)

(i) 2nd term in 1st line obtained from 1st term by $(b, \nu) \to (c, \rho) \to (d, \sigma) \to (b, \nu)$

(ii) 2nd, 3rd lines obtained by the same cycling of 1st line

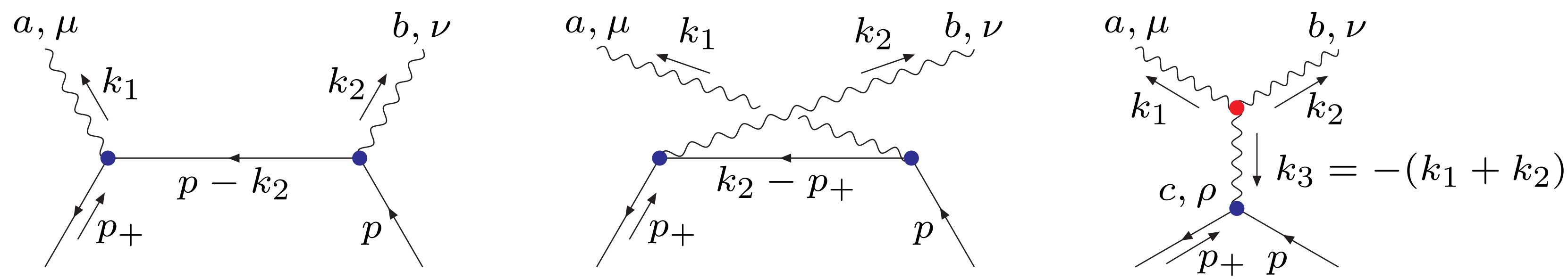
2. A check on the necessity of equality of coupling constants

Same g appears in different vertices: a result of gau. inva.

Now we check the necessity of this in a sample process, $\psi\bar{\psi} \to AA$:

$$i\mathcal{M} = i\mathcal{M}^{\mu\nu}\varepsilon^*_\mu(k_1)\varepsilon^*_\nu(k_2), \quad \varepsilon_\mu(k_1),\ \varepsilon_\nu(k_2):\ \text{polarization vectors, } k_i^\alpha\varepsilon_\alpha(k_i) = 0$$

$$i\mathcal{M}^{\mu\nu} = i\mathcal{M}^{\mu\nu}_{(1)} + i\mathcal{M}^{\mu\nu}_{(2)} + i\mathcal{M}^{\mu\nu}_{(3)}$$



(1) (2) (3)

Feynman rules give

$$
\begin{aligned}
i\mathcal{M}^{\mu\nu}_{(1)+(2)} &= \bar{\nu}(p_+) ig\gamma^\mu t^a \frac{i}{\not{p} - \not{k}_2 - m} ig\gamma^\nu t^b u(p) + \bar{\nu}(p_+) ig\gamma^\nu t^b \frac{i}{\not{k}_2 - \not{p}_+ - m} ig\gamma^\mu t^a u(p) \\
&= i^3 g^2 \bar{\nu}(p_+) \Big(\gamma^\mu \frac{1}{\not{p} - \not{k}_2 - m} \gamma^\nu t^a t^b + \gamma^\nu \frac{1}{\not{k}_2 - \not{p}_+ - m} \gamma^\mu t^b t^a \Big) u(p), \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
i\mathcal{M}^{\mu\nu}_{(3)} &= \bar{\nu}(p_+) ig\gamma_\rho t^c u(p) \frac{-i}{k_3^2} \\
&\quad \times (-g f^{abc}) [g^{\mu\nu}(k_1 - k_2)^\rho + g^{\nu\rho}(k_2 - k_3)^\mu + g^{\rho\mu}(k_3 - k_1)^\nu] \\
&= \frac{-g^2 f^{abc}}{k_3^2} \bar{\nu}(p_+) \gamma_\rho t^c u(p) [g^{\mu\nu}(k_1 - k_2)^\rho + g^{\nu\rho}(k_2 - k_3)^\mu + g^{\rho\mu}(k_3 - k_1)^\nu]. \qquad (9)
\end{aligned}
$$

In QED, we have Ward identities (or W.-T. identities for Green's functions). Now we examine their analogues in non-Abelian gau. theories:

$$i\mathcal{M}^{\mu\nu}k_{2\nu} = 0?$$

Compute

$$\begin{aligned}
i\mathcal{M}^{\mu\nu}_{(1)+(2)}k_{2\nu} &= -ig^2\bar{v}(p_+)\Big(\gamma^\mu \frac{1}{\not{p}-\not{k}_2-m}\underset{\displaystyle \not{k}_2-\not{p}+m}{\underset{\downarrow}{\not{k}_2}} t^a t^b + \underset{\displaystyle \not{k}_2-\not{p}_+-m}{\underset{\downarrow}{\not{k}_2}}\frac{1}{\not{k}_2-\not{p}_+-m}\gamma^\mu t^b t^a\Big)u(p) \quad \text{on-shell fermions} \\
&= -ig^2\bar{v}(p_+)\gamma^\mu[t^b,t^a]u(p) \\
&= -g^2 f^{abc}\bar{v}(p_+)\gamma^\mu t^c u(p) \quad \text{non-Abelian; would vanish in Abelian case} \\
i\mathcal{M}^{\mu\nu}_{(3)}k_{2\nu} &= -g^2 f^{abc}\bar{v}(p_+)\gamma_\rho t^c u(p)[k_1^\rho k_1^\mu + k_3^\rho k_3^\mu + g^{\rho\mu}(k_1^2 - k_3^2)]\frac{1}{k_3^2} \\
\Rightarrow i\mathcal{M}^{\mu\nu}_{(1)+(2)+(3)}k_{2\nu} &= -g^2 f^{abc}\bar{v}(p_+)\gamma_\rho t^c u(p)\frac{1}{k_3^2}[k_1^\rho k_1^\mu + k_3^\rho k_3^\mu + g^{\rho\mu}k_1^2] \\
&= -g^2 f^{abc}\bar{v}(p_+)\gamma_\rho t^c u(p)\frac{1}{k_3^2}\big(k_1^\rho k_1^\mu + g^{\rho\mu}k_1^2\big) \neq 0 \quad \text{on-shell fermions}
\end{aligned}$$

Discussions:

(1) QED: $i\mathcal{M}^{\mu\nu}k_{2\nu} = 0$ for on-shell fermions, without restrictions on photons.

Non-Abelian: $i\mathcal{M}^{\mu\nu}k_{2\nu} \neq 0$ when only fermions are required to be physical.

In more detail, consider the remaining terms:

$k_1^\rho k_1^\mu$: vanishing for physical polar. of 1st gau. boson: $k_1^\mu \varepsilon_\mu^*(k_1) = 0$

$g^{\rho\mu}k_1^2$: vanishing for on-shell 1st gau. boson: $k_1^2 = 0$

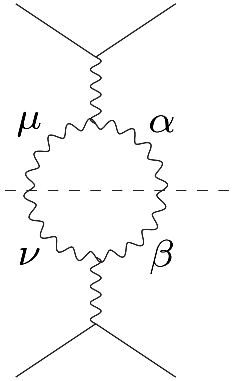
$\Rightarrow \mathcal{M}^{\mu\nu}k_{2\nu} = 0$ if 1st gau. boson is also physical.

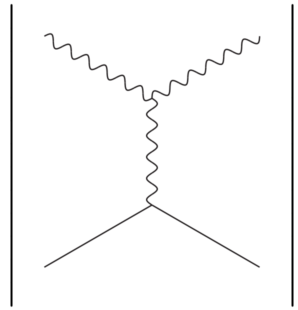
(2) Consequences of the above:

QED: no unphysical polar. states can be produced, due to Ward identities.

Non-Abelian: no threats on phys. amplitudes at tree level, but inconsistency could appear at a higher order

Consider the following one-loop contribution in optical theorem:

2Im 

$\stackrel{?}{=} \int d\Pi_2$  ^{2}

LHS: cut gau. bosons involve $g_{\mu\nu}$ and $g_{\alpha\beta}$; e.g.,

$$g_{\mu\nu} \quad = \quad \varepsilon^{-}_{\mu}\varepsilon^{+*}_{\nu} + \varepsilon^{+}_{\mu}\varepsilon^{-*}_{\nu} - \sum_{i=1}^{2} \varepsilon^{T}_{i\mu}\varepsilon^{T*}_{j\nu} \tag{10}$$

unphysical polarizations
(time-like & longitudinal)

transverse polarizations

A convenient choice for an on-shell gauge boson of momentum $k^\mu = (k^0, \mathbf{k})$:

$$\epsilon^{+\mu}(k) = 1/(\sqrt{2}|\mathbf{k}|)(k^0, \mathbf{k}), \quad \epsilon^{-\mu}(k) = 1/(\sqrt{2}|\mathbf{k}|)(k^0, -\mathbf{k})$$

RHS: external physical gau. bosons involve only physical polarizations

Two sides do not match $\Rightarrow$ unitarity violation in S-matrix?

[finished in 3 units on Nov 16, 2012.]

5.2. The Faddeev-Popov Lagrangian

We follow procedure in §1.3 to treat the freedom introduced by gau. inva.

We'll find that our treatment in §5.1 based on 'guess work' is incomplete due to non-Abelian character of gau. symmetries.

Start with pure gauge field case:

$$\int \mathcal{D}A \exp\left[i \int d^4x \left(-\frac{1}{4}\right) (F^a_{\mu\nu})^2\right] \tag{11}$$

Over-counting of physically indept. A
$\Rightarrow$ must factor out contributions in the directions of sym. transfs.

Gauge-fixing: $G(A) = 0$

Faddeev-Popov's trick: inserting following identity into PI

$$1 = \int \mathcal{D}\alpha(x)\ \delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) \tag{12}$$

where A^α is transformed from A by the group parameter α:

$$(A^\alpha)^a_\mu t^a = e^{i\alpha^d t^d}(A^b_\mu t^b + ig^{-1}\partial_\mu)e^{-i\alpha^c t^c} \tag{13}$$

or, for infinitesimal α^a,

$$(A^\alpha)^a_\mu = A^a_\mu + g^{-1}\partial_\mu \alpha^a + f^{abc}A^b_\mu \alpha^c = A^a_\mu + g^{-1}(D_\mu \alpha)^a \quad \text{gau. cov. derivative for a field in adj. rep.} \tag{14}$$

For $G(A)$ that is linear in A (assumed below), $\frac{\delta G(A^\alpha)}{\delta\alpha}$ is indept. of α.

$$\begin{aligned}
(11) &= \int \mathcal{D}A\ \exp\Big[-\frac{i}{4}\int d^4x\ (F^a_{\mu\nu})^2\Big] \int \mathcal{D}\alpha\ \delta(G(A^\alpha))\det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big) \\
&= \int \mathcal{D}\alpha \int \mathcal{D}A\ \delta(G(A^\alpha))\det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big)\exp\Big[-\frac{i}{4}\int d^4x\ ((F^\alpha)^a_{\mu\nu})^2\Big] \quad \text{gau. inva. of } F^2 \\
&= \int \mathcal{D}\alpha \int \mathcal{D}A^\alpha\ \delta(G(A^\alpha))\det\Big(\cdots\Big)\exp\Big[\cdots\Big] \quad \text{PI measure not changed by (14)} \\
&= \Big(\int \mathcal{D}\alpha\Big)\int \mathcal{D}A\ \delta(G(A))\det\Big(\frac{\delta G(A^\alpha)}{\delta\alpha}\Big)\exp\Big[-\frac{i}{4}\int d^4x\ (F^a_{\mu\nu})^2\Big]
\end{aligned} \tag{15}$$

where

$$\det(\cdots): \ \text{indept. of } \alpha \Leftrightarrow \text{renaming of integration variables}: \ A^\alpha \to A$$

$$\int \mathcal{D}\alpha \text{ cancelled finally, dropped below}$$

Choose

$$G(A) = \partial^\mu A_\mu^a(x) - \omega^a(x) \tag{16}$$

(15) holds for any $\omega^a(x)$ →introduce a Gaussian weighting centered at $\omega = 0$:

$$\begin{aligned}(11) = {} & N(\xi) \int \mathcal{D}\omega(x) \exp\left[-i \int \frac{1}{2\xi} \omega^2(x) d^4x\right] \int \mathcal{D}A \quad \text{normalization factor, cancelled finally} \\ & \times \delta(G(A)) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) \exp\left[-\frac{i}{4} \int d^4x (F_{\mu\nu}^a)^2\right] \\ = {} & N(\xi) \int \mathcal{D}A \ \det(\cdots) \exp i \int d^4x \left[-\frac{1}{4}(F_{\mu\nu}^a)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2\right] \end{aligned} \tag{17}$$

Quadratic terms give gauge boson propagator as for photon:

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle \quad = \quad \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + i\epsilon} \left[g_{\mu\nu} - (1 - \xi)\frac{k_\mu k_\nu}{k^2}\right] e^{-ik\cdot(x-y)} \delta^{ab} \tag{18}$$

$$\xi = 1, \ \text{'t Hooft-Feynman}; \ \xi = 0, \ \text{Landau}$$

New: $\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$ depends on A in non-Abelian case because

$$\frac{\delta G(A^\alpha)}{\delta\alpha} = \frac{1}{g}\partial^\mu D_\mu \quad \text{gau. cov. derivative for adj. fields} \tag{19}$$

It cannot be ignored as a normalization factor as in Abelian case!
It is inconvenient to use (17) directly for computation because of det.
It would be better to represent its contributions by new terms in Lagrangian.

$$\det(\cdots) \quad \overset{?}{\longleftrightarrow} \quad \int(\mathcal{D}\text{ fields}) \times \text{Gaussian}$$

must be scalar under Lorentz transf. must be anticommuting like ψ

$$\underbrace{\hspace{10em}}_{\text{violation of spin-statistics relation: ghost fields}}$$

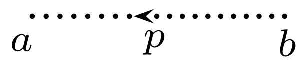
$$\Rightarrow \quad \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \int\mathcal{D}c\mathcal{D}\bar{c}\ \exp\left[i\int d^4x\ \overline{c^a}(-\partial^\mu D_\mu c)^a\right] \quad \text{acting on all fields on rhs}$$

$$\text{i.e., } \mathcal{L}_{\text{ghost}} = \overline{c^a}\Big(-\partial^2\delta^{ac} - g\partial^\mu f^{abc}A^b_\mu\Big)c^c \quad \text{as new terms in } \mathcal{L} \tag{20}$$

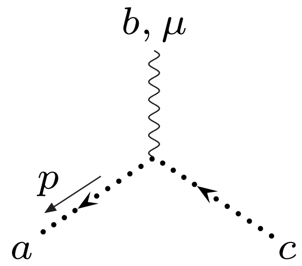
$\overline{c^a}$, c^c : both in adj. rep. of gau. group (only global transfs. relevant here; local ones lost by gau. fixing!

Quadratic terms yield ghost propagator:

$$\langle c^a(x)\bar{c}^b(y)\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2}e^{-ik\cdot(x-y)}\delta^{ab} \tag{21}$$

anticommuting $\Rightarrow$ carry an arrow of ghost-number flow: 

Trilinear term gives a vertex:



$$= -gf^{abc}p_\mu$$

Summary:

(1) Including fermions (scalars similar), our working Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \overline{\psi}(i\underset{\substack{\uparrow \\ \text{for fermion's rep.}}}{\not{D}} - m)\psi + \overline{c^a}(-\partial^\mu \underset{\substack{\uparrow \\ \text{for adj. rep.}}}{D^{ac}_\mu})c^c \tag{22}$$

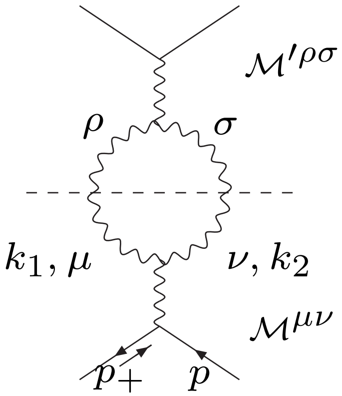
(2) Above $\mathcal{L}$ is correct for computing correlation functions of gau. inva. operators, and also correct for obtaining gau. inva. S-matrix (needs the technique of BRS(T) invariance, skipped here).

5.3. Ghosts and unitarity

1. Recovery of unitarity

Back to the example of unitarity check in optical theorem in §5.1.

Gauge boson loop



To pick up the imaginary part, cut gau. boson lines become

$$[(-ig_{\mu\rho})(-2\pi i)\delta(k_1^2)][(-ig_{\nu\sigma})(-2\pi i)\delta(k_2^2)]$$

convert $\int d^4k$ into phase-space integral

for 2 massless particles

$$\therefore \qquad \frac{1}{2}\cdot i\mathcal{M}^{\mu\nu} i\mathcal{M}'^{\rho\sigma} g_{\mu\rho} g_{\nu\sigma} \tag{23}$$

sym. factor from the loop, or
sym. factor for 2 identical particles in PS integral

Application of decomposition (10) in (23):

$$
\begin{aligned}
g_{\mu\rho}g_{\nu\sigma} \quad &= \quad [\varepsilon^{-}_{\mu}(k_1)\varepsilon^{+*}_{\rho}(k_1) + \varepsilon^{+}_{\mu}(k_1)\varepsilon^{-*}_{\rho}(k_1) - \sum_i \varepsilon^{T}_{i\mu}(k_1)\varepsilon^{T*}_{i\rho}(k_1)] \\
&\times \quad [\varepsilon^{-}_{\nu}(k_2)\varepsilon^{+*}_{\sigma}(k_2) + \varepsilon^{+}_{\nu}(k_2)\varepsilon^{-*}_{\sigma}(k_2) - \sum_i \varepsilon^{T}_{i\nu}(k_2)\varepsilon^{T*}_{i\sigma}(k_2)]
\end{aligned}
$$

$TT \times TT$: OK, matches RHS of optical theorem

$TT \times (+-)$: vanishing, one example shown in §5.1.

$(\pm\mp) \times (\pm\mp)$: vanishing, check by yourself

$(\pm\mp) \times (\mp\pm)$: nonvanishing and equal, with sum given by

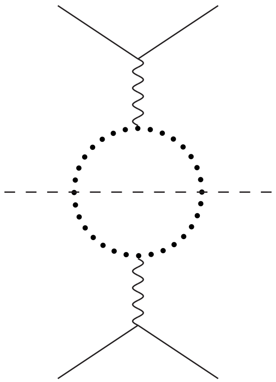
$$
\left[ig\bar{v}(p_+)\gamma_\mu t^c u(p)\frac{-i}{(k_1{+}k_2)^2}(-gf^{abc}k_1^\mu)\right]\left[ig\bar{u}(p)\gamma_\rho t^d v(p_+)\frac{-i}{(k_1{+}k_2)^2}(-gf^{abd}(-k_2)^\rho)\right] \quad (24)
$$

where the following unphys polarizations are used:

$$
\text{forward}: \ \epsilon^{+\mu}(k) = \frac{1}{\sqrt{2}|\mathbf{k}|}(k^0, +\mathbf{k}); \quad \text{backward}: \ \epsilon^{-\mu}(k) = \frac{1}{\sqrt{2}|\mathbf{k}|}(k^0, -\mathbf{k}).
$$

Ghost loop

additional contri. from careful treatment of gau. fixing in non-Abelian case



exactly same contribution to Im as (24)

except for an additional (-1) from a Grassmannian loop!

$\Rightarrow$ cancelation of unphysical pol. contri. by ghost contri.

2. Rules for practical calculations

(1) Wherever a gauge boson loop appears, its accompanying ghost loop must be included with a minus sign.

(2) For S-matrix elements involving external gauge bosons, there are 2 equivalent procedures to follow:

(i) Use only physical pol. vectors for external gauge bosons (at $\mathcal{A}$ level), or

(ii) Apply pol. sum $\sum \epsilon_\mu \epsilon_\nu^* \to -g_{\mu\nu}$ to each external gauge boson, and then remove unphysical pol. contributions by subtracting (at σ level) all possible contributions of ghosts that replace external gauge bosons.

5.4. Outline of one-loop divergences of non-Abelian gauge theory

Working Lagrangian in (22), Feynman rules in §5.1 & §5.2.

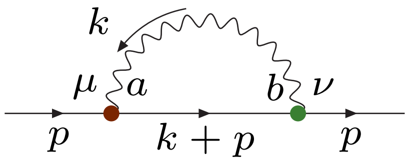
3 fields – gauge bosons, fermions, and ghosts

2 parameters – gauge coupling and fermion mass

(gauge parameter ξ – can choose not to renormalize it).

$\mathcal{L}$ contains only dimensionless couplings and includes all terms consistent with gauge symmetry $\Rightarrow$ renormalizable

1. Fermion self-energy ($\xi = 1$ gauge, group indices of fermions suppressed)



$$= \int \frac{d^4k}{(2\pi)^4} igt^b\gamma_\nu \frac{i}{\not{k} + \not{p} - m} igt^a\gamma_\mu \frac{-ig^{\mu\nu}\delta^{ab}}{k^2}$$

$$= t^a t^a \int \frac{d^4k}{(2\pi)^4} ig\gamma^\mu \frac{i}{\not{k} + \not{p} - m} ig\gamma_\mu \frac{-i}{k^2}$$

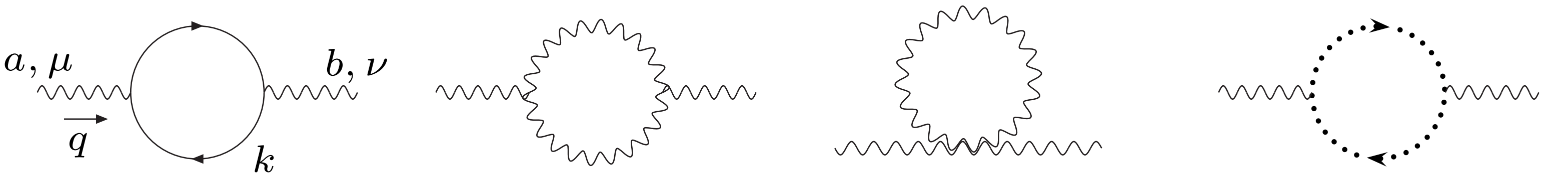
$C_2(r)$ same as in QED, §2.3

For later use, we record UV div. part for fermion wavefunc. renor.:

$$(\delta_2)_{\rm div} = C_2(r)(-1)\frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}, \text{ dim. reg. with } d = 4-2\epsilon \qquad \text{same as in QED, §2.3} \tag{25}$$

δ_m skipped.

2. Gauge boson self-energy



(a) (b) (c) (d)

- Gauge boson self-energy automatically transverse as in QED:

(a) as contri. from matter fields is transverse by itself;

$(b), (c), (d)$ only sum to a transverse result (unphysical pol. and ghosts)

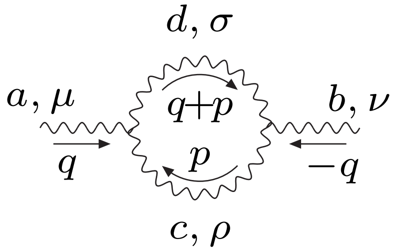
- Must use a regularization that keeps gau. sym.: DR most convenient

$$
\begin{aligned}
(a) &= (-1)\int\frac{d^dk}{(2\pi)^d}\text{Tr}\left(ig\gamma_\nu t^b\frac{i}{\not k+\not q-m}ig\gamma_\mu t^a\frac{i}{\not k-m}\right) \quad \text{in both spinor \& internal spaces}\\
&= \text{Tr}\,(t^bt^a)(-1)\int\frac{d^dk}{(2\pi)^d}\text{Tr}\left(ig\gamma_\nu\frac{i}{\not k+\not q-m}ig\gamma_\mu\frac{i}{\not k-m}\right) \quad \text{QED}\\
&= C(r)\delta^{ab}\left[i(q^2g_{\mu\nu}-q_\mu q_\nu)\frac{-g^2}{(4\pi)^2}\frac{4}{3}\frac{1}{\epsilon}+\text{UV finite}\right].
\end{aligned}
$$

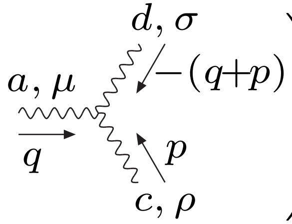
For n_f species of fermions in same rep. r, total contribution (UV part) is (counted by quan. numbers other than the one for considered gau. group)

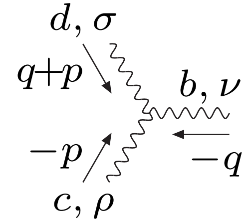
$$
C(r)\delta^{ab}i(q^2g_{\mu\nu}-q_\mu q_\nu)\frac{-g^2}{(4\pi)^2}\frac{4}{3}n_f\frac{1}{\epsilon} \tag{26}
$$

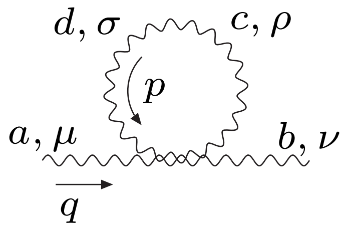
(b), (c), (d) are new:



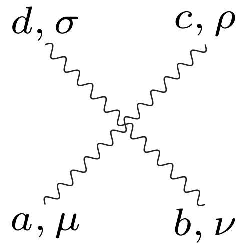
$$= \int \frac{d^dp}{(2\pi)^d} \frac{-i}{p^2} \frac{-i}{(q+p)^2} \cdot \Bigg(\quad \Bigg) \cdot \Bigg(\quad \Bigg) \cdot \frac{1}{2}$$

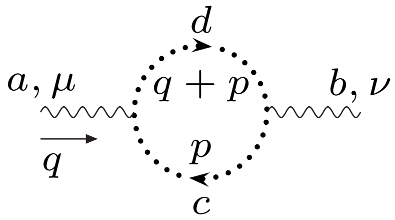






$$= \int \frac{d^dp}{(2\pi)^d} \frac{-i}{p^2} g^{\rho\sigma} \delta^{cd} \cdot \Bigg(\quad \Bigg) \cdot \frac{1}{2}$$





$$= (-1) \int \frac{d^dp}{(2\pi)^d} \frac{i}{p^2} \frac{i}{(p+q)^2} \cdot (-g) f^{dac} (q+p)_\mu (-g) f^{cbd} p_\nu$$

Group factors:

$$(b) \quad : \quad f^{acd} f^{bcd} = C_2(G)\delta^{ab} \text{ see (4.48)}$$

$(c) \quad :$ one combination of ff vanishes, while the other two are the same as in (b)

$$(d) \quad : \quad f^{dac} f^{cbd} = -C_2(G)\delta^{ab}$$

Lorentz structure:

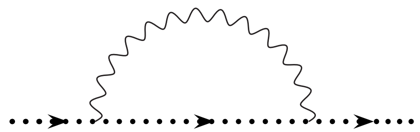
Need some algebraic tricks to combine (b), (c), (d) into a transverse form *before* Feynman parameter integration is finished.

$$(b) + (c) + (d) \Rightarrow i(q^2 g_{\mu\nu} - q_\mu q_\nu)\frac{+g^2}{(4\pi)^2}\frac{5}{3}C_2(G)\delta^{ab}\frac{1}{\epsilon}, \quad \left(\frac{13}{6} - \frac{1}{2}\xi\right) \text{ for arbitrary } \xi \tag{27}$$

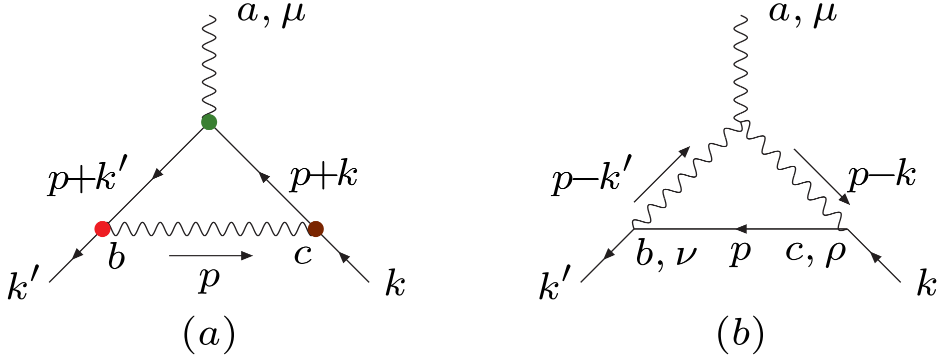
$(a) + (b) + (c) + (d)$ gives gauge boson wavefunction renormalization:

$$(\delta_3)_{\text{div}} = \frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}\left[\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r)\right] \tag{28}$$

3. Ghost self-energy: skipped



4. Fermion-gauge boson vertex



$$(a) \quad = \quad \int \frac{d^dp}{(2\pi)^d} igt^b\gamma_\nu \frac{i}{\not p + \not k' - m} igt^a\gamma_\mu \frac{i}{\not p + \not k - m} igt^c\gamma^\nu \frac{-i}{p^2}\delta^{bc}$$

$$= \quad t^bt^at^b \int \frac{d^dp}{(2\pi)^d} ig\gamma_\nu \frac{i}{\not p + \not k' - m} ig\gamma_\mu \frac{i}{\not p + \not k - m} ig\gamma^\nu \frac{-i}{p^2}$$

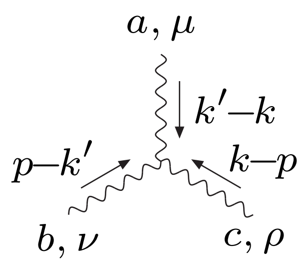
same as in QED upon $-e \to g$

$$t^bt^at^b \quad = \quad t^bt^bt^a + t^b[t^a, t^b] = C_2(r)t^a + if^{abc}t^bt^c = C_2(r)t^a + \frac{i}{2}f^{abc}[t^b, t^c]$$

$$= \quad C_2(r)t^a - \frac{1}{2}f^{abc}f^{bcd}t^d = \Big(C_2(r) - \frac{1}{2}C_2(G)\Big)t^a$$

$$\therefore (a) \quad \Rightarrow \quad \Big(C_2(r) - \frac{1}{2}C_2(G)\Big)t^a \cdot \frac{i}{(4\pi)^2}g^3\gamma_\mu\frac{1}{\epsilon}$$

$$(b) = \int \frac{d^4p}{(2\pi)^4} igt^b\gamma^\nu \frac{i}{\not{p} - m} igt^c\gamma^\rho \frac{-i}{(p-k)^2}\frac{-i}{(p-k')^2} \times$$



Group factors:

$$t^b t^c f^{abc} = \frac{1}{2} f^{abc}[t^b, t^c] = \frac{i}{2} f^{abc} f^{bcd} t^d = \frac{i}{2} C_2(G) t^a$$

Numerator: at most quadratic in p

$$\therefore\ (b) \Rightarrow \frac{ig^3}{(4\pi)^2}\frac{3}{2}C_2(G)t^a\gamma_\mu\frac{1}{\epsilon}$$

Then, $(a) + (b)$ gives renor. const. for gauge boson-fermion vertex:

$$(\delta_1)_{\text{div}} = -\frac{g^2}{(4\pi)^2}\frac{1}{\epsilon}[C_2(r) + C_2(G)] \tag{29}$$

5. AAA, $AAAA$, $c\bar{c}A$ vertices: skipped

6. β function for non-Abelian gauge theories

$(\delta_1)_{\rm div} \neq (\delta_2)_{\rm div}$ due to $C_2(G)$ term in $(\delta_1)_{\rm div} \Rightarrow$ $Z_1 \neq Z_2$ in non-Abelian theory!
QED formula for β function not applicable here!

$$\text{DR in } d = 4 - 2\epsilon: \;\; g = g_0 Z_1^{-1} Z_2 Z_3^{\frac{1}{2}} \mu^{-\epsilon}, \;\; \mu = \text{renormalization scale}, \;\; Z_i = 1 + \delta_i$$

$$\text{MS: } Z_i = 1 + \sum_{\nu=1}^{\infty} Z_i^{(\nu)}(g)\epsilon^{-\nu}, \;\; \mu\text{-dependence only through } g$$

$$\beta(g,\epsilon) = \mu \frac{dg}{d\mu} = -\epsilon g + \mu^{-\epsilon} g_0 \beta(g,\epsilon) \frac{d}{dg}\left(Z_1^{-1} Z_2 Z_3^{1/2}\right)$$

$$= -\epsilon g + g\beta(g,\epsilon)\left[-Z_1^{-1}\frac{dZ_1}{dg} + Z_2^{-1}\frac{dZ_2}{dg} + \frac{1}{2} Z_3^{-1}\frac{dZ_3}{dg}\right]$$

$$\text{i.e., } [\beta(g,\epsilon) + \epsilon g] Z_1 Z_2 Z_3 = g\beta(g,\epsilon)\left[-Z_2 Z_3 \frac{dZ_1}{dg} + Z_3 Z_1 \frac{dZ_2}{dg} + \frac{1}{2} Z_1 Z_2 \frac{dZ_3}{dg}\right] \tag{30}$$

For a renormalizable theory, $\beta(g,\epsilon) = \sum_{\nu=0}^{\infty} \beta_{(\nu)}\epsilon^{\nu}$, $\beta = \lim_{\epsilon\to 0}\beta(g,\epsilon) = \beta_{(0)}$.
It can be shown that $\beta_{(n)} = 0$ for $n \geqslant 2$, thus $\beta(g,\epsilon) = \beta_{(0)} + \beta_{(1)}\epsilon$.

Eq. (30) as powers in ϵ:

$$\begin{aligned}
\mathcal{O}(\epsilon^1) &: \quad \beta_{(1)} + g = 0 \;\Rightarrow\; \beta_{(1)} = -g \\
\mathcal{O}(\epsilon^0) &: \quad \beta_{(0)} + (\beta_{(1)} + g)(Z_1^{(1)} + Z_2^{(1)} + Z_3^{(1)}) = g\beta_{(1)}\frac{d}{dg}\left(-Z_1^{(1)} + Z_2^{(1)} + \frac{1}{2}Z_3^{(1)}\right) \\
&\quad \Rightarrow \beta_{(0)} = g^2\frac{d}{dg}\left(Z_1^{(1)} - Z_2^{(1)} - \frac{1}{2}Z_3^{(1)}\right)
\end{aligned}$$

At one loop, we thus have

$$\begin{aligned}
\beta(g) &= \beta_{(0)}(g) = -\frac{2g^3}{(4\pi)^2}\left\{\left[C_2(r) + C_2(G)\right] - C_2(r) + \frac{1}{2}\left[\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r)\right]\right\} \\
&= -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right] \qquad (31)
\end{aligned}$$

For n_f not very large, $\beta(g) < 0$

$\Rightarrow$ Non-Abelian gauge theories are asymptotically free!

7. Summary

- 5 renor. consts. (not counting that of ξ) remove all UV div.

  $\Rightarrow$ reflection of renormalizability
- Same g appears in different vertices due to gau. inva.

  $\Rightarrow$ renormalization of different vertices is related

  $\Rightarrow$ same $\beta(g)$ can be obtained from any of them
- Although $Z_1 \neq Z_2$, renormalization of a vertex and its fields is simply related among different vertices.

Homework: 16.2, 16.3

[finished in 2 units on Nov 23, 2012.]

Chapter 6 Quantum Chromodynamics

Application of non-Abelian gau. theory to strong interactions.

Basic processes and experimental techniques on strong interactions.

6.1. Quark model and QCD

6.2. e^+e^- annihilation into hadrons

6.3. Deep inelastic scattering

6.4. Hard-scattering processes in hadron collisions

6.1. Quark model and QCD

1. Quark model and color quantum number

Gell-Mann & Zweig: hadrons are made of quarks

$q = u,\ d,\ s$; later also $c,\ b,\ t \leftarrow$ 'flavor'

e.g., mesons $= (q\bar{q}')$, baryons $= (qq'q'')$

$\pi^+ \sim (u\bar{d}),\ K^- \sim (s\bar{u}),\ p \sim (uud), n \sim (udd)$

quarks carry fractional charges:

$Q = +\frac{2}{3}$ for u, c, t; $Q = -\frac{1}{3}$ for d, s, b

Very successful in hadron spectroscopy, and in EM and weak transitions.

Incorporates naturally isospin $SU(2)$ sym.: strong interactions are inva. under $SU(2)$ transf. under which $u,\ d$ constitute a doublet in the limit $m_u = m_d$.

Gell-Mann & Ne'eman: approximate $SU(3)$ sym. of strong interactions of which $u,\ d,\ s$ constitute a triplet in the limit $m_u = m_d = m_s$.

Difficulties with quark model and the need for color quantum number

- Free quarks with fractional electric charges never observed
(⇒ color confinement in QCD)
- Apparent violation of spin-statistics with baryons

e.g., $\Delta^{++} \sim (uuu)$, spin $\frac{3}{2}$, charge $+2$

$$\left.\begin{array}{l}\text{orbital angular momentum} = 0 \\ \Rightarrow \text{symmetric orbital wave-func} \\ \text{spin } \frac{3}{2} \Rightarrow \text{symmetric spin wave-func} \\ \text{all } u \Rightarrow \text{symmetric flavor wave-func}\end{array}\right\} \Rightarrow \text{overall symmetric wave-func?}$$

Han & Nambu, Greenberg, Gell-Mann:

quarks carry an unobserved quantum number, 'color'

e.g., meson $\sim (q_i \bar{q}^i)$, baryons $\sim (\epsilon^{ijk} q_i q_j q_k)$

color index, summed over

3 colors → antisymmetric color wave-func
⇒ overall antisym. wave-func!

Simplest choice:

Each *flavor* of quark comes in 3 colors and forms funda. rep. of $SU(3)_{\text{color}}$
Mesons and baryons are singlets under $SU(3)_{\text{color}}$.

Further questions: Why color quantum number? Why are hadrons color-singlets? Some dynamics is called for!

Phenomenologically, parton-model (hadrons as loose bound states of quarks) is successful in deep inelastic scattering $\rightarrow$ asymptotic freedom $\leftarrow$ non-Abelian gau. theory has the property!

$\Rightarrow$ 'One stone for two birds': $SU(3)_{\rm color}$ is the gau. group of a gau. theory!

2. Quantum chromodynamics (QCD)

gau. bosons = gluons, in adj. rep. of $SU(3)_{\rm color}$ (unrelated to $SU(3)_{\rm flavor}$!)

quarks in funda. rep., e.g., (u_1, u_2, u_3) (warning: flavors vs colors)

asymptotic freedom $\Rightarrow$ perturbative QCD for hard processes (here!)
(large momentum transfer or short-distance)

Strong-int. regime can be investigated by lattice approach. Wilson showed color confinement in QCD and answered why only color singlets are observed.

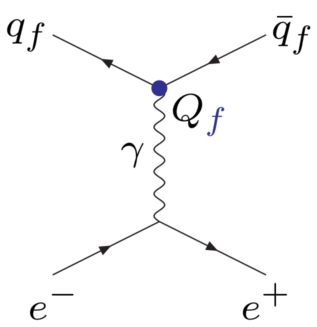
6.2. e^+e^- annihilation into hadrons

1. Total cross section

§5.1 in QFT:

$\sigma(e^+e^- \to \text{hadrons})$ at high energy can be well described by $\sigma(e^+e^- \to \sum q\bar{q})$

Tree level



$$\sigma(e^+e^- \to \text{hadrons}) = \sigma_0 \cdot 3 \cdot \sum_f Q_f^2 \qquad \sqrt{s} \gg m_{q_f} \qquad (1)$$

each flavor in 3 colors   flavor

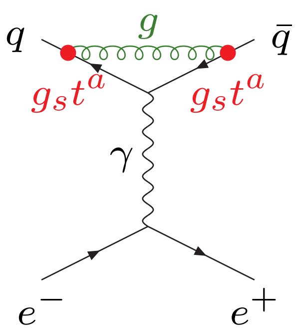
$$\sigma_0 = \sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s} \qquad \sqrt{s} \gg m_\mu \qquad (2)$$

Expts. confirmed the factor 3 in (1)!

Warning: additional (SM) contri. to amplitude at sufficiently high energies.

First QCD corrections

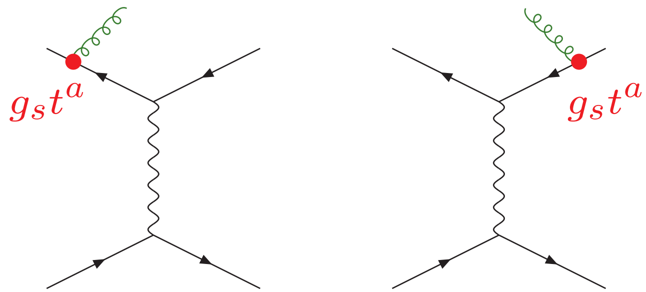
Virtual correction



$+$ c.t.

to be summed with tree-level amplitude $\Rightarrow$ correction terms of $\mathcal{O}(g_s^2)$

Real radiation of gluons



A separate process with $\sigma \sim \mathcal{O}(g_s^2)$; must be included due to finite expt. precision in measuring gluon's energy; essential for cancellation of IR div.

g_s: coupling of QCD

t^a: generator matrix for funda. rep.

Non-Abelian properties inessential for this correction except for color factor:

$$\text{Tr}\,(t^a t^a) = C_2(r) \cdot \text{Tr}\,\mathbf{1}_{3\times 3} = \frac{4}{3} \cdot 3 \quad \text{funda. rep.} \tag{3}$$

Final result with first QCD corrections:

$$\sigma(e^+e^- \to \text{hadrons}) = \sigma_0 \cdot 3 \cdot \sum_f Q_f^2 \left[1 + \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right], \;\; \alpha_s \equiv \frac{g_s^2}{4\pi} \tag{4}$$

Renormalization group improvement

QED: on-shell renor. reasonable and convenient for defining α

QCD: on-shell quarks make not much sense (for defining α_s) since no free quarks are observed. What are observed are color-singlet hadrons.

$*$ α_s defined at some scale M, but σ must be indept. of M.

$*$ for Q^2 (characteristic scale of process considered) very different from M^2, n-loop corrections $\sim \left[\alpha_s \ln(Q^2/M^2)\right]^n$ could be very large.

$\Rightarrow$ RG improved perturbation theory

At leading order of QCD corrections,

$$\sigma(e^+e^- \to \text{hadrons}) = \sigma_0 \cdot 3 \cdot \sum Q_f^2 \left[1 + \alpha_s(\sqrt{s})/\pi + \mathcal{O}(\alpha_s^2(\sqrt{s}))\right] \quad (5)$$

where $\alpha_s(Q) = \bar{g}_s^2/(4\pi)$ is the running coupling const.:

$$\frac{d\bar{g}_s}{d\ln(Q/M)} = \beta(\bar{g}_s),\ \bar{g}_s(M) = g_s,\ \beta(\bar{g}_s) = -\frac{1}{(4\pi)^2} b_0 \bar{g}_s^3,\ b_0 = 11 - \frac{2}{3} n_f \leftarrow (5.31) \quad (6)$$

$$\Rightarrow \quad \alpha_s(Q) = \frac{\alpha_s}{1 + b_0\alpha_s/(2\pi)\ln(Q/M)}, \quad \alpha_s = \frac{g_s^2}{4\pi} \quad (7)$$

Sometimes, it is convenient to trade $\alpha_s(M)$ for a mass scale $\Lambda_{\rm QCD}$ defined by

$$
\begin{aligned}
1 &= \frac{b_0\alpha_s}{2\pi}\ln(M/\Lambda_{\rm QCD}) \\
\Rightarrow \quad \alpha_s(Q) &= \frac{2\pi}{b_0\ln(Q/\Lambda_{\rm QCD})}
\end{aligned}
\tag{8}
$$

i.e., $\Lambda_{\rm QCD}$ is the scale at which QCD becomes strongly interacting.
Experimentally, $\Lambda_{\rm QCD} \sim 200\ {\rm MeV}$

Measurements of α_s

Precision not comparable to α of QED due to soft processes of hadronization.

To compare α_s extracted from different processes, we must choose a common renor. scheme and choose a common renor. scale

It becomes conventional now to work with MS or $\overline{\text{MS}}$ and choose m_Z as the common renor. scale: $\alpha_s^{\text{MS}}(m_Z)$ or $\alpha_s^{\overline{\text{MS}}}(m_Z)$ (mass of neutral weak gau. boson Z)

Different measurements show consistent values: $\alpha_s^{\overline{\text{MS}}}(m_Z) \approx 0.11 \sim 0.12$

2. Hadronic jets

QED: IR div. cancelled between

$e^+e^- \to \mu^+\mu^-\gamma$ and virtual radiative corrections to $e^+e^- \to \mu^+\mu^-$.

$*$ limited experimental energy resolution;

$*$ Sudakov double logarithms for large momentum transfers.

QCD: IR div. (due to massless gluons) also cancelled between $e^+e^- \to q\bar{q}g$ and radiative virtual corrections $e^+e^- \to q\bar{q}$.

But physical understanding or interpretation is different:

$*$ It makes no experimental sense to speak of a soft, collinear gluon because $q\bar{q}$ will hadronize to 2 back-to-back jets of hadrons where there is no way to isolate a soft gluon.

$*$ When radiated real gluon is hard and has a large enough transverse momentum relative to the $q\bar{q}$ system, we will observe 3 jets of hadrons.

6.3. Deep inelastic scattering

1. Parton model and Bjorken scaling

Experimental facts & difficulties

- $p\bar{p}$ collision at high energies: produce large number of π's mainly collinear with collision axis; very rare π's with large transverse momenta.

  $\Rightarrow$ suggesting that hadrons look as loosely bound states of constituents

  At high energies, constituents have almost light-like momenta. $|q^2|$ very small though components of q (momentum transfer) could be very large. $\rightarrow$ large transverse momentum is rare.
- ep deep inelastic scattering: large rates for hard scattering observed!

Conflict between the two?

Parton model by Bjorken and Feynman:

$$p \text{ is a loosely bound state of partons} \begin{cases} \text{fermions (quarks \& anti-quarks)} \\ \text{binding particles (gluons)} \end{cases}$$

• In $p\bar{p}$ collision, no large momentum transfer occurs via strong interactions between partons $\longrightarrow$ collinear jets.

• In ep collision, e interacting with quarks via QED interactions can knock out a quark from $p \Rightarrow$ hard scattering between e and q, followed by soft hadronization $\rightarrow$ collinear jet in the direction of the knocked-out quark.

• Parton model consist. with later discovery of asymptotic freedom of QCD.

Bjorken scaling: $ep \rightarrow eX$, hard QED process in deep inelastic regime
When σ for QED subprocess is removed, no dependence on q^2 remains in differential cross section for $ep \rightarrow eX$

Violation of Bjorken scaling when perturbative QCD corrections are included.

2. Deep inelastic scattering with electrons

Structure of proton as probed by EM interactions.

Deep inelastic:
$q^2 = (k' - k)^2$ large negative, i.e., $Q^2 = -q^2$ large positive (measurable)

Kinematics

Ignore $m_p,\ m_e$ at high energy: $P^2 \sim 0,\ k^2 \sim k'^2 \sim 0$, etc.

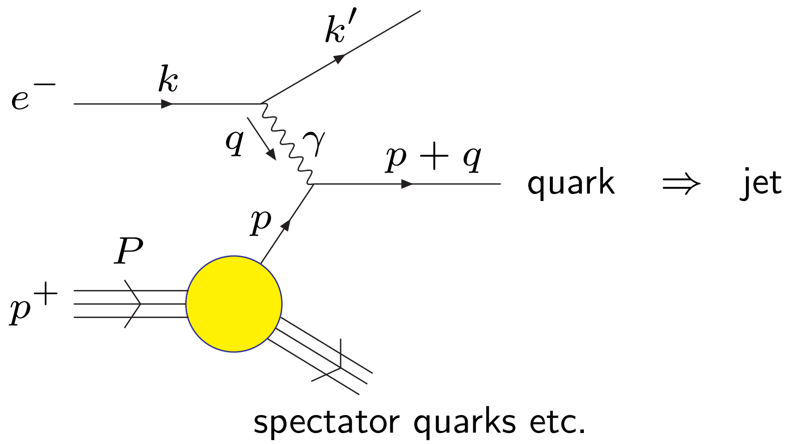
Struck quark: carries a fraction ξ of the parent proton's momentum, i.e.,

$$p = \xi P,\ \xi \in [0,1] \quad \text{large transverse momentum (via exchange of hard gluons among partons) suppressed by small } \alpha_s \tag{9}$$

Mandelstam variables $\hat{s},\ \hat{t},\ \hat{u}$ for subprocess:

$$\hat{s} = (k+p)^2 = 2k\cdot p = 2\xi P\cdot k = \xi s,\ \ s = (k+P)^2$$

$$\hat{t} = (k'-k)^2 = -2k\cdot k' = -Q^2, \quad \hat{u} = -(\hat{s}+\hat{t})$$



Cross section for subprocess

Ignore QCD effects. Obtained from $e\mu \to e\mu$ (via crossing of $e^- e^+ \to \mu^- \mu^+$):

$$\frac{d\sigma}{d\hat{t}}(e^- q \to e^- q) = \frac{2\pi\alpha^2 Q_q^2}{\hat{s}^2} \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} \qquad \text{QFT: §5.4} \tag{10}$$

Cross section for the whole process

Parton distribution function (PDF):

$f_f(\xi)d\xi =$ probability of finding parton f (in proton) with momentum fraction between ξ and $\xi + d\xi$

soft phys, incomputable in pert. theory but universal for all processes

$$\begin{aligned} & \frac{d\sigma}{dQ^2}\Big(e^-(k)p^+(P) \to e^-(k')X\Big) \qquad \hat{s} = Q^2 - \hat{u} \geqslant Q^2 \\ = \quad & \int_0^1 d\xi \sum_q f_q(\xi) \cdot \frac{d\sigma}{dQ^2}\Big(e^-(k)q(\xi P) \to e^-(k')q(\xi P + q)\Big) \cdot \theta(\xi s - Q^2) \end{aligned} \tag{11}$$

Further manipulations

$$\text{massless quark: } 0 \approx (p+q)^2 \approx 2p \cdot q + q^2 = 2\xi P \cdot q - Q^2 \Rightarrow \xi = \frac{Q^2}{2P \cdot q} \equiv x \in [0,1] \tag{12}$$

Define

$$y \equiv \frac{P \cdot q}{P \cdot k} = \begin{cases} \dfrac{q^0}{k^0} \text{ in proton's rest frame, i.e., fraction of initial} \\ e^- \text{ energy transferred to the hadronic system} \\ = \dfrac{2p \cdot (k - k')}{2p \cdot k} = \dfrac{\hat{s} + \hat{u}}{\hat{s}} \Rightarrow \dfrac{\hat{u}}{\hat{s}} = y - 1 \leq 0 \to y \in [0, 1] \end{cases} \tag{13}$$

Then

$$Q^2 = x\ 2P \cdot q = xy\ 2P \cdot k = xys$$

Trade Q^2 and ξ for y and x:

$$d\xi dQ^2 = dx \cdot \left.\frac{dQ^2}{dy}\right|_x dy = sxdxdy$$

In terms of x and y, the differential cross section is

$$\frac{d^2\sigma}{dxdy}\left(e^- p^+ \to e^- X\right) = \left(\sum_q x f_q(x) Q_q^2\right) \frac{2\pi\alpha^2 s}{Q^4}\left[1 + (1-y)^2\right] \tag{14}$$

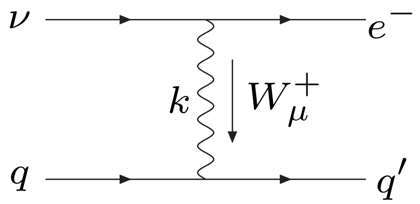
Bjorken scaling photon prop. in QED subpro.

3. Deep inelastic scattering (DIS) with neutrinos

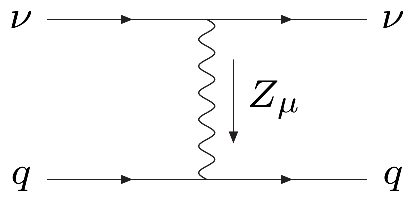
DIS with e sensitive to PDF in the combination $\sum_q x f_q(x) Q_q^2 \leftarrow$ QED

DIS with ν (only weakly interacting) sensitive to different combinations

Subprocesses in DIS with ν (details on standard model (SM) in last chapter):



charged-current (CC) weak int.
(nuclear β decay)



neutral-current (NC) weak int.
(discovered in 1970's)

We consider DIS induced by CC interactions.

For $|k^2| \ll m_W^2$, CC interactions can be described by effective 4-fermion int.:

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} \bar{\ell}\gamma^\mu (1-\gamma_5)\nu \cdot \bar{u}\gamma_\mu(1-\gamma_5) d + \text{h.c.} \qquad (15)$$

charged lepton associated with ν
up and down quarks

where

Fermi const. $G_F \sim 10^{-5}\ \mathrm{GeV}^{-2}$; SM: $\dfrac{G_F}{\sqrt{2}} = \dfrac{g^2}{8m_W^2}$, $g = SU(2)_L$ coupling

Only LH fields enter in CC interactions!

Cross section for subprocesses can be worked out *ab initio* using (15) as $\mathcal{L}_{\mathrm{int}}$

But easier derivation possible, based on DIS results with e:

with e: (10)	with ν
photon propagator $(\hat{t})^{-2}$	$W^{\pm}$ propagator $(m_W^2)^{-2}$ for small momentum tranf.
couplings $e^4 Q_q^2$	$(g/\sqrt{2})^4$
$\hat{s}^2 + \hat{u}^2 \begin{cases} \hat{s}^2: & \text{LL}\rightarrow\text{, or RR}\rightarrow \\ \hat{u}^2: & \text{other helicities} \end{cases}$	$\left.\begin{array}{l}\text{remains}\\ \text{drops}\end{array}\right\}$ due to pure LL coupling
spin average over initial e	no spin average over ν, again due to pure LH

$$\Rightarrow \frac{d\sigma}{d\hat{t}}(\nu_\mu d \to \mu^- u) = \frac{\pi g^4}{2(4\pi)^2 \hat{s}^2} \frac{\hat{s}^2}{m_W^4} = \frac{G_F^2}{\pi} \tag{16}$$

With $\overline{\nu}$, LH field in $\mathcal{L}_{\text{eff}}$ corresponds to RH particle $\overline{\nu}$; thus $\hat{u}^2$ (instead of $\hat{s}^2$) term contributes:

$$\frac{d\sigma}{d\hat{t}}(\overline{\nu}_\mu u \to \mu^+ d) \quad = \quad \frac{\pi g^4}{2(4\pi)^2 \hat{s}^2} \frac{\hat{u}^2}{m_W^4} = \frac{G_F^2}{\pi}(1-y)^2 \tag{17}$$

Similarly,

$$\frac{d\sigma}{d\hat{t}}(\nu_\mu \overline{u} \to \mu^- \overline{d}) = \frac{d\sigma}{d\hat{t}}(\overline{\nu}_\mu u \to \mu^+ d), \quad \frac{d\sigma}{d\hat{t}}(\overline{\nu}_\mu \overline{d} \to \mu^+ \overline{u}) = \frac{d\sigma}{d\hat{t}}(\nu_\mu d \to \mu^- u) \tag{18}$$

Cross section for DIS with ν:

proton $\sim (uud)$, also some $\overline{q}$ and gluons g.

Same kinematics as for DIS with e, thus (14) applies with some replacements in cross section for subprocesses:

$$\begin{aligned}
\frac{d^2\sigma}{dx\,dy}(\nu_\mu p \to \mu^- X) \quad &= \quad \frac{sG_F^2}{\pi}\left[x f_d(x) + x f_{\bar{u}}(x)(1-y)^2\right] \text{ from (16) or (17), dominant} \\
\frac{d^2\sigma}{dx\,dy}(\overline{\nu}_\mu p \to \mu^+ X) \quad &= \quad \frac{sG_F^2}{\pi}\left[x f_u(x)(1-y)^2 + x f_{\bar{d}}(x)\right] \text{ from (18), sub-dominant}
\end{aligned} \tag{19}$$

4. Relations for PDF

(1) General features for PDF's of proton and neutron:

$f_{u,d}(x)$ dominant	$f_{\overline{q}}(x)$ and $f_g(x)$ small
peaking at $x \sim 1/3$	blowing up at very small x

weak dependence on Q^2 via higher-order QCD corrections $\rightarrow$ scaling violation

(2) Momentum conservation

$$\int_0^1 dx\; x\Big[\overbrace{f_u(x) + f_d(x) + f_{\overline{u}}(x) + f_{\overline{d}}(x)}^{\sim 50\%} + f_g(x) + \cdots\Big] = 1 \tag{20}$$

The proton's momentum is carried by its partons whose transverse momentum is ignored in high energy scattering.

(3) $p \sim (uud)$:

$$\int dx\Big[f_u(x) - f_{\overline{u}}(x)\Big] = 2, \;\; \int dx\Big[f_d(x) - f_{\overline{d}}(x)\Big] = 1 \tag{21}$$

(4) Neutron's PDF can be obtained from proton's by isospin sym.:

$$f_u^n(x) = f_d(x),\ f_d^n(x) = f_u(x),\ \text{etc.} \tag{22}$$

(5) Anti-proton's PDF can be obtained by charge-conjugation sym.:

$$f_u^{\overline{p}}(x) = f_{\overline{u}}(x),\ f_{\overline{d}}^{\overline{p}}(x) = f_d(x),\ \text{etc.} \tag{23}$$

[finished in 2.5 units on Nov 27, 2012. Next section self-learning.]

6.4. Hard-scattering processes in hadron collisions

Most collisions of two hadrons involve soft interactions of their constituents: pert. QCD not applicable.

In some kinematical regions where large momentum transfers occur (hard scattering), we can apply pert. QCD to compute cross sections.

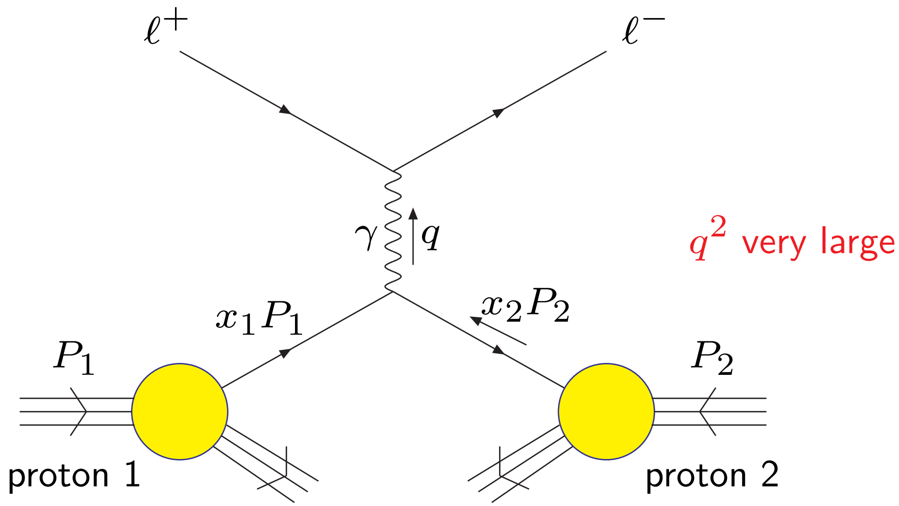
For example, if hard scattering occurs via the subprocess $q + \overline{q} \to X$, the leading pert. QCD result is

$$
\begin{aligned}
\sigma(p(P_1) + p(P_2) \to Y + X) &= \int_0^1 dx_1 \int_0^1 dx_2 \sum_f f_f(x_1) f_{\bar{f}}(x_2) \quad \begin{array}{l} q_f \text{ PDF in proton 1} \\ \bar{q}_f \text{ PDF in proton 2} \end{array} \\
&\times \quad \sigma(q_f(x_1 P_1) + \overline{q}_f(x_2 P_2) \to X) \quad \begin{array}{l} \text{cross section} \\ \text{for subprocess} \end{array} \qquad (24)
\end{aligned}
$$

1. Drell-Yan process

large invariant-mass lepton pair production in pp or $p\bar{p}$ collision

Kinematics:



The lepton pair $\ell^+\ell^-$ is produced by $q\overline{q}$ annihilation via QED interaction.

Work in CM of the parent protons.
Ignore all masses and transverse momenta of parton quarks.

$$P_1 = (E, 0, 0, E),\ P_2 = (E, 0, 0, -E) \to s = (P_1 + P_2)^2 = 4E^2$$

The parton quark and anti-quark carry momentum fractions $x_1,\ x_2$ of their parent protons; then

$$q = x_1 P_1 + x_2 P_2 = \Big((x_1 + x_2)E, 0, 0, (x_1 - x_2)E\Big) \to q^2 = x_1 x_2 s$$

Define

$$M^2 = q^2,\ q^0 = M\cosh Y,\ Y:\ \text{rapidity of the virtual photon} \tag{25}$$

Trade $x_1,\ x_2$ for $M^2,\ Y$:

$$\left.\begin{array}{l} M^2 = x_1 x_2 s \\ \cosh Y = \dfrac{(x_1+x_2)E}{2E\sqrt{x_1x_2}} = \dfrac{1}{2}\left(\sqrt{\dfrac{x_1}{x_2}} + \sqrt{\dfrac{x_2}{x_1}}\right) \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} x_1 = \dfrac{M}{\sqrt{s}}e^{Y} \\ x_2 = \dfrac{M}{\sqrt{s}}e^{-Y} \end{array}\right. \quad \begin{array}{l} x_1 \leftrightarrow x_2 \\ \text{also OK} \end{array}$$

Subprocess $q_f\overline{q}_f \to \ell^+\ell^-$:

$$\sigma(q_f\overline{q}_f \to \ell^+\ell^-) = \sigma_0 \cdot \left(\frac{1}{3}\right)^2 \cdot 3 \cdot Q_f^2 = \frac{1}{3}Q_f^2\sigma_0,\ \text{color}$$

$$\sigma_0 = \sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3\hat{s}},\ \hat{s} = q^2 = M^2 \tag{26}$$

Whole process

Insert (26) into (24) and change variables $x_1,\ x_2$ to $M^2,\ Y$:

$$\frac{d\sigma}{dM^2dY}\big(pp \to \ell^+\ell^- + \text{anything}\big) = \sum_f x_1 f_f(x_1)\ x_2 f_{\bar{f}}(x_2)\ \frac{1}{3}Q_f^2\frac{4\pi\alpha^2}{3M^4} \tag{27}$$

2. Jet pair production

Dominant reactions in hadron collisions are those of QCD. For $2 \to 2$ parton reactions, they are: $qq \to qq,\ q\overline{q} \to q\overline{q},\ q\overline{q} \to gg, qg \to qg,\ gg \to gg$, etc.

It is experimentally difficult to separate quark-initialed jets from gluon-initialed jets or identify the initial partons in the subprocesses.
What is observed is a sum of cross sections over all channels of jets.

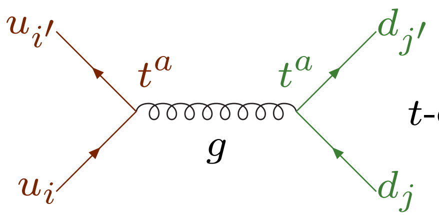
We consider leading QCD contributions and concentrate on color factors. For more discussions, see:

T. Gottschalk, D. Sivers, PRD**21** (1980) 102-130;

R. Cutler, D. Sivers, PRD**17** (1978) 196-211;

J.F. Owens *et al.*, PRD**18** (1978) 1501-1514.

(1) $u + d \to u + d$



t-channel

Prototype in QED: $e^- \mu^- \to e^- \mu^-$, $\dfrac{d\sigma}{d\hat{t}} = \dfrac{2\pi\alpha^2}{\hat{s}^2} \dfrac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}$

Differences: $\alpha \to \alpha_s$ and color factors

$$\mathcal{M} \quad : \quad (t^a)_{ii'} (t^a)_{jj'} \quad u\text{-line} \quad d\text{-line}$$

$$\sum \overline{|\mathcal{M}|^2} \quad : \quad \left(\frac{1}{3}\right)^2 \cdot (t^a)_{ii'} (t^a)_{jj'} \cdot (t^b)_{i'i} (t^b)_{j'j} = \left(\frac{1}{3}\right)^2 \mathrm{tr}(t^a t^b)\ \mathrm{tr}(t^a t^b)$$

$$= \left(\frac{C(r)}{3}\right)^2 \cdot \delta^{ab} \delta^{ab} = \frac{2}{9}$$

$$\therefore \quad \frac{d\sigma}{d\hat{t}} = \frac{2}{9} \frac{2\pi\alpha_s^2}{\hat{s}^2} \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}$$

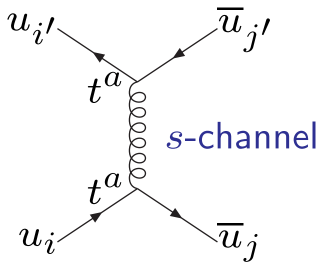
Crossed channel: $u\bar{u} \to d\bar{d}$ by $\hat{s} \leftrightarrow \hat{t}$ in $|\mathcal{M}|^2$

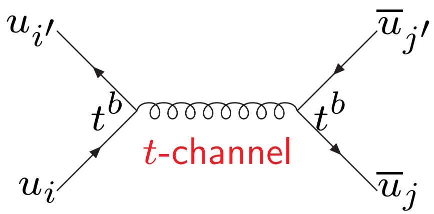
(2) $u\overline{u} \to u\overline{u}$

Prototype in QED: $e^- e^+ \to e^- e^+$ (Bhabha scattering)

$$\begin{aligned}\frac{d\sigma}{d\hat{t}} &= \frac{2\pi\alpha^2}{\hat{s}^2}\left[\left(\frac{\hat{s}}{\hat{t}}\right)^2 + \left(\frac{\hat{t}}{\hat{s}}\right)^2 + \hat{u}^2\left(\frac{1}{\hat{s}} + \frac{1}{\hat{t}}\right)^2\right] \\ &= \frac{2\pi\alpha^2}{\hat{s}^2}\left[\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} + \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2} + \frac{2\hat{u}^2}{\hat{s}\hat{t}}\right]\end{aligned}$$

t-channel s-channel interference





$\mathcal{M}$: s-channel : $(t^a)_{ji}(t^a)_{i'j'}$ t-channel : $(t^b)_{i'i}(t^b)_{jj'}$

Color factors in $\sum \overline{|\mathcal{M}|^2}$:

s-channel	$\left(\frac{1}{3}\right)^2 (t^a)_{ji}(t^a)_{i'j'} \cdot (t^c)_{ij}(t^c)_{j'i'} = \left(\frac{1}{3}\right)^2 \mathrm{tr}(t^a t^c)\mathrm{tr}(t^a t^c)$	same as for $ud \to ud$
t-channel	$\left(\frac{1}{3}\right)^2 (t^b)_{i'i}(t^b)_{jj'} \cdot (t^d)_{ii'}(t^d)_{j'j} = \left(\frac{1}{3}\right)^2 \mathrm{tr}(t^b t^d)\mathrm{tr}(t^b t^d)$	same as ...
interference	$\left(\frac{1}{3}\right)^2 (t^a)_{ji}(t^a)_{i'j'} \cdot (t^b)_{ii'}(t^b)_{j'j} = \left(\frac{1}{3}\right)^2 \mathrm{tr}(t^a t^b t^a t^b) = -\frac{2}{27}$	

More details for the interference factor:

$$
\begin{aligned}
t^a t^b t^a t^b &= t^a \Big([t^b, t^a] + t^a t^b \Big) t^b = -i f^{abc} t^a t^c t^b + t^a t^a t^b t^b \\
&= -\frac{i}{2} f^{abc} i f^{cbd} t^a t^d + [C_2(r)]^2 \mathbf{1} = \Big(-\frac{1}{2} C_2(G) + C_2(r) \Big) C_2(r) \cdot \mathbf{1} \\
\mathrm{tr}(t^a t^b t^a t^b) &= \Big(C_2(r) - \frac{1}{2} C_2(G) \Big) C_2(r) \cdot 3 = \Big(\frac{4}{3} - \frac{3}{2} \Big) \frac{4}{3} \cdot 3 = -\frac{2}{3}
\end{aligned}
$$

Crossed channel: $uu \to uu$ by $\hat{s} \leftrightarrow \hat{u}$ in $|\mathcal{M}|^2$

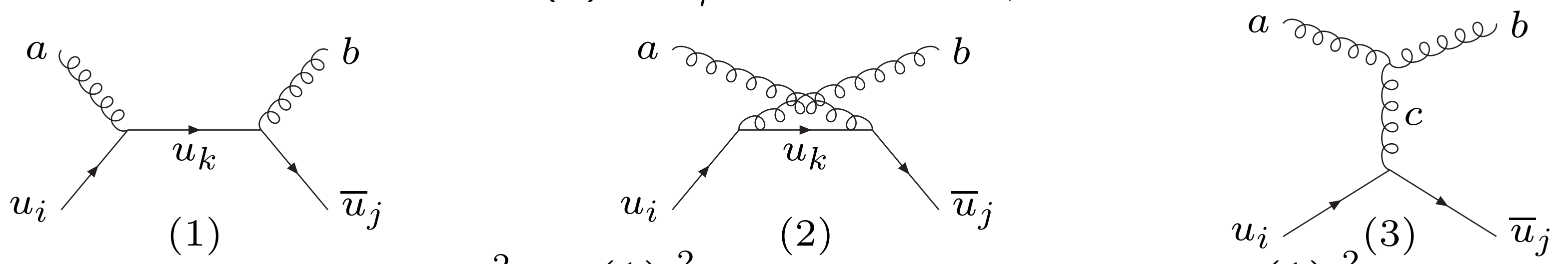
(3) $u\overline{u} \to gg$

Prototype in QED: $e^+e^- \to \gamma\gamma$. Differences:

(1) Additional diagram in QCD due to gluon self-interaction

(2) Be careful with pol. sum in QCD, 2 equivalent procedures:

(i) Use only physical pol.; (ii) $\sum \varepsilon^*_\mu(k)\varepsilon_\nu(k) \to -g_{\mu\nu}$ and subtract $\sigma(u\overline{u} \to c_a\overline{c}_a)$



$$\mathcal{M}_{(1)}: \ (t^b)_{jk}(t^a)_{ki} \quad \xrightarrow{|\cdots|^2} \quad \left(\frac{1}{3}\right)^2 (t^b)_{jk}(t^a)_{ki} \cdot (t^a)_{im}(t^b)_{mj} = \left(\frac{1}{3}\right)^2 \mathrm{tr}(t^a t^b t^a t^b)$$

$$= \left(\frac{C_2(r)}{3}\right)^2 \cdot 3$$

$\mathcal{M}_{(2)}: \ (t^a)_{jk}(t^b)_{ki}$ same as above

$$\mathcal{M}_{(3)}: \ if^{abc}(t^c)_{ji} \qquad \left(\frac{1}{3}\right)^2 f^{abc}(t^c)_{ji} \cdot f^{abd}(t^d)_{ij} = \left(\frac{1}{3}\right)^2 C_2(G)C_2(r) \cdot 3$$

Color factors for the interference terms, e.g.:

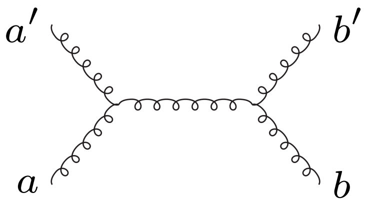
$$\mathcal{M}_{(1)}\mathcal{M}^*_{(3)}: \qquad \left(\frac{1}{3}\right)^2 (t^b)_{jk}(t^a)_{ki}\cdot(-i)f^{abc}(t^c)_{ij} = \left(\frac{1}{3}\right)^2 (-i)f^{abc}\mathrm{tr}(t^b t^a t^c)$$

$$= \left(\frac{1}{3}\right)^2\left(-\frac{3}{2}\right) C_2(G)C_2(r)$$

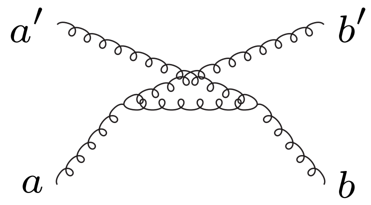
More complete computation by yourselves! Crossed channel: $ug \to ug$.

(4) $gg \to gg$

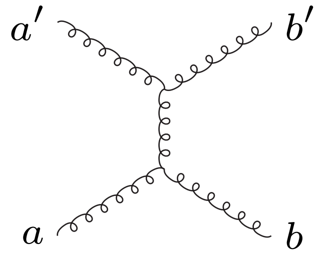
Most complicated $2 \to 2$ process, characteristic of non-Abelian gau. theories.



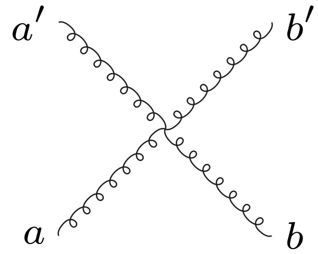
t-channel



u-channel



s-channel



contact interaction

Homework: 17.3

Chapter 7 Perturbation theory anomalies

Quantum effects can destroy symmetries of classical theories – anomaly

Example: massless Dirac field ψ, free, or in QED or QCD

$$\mathcal{L} \text{ is inva. under separate transf.} \begin{cases} \psi_L \to U_L \psi_L, & U_L = \exp(i\alpha_L) \\ \psi_R \to U_R \psi_R, & U_R = \exp(i\alpha_R) \end{cases}$$

$\Rightarrow$ Noether-currents $j^\mu_{L,R} = \overline{\psi}\gamma^\mu \frac{1}{2}(1 \mp \gamma_5)\psi$ are conserved: $\partial_\mu j^\mu_{L,R} = 0$

or, the vector and axial vector currents are conserved:

$$\partial_\mu j^\mu = 0,\ \partial_\mu j^{\mu 5} = 0;\ j^\mu = \overline{\psi}\gamma^\mu\psi,\ j^{\mu 5} = \overline{\psi}\gamma^\mu\gamma^5\psi$$

The result is altered by quantum effects. Important consequences include:

(1) $\partial_\mu j^{\mu 5} = 0$ is incompatible with gau. symmetries.

(2) Chiral gau. symmetries must not have anomalies of gau. theories.

(3) Occurrence of Goldstone bosons in QCD due to spontaneous breaking of chiral sym.

7.1. The axial current in four dimensions

7.2. Goldstone bosons and chiral symmetries in QCD

7.3. Chiral anomalies and chiral gau. theories

7.1. The axial current in four dimensions

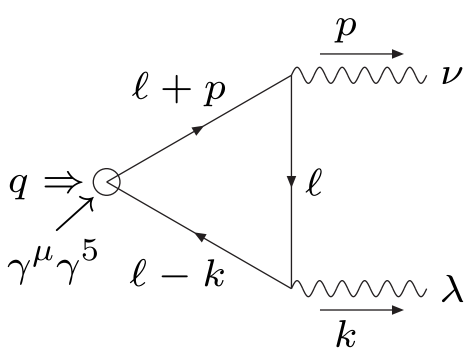
Chiral anomalies first studied by Adler, Bell & Jackiw – ABJ anomalies

1. Calculation in terms of triangle diagrams

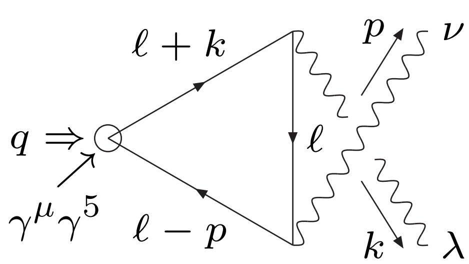
Consider the coupling of $j^{\mu 5}$ to two photons in massless QED:

$$\int d^4x\, e^{-iq\cdot x}\langle p,k|j^{\mu 5}(x)|0\rangle = (2\pi)^4\delta^4(p+k-q)\varepsilon^*_\nu(p)\varepsilon^*_\lambda(k)\mathcal{M}^{\mu\nu\lambda}(p,k) \tag{1}$$

The lowest order contribution comes from the one-loop diagrams:



(1)



(2)

The amplitudes are

$$
\begin{aligned}
\mathcal{M}^{\mu\nu\lambda}_{(1)} &= (-1)(-ie)^2 \int \frac{d^4\ell}{(2\pi)^4}\, \mathrm{Tr}\Big(\gamma^\mu\gamma^5 \frac{i}{\not\ell - \not k}\gamma^\lambda \frac{i}{\not\ell}\gamma^\nu \frac{i}{\not\ell + \not p}\Big) \\
&= -ie^2 \int \frac{d^4\ell}{(2\pi)^4}\, \mathrm{Tr}\Big(\gamma^\mu\gamma^5 (\not\ell - \not k)\gamma^\lambda \not\ell \gamma^\nu (\not\ell + \not p)\Big) \Big[(\ell-k)^2\ell^2(\ell+p)^2\Big]^{-1} \qquad (2) \\
\mathcal{M}^{\mu\nu\lambda}_{(2)} &= \mathcal{M}^{\mu\nu\lambda}_{(1)}\Big|_{\substack{p\leftrightarrow k \\ \nu\leftrightarrow\lambda}}
\end{aligned}
$$

We concentrate on $\mathcal{M}_{(1)}$.

$$
\left.\begin{array}{r} \text{apparently linearly UV div.} \longrightarrow \text{need regularization.} \\ \text{must preserve gau. sym.} \end{array}\right\} \Rightarrow \text{dim. reg.}
$$

New problem with γ_5 in DR:

$$
\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \qquad (3)
$$

properly defined only in 4-dims, no generalization to $d = 4 - \epsilon$ dims!

't Hooft & Veltman:

γ_5, external momenta & external indices defined in 4-dim.

loop momenta & contracted internal indices defined in d-dim.

Denote $\|$ for directions in $0,1,2,3$ & $\perp$ for directions in extra dims. Then,

$$\{\gamma_5,\gamma_{\|}\}=0,\ [\gamma_5,\gamma_\perp]=0,\ \ell=\ell_{\|}+\ell_\perp,\ \text{etc.} \tag{4}$$

Compute in DR augmented by the recipe of 't Hooft-Veltman:

$$\begin{aligned} i q_\mu \mathcal{M}^{\mu\nu\lambda}_{(1)} &= e^2\int\frac{d^d\ell}{(2\pi)^d}\mathrm{Tr}\Big(\not{q}\gamma^5\frac{1}{\not{\ell}-\not{k}}\gamma^\lambda\frac{1}{\not{\ell}}\gamma^\nu\frac{1}{\not{\ell}+\not{p}}\Big) \\ &= e^2\int\frac{d^d\ell}{(2\pi)^d}\mathrm{Tr}\Big(\frac{1}{\not{\ell}+\not{p}}(\not{p}+\not{k})\gamma^5\frac{1}{\not{\ell}-\not{k}}\gamma^\lambda\frac{1}{\not{\ell}}\gamma^\nu\Big)\end{aligned}$$

Decompose as follows:

$$\begin{aligned}(\not{p}+\not{k})\gamma^5 &= (\not{\ell}+\not{p})\gamma^5+(\not{k}-\not{\ell})\gamma^5=(\not{\ell}+\not{p})\gamma^5-\gamma^5\not{k}+\gamma^5\not{\ell}_{\|}-\gamma^5\not{\ell}_\perp \\ &= (\not{\ell}+\not{p})\gamma^5+\gamma^5(\not{\ell}-\not{k})-2\gamma^5\not{\ell}_\perp\end{aligned}$$

Then,

$$\mathrm{Tr}\Big(\frac{1}{\not\ell+\not p}\cdots\gamma^\nu\Big) = \mathrm{Tr}\Big(\gamma^5\frac{1}{\not\ell-\not k}\gamma^\lambda\frac{1}{\not\ell}\gamma^\nu + \underset{\leftrightarrow}{\gamma^5\gamma^\lambda}\frac{1}{\not\ell}\gamma^\nu\frac{1}{\not\ell+\not p} - 2\frac{1}{\not\ell+\not p}\gamma^5\not\ell_\perp\frac{1}{\not\ell-\not k}\gamma^\lambda\frac{1}{\not\ell}\gamma^\nu\Big)$$

$$= \mathrm{Tr}\Big(\gamma^5\frac{1}{\not\ell-\not k}\gamma^\lambda\frac{1}{\not\ell}\gamma^\nu_{①} - \gamma^5\frac{1}{\not\ell}\gamma^\nu\frac{1}{\not\ell+\not p}\gamma^\lambda_{②} - 2\frac{1}{\not\ell+\not p}\gamma^5\not\ell_\perp\frac{1}{\not\ell-\not k}\gamma^\lambda\frac{1}{\not\ell}\gamma^\nu_{③}\Big)$$

$$iq_\mu\mathcal{M}^{\mu\nu\lambda}_{(1)} = e^2\int\frac{d^d\ell}{(2\pi)^d}\,\mathrm{Tr}(① + ② + ③)$$

Each of ①, ②, ③ is well-defined in DR.
Make $\ell\to\ell+k$ in ①. Then, ① + ② is *anti-symmetric* under $p\leftrightarrow k$, $\nu\leftrightarrow\lambda$, and is thus cancelled by the corresponding term in $iq_\mu\mathcal{M}^{\mu\nu\lambda}_{(2)}$.

Now compute ③:

$$(-2)\int\frac{d^d\ell}{(2\pi)^d}\mathrm{Tr}\Big(\gamma^5\not\ell_\perp\frac{1}{\not\ell-\not k}\gamma^\lambda\frac{1}{\not\ell}\gamma^\nu\frac{1}{\not\ell+\not p}\Big) = \int\frac{d^d\ell}{(2\pi)^d}\frac{(-2)N^{\lambda\nu}}{D},$$

$$\frac{1}{D} = 2\int_0^1 dx\int_0^1 dy\ (1-y)\Big[(\ell-P)^2-\Delta\Big]^{-3},\quad \begin{aligned} P &= kx(1-y)-py,\\ \Delta &= -p^2y(1-y)-2k\cdot pxy(1-y)\\ &\quad -k^2x(1-y)\big(1-x(1-y)\big)\end{aligned}$$

$$N^{\lambda\nu} = \mathrm{Tr}(\gamma^5 \not{\ell}_\perp (\not{\ell} - \not{k})\gamma^\lambda \not{\ell} \gamma^\nu (\not{\ell} + \not{p}))$$ cannot contri., e.g.:

$$\not{\ell}\gamma^\lambda \not{\ell} = \not{\ell}(-\not{\ell}\gamma^\lambda + 2\ell^\lambda) = -\ell^2\gamma^\lambda + 2\ell^\lambda \not{\ell},$$ no enough $\gamma_\parallel$ to survive $\mathrm{Tr}(\gamma^5 \cdots)$

$$\Rightarrow N^{\lambda\nu} = -\mathrm{Tr}(\gamma^5 \not{\ell}_\perp \not{k}\gamma^\lambda \not{\ell}\gamma^\nu \not{p})$$ all $\parallel \Rightarrow$ $\not{\ell}$ must be $\perp$ to avoid vanishing

$$= -\mathrm{Tr}(\gamma^5 \not{\ell}_\perp \not{k}\gamma^\lambda \not{\ell}_\perp \gamma^\nu \not{p}) = -\ell_\perp^2 \mathrm{Tr}(\gamma^5 \not{k}\gamma^\lambda\gamma^\nu \not{p}) = i4\ell_\perp^2 \epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta$$ no symm. integration thus far!

Therefore, upon $\ell \to \ell + P$ which causes no changes in $N^{\lambda\nu}$ since $P = P_\parallel$:

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{N^{\lambda\nu}}{D} = 2\int_0^1 dx \int_0^1 dy\ (1-y) \cdot i4\epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell_\perp^2}{(\ell^2 - \Delta)^3}.$$

Symmetric integration:

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell_\perp^2}{(\ell^2-\Delta)^3} = \frac{d-4}{d} \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta)^3} = \frac{d-4}{d} \frac{i}{(4\pi)^2} \Big(\frac{4\pi}{\Delta}\Big)^{\frac{4-d}{2}} \Gamma\Big(\frac{4-d}{2}\Big) \frac{d}{4}$$

$$\xrightarrow{d \to 4} -\frac{1}{2}\frac{i}{(4\pi)^2}.$$

Therefore, dropping the anti-sym. part that is cancelled by diagram (2),

$$iq_\mu \mathcal{M}^{\mu\nu\lambda}_{(1)} = e^2(-2)\cdot i4\epsilon^{\alpha\lambda\nu\beta}k_\alpha p_\beta\Big(-\frac{1}{2}\Big)\frac{i}{(4\pi)^2} = -\frac{e^2}{4\pi^2}\epsilon^{\alpha\lambda\nu\beta}k_\alpha p_\beta,$$

which is symmetric under $p \leftrightarrow k,\ \nu \leftrightarrow \lambda$, and doubled by diagram (2):

$$\therefore \quad iq_\mu \mathcal{M}^{\mu\nu\lambda} = \frac{e^2}{2\pi^2}\epsilon^{\alpha\lambda\beta\nu}k_\alpha p_\beta \tag{5}$$

Alternative form in coordinate space

Multiply (1) by iq_μ:

$$\begin{aligned}
\text{lhs} &= \int d^4x\ e^{-iq\cdot x} iq_\mu \langle p,k|j^{\mu 5}(x)|0\rangle = \int d^4x\ e^{-iq\cdot x}\langle p,k|\partial_\mu j^{\mu 5}(x)|0\rangle \\
&= \int d^4x\ e^{i(-q+p+k)\cdot x}\langle p,k|\partial_\mu j^{\mu 5}(0)|0\rangle = (2\pi)^4\delta^4(p+k-q)\langle p,k|\partial_\mu j^{\mu 5}(0)|0\rangle \\
\text{rhs} &= (2\pi)^4\delta^4(p+k-q)\varepsilon^*_\nu(p)\varepsilon^*_\lambda(k)\cdot\frac{e^2}{2\pi^2}\epsilon^{\alpha\lambda\beta\nu}k_\alpha p_\beta \\
&= (2\pi)^4\delta^4(p+k-q)\frac{e^2}{2\pi^2}\Big(-\frac{1}{8}\Big)\langle p,k|\epsilon^{\alpha\lambda\beta\nu}F_{\alpha\lambda}F_{\beta\nu}(0)|0\rangle.
\end{aligned}$$

Thus,

$$\langle p,k|\partial_\mu j^{\mu 5}(0)|0\rangle = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\lambda\beta\nu}\langle p,k|F_{\alpha\lambda}F_{\beta\nu}(0)|0\rangle. \tag{6}$$

Comments

- Careless manipulation would lead to conservation of the axial current:

$$\begin{aligned}
iq_\mu \mathcal{M}^{\mu\nu\lambda}_{(1)} &\overset{(2)}{=} e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\Big(\frac{1}{\not\ell+\not p}\not q\gamma^5\frac{1}{\not\ell-\not k}\gamma^\lambda\frac{1}{\not\ell}\gamma^\nu\Big) && \begin{array}{l}(\not p+\not\ell+\not k-\not\ell)\gamma^5\\ \overset{?}{=}(\not\ell+\not p)\gamma^5+\gamma^5(\not\ell-\not k)\end{array}\\
&\overset{?}{=} e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\Big(\gamma^5\frac{1}{\not\ell-\not k}\gamma^\lambda\frac{1}{\not\ell}\gamma^\nu+\gamma^5\gamma^\lambda\frac{1}{\not\ell}\gamma^\nu\frac{1}{\not\ell+\not p}\Big) && \begin{array}{l}\ell\to\ell+k?\\ \{\gamma^5,\gamma^\rho\}\overset{?}{=}0\end{array}\\
&\overset{?}{=} e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\Big(\gamma^5\frac{1}{\not\ell}\gamma^\lambda\frac{1}{\not\ell+\not k}\gamma^\nu-\gamma^5\frac{1}{\not\ell}\gamma^\nu\frac{1}{\not\ell+\not p}\gamma^\lambda\Big) && \begin{array}{l}\text{anti-sym. in } p\leftrightarrow p,\nu\leftrightarrow\lambda\\ \text{cancelled by diagram (2)!}\end{array}\\
\Longrightarrow\quad & iq_\mu\Big(\mathcal{M}^{\mu\nu\lambda}_{(1)}+\mathcal{M}^{\mu\nu\lambda}_{(2)}\Big)\overset{?}{=}0
\end{aligned}$$

Problem: ill-defined integrals do not allow shift of loop momentum; regularization is required!

- Operator form of eqn.(6) holds:

$$\partial_\mu j^{\mu 5} = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}\ , \text{ ABJ anomaly} \tag{7}$$

It is correct to all orders in perturbation theory!

2. Other approaches to chiral anomaly

(1) Point-splitting regularization of the axial current as a composite operator

(2) Path integral approach: integration measure not invariant under chiral transf. even if the action is!

7.2. Goldstone bosons and chiral symmetries in QCD

1. Chiral symmetries of QCD: classical case

$$\mathcal{L}_{\text{QCD}} = \bar{u} i \not{D} u + \bar{d} i \not{D} d - m_u \bar{u} u - m_d \bar{d} d + \cdots \quad \begin{array}{l} \bar{u}_L \gamma_\mu u_L + \bar{u}_R \gamma_\mu u_R \\ \bar{u}_L u_R + \bar{u}_R u_L \end{array} \tag{8}$$

$$D_\mu = \partial_\mu - i g_s G^a_\mu t^a$$

QCD coupling, gluon; quark in fund. rep.; a : color index: $1 \to 8$

For $m_{u,d} \ll \Lambda_{\text{QCD}}$ where QCD becomes strong, it is good to set

$$m_u = m_d = 0 \quad \text{(to a less precise extent, } m_s = 0) \tag{9}$$

Then, $\mathcal{L}_{\text{QCD}}$ is inva. under

$$Q_L \to U_L Q_L, \; Q_R \to U_R Q_R, \tag{10}$$

$$Q \equiv \begin{pmatrix} u \\ d \end{pmatrix}, Q_L \equiv \frac{1}{2}(1-\gamma_5) \begin{pmatrix} u \\ d \end{pmatrix} \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \; Q_R \equiv \frac{1}{2}(1+\gamma_5) \begin{pmatrix} u \\ d \end{pmatrix} \equiv \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \tag{11}$$

where

$$\left. \begin{array}{l} U_L \in SU(2)_L \times U(1)_L \\ U_R \in SU(2)_R \times U(1)_R \end{array} \right\} \Rightarrow \text{QCD flavor sym: } SU(2)_L \times U(1)_L \times SU(2)_R \times U(1)_R$$

Its Noether currents are

$$\text{iso-singlet} \qquad j_L^\mu = \overline{Q}_L \gamma^\mu 1_2 Q_L = \bar{u}_L \gamma^\mu u_L + \bar{d}_L \gamma^\mu d_L, \; j_R^\mu = \overline{Q}_R \gamma^\mu 1_2 Q_R$$

$$\text{iso-triplet} \qquad j_{La}^\mu = \overline{Q}_L \gamma^\mu \frac{\sigma^a}{2} Q_L, \;\; j_{Ra}^\mu = \overline{Q}_R \gamma^\mu \frac{\sigma^a}{2} Q_R \; (a = 1, 2, 3) \tag{12}$$

or their linear combinations

$$\text{vector} \qquad j^\mu = \overline{Q} \gamma^\mu Q \; (\text{baryon number}), \; j_a^\mu = \overline{Q} \gamma^\mu \frac{\sigma^a}{2} Q \; (\text{isospin})$$

$$\text{axial vector:} \qquad j^{\mu 5} = \overline{Q} \gamma^\mu \gamma^5 Q, \;\; j_a^{\mu 5} = \overline{Q} \gamma^\mu \gamma^5 \frac{\sigma^a}{2} Q \tag{13}$$

These currents are *classically conserved*.

2. Spontaneous breaking of chiral symmetries

Vector currents correspond to conservation laws in strong interactions.

But no similar conservation laws corresponding to $j^{\mu 5}$, $j_a^{\mu 5}$. Why?

Nambu & Jona-Lasinio:
Chiral symmetries are *spontaneously broken* by quark condensate:

$$\langle 0|\overline{Q}Q|0\rangle = \langle 0|\overline{Q}_L Q_R + \overline{Q}_R Q_L|0\rangle \neq 0 \Rightarrow U_L(2) \times U_R(2) \xrightarrow{\text{SSB}} U(1)_V \times SU(2)_V \tag{14}$$

$\Rightarrow$ offering some kind of effective mass for u, d though $m_u = m_d = 0$ in $\mathcal{L}_{\rm QCD}$ (major part of the nucleon mass)

Goldstone theorem:
to each spontaneously broken *continuous* sym. corresponds a massless particle with quantum numbers of the broken sym. current (more later)

$\Rightarrow$ 3+1 Goldstone bosons with $J^P = 0^-$ in QCD for $m_u = m_d = 0$.

But there are only 3 low-lying pseudoscalars ($\pi^{\pm,0}$, iso-triplet $(I = 1)$) in nature $\leftrightarrow$ SSB of $SU(2)_A$

$U(1)_A$ does not break spontaneously.
What happens to it? – to be broken by anomaly!

Consider $j_a^{\mu 5}$ and associated pions:

$$\langle 0|j_a^{\mu 5}(x)|\pi^b(p)\rangle = -ip^\mu f_\pi \delta^{ab} e^{-ip\cdot x} \quad \text{Lorentz sym} \quad \text{isospin sym} \quad \text{transl. inva.} \tag{15}$$

$$f_\pi : \text{ 'pion decay const.'} \sim 94 \text{ MeV}, \text{ appearing in } \pi \to \mu\nu$$

$$\partial_\mu j_a^{\mu 5} = 0 \leftrightarrow p^2 = 0 \tag{16}$$

Low-energy dynamics of pions may be studied order by order in p based on the fact that they are Goldstone bosons of spontaneously broken chiral sym.

Small effects of $m_{u,d}$ (and thus m_π) can be incorporated as well.

$\Rightarrow$ Chiral Perturbation Theory of Weinberg, Gasser, Leutwyler, ...

3. Anomalies of chiral currents

Treatment in §7.1 may be generalized to the case of chiral sym. in QCD.

(1) Anomalies from QCD interactions

Differences from QED case

- Gauge inva. combination $\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \longrightarrow \epsilon^{\mu\nu\alpha\beta}G^c_{\mu\nu}G^d_{\alpha\beta}$ (non-linear!)
- QED couplings $\longrightarrow$ trace over QCD couplings $\mathrm{Tr}(t^c t^d)$

- Non-trivial flavor (isospin) structure in the current: $\mathrm{Tr}\frac{\sigma^a}{2}$ or $\mathrm{Tr}\ 1_2$

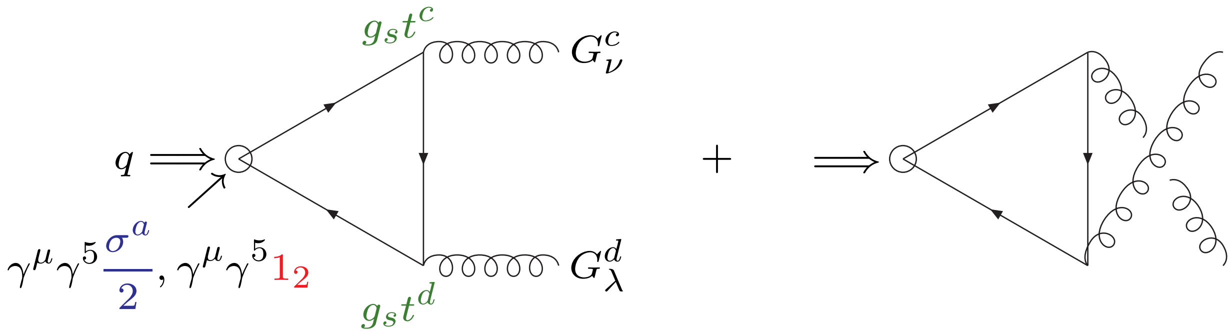
$\Rightarrow$ Iso-triplet axial currents have no anomalies from QCD interactions:

$$\partial_\mu j_a^{\mu 5} = -\frac{g_s^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}G^c_{\mu\nu}G^d_{\alpha\beta}\ \mathrm{Tr}\frac{\sigma^a}{2}\cdot\mathrm{Tr}(t^c t^d) = 0 \quad \text{color space} \tag{17}$$

but iso-singlet axial current has anomalies from QCD interactions:

$$\partial_\mu j^{\mu 5} = -\frac{g_s^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}G^c_{\mu\nu}G^d_{\alpha\beta}\ \mathrm{Tr}1_2\cdot\mathrm{Tr}(t^c t^d) = -\frac{n_f g_s^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}G^c_{\mu\nu}G^c_{\alpha\beta} \quad n_f=2, \text{number of flavors} \tag{18}$$

No $U(1)_A$ sym. at all due to the anomaly!



(2) Anomalies from QED interactions

Iso-triplet:

$$\partial_\mu j_a^{\mu 5} = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\ \mathrm{Tr}\Big(\frac{\sigma^a}{2}Q^2\Big)\cdot \mathrm{Tr}\ 1_3 = -\frac{N_c e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\ \mathrm{Tr}\Big(\frac{\sigma^a}{2}Q^2\Big) \tag{19}$$

$$Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix},\ \text{electric charges of } u,\ d \text{ quarks} \qquad N_c = 3,\ \text{number of colors},$$

non-vanishing only for $a = 3$, corresponding to π^0:

$$\partial_\mu j_3^{\mu 5} = -\frac{N_c e^2}{96\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{20}$$

For iso-singlet current there is a similar contribution to rhs of (18):

$$-\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\ \mathrm{Tr}\Big(1_2\cdot Q^2\Big)\cdot \mathrm{Tr}\ 1_3 = -\frac{5e^2}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\ \ (n_f = 2,\ N_c = 3)$$

An important consequence of (20):

π^0 decays dominantly into 2 photons due to the anomaly!

First determine decay amplitude in chiral limit (i.e., $m_u = m_d = 0$). Define

$$i\mathcal{M}(\pi^0(q) \to \gamma(p)\gamma(k)) \equiv i{\color{red}A}\ \varepsilon^*_\nu(p)\varepsilon^*_\lambda(k)\epsilon^{\nu\lambda\alpha\beta}p_\alpha k_\beta \qquad {\color{red}A:\ \text{const.}} \tag{21}$$

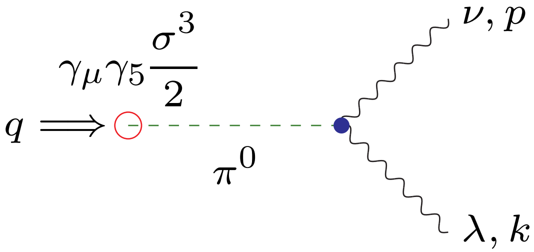
gau. inva.
Lorentz cova.
parity conservation

Define

$$\langle p, k|j_3^{\mu 5}(0)|0\rangle \equiv \varepsilon^*_\nu(p)\varepsilon^*_\lambda(k)\mathcal{M}^{\mu\nu\lambda}(p,k)$$

$$\overset{(20)}{\Longrightarrow} \quad iq_\mu\mathcal{M}^{\mu\nu\lambda}(p,k) = -\frac{N_c e^2}{12\pi^2}\epsilon^{\nu\lambda\alpha\beta}p_\alpha k_\beta, \quad q = p + k \tag{22}$$

Dominant contribution to $\langle p, k|j_3^{\mu 5}(0)|0\rangle$ comes from π^0-pole diagram:



$${\color{red}iq^\mu f_\pi}\ {\color{green}\frac{i}{q^2}}\ {\color{blue}iA\varepsilon^*_\nu\varepsilon^*_\lambda\epsilon^{\nu\lambda\alpha\beta}p_\alpha k_\beta} = -\frac{if_\pi A}{q^2}q^\mu\varepsilon^*_\nu\varepsilon^*_\lambda\epsilon^{\nu\lambda\alpha\beta}p_\alpha k_\beta$$

(15)

Multiplying the above by iq_μ and comparing with (22) gives

$$A = -\frac{N_c e^2}{12\pi^2}\frac{1}{f_\pi} \tag{23}$$

Then compute decay rate using (21), (23).
We must keep $m_\pi \neq 0$ for kinematics:

$$\Gamma(\pi^0 \to 2\gamma) = \underbrace{\frac{1}{2m_\pi}}_{\text{normalization}} \cdot \underbrace{\frac{1}{8\pi}}_{\text{2-body PS}} \cdot \underbrace{\frac{1}{2!}}_{\text{identical particles}} \sum_{\text{pol.}} |\mathcal{M}(\pi^0 \to 2\gamma)|^2 = \frac{A^2 m_\pi^3}{64\pi} = \frac{N_c^2 \alpha^2 m_\pi^3}{576\pi^3 f_\pi^2} \tag{24}$$

$N_c = 3$ is confirmed experimentally to good precision.

7.3. Chiral anomalies and chiral gauge theories

• Gauge currents (coupled to gauge fields) must not have anomalies.

Otherwise, violation of Ward identities ⇒ spoiling renormalizatility and unitarity in S matrix.

• Vector-like gauge theories (in which gauge fields couple only to vector currents, like QED and QCD) have no chiral anomalies with gauge currents.

• Weak interactions require chiral gauge couplings, i.e., LH and RH currents interact differently; e.g., in DIS we displayed effective 4-Fermi interaction:

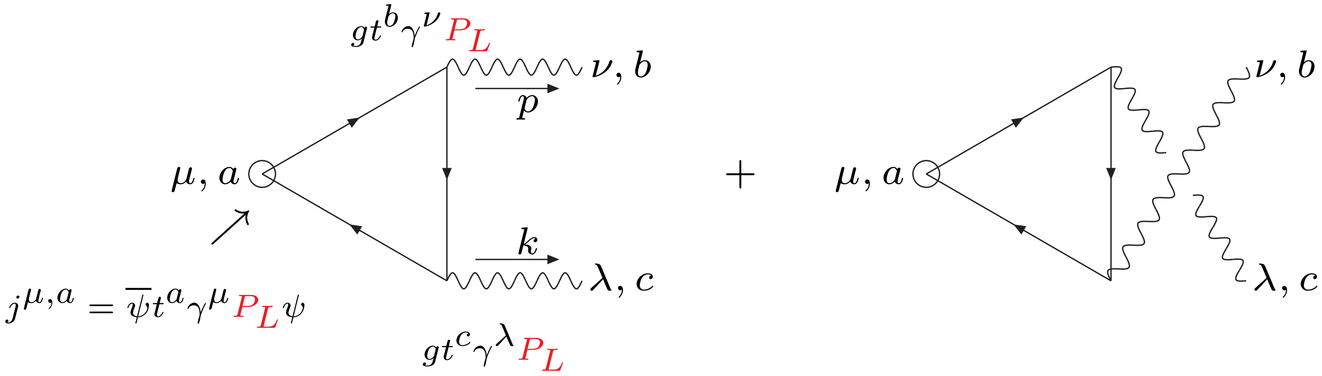
$$\mathcal{L}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}}\bar{\ell}\gamma^{\mu}\frac{1}{2}(1-\gamma_5)\nu \cdot \bar{u}\gamma_{\mu}\frac{1}{2}(1-\gamma_5)d + \text{h.c.} \quad \text{LH} \times \text{LH via } W^{\pm} \text{ exchange} \tag{6.15}$$

• Conservation of vector currents can be preserved by choosing appropriate regularization. This must be guaranteed because of QED at least.

Then, possible anomalies of axial vector (gauge) currents must be cancelled among different fermions.

• Chiral anomalies do not depend on fermion mass. Thus we can evaluate separate contributions of LH and RH Weyl fermions even if they are paired into massive Dirac fermions. They contribute oppositely.

- Generalization of §7.1, §7.2 to non-Abelian gau. theory of LH fermions:



$P_L = \frac{1}{2}(1 - \gamma_5)$

Three vertices are actually symmetric. Take divergence of vertex (μ, a):

$$\langle p, \nu, b; k, \lambda, c | \partial_\mu j^{\mu,a} | 0 \rangle \overset{(5)}{=} -\frac{1}{2}\frac{g^2}{4\pi^2}\epsilon^{\alpha\nu\beta\lambda} p_\alpha k_\beta \cdot A^{abc} \quad \Leftarrow (P_L)^3 = P_L = \frac{1}{2}(1 - \gamma_5) \tag{25}$$

$$A^{abc} = \mathrm{Tr}\Big(t^a \{t^b, t^c\}\Big), \text{ totally symmetric in } (a, b, c) \quad \text{sum of 2 graphs} \tag{26}$$

For RH chiral currents and gau. couplings, the sign of rhs in (25) is reversed.

In chiral gauge theory, different species of fermions with either chirality may couple to gauge bosons. If A^{abc} sums to zero, we say anomaly free.

Some special cases:

(1) $SU(2)$ is anomaly free for fermions in any rep. of it.
Reason: no invariant with totally symmetric 3 indices in adj. rep.
Example: fundamental rep. $t^a = \dfrac{\sigma^a}{2}$

$$A^{abc} = \mathrm{Tr}\Big(\frac{\sigma^a}{2}\Big\{\frac{\sigma^b}{2}, \frac{\sigma^c}{2}\Big\}\Big) = \frac{1}{2}\delta^{bc}\,\mathrm{Tr}\,\frac{\sigma^a}{2} = 0$$

(2) Given rep $R:\ t^a_R,\ A^{abc}(R) = \mathrm{Tr}\Big(t^a_R\{t^b_R, t^c_R\}\Big)$, its conjugate rep $\bar{R}$ has

$$t^a_{\bar{R}} = -(t^a_R)^T,\ A^{abc}(\bar{R}) = \mathrm{Tr}(-t^{a\,T}_R)\Big\{(-t^{b\,T}_R), (-t^{c\,T}_R)\Big\} = -\mathrm{Tr}\Big(t^a_R\{t^b_R, t^c_R\}\Big) = -A^{abc}(R)$$

If R is real, R, $\bar{R}$ are related by unitary transf., i.e., $A^{abc}(\bar{R}) = A^{abc}(R)$.

$\Rightarrow\ A^{abc}(R) = 0$;
i.e., fermions in real rep. of gauge group do not contribute to its anomaly

[finished in 3 units on Nov 30, 2012.]

Chapter 8 Gauge theories with spontaneous symmetry breaking

$U(1)$ and $SU(3)$ gauge symmetries of QED and QCD are exact symmetries. Photons and gluons are strictly massless gauge bosons.

Weak interactions are short-distance forces $\longrightarrow$ carriers of force are massive. To incorporate weak interactions into the framework of gauge theories, the gauge bosons must be massive, which is however forbidden by gauge sym.

Spontaneous symmetry breaking (SSB) offers a new and consistent way to realize symmetries, and via Higgs mechanism to generate masses for gauge bosons.

Status 2012:

new resonance (mass ~ 125 GeV) discovered at LHC, with properties similar to the SM Higgs particle

8.1. Spontaneous symmetry breaking (SSB)

8.2. The Higgs mechanism

8.3. The Glashow-Weinberg-Salam theory of weak interactions (standard model, SM)

8.4. The R_ξ gauges

8.5. Examples of calculations in SM

Suggested reading:

P. Langacker, Introduction to SM and electroweak physics, arXiv: 0901.0241 [hep-ph]

C. Quigg, Unanswered questions in electroweak theory, arXiv: 0905.3187 [hep-ph]

8.1. Spontaneous symmetry breaking

Possible ways of symmetry realization:

(1) Manifest or Wigner mode: particles constitute reps of sym. group e.g., isospin $SU(2)$: (p,n) as a doublet, (π^+,π^0,π^-) as a triplet

(2) Localized and exact: QED $U(1)$, QCD $SU(3)$

(3) Nambu-Goldstone mode of sym. realization:
sym. manifest in $\mathcal{L}$ but hidden or spontaneously broken in dynamics
e.g., chiral sym. in QCD: $SU(2)_L \times SU(2)_R \xrightarrow{\text{SSB}} SU(2)_{\text{isospin}}$

(4) Spontaneous breaking of localized sym. $\longrightarrow$ Higgs mechanism in SM

1. Spontaneous breaking of a discrete sym.

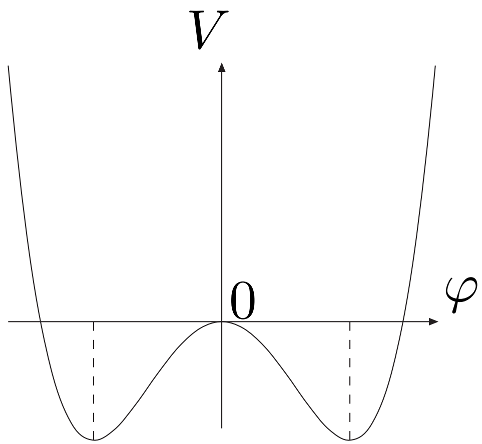
Real φ^4 with a mass term of wrong sign:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu\varphi)^2 + \frac{1}{2}\mu^2\varphi^2 - \frac{1}{4}\lambda\varphi^4,\ \mu^2 > 0,\ \lambda > 0;\ \text{invar. under } \varphi \to -\varphi \qquad (1) \\ \Rightarrow H &= \int d^3\mathbf{x}\,\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}\mu^2\varphi^2 + \frac{1}{4}\lambda\varphi^4\right) \end{aligned}$$

Field configuration of lowest energy:

$$\varphi(x) = \varphi_0 = \text{const.}$$

which is fixed by minimizing the potential



$$V(\varphi) = -\frac{1}{2}\mu^2\varphi^2 + \frac{1}{4}\lambda\varphi^4$$

$$\Rightarrow \begin{cases} \left.\dfrac{dV}{d\varphi}\right|_{\varphi_0} = 0 \to \varphi_0 = 0 \text{ or } \varphi_0^2 = \dfrac{\mu^2}{\lambda} \\ \left.\dfrac{d^2V}{d\varphi^2}\right|_{\varphi_0} = 3\lambda\varphi_0^2 - \mu^2 \end{cases}$$

$\varphi_0 = 0 :$ local maximum

$\varphi_0 = \pm\sqrt{\mu^2/\lambda} :$ local minima

Stable perturbation can only be built upon a minimum.

Expand φ at $\varphi = v = \sqrt{\mu^2/\lambda}$ (equivalent to expansion at $\varphi = -v$):

$$\varphi(x) = \sigma(x) + v \quad \text{vacuum expectation value (VEV)} \tag{2}$$

$$\Rightarrow \qquad \mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 - \mu^2\sigma^2 - \sqrt{\lambda\mu^2}\sigma^3 - \frac{1}{4}\lambda\sigma^4, \quad m_\sigma^2 = 2\mu^2 \tag{3}$$

No manifest discrete sym. in $\mathcal{L}$, but sym. hidden in relations of 3 terms.

2. Spontaneous breaking of a continuous sym.

Consider as an example the linear σ model of N real scalars:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi^i)^2 + \frac{1}{2}\mu^2(\varphi^i)^2 - \frac{1}{4}\lambda\big[(\varphi^i)^2\big]^2, \ (i = 1, \cdots, N \text{ summed}), \ \mu^2 > 0, \ \lambda > 0 \tag{4}$$

inva. under $O(N)$ group transf.:

$$\varphi^i \longrightarrow R^{ij}\varphi^j, \ R \in O(N): \ \text{rotation in } N \text{ dims} \tag{5}$$

Field configuration of lowest energy, $\varphi^i(x) = \varphi_0^i = \text{const.}$, is determined by minimizing the potential:

$$V(\varphi^i) = -\frac{1}{2}\mu^2(\varphi^i)^2 + \frac{1}{4}\lambda\big[(\varphi^i)^2\big]^2 \Rightarrow (\varphi_0^i)^2 = \frac{\mu^2}{\lambda}: \ \begin{array}{l}\text{length fixed but directions}\\ \text{in } N \text{ dims arbitrary}\end{array}$$

Must choose a direction about which we can do perturbation.

For an arbitrarily chosen φ_0^i, we can always rotate it into the N-th axis:

$$\varphi_0^i = (0, 0, \cdots, 0, v), \ v = \sqrt{\frac{\mu^2}{\lambda}} \tag{6}$$

Rename fields by separating out vacuum expectation value (VEV):

$$\varphi^i(x) = \big(\pi^k(x), v + \sigma(x)\big),\ \ k = 1, \cdots, N-1 \tag{7}$$

$$\begin{aligned}
\Rightarrow \quad \mathcal{L} = & + \Big[\frac{1}{2}(\partial_\mu \pi^k)^2 - \frac{1}{4}\lambda\big[(\pi^k)^2\big]^2\Big] \\
& + \Big[\frac{1}{2}(\partial_\mu \sigma)^2 - \mu^2\sigma^2 - \sqrt{\lambda\mu^2}\sigma^3 - \frac{1}{4}\lambda\sigma^4\Big] \quad \text{same as (3)} \\
& + \Big[- \sqrt{\lambda\mu^2}(\pi^k)^2\sigma - \frac{1}{2}\lambda(\pi^k)^2\sigma^2\Big] \quad \text{mixed interactions between } \sigma \ \&\ \pi
\end{aligned} \tag{8}$$

All $(N-1)$ π^k fields are massless while σ is massive!

Fate of symmetry:

$$\begin{array}{c} O(N) \text{ mixing all of } \pi^k \ \&\ \sigma \\ \text{in original } \mathcal{L} \text{ in (4)} \end{array} \xrightarrow[O(N-1) \text{ mixing } \pi^k]{\text{vacuum (6) only has}} O(N-1) \text{ in final } \mathcal{L} \text{ in (8)}$$

$N(N-1)/2$ generators $\qquad\qquad\qquad\qquad$ $(N-1)(N-2)/2$ generators

$$\begin{aligned}
\text{No. of generators for broken symmetries} &= N(N-1)/2 - (N-1)(N-2)/2 = N-1 \\
&= \text{No. of massless } \pi^k \ (\text{Nambu-Goldstone bosons})
\end{aligned}$$

Analogue in condensed matter: magnetization of a ferromagnet

$T > T_c$: no magnetization due to random orientations of spins
$\longrightarrow$ rotation sym.

$T < T_c$: magnetization in some definite direction which however is arbitrary
$\longrightarrow$ sym. hidden in lowest-energy state or SSB

3. Goldstone theorem

To each spontaneously broken continuous symmetry, corresponds a massless particle (Nambu-Goldstone particle) with the quantum numbers of the broken sym. current.

Points:

- Original $\mathcal{L}$ has a global continuous sym. G.
- States of lowest energy have non-vanishing VEV of fields that form a nontrivial rep. of G. These states transform into each other under G and are completely equivalent.
- To proceed, we must specify a unique vacuum above which excitations (particles after quantization) are analyzed for dynamics.

• Any choice of a vacuum will do. But once a choice made, the vacuum will not preserve G; but usually it still preserves a subgroup H of G.

• No. of broken symmetries = dim(G)-dim(H) = No. of NG particles

Consider

$$\mathcal{L} = (\text{derivative terms}) - V(\varphi), \text{ respecting sym. group } G \tag{9}$$

Configuration of lowest energy: $\varphi^a(x) = \varphi_0^a =$ const., i.e.,

$$\left.\frac{\partial V}{\partial \varphi^a}\right|_{\varphi_0^c} = 0, \quad \left.\frac{\partial^2 V}{\partial \varphi^a \partial \varphi^b}\right|_{\varphi_0^c} = m_{ab}^2 \text{ positive-semi-definite} \tag{10}$$

Expand V about φ_0:

$$V(\varphi) = V(\varphi_0) + 0 + \frac{1}{2}(\varphi - \varphi_0)^a (\varphi - \varphi_0)^b m_{ab}^2 + \cdots$$

will give mass^2 matrix
upon shifting VEV φ_0

Under an infinitesimal sym. transf.,

$$\varphi^a \longrightarrow \varphi^a + \alpha\Delta^a(\varphi) \qquad \alpha \ll 1 \qquad \Delta^a \text{depends on } G \text{ \& rep.} \tag{11}$$

$V(\varphi)$ is invar.; i.e., $V(\varphi) = V(\varphi + \alpha\Delta(\varphi))$, or to lowest order in α:

$$\Delta^a(\varphi)\frac{\partial V(\varphi)}{\partial \varphi^a} = 0 \xrightarrow[\varphi\to\varphi_0]{\partial/\partial\varphi^b} \left(\frac{\partial \Delta^a(\varphi)}{\partial \varphi^b}\right)_{\varphi_0} \left(\frac{\partial V(\varphi)}{\partial \varphi^a}\right)_{\varphi_0} + \Delta^a(\varphi_0) m^2_{ab} = 0 \quad \text{vanishing}$$

Thus (reminder: a summed)

$$\Delta^a(\varphi_0) m^2_{ab} = 0 \tag{12}$$

For some a, $\Delta^a(\varphi_0) = 0$, i.e., vacuum preserves that sym.
$\Rightarrow$ all those a's form a subgroup H of G: nothing out of (12).
For some a, $\Delta^a(\varphi_0) \neq 0$, i.e., vacuum breaks that sym.:

$$(12) \leftrightarrow m^2_{ba}\Delta^a(\varphi_0) = 0 \leftrightarrow \left(m^2\right)\begin{pmatrix}\vdots\end{pmatrix} = 0\begin{pmatrix}\vdots\end{pmatrix}, \text{ i.e., eigenvalue} = 0,$$

i.e., $\Delta^a(\varphi_0)$ is actually the eigenvector for the zero eigenvalue of (m^2_{ab}).
Degeneracy of zero eigenvalues $= \dim(G) - \dim(H)$.

4. Final remarks

(1) Above discussions are classical.

Read original papers for better understanding.

(2) Spontaneously broken sym. (or hidden sym.) is completely different from explicitly broken sym. Consider $\mathcal{L}$ in (8):

$(N-1)+1$ real fields with 5 interaction terms but with only 2 parameters!

Can you imagine it is renormalizable?

$\Rightarrow$ repeat calculations of Peskin!

8.2. The Higgs mechanism

1. The Abelian case

$U(1)$ gauge theory with a complex scalar:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\varphi)^\dagger(D^\mu\varphi) - V(\varphi^\dagger\varphi) \tag{13}$$

$$D_\mu\varphi = \partial_\mu\varphi + igA_\mu\varphi,\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$V = -\mu^2\varphi^\dagger\varphi + \frac{1}{2}\lambda(\varphi^\dagger\varphi)^2,\ \mu^2 > 0,\ \lambda > 0 \tag{14}$$

The original $\mathcal{L}$ has the gauge sym.:

$$\varphi(x) \longrightarrow e^{i\alpha(x)}\varphi(x),\ A_\mu(x) \longrightarrow A_\mu(x) - g^{-1}\partial_\mu\alpha(x) \tag{15}$$

But global $U(1)$ is spontaneously broken due to non-vanishing VEV:

$$\langle\varphi^\dagger\varphi\rangle \equiv \varphi_0^\dagger\varphi_0 = \mu^2/\lambda \tag{16}$$

Without loss of generality, we can choose (by applying a global $U(1)$ transf.)

$$\varphi_0 = \sqrt{\mu^2/\lambda} \tag{17}$$

Writing

$$\varphi(x) = \varphi_0 + \frac{1}{\sqrt{2}}\big(\varphi_1(x) + i\varphi_2(x)\big)$$

φ_1 becomes massive while φ_2 is massless (NGB from SSB of global $U(1)$):

$$V = \text{const.} + \mu^2\varphi_1^2 + \mathcal{O}(\varphi_i^3)$$

New feature after $U(1)$ is gauged is that gauge field becomes massive too

$$\begin{aligned}
D_\mu\varphi &= \frac{1}{\sqrt{2}}(\partial_\mu\varphi_1 + i\partial_\mu\varphi_2) + igA_\mu\varphi_0 + ig\frac{1}{\sqrt{2}}A_\mu(\varphi_1 + i\varphi_2) \\
&= \frac{1}{\sqrt{2}}(\partial_\mu\varphi_1 + igA_\mu\varphi_1) + \frac{i}{\sqrt{2}}(\partial_\mu\varphi_2 + igA_\mu\varphi_2) + igA_\mu\varphi_0 \\
(D_\mu\varphi)^\dagger(D^\mu\varphi) &= g^2\varphi_0^\dagger\varphi_0 A_\mu A^\mu + (\text{terms involving } \varphi_i) = g^2\frac{\mu^2}{\lambda}A_\mu A^\mu + (\cdots)
\end{aligned}$$

Therefore,

$$m_A^2 = 2g^2\frac{\mu^2}{\lambda} \qquad \text{Higgs mechanism}$$

An apparent problem with counting of physical degrees of freedom:

Before SSB, 2 real scalars + 2 helicity states of a massless gauge boson;

After SSB, 2 real scalars + 3 helicity states of a massive gauge boson?

Solution: after SSB, φ_2 is *not* an indept. dynamical field because

$$(D_\mu\varphi)^\dagger(D^\mu\varphi) = \frac{1}{2}(\partial_\mu\varphi_1)^2 + \frac{1}{2}(\partial_\mu\varphi_2)^2 + \frac{1}{2}m_A^2 A_\mu A^\mu + \sqrt{2}g\varphi_0 A_\mu\partial^\mu\varphi_2 + \text{(cubic/quartic terms)}$$

bilinear in fields, mixing A_μ and φ_2
to be diagonalized into a canonical form!

i.e., φ_2: would-be GB, is eaten up by gauge boson to become its longitudinal component.

Unitarity gauge

The easiest way to see the physical spectrum is to work in a special gauge. By choosing $\alpha(x)$ properly, we can make the transformed $\varphi(x)$ real:

$$\varphi \equiv \rho e^{i\theta}, \ \alpha = -\theta$$

$$\varphi \to e^{i\alpha}\varphi = \rho, \ A_\mu \to A_\mu + g^{-1}\partial_\mu\theta \equiv B_\mu$$

Then,

$$
\begin{aligned}
& D_\mu\varphi \to e^{-i\theta}D_\mu\varphi = e^{-i\theta}\left[e^{i\theta}(\partial_\mu\rho + i\rho\partial_\mu\theta) + ige^{i\theta}A_\mu\rho\right] = \partial_\mu\rho + i\rho g B_\mu \\
& F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \\
\Rightarrow \quad & \mathcal{L} = -\frac{1}{4}(B_{\mu\nu})^2 + (\partial_\mu\rho - i\rho g B_\mu)(\partial^\mu\rho + i\rho g B^\mu) - V(\rho^2) \\
& \quad = -\frac{1}{4}(B_{\mu\nu})^2 + (\partial_\mu\rho)^2 + g^2 B_\mu B^\mu \rho^2 - V(\rho^2) \qquad \text{no } \varphi_2 \text{ any more!} \qquad (18)
\end{aligned}
$$

For $\langle\rho^2\rangle \neq 0,\ B_\mu$ gets a mass: $m_B^2 = 2g^2\langle\rho^2\rangle$.

No bilinear mixing occurs between B_μ and scalars.

Actually, φ_2 disappears completely! Both ρ & B_μ are physical.

Point – Physical spectrum is manifest in unitarity gauge, but it is delicate or dangerous to calculate in unitarity gauge beyond tree level.

Most recent discussion concerning delicacy of unitarity gauge:

following citations to R. Gastmans et al, arXiv: 1108.5322 [hep-ph]

see also our earlier work: J.-P. Bu et al, PLB665 (2008)39, para following eq (16)

[finished in 3 units on Dec 07, 2012.]

2. The non-Abelian case

Example 1: SU(2) spontaneously broken by a scalar doublet VEV

Gauge fields: A_μ^a $(a = 1, 2, 3)$ in adj. rep. of SU(2)

Scalar fields: $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ in funda. rep., $D_\mu \varphi = \partial_\mu \varphi - igA_\mu^a \frac{\sigma^a}{2} \varphi$

Assume $\varphi^\dagger \varphi$ develops a VEV. Use global $SU(2)$ transf. to arrive at

$$\langle \varphi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad v > 0 \quad \text{conventional} \tag{19}$$

It breaks $SU(2)$ completely because no transf. can leave $\langle \varphi \rangle$ invariant, i.e., no generator can annihilate $\langle \varphi \rangle$: $t^i \langle \varphi \rangle = 0$.

Gauge fields get mass from the kinetic terms for φ:

$$D_\mu \varphi = D_\mu \underbrace{(\varphi - \langle \varphi \rangle)}_{\text{shifted fields without VEV}} - igA_\mu \frac{\sigma^a}{2} \langle \varphi \rangle = D_\mu (\varphi - \langle \varphi \rangle) - igA_\mu \frac{\sigma^a}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$
\begin{aligned}
(D_\mu\varphi)^\dagger(D^\mu\varphi) &= \frac{1}{2}g^2v^2A^a_\mu A^{b,\mu}(0,1)\frac{\sigma^a}{2}\frac{\sigma^b}{2}\begin{pmatrix}0\\1\end{pmatrix}+\cdots \\
&= \frac{1}{4}g^2v^2A^a_\mu A^{b,\mu}(0,1)\left\{\frac{\sigma^a}{2},\frac{\sigma^b}{2}\right\}\begin{pmatrix}0\\1\end{pmatrix}+\cdots \\
&= \frac{1}{8}g^2v^2A^a_\mu A^{a,\mu}+\cdots \\
\Rightarrow \qquad & m_A^2=\frac{1}{4}g^2v^2 \quad \text{for all 3 gauge bosons}
\end{aligned} \tag{20}
$$

Example 2: $SU(2)$ spontaneously broken by a real scalar triplet VEV

Gauge bosons as above. Scalar fields in adj. rep. as well: $(t^a_G)_{bc}=if^{bac}=i\epsilon^{bac}$

in components $\quad (D_\mu\varphi)_a=\partial_\mu\varphi_a+g\epsilon_{abc}A^b_\mu\varphi_c \quad \leftrightarrow \quad D_\mu\varphi=(\partial_\mu-igA^a_\mu t^a_G)\varphi \quad$ in matrix

Assume $\varphi^\dagger\varphi$ develops a VEV. Use global $SU(2)$ transf. to arrive at

$$
\langle\varphi\rangle=\begin{pmatrix}0\\0\\v\end{pmatrix}, \; v>0 \tag{21}
$$

Sym. tranfs. generated by t^3_G leave $\langle\varphi\rangle$ invar. because

$t^3_G\langle\varphi\rangle = i \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \langle\varphi\rangle = 0$. But those by t^1_G & t^2_G do not.

$\Rightarrow SU(2) \xrightarrow[\text{by triplet VEV}]{\text{SSB}} U(1)$: expect 2 massive and 1 massless gau. bosons

Gauge boson mass terms from $\frac{1}{2}(D_\mu\varphi)^\dagger(D^\mu\varphi)$:

$$\frac{1}{2}\Big(g\epsilon_{abc}A^b_\mu\langle\varphi_c\rangle\Big)^2 = \frac{1}{2}g^2v^2\Big(\epsilon_{ab3}A^b_\mu\Big)^2 = \frac{1}{2}g^2v^2\Big((A^1_\mu)^2 + (A^2_\mu)^2\Big)$$

$$\Longrightarrow \quad m^2_{A^1} = m^2_{A^2} = g^2v^2, \; m^2_{A^3} = 0 \quad \text{gau. boson for unbroken } U(1) \tag{22}$$

General case:

gau. group G, scalars φ in rep. r of G with generators t^a.

Assume φ develops a VEV, $\langle\varphi_\ell\rangle \equiv \varphi^0_\ell$.

Symmetries generated by t^i are unbroken if $t^i\langle\varphi\rangle = 0$.
They form a subgroup H of G because if $t^i\langle\varphi\rangle = 0$, $t^j\langle\varphi\rangle = 0$, we have $[t^i, t^j]\langle\varphi\rangle = 0 = if^{ijk}t^k\langle\varphi\rangle$, i.e.,$t^k\langle\varphi\rangle = 0$ as well. ?

Gauge boson mass terms:

$$
\begin{aligned}
&(D_\mu\varphi)^\dagger(D^\mu\varphi) \to \big(-igA_\mu^a t^a\langle\varphi\rangle\big)^\dagger\big(-igA_\mu^b t^b\langle\varphi\rangle\big) \equiv \frac{1}{2}m_{ab}^2 A_\mu^a A^{b,\mu} \\
&m_{ab}^2 = g^2(t^a\langle\varphi\rangle)^\dagger(t^b\langle\varphi\rangle) + (a \leftrightarrow b)
\end{aligned}
\tag{23}
$$

It is symmetric and positive semi-definite because for any real vector x^a,

$$
\frac{1}{2}x^T m^2 x = g^2(x^a t^a\langle\varphi\rangle)^\dagger(x^b t^b\langle\varphi\rangle) \geqslant 0
$$

Gauge bosons associated with generators t^i of subgroup H do not appear in the above and are thus massless.

Masses of all other gauge bosons are obtained by diagonalizing (m_{ab}^2).

3. Final remarks

Arranging scalars to develop a VEV is only one of ways to trigger SSB.

We can also do without introducing scalars:
e.g., chiral symmetry breaking in QCD $\to$ dynamical sym. breaking.

Its generalization to electroweak theories: technicolor models.

8.3. The Glashow-Weinberg-Salam theory of weak interactions (SM)

The history of weak interaction started with study of β decays of nuclei.

First successful phenomenological model at low energies:
4-Fermi interactions of LH fields.

GWS theory is a fundamental theory that

- unifies weak and EM interactions
- is renormalizable and quantum mechanically consistent, and
- passes all experimental tests up to now, in particular
  (1) a resonance at $\sim 125\ \mathrm{GeV}$ discovered recently at LHC, with properties similar to the Higgs particle, and
  (2) neutrino mass and mixing can be incorporated, albeit unnaturally.

Most important expts that verified SM include SLC, LEP, Tevatron, etc.

LHC (Large Hadron Collider) is exploring phys in SM and beyond.

Advice: learn SM by reproducing all following results

1. Basic structure of SM

Gauge group: $G_{\rm SM} = SU(2)_L \times U(1)_Y$ $\;(\,_Y:\;\text{hypercharge},\;\,_L:\;\text{LH})$

Fundamental fields form irre. reps. of $G_{\rm SM}$

gau. fields	:	A^a_μ of $SU(2)_L$, B_μ of $U(1)_Y$
fermions	:	doublets or singlets under $SU(2)_L$, carrying appropriate charges of $U(1)_Y$
scalars (Higgs)	:	responsible for SSB of $SU(2)_L \times U(1)_Y \longrightarrow U(1)_Q$ ($_Q$: electric charge)
		and via Higgs mechanism / Yukawa coupling giving mass to weak gau. bosons / fermions

(1) Gauge boson sector

$$\mathcal{L}_F = -\frac{1}{4}A^a_{\mu\nu}A^{a,\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{24}$$

$$A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_2\epsilon^{abc}A^b_\mu A^c_\nu \qquad g_2:\ \text{coupling of } SU(2)_L$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \qquad g_1:\ \text{coupling of } U(1)_Y$$

Mass terms forbidden! Theory of weak interactions not yet in sight.

(2) Higgs sector

$3+1$ gau. fields $\xrightarrow{\text{desired}}$ 3 massive for weak interaction $\rightarrow$ 3 would-be GB's needed
1 massless for EM

With 3 real scalars in adj. rep. of $SU(2)_L$ (see Example 2 in §8.2.2), we cannot reproduce phenomenology correctly.

Next simplest: 4 real scalars form a complex doublet of $SU(2)_L$:

$$\varphi = \begin{pmatrix} \varphi_+ \\ \varphi_0 \end{pmatrix}, \; Y = \frac{1}{2} \quad \text{(prepared for fermions later)} \quad \text{(both } \varphi_+ \text{ and } \varphi_0 \text{ complex)} \tag{25}$$

It transforms under $G_{\rm SM}$ as

$$\varphi \rightarrow \exp\left(i\alpha^a \frac{\sigma^a}{2}\right) \exp\left(i\frac{1}{2}\beta\right)\varphi, \quad SU(2)_L,\; U(1)_Y \text{ factors interchangeable} \tag{26}$$

The Lagrangian for φ is

$$\begin{aligned} \mathcal{L}_\varphi &= (D_\mu\varphi)^\dagger(D^\mu\varphi) + \mu^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2, \; \mu^2 > 0, \; \lambda > 0 \\ D_\mu\varphi &= \partial_\mu\varphi - ig_2 A_\mu^a \frac{\sigma^a}{2}\varphi - ig_1\frac{1}{2}B_\mu\varphi \end{aligned} \tag{27}$$

The 'wrong sign' of μ^2 term triggers SSB: $\langle\varphi^\dagger\varphi\rangle = \frac{1}{2}\frac{\mu^2}{\lambda} \equiv \frac{1}{2}v^2$.
Using a global transf. of $G_{\rm SM}$, we can always choose

$$\langle\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix},\ v>0 \tag{28}$$

VEV results in mass terms for gau. bosons via Higgs mechanism:

$$\begin{aligned}
D_\mu\varphi &= \left[-ig_2\begin{pmatrix}A^3_\mu & A^1_\mu - iA^2_\mu\\ A^1_\mu + iA^2_\mu & -A^3_\mu\end{pmatrix} - ig_1\begin{pmatrix}B_\mu & 0\\ 0 & B_\mu\end{pmatrix}\right]\frac{1}{2}\frac{v}{\sqrt{2}}\begin{pmatrix}0\\ 1\end{pmatrix} + \cdots\\
&= -i\frac{v}{2}\begin{pmatrix}\frac{1}{\sqrt{2}}(g_2A^3_\mu + g_1B_\mu) & g_2W^+_\mu\\ g_2W^-_\mu & \frac{1}{\sqrt{2}}(-g_2A^3_\mu + g_1B_\mu)\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} + \cdots\\
W^\pm_\mu &\equiv \frac{1}{\sqrt{2}}(A^1_\mu \mp iA^2_\mu)
\end{aligned} \tag{29}$$

Then, gau. boson mass terms are

$$+\frac{v^2}{4}\Big[W^+_\mu W^{-\mu}g_2^2 + \big(-g_2A^3_\mu + g_1B_\mu\big)^2\frac{1}{2}\Big]$$

i.e., A_μ^1, A_μ^2 (or $W_\mu^\pm$) and a linear combination of $(A_\mu^3,\ B_\mu)$ become massive.
Introduce orthogonal fields of definite mass (mass eigenstates):

$$\begin{pmatrix} Z_\mu^0 \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A_\mu^3 \\ B_\mu \end{pmatrix}, \qquad \underbrace{\sin\theta_W \equiv \frac{g_1}{\sqrt{g_1^2+g_2^2}}}_{\equiv s_W}, \quad \underbrace{\cos\theta_W \equiv \frac{g_2}{\sqrt{g_1^2+g_2^2}}}_{\equiv c_W} \tag{30}$$

θ_W : weak mixing angle or Weinberg angle

$\Rightarrow$ gau. boson mass terms

$$= +\frac{1}{2}m_Z^2(Z_\mu^0)^2 + m_W^2 W_\mu^+ W^{-\mu}, \qquad m_Z = \frac{1}{2}v\sqrt{g_1^2+g_2^2},\ m_W = \frac{1}{2}g_2 v,\ m_A = 0 \tag{31}$$

Comments:

- $m_W = m_Z c_W$, resulting from doublet scalar VEV, verified experimentally.
- $A_\mu^1,\ A_\mu^2 \to W_\mu^\pm$ for conventional QED interactions later.
- 3 massive gau. bosons for weak interactions
  1 massless gau. boson to be hopefully identified with QED photon

Would-be GB's

Expand original scalar field about our chosen VEV in eq. (28):

$$\varphi = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(iG^0 + h + v) \end{pmatrix}, \text{ containing 3 would-be GB's \& 1 physical scalar field} \tag{32}$$

The easiest way to separate would-be GB's from the physical scalar is by inspecting bilinear mixing terms between gauge fields and φ:

$$G^\pm: \text{ would-be GB's for } W^\pm_\mu; \quad G^0: \text{ would-be GB for } Z_\mu; \quad h: \text{ physical scalar field} \qquad \text{check!}$$

Plugging (32) into (27) generates all scalar-gau. boson interaction terms. To move to the unitary gauge, simply set $G^\pm = G^0 = 0$.

Unbroken sym.

$\langle\varphi\rangle$ in (28) seems to break all sym. of $SU(2)_L \times U(1)_Y$ because

$$\frac{\sigma^a}{2}\langle\varphi\rangle \neq 0, \ \frac{1}{2}\langle\varphi\rangle \neq 0$$

But actually,

$$\left(\frac{\sigma^3}{2}+\frac{1}{2}\right)\langle\varphi\rangle = \frac{1}{2}\begin{pmatrix}1 & \\ & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} = 0 \tag{33}$$

$$\Rightarrow \text{ sym. generated by } \frac{\sigma^3}{2}+\frac{1}{2}, \text{ i.e., } T^3+Y, \text{ is left unbroken!}$$

$$T^3: \text{ diagonal generator of } SU(2)_L; \; Y: \text{ generator of } U(1)_Y.$$

We wish it corresponds to the EM charge operator:

$$Q = T^3 + Y \qquad \text{Warning: sometimes using different normalization} + \frac{1}{2}Y \tag{34}$$

(3) Fermion sector

From 4-Fermi interactions, we know only LH quarks & leptons participate weak interactions via $W_\mu^\pm$-exchange. The simplest arrangement is

$$E_L \equiv \begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L, \; Q_L \equiv \begin{pmatrix}u\\ d\end{pmatrix}_L, \text{ both are doublets of } SU(2)_L$$

$$e_R^-, \; u_R, \; d_R \text{ as singlets of } SU(2)_L \text{ (hence the subscript } _L) \tag{35}$$

To reproduce the EM charges, we must choose Y according to (34):

$$T^3(u_L) = +\frac{1}{2},\ Q(u_L) = +\frac{2}{3} \Rightarrow Y(u_L) = +\frac{1}{6}$$

$$T^3(d_L) = -\frac{1}{2},\ Q(d_L) = -\frac{1}{3} \Rightarrow Y(d_L) = Y(u_L),\ \text{as required for components of same doublet}$$

$$T^3(e^-_L) = -\frac{1}{2},\ Q(e^-_L) = -1,\ T^3(\nu_{eL}) = \frac{1}{2},\ Q(\nu_{eL}) = 0 \Rightarrow Y(e^-) = Y(\nu_e) = -\frac{1}{2}$$

$$T^3(u_R) = T^3(d_R) = T^3(e^-_R) = 0 \Rightarrow Y(u_R) = Q(u_R) = +\frac{2}{3},\ Y(d_R) = Q(d_R) = -\frac{1}{3}$$

$$Y(e^-_R) = Q(e^-_R) = -1 \tag{36}$$

Then gau. covariant derivatives for fermions are fixed:

$$D_\mu \psi = (\partial_\mu - ig_2 T^a A^a_\mu - ig_1 Y B_\mu)\psi \tag{37}$$

$$\text{e.g.,} \qquad D_\mu Q_L = (\partial_\mu - ig_2 \frac{\sigma^a}{2} A^a_\mu - ig_1 \frac{1}{6} B_\mu) Q_L,$$

$$D_\mu u_R = (\partial_\mu - \quad 0 \quad - ig_1 \frac{2}{3} B_\mu) u_R,$$

$$D_\mu e_R = (\partial_\mu - \quad 0 \quad - ig_1 (-1) B_\mu) e_R$$

Organize terms in terms of physical gau. bosons $W_\mu^\pm,\ Z_\mu,\ A_\mu$:

$$D_\mu = \partial_\mu - ig_2 T^a A_\mu^a - ig_1 Y B_\mu \qquad T^\pm = T^1 \pm iT^2$$

$$\overset{(29,30)}{=} \partial_\mu - i\frac{g_2}{\sqrt{2}}(T^+W_\mu^+ + T^-W_\mu^-) - ig_2T^3(c_W Z_\mu^0 + s_W A_\mu) - ig_1 Y(-s_W Z_\mu^0 + c_W A_\mu)$$

$$\overset{(30,34)}{=} \partial_\mu - i\frac{g_2}{\sqrt{2}}(T^+W_\mu^+ + T^-W_\mu^-) - i\sqrt{g_1^2+g_2^2}(T^3 - Qs_W^2)Z_\mu^0 - ieQA_\mu \tag{38}$$

where EM coupling (absolute magnitude of electron charge) is defined as

$$e = g_1 c_W = g_2 s_W = g_1 g_2 (g_1^2 + g_2^2)^{-1/2},\ \text{i.e., } e^{-2} = g_1^{-2} + g_2^{-2} \tag{39}$$

$$\because T^\pm = \sigma^\pm,\ T^3 = \frac{1}{2}\sigma^3 \text{ for } (E_L, Q_L);\ T^\pm = T^3 = 0 \text{ for } (e_R, u_R, d_R);\ Q: \text{ same for } \psi_{L,R}$$

$$\therefore \mathcal{L}_\psi^{\text{k.e.}} = \sum_{\psi=u,d,e,\nu_e} \overline{\psi} i\not{D}\psi$$

$$= (\overline{\nu_{eL}} i\not\partial \nu_{eL} + \overline{e_L} i\not\partial e_L + \overline{u_L} i\not\partial u_L + \overline{d_L} i\not\partial d_L) + (\overline{e_R} i\not\partial e_R + \overline{u_R} i\not\partial u_R + \overline{d_R} i\not\partial d_R)$$
$$+ g_2\big(W_\mu^+ J_W^{+\mu} + W_\mu^- J_W^{-\mu} + Z_\mu^0 J_Z^\mu\big) + eA_\mu J_{\text{EM}}^\mu \tag{40}$$

weak interactions EM interactions

where the charged (CC) and neutral (NC) weak currents are

$$\begin{aligned} J_W^{+\mu} &= \frac{1}{\sqrt{2}}(\overline{\nu_L}\gamma^\mu e_L + \overline{u_L}\gamma^\mu d_L), \quad (J_W^{+\mu})^\dagger = J_W^{-\mu} \\ c_W J_Z^\mu &= \overline{\nu_L}\Big(\frac{1}{2}\Big)\gamma^\mu \nu_L + \overline{e_L}\Big(-\frac{1}{2} + s_w^2\Big)\gamma^\mu e_L + \overline{e_R}(s_w^2)\gamma^\mu e_R \\ &\quad + \overline{u_L}\Big(\frac{1}{2} - \frac{2}{3}s_w^2\Big)\gamma^\mu u_L + \overline{d_L}\Big(-\frac{1}{2} + \frac{1}{3}s_w^2\Big)\gamma^\mu d_L \\ &\quad + \overline{u_R}\Big(-\frac{2}{3}s_w^2\Big)\gamma^\mu u_R + \overline{d_R}\Big(\frac{1}{3}s_w^2\Big)\gamma^\mu d_R, \end{aligned} \tag{41}$$

and the EM current is

$$J^\mu_{\mathrm{EM}} = \overline{e}(-1)\gamma^\mu e + \overline{u}\Big(\frac{2}{3}\Big)\gamma^\mu u + \overline{d}\Big(-\frac{1}{3}\Big)\gamma^\mu d.$$

No ν_R in SM. If introduced, they would be neutral under $SU(2)_L \times U(1)_Y$.
[finished in 2.5 units on Dec 11, 2012.]

Fermion masses via Higgs mechanism and Yukawa coupling
φ transforms as a doublet of $SU(2)_L$ with $Y = +1/2$
$\Rightarrow$ So does $\tilde{\varphi} = i\sigma^2\varphi^*$ but with $Y = -1/2$ (see §3.2 QFT)

We can form terms involving fermions and φ or $\tilde{\varphi}$ via Yukawa coupling:

$$\begin{aligned} \mathcal{L}_\psi^{\text{Yukawa}} &= -\lambda_e \overline{E_L}\, \varphi\, e_R \;-\lambda_d \overline{Q_L}\, \varphi\, d_R - \lambda_u \overline{Q_L}\, \tilde{\varphi}\, u_R + \text{h.c.} \\ Y &: \quad +\frac{1}{2}+\frac{1}{2}-1 \quad -\frac{1}{6}+\frac{1}{2}-\frac{1}{3} \quad -\frac{1}{6}-\frac{1}{2}+\frac{2}{3} \end{aligned} \tag{42}$$

where *each term is inva.* under $G_{\rm SM}$. By redefining fields, $\lambda_{e,u,d}$ can be made real positive. After SSB, $\mathcal{L}_\psi^{\text{Yukawa}}$ contains the mass term:

$$\begin{aligned} \mathcal{L}_\psi^{\text{Yukawa}} &= -\lambda_e(\overline{\nu_{eL}}\ \ \overline{e_L})\frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} e_R - \lambda_d(\overline{u_L}\ \ \overline{d_L})\frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} d_R - \lambda_u(\overline{u_L}\ \ \overline{d_L})\frac{v}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} u_R \\ &\quad +\text{h.c.} + \cdots \\ &= -\frac{\lambda_e v}{\sqrt{2}}(\overline{e_L}e_R + \overline{e_R}e_L) - \frac{\lambda_d v}{\sqrt{2}}(\overline{d_L}d_R + \overline{d_R}d_L) - \frac{\lambda_u v}{\sqrt{2}}(\overline{u_L}u_R + \overline{u_R}u_L) + \cdots \\ &= -m_e\bar{e}e - m_d\bar{d}d - m_u\bar{u}u + \cdots \\ &\quad m_\psi = \lambda_\psi v/\sqrt{2},\ \ \psi = e, d, u \end{aligned} \tag{43}$$

$\mathcal{L}_\psi^{\text{Yukawa}}$ also contains interaction terms of fermions with $G^{\pm,0}$ and h which are proportional to fermion masses.

2. Anomaly cancellation

- Must include gauge currents of QCD (and those of gravity).
- Check cancellation either with gau. bosons before SSB (A^a_μ & B_μ) or with those after SSB ($W^\pm_\mu$, Z^0_μ & A_μ) because they are related by linear transf.

  Easier with A^a_μ & B_μ due to definite group properties.
- Anomaly indept. of fermion masses $\longrightarrow$ separate contri. from LH ($\frac{1}{2}(1-\gamma_5)$) & RH ($\frac{1}{2}(1+\gamma_5)$) fermion fields with a relative minus sign:

  $\mathrm{Tr}(\gamma_5\, t^a\{t^b, t^c\}) \triangleq 0,\ t^a$: generators of $SU(3)_{\text{color}} \times SU(2)_L \times U(1)_Y$
- Simplifications for triangle diagrams:

(1) $SU(3)_{\text{color}}$: vector-like; no anomaly for $[SU(3)_{\text{color}}]^3$

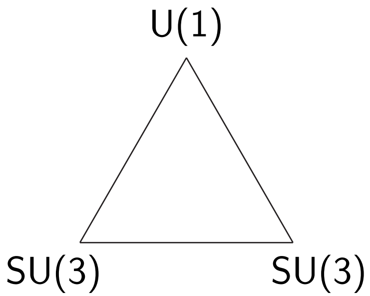
(2) $SU(2)$: anomaly free due to group structure; no anomaly for $[SU(2)_L]^3$

(3) diagrams with one $SU(3)_{\text{color}}$ vertex or one $SU(2)_L$ vertex anomaly free:

$\propto \mathrm{Tr}\ t^a = \mathrm{Tr}\ \lambda^a = 0$ for $SU(3)_{\mathrm{color}}$, and $\propto \mathrm{Tr}\ t^a = \mathrm{Tr}\ \frac{\sigma^a}{2} = 0$ for $SU(2)_L$

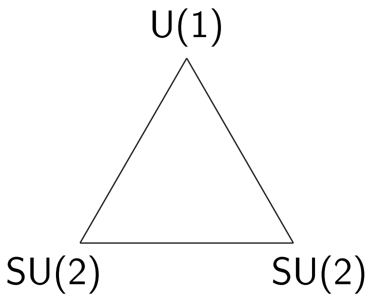
- Remaining diagrams to be checked are

(1)



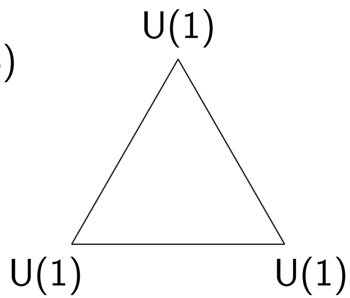
$$
\begin{aligned}
&\propto \quad \mathrm{Tr}\, t^a t^b Y = \frac{1}{2}\delta^{ab} \sum_q Y_q (-1)^{\mathsf{LH}} \\
&= \quad \frac{1}{2}\delta^{ab}\Big[\underset{\substack{\uparrow \\ u_L \,\&\, d_L}}{2} \cdot \underset{\substack{\uparrow \\ \mathsf{LH}}}{\Big(-\frac{1}{6}\Big)} + \underset{\substack{\uparrow \\ u_R}}{\frac{2}{3}} - \underset{\substack{\uparrow \\ d_R}}{\frac{1}{3}}\Big] = 0
\end{aligned}
$$

(2)



$$
\begin{aligned}
&\propto \quad \mathrm{Tr}\, \frac{\sigma^a}{2}\frac{\sigma^b}{2} Y = \frac{1}{2}\delta^{ab} \sum_{\psi_L} Y_{\psi_L} \times (-1) \\
&= \quad \frac{1}{2}\delta^{ab}\Big[\underset{\substack{\uparrow \\ E_L}}{\Big(-\frac{1}{2}\Big)} + \underset{\substack{\uparrow \\ N_c}}{3} \cdot \underset{\substack{\uparrow \\ Q_L}}{\frac{1}{6}}\Big] \times \underset{\substack{\uparrow \\ \mathsf{LH}}}{(-1)} = 0
\end{aligned}
$$

(3)



$$\propto \quad \mathrm{Tr}\, Y^3$$

$$= \underset{\substack{\uparrow \\ \nu_{eL}\ \&\ e_L}}{\overset{\substack{\text{LH} \\ \downarrow}}{-2}}\Big(-\frac{1}{2}\Big)^3 + \underset{\substack{\uparrow \\ e_R}}{(-1)^3} + \underset{\substack{\uparrow \\ N_c}}{3}\Big[\underset{\substack{\uparrow \\ u_L\ \&\ d_L}}{\overset{\substack{\text{LH} \\ \downarrow}}{-2}} \times \Big(\frac{1}{6}\Big)^3 + \underset{\substack{\uparrow \\ u_R}}{\Big(\frac{2}{3}\Big)^3} + \underset{\substack{\uparrow \\ d_R}}{\Big(-\frac{1}{3}\Big)^3}\Big] = 0$$

Important: anomaly cancellation occurs only for a complete generation of leptons & quarks!

3. Quark mixing for 3 generations: CKM matrix [skipped in 2008, 2009]

We know experimentally 6 leptons and 6 quarks classified into 3 generations (families): $\nu_e,\ e,\ u,\ d;\ \nu_\mu,\ \mu,\ c,\ s;\ \nu_\tau,\ \tau,\ t,\ b$

From theoretical point of view, quarks of same charge can mix $\to$ CKM
so can leptons of same charge (in extended SM) $\to$ PMNS

Quark mixing

Denote original quark fields (forming irre. rep. of G_{SM}, gauge eigenstates) by a prime.

$$\begin{aligned}
\mathcal{L}_{\psi}^{\text{Yukawa}} &= -\lambda_d^{ij}\overline{Q_L'^i}\varphi d_R'^j - \lambda_u^{ij}\overline{Q_L'^i}\tilde{\varphi} u_R'^j + (\text{leptons}) + \text{h.c.} \quad i,j = 1,2,3 \text{ for generations} \\
&= -\frac{v}{\sqrt{2}}\lambda_d^{ij}\overline{d_L'^i}d_R'^j - \frac{v}{\sqrt{2}}\lambda_u^{ij}\overline{u_L'^i}u_R'^j + \text{h.c.} + \cdots + (\text{leptons}) \qquad (44)
\end{aligned}$$

Generally, λ_d, λ_u are not diagonal. Denote mass matrices in primed basis:

$$m_d' = \frac{v}{\sqrt{2}}\lambda_d,\ m_u' = \frac{v}{\sqrt{2}}\lambda_u \qquad (45)$$

An arbitrary complex matrix can always be diagonalized by *biunitary* transfs to a real positive semi-definite form:

$$U_{dL}^{\dagger} m_d' U_{dR} \equiv m_d,\ U_{uL}^{\dagger} m_u' U_{uR} \equiv m_u, \qquad (46)$$

where $m_d = \text{diag}(m_d, m_s, m_b)$ and $m_u = \text{diag}(m_u, m_c, m_t)$.
The transformations amount to

$$u_L' = U_{uL}u_L,\ u_R' = U_{uR}u_R,\ d_L' = U_{dL}d_L,\ d_R' = U_{dR}d_R,$$

and the mass matrices are diagonal and positive semi-definite in the basis of unprimed fields (mass eigenstates):

$$\mathcal{L}_\psi^{\text{Yukawa}} = -\sum_{q=u,c,t}(m_q\overline{q_L}q_R + \text{h.c.}) - \sum_{q=d,s,b}(m_q\overline{q_L}q_R + \text{h.c.}) + \cdots + (\text{leptons})$$

What else changes in $\mathcal{L}_\psi$?

k.e. terms: $\overline{\psi}i\not{D}\psi$

$$D_\mu \overset{(38)}{=} \underbrace{\partial_\mu}_{\substack{\uparrow \\ \psi_L \text{ of same charge} \\ \psi_R \text{ of } \cdots}} - i\frac{g_2}{\sqrt{2}}\underbrace{(T^+W_\mu^+ + T^-W_\mu^-)}_{\substack{\uparrow \\ \text{relating } u_L \text{ to } d_L}} - i\sqrt{g_1^2+g_2^2}\underbrace{(T^3 - Qs_w^2)Z_\mu^0}_{\substack{\uparrow \\ \text{relating } u_L \text{ of same } T_3 \,\&\, Q \\ d_L \text{ of } \cdots \\ u_R \text{ of same } Q \\ d_R \text{ of } \cdots}} - \underbrace{ieQA_\mu}_{\substack{\nwarrow \\ \text{relating } u_{L,R} \text{ to } u_{L,R} \\ d_{L,R} \text{ to } d_{L,R}}}$$

In all terms except the $W_\mu^\pm$, we always have the structure:

$$\overline{u'_L}\cdots u'_L = \overline{u_L}\cdots U_{uL}U_{uL}^\dagger u_L = \overline{u_L}\cdots u_L, \text{ etc.}$$

i.e., U_{uL}, U_{dL}, U_{uR}, U_{dR} disappear from all these terms.

For $W_\mu^\pm$ terms, we have

$$+\frac{g_2}{\sqrt{2}} W_\mu^+ \overline{u_L'^i} \gamma^\mu d_L'^i + \text{h.c.} = +\frac{g_2}{\sqrt{2}} W_\mu^+ \overline{u_L^i} \gamma^\mu d_L^j (U_{uL}^\dagger U_{dL})_{ij} + \text{h.c.}$$

Generally,

$$V_{\text{CKM}} \equiv U_{uL}^\dagger U_{dL} \neq \mathbf{1}_{3\times 3}, \text{ Cabibbo-Kobayashi-Maskawa}$$

A general 3×3 unitary matrix has 3^2 real parameters: 3 mixing angles, 6 phases. But V_{CKM} has only 4 physical parameters: 3 mixing angles and 1 CP-violating phase; other phases nonphys. Vital difference to 2 families!

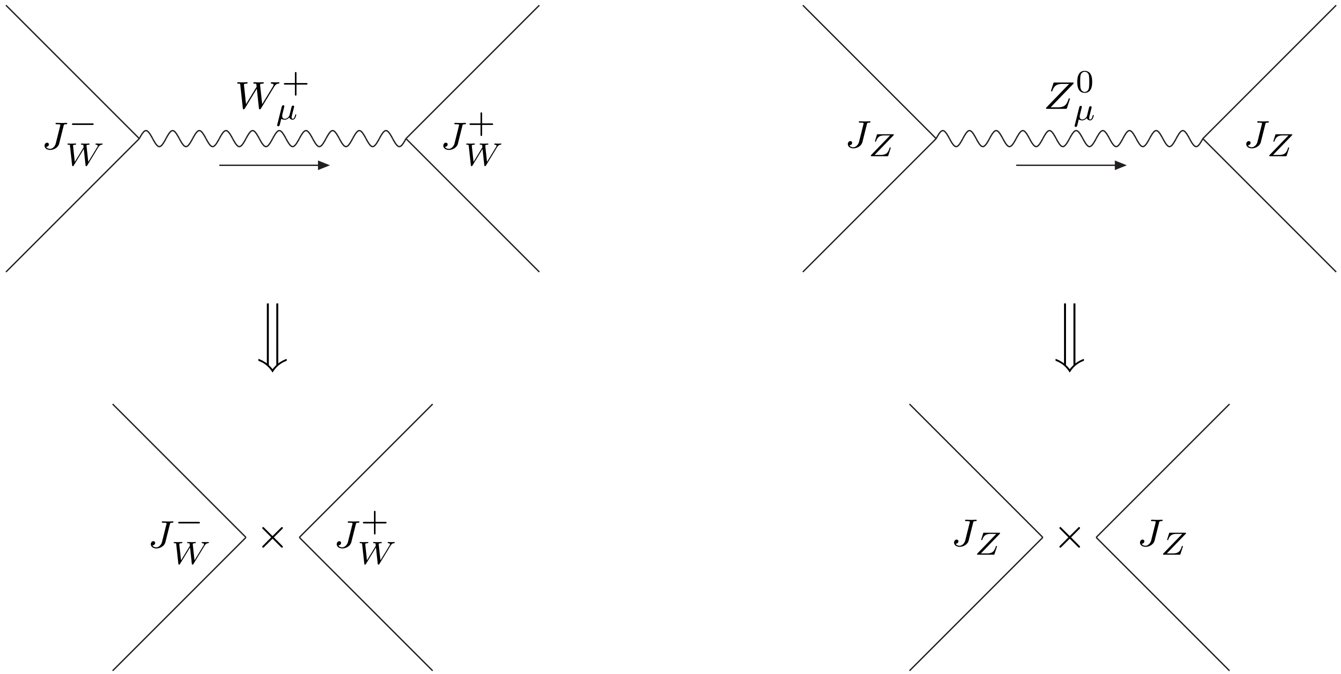
Yukawa terms:

V_{CKM} appears in $u - d - G^\pm$ couplings but not in others.

4. Low energy approximation of weak interactions

Basic $\psi - \psi - \text{gau. boson}$ interactions are shown in (40), (41).
For $|\text{(momentum transfer)}^2| \ll m_{W,Z}^2$, approximate gau. boson propagators:

$$\frac{-ig_{\mu\nu}}{k^2 - m_{W,Z}^2} \approx \frac{-ig_{\mu\nu}}{-m_{W,Z}^2}$$



$$\begin{aligned}
i\mathcal{A}_W &\approx ig_2 J_W^{+\mu} \frac{ig_{\mu\nu}}{m_W^2} ig_2 J_W^{-\nu} = -\frac{ig_2^2}{m_W^2} J_W^{+\mu} J_{W,\mu}^- \\
i\mathcal{A}_Z &\approx ig_2 J_Z^{\mu} \frac{ig_{\mu\nu}}{m_Z^2} ig_2 J_Z^{\nu} = -\frac{ig_2^2}{m_Z^2} J_Z^{\mu} J_{Z,\mu}
\end{aligned}$$

They can be reproduced by the effective 4-Fermi interactions at low energies:

$$
\begin{aligned}
\mathcal{L}_W^{\rm eff} &= -\frac{g_2^2}{m_W^2}\Big(\frac{1}{\sqrt{2}}\Big)^2\Big[(\overline{\nu_{eL}}\gamma^\mu e_L + \cdots) + (V_{\rm CKM})_{ij}\overline{u_L^i}\gamma^\mu d_L^j\Big] \\
&\times \Big[(\overline{e_L}\gamma_\mu \nu_{eL} + \cdots) + (V_{\rm CKM})^*_{ij}\overline{d_L^j}\gamma_\mu u_L^i\Big] \\
\mathcal{L}_Z^{\rm eff} &= -\frac{g_2^2}{m_Z^2}\Big(\frac{1}{c_W}\Big)^2\Big[\overline{\nu_{eL}}\frac{1}{2}\gamma^\mu \nu_{eL} + \cdots\Big]\Big[\overline{\nu_{eL}}\frac{1}{2}\gamma_\mu \nu_{eL} + \cdots\Big]
\end{aligned}
$$

Consequences

$*$ $\dfrac{G_F}{\sqrt{2}} = \dfrac{g_2^2}{8m_W^2}$, compared to eq.(6.15).

$*$ SM predicts NC 4-Fermi interactions that were later confirmed in DIS.

8.4. The R_ξ gauges

Previous discussion of SSB gau. theories is basically classical.
Now we do quantization using Faddeev-Popov trick in PI.
Main goal: derive propagators for gau. bosons and would-be GB's in SM.

1. Abelian case

Consider SSB $U(1)$ gau. theory of a complex scalar, defined in eqns. (13-18):

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\varphi)^\dagger(D^\mu\varphi) - V(\varphi^\dagger\varphi) \tag{13}$$

$$D_\mu\varphi = \partial_\mu\varphi + igA_\mu\varphi,\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$V = -\mu^2\varphi^\dagger\varphi + \frac{1}{2}\lambda(\varphi^\dagger\varphi)^2,\ \mu^2 > 0,\ \lambda > 0 \tag{14}$$

In terms of real scalars introduced by $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$, infin. gau. transf. is:

$$\delta\varphi_1 = -\alpha\varphi_2,\ \delta\varphi_2 = \alpha\varphi_1,\ \delta A_\mu = -\frac{1}{g}\partial_\mu\alpha \tag{51}$$

$\langle\varphi\rangle = v/\sqrt{2} \neq 0$ triggers SSB.

Introduce physical field h and would-be GB π via

$$\varphi_1 = v + h,\ \varphi_2 = \pi \tag{52}$$

Then, (13) and (51) become

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}(\partial_\mu h - gA_\mu \pi)^2 + \frac{1}{2}\big(\partial_\mu \pi + gA_\mu(v+h)\big)^2 - V(\varphi^\dagger\varphi) \tag{53}$$

$$\delta h = -\alpha\pi,\ \delta\pi = \alpha(v+h),\ \delta A_\mu = -\frac{1}{g}\partial_\mu\alpha \tag{54}$$

Since (53) has sym. (54), we must do gau. fixing in PI quantization:

$$\begin{aligned}
Z &= \int \mathcal{D}A\mathcal{D}h\mathcal{D}\pi\ e^{i\int \mathcal{L}[A,h,\pi]} \\
&\quad \Big\downarrow \text{ splitting infinite factor of integral in group space, by inserting a gau. fixing condition} \\
&= \text{const.}\cdot \int \mathcal{D}A\mathcal{D}h\mathcal{D}\pi\ e^{i\int \mathcal{L}[A,h,\pi]}\ \underbrace{\delta(G[A,h,\pi])}_{\text{gau. fixing}} \det\Big(\underbrace{\frac{\delta G(A^\alpha,\cdots)}{\delta\alpha}}_{\text{variation under (54)}}\Big) \\
&\quad \Big\downarrow \text{ Gaussian weighting over gau. fixing} \\
&= \text{const.}\cdot \int \mathcal{D}A\mathcal{D}h\mathcal{D}\pi\ \exp i\int\Big(\mathcal{L}[A,h,\pi] - \frac{1}{2}(G)^2\Big)\ \det\Big(\frac{\delta G(A^\alpha,\cdots)}{\delta\alpha}\Big)
\end{aligned} \tag{55}$$

How to choose G?

In principle arbitrary, but in practice better to facilitate calculations.

Two main concerns:

$*$ obtain a propagator for gau. bosons $\to$ natural generalization of QED

$*$ remove quadratic mixing between gau. bosons & would-be GB in $\mathcal{L}$

$$\therefore G = \frac{1}{\sqrt{\xi}}(\partial_\mu A^\mu - \xi g v \pi) \tag{56}$$

G^2 modifies quadratic terms in $\mathcal{L}$ so that

$$\begin{aligned} & \mathcal{L} - \frac{1}{2}G^2 \\ = \quad & -\frac{1}{2}A_\mu\Big[-g^{\mu\nu}\partial^2 + (1-\xi^{-1})\partial^\mu\partial^\nu - m_A^2 g^{\mu\nu}\Big]A_\nu \longleftarrow \text{integration by parts} \\ & +\Big[\frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}m_h^2 h^2\Big] + \Big[\frac{1}{2}(\partial_\mu \pi)^2 - \frac{1}{2}\xi m_A^2 \pi^2\Big] + m_A \partial_\mu(A^\mu \pi) \end{aligned} \tag{57}$$

would-be GB π has mass: $m_\pi^2 = \xi m_A^2$ total derivative ignorable

$+$(higher order terms)

Propagators:

$$\text{- - - - - - - - - -}\overset{h}{\underset{p}{}}\ = \frac{i}{p^2 - m_h^2 + i\epsilon} \qquad \text{- - - - - - - - - -}\overset{\pi}{\underset{p}{}}\ = \frac{i}{p^2 - m_\pi^2 + i\epsilon}$$

Quadratic terms in A_μ imply 1PI 2-point vertex of A_μ–A_ν in momen. space:

$$-i\Big[g^{\mu\nu}p^2 - (1 - \xi^{-1})p^\mu p^\nu - m_A^2 g^{\mu\nu}\Big] \quad \text{k.e., gauge fixing, mass}$$

$$= \quad -i\Big\{\Big(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\Big)(p^2 - m_A^2) + p^\mu p^\nu(\xi^{-1} - \frac{m_A^2}{p^2})\Big\} \equiv -i\Gamma^{\mu\nu}$$

whose inverse should give the minus propagator $-\tilde{D}_F^{\mu\nu}(p)$:

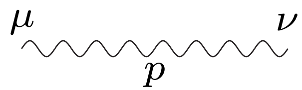
$$(-i\Gamma^{\mu\nu})(-\tilde{D}_F)_{\nu\alpha} = g^\mu_\alpha$$

To find $\tilde{D}_F$, decompose it in the form

$$(\tilde{D}_F)_{\nu\alpha} = -i\Big[\Big(g_{\nu\alpha} - \frac{p_\nu p_\alpha}{p^2}\Big)A(p^2) + \frac{p_\nu p_\alpha}{p^2}B(p^2)\Big]$$

and plug it into the above eqn:

$$
\begin{aligned}
&\Big(g^\mu_\alpha - \frac{p^\mu p_\alpha}{p^2}\Big)(p^2 - m_A^2)A(p^2) + \frac{p^\mu p_\alpha}{p^2}(\xi^{-1} - \frac{m_A^2}{p^2})B(p^2) = g^\mu_\alpha \\
\Rightarrow\quad & A(p^2) = (p^2 - m_A^2)^{-1},\ B(p^2) = \xi(p^2 - \xi m_A^2)^{-1} \\
\Rightarrow\quad & \tilde{D}_F^{\mu\nu}(p) = -i\Big[\Big(g^{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big)\frac{1}{p^2 - m_A^2} + \frac{\xi}{p^2 - \xi m_A^2}\frac{p^\mu p^\nu}{p^2}\Big] \\
& \qquad\qquad = -i\left[g^{\mu\nu} + (\xi - 1)\frac{p^\mu p^\nu}{p^2 - \xi m_A^2}\right]\frac{1}{p^2 - m_A^2} \quad \longleftrightarrow
\end{aligned}
$$



(58)

Ghosts

Back to (55): represent $\det\Big(\frac{\delta G(A^\alpha, \cdots)}{\delta\alpha}\Big)$ by the Gaussian integral of the Grassmannian fields c, $\bar{c}$ (ghosts),

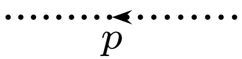
$$G(A^\alpha, \cdots) = \frac{1}{\sqrt{\xi}}\Big[\partial_\mu (A_{\alpha})^\mu - \xi m_A \pi_\alpha\Big] \quad \alpha \text{ refers to fields after infin. transf. in (54)}$$

$$\Rightarrow\quad \frac{\delta G(A^\alpha, \cdots)}{\delta\alpha} = \frac{1}{\sqrt{\xi}}\Big[-g^{-1}\partial^2 - \xi m_A(v+h)\Big] = +\frac{1}{g\sqrt{\xi}}\Big[-\partial^2 - \xi g m_A(v+h)\Big]$$

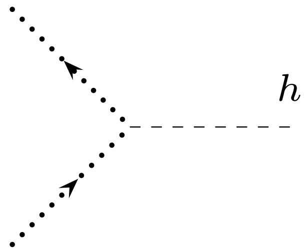
Up to an irrelevant const. factor in 1PI, det is reproduced by PI over c, $\bar{c}$ of an additional term in $\mathcal{L}$:

$$\mathcal{L}_{\text{ghost}} = \bar{c}\Big[-\partial^2 - \xi m_A^2 (1 + h/v) \Big] c \tag{59}$$

$\longleftrightarrow$



$$= \frac{i}{p^2 - \underbrace{\xi m_A^2}_{m_\pi^2} + i\epsilon}$$



$$= -i\xi \frac{m_A^2}{v} \tag{60}$$

Summary

$*$ Our final working Lagrangian is

$$\mathcal{L}_{\text{eff}} = (53) - \frac{1}{2} G^2 + \mathcal{L}_{\text{ghost}} \tag{61}$$

$*$ Unphysical poles at $m_\pi^2 = \xi m_A^2$ in propagators of A_μ, π, c must not contribute to S-matrix $\leftrightarrow$ S-matrix is indept. of ξ

* Polarization sum of massive gau. bosons

  Physical spectrum is manifest in unitarity gauge ($\xi \to \infty$) where unphysical states decouple due to their infinite mass.

  The gau. boson propagator in unitarity gauge becomes

$$\frac{i}{p^2 - m_A^2}\Big(\underbrace{-\, g^{\mu\nu} + \frac{p^\mu p^\nu}{m_A^2}} \Big) \tag{62}$$

$\uparrow$
pol. sum for a physical gau. boson:

$$\sum \varepsilon^\mu(p)\varepsilon^{\nu *}(p) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m_A^2}$$

* R_ξ gauge: suitable for loop computation due to better high-p behaviour of the gau. boson propagator than in unitarity gauge.

  Especially convenient is the $\xi = 1$ gauge ('t-Hooft-Feynman gauge)

[almost finished in 3 units on Dec 21, 2012.]

2. Non-Abelian case: SM

Discussion for the Abelian case can be generalized to the non-Abelian case. All formal manipulations are similar.

We concentrate on SM. From §8.3 (especially the Higgs sector), the bilinear mixing terms between gau. bosons and scalars are

$$\begin{aligned}
& \partial^\mu \varphi^\dagger D_\mu \langle\varphi\rangle + \text{h.c.} \\
= \quad & \partial^\mu \varphi^\dagger \Big[- i g_2 A^a_\mu \frac{\sigma^a}{2} \langle\varphi\rangle - i g_1 \frac{1}{2} B_\mu \langle\varphi\rangle \Big] + \text{h.c.} \\
= \quad & i g_2 A^a_\mu \Big(\langle\varphi\rangle^\dagger \frac{\sigma^a}{2} \partial^\mu \varphi - \partial^\mu \varphi^\dagger \frac{\sigma^a}{2} \langle\varphi\rangle \Big) + i g_1 B_\mu \Big(\frac{1}{2} \langle\varphi\rangle^\dagger \partial^\mu \varphi - \frac{1}{2} \partial^\mu \varphi^\dagger \langle\varphi\rangle \Big)
\end{aligned}$$

which is equivalent in the action to

$$-i g_2 (\partial^\mu A^a_\mu) \Big(\langle\varphi\rangle^\dagger \frac{\sigma^a}{2} \varphi - \varphi^\dagger \frac{\sigma^a}{2} \langle\varphi\rangle \Big) - i g_1 (\partial^\mu B_\mu) \Big(\frac{1}{2} \langle\varphi\rangle^\dagger \varphi - \frac{1}{2} \varphi^\dagger \langle\varphi\rangle \Big)$$

Gau. fixing functions are chosen to cancel above mixing terms in final $\mathcal{L}$:

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{2} (G^a)^2 - \frac{1}{2} (G)^2 \tag{63}$$

where

$$
\begin{aligned}
G^a &= \frac{1}{\sqrt{\xi}}\Big[\partial^\mu A^a_\mu - ig_2\xi\Big(\langle\varphi\rangle^\dagger\frac{\sigma^a}{2}\varphi - \varphi^\dagger\frac{\sigma^a}{2}\langle\varphi\rangle\Big)\Big] \\
G &= \frac{1}{\sqrt{\xi}}\Big[\partial^\mu B_\mu - ig_1\xi\Big(\frac{1}{2}\langle\varphi\rangle^\dagger\varphi - \frac{1}{2}\varphi^\dagger\langle\varphi\rangle\Big)\Big] \qquad (64)
\end{aligned}
$$

To find ghost terms, consider variation of G^a, G under infin. gau. transf.:

$$
SU(2)_L:\ A^a_\mu \to (A_{\alpha a})^a_\mu = A^a_\mu + g_2^{-1}\partial_\mu\alpha^a - \epsilon^{abc}\alpha^b A^c_\mu, \quad \varphi \to \varphi_{\alpha a} = \varphi + i\alpha^a\frac{\sigma^a}{2}\varphi \qquad (65)
$$

$$
U(1)_Y:\ B_\mu \to (B_\alpha)_\mu = B_\mu + g_1^{-1}\partial_\mu\alpha, \quad \varphi \to \varphi_\alpha = \varphi + i\frac{1}{2}\alpha\varphi \qquad (66)
$$

where φ includes $\langle\varphi\rangle$. Thus,

$$
\begin{aligned}
\frac{\delta G^a(A_\alpha,\cdots)}{\delta\alpha^b} &= \frac{1}{\sqrt{\xi}}\Big\{g_2^{-1}\partial^2\delta^{ab} - \epsilon^{abc}\partial^\mu A^c_\mu - ig_2\xi\Big(\langle\varphi\rangle^\dagger\frac{\sigma^a}{2}i\frac{\sigma^b}{2}\varphi + \varphi^\dagger i\frac{\sigma^b}{2}\frac{\sigma^a}{2}\langle\varphi\rangle\Big)\Big\} \equiv \frac{-X^{ab}}{g_2\sqrt{\xi}} \\
\frac{\delta G^a(A_\alpha,\cdots)}{\delta\alpha} &= -ig_2\sqrt{\xi}\frac{i}{2}\Big(\langle\varphi\rangle^\dagger\frac{\sigma^a}{2}\varphi + \varphi^\dagger\frac{\sigma^a}{2}\langle\varphi\rangle\Big) = \frac{1}{2}g_2\sqrt{\xi}\Big(\langle\varphi\rangle^\dagger\frac{\sigma^a}{2}\varphi + \varphi^\dagger\frac{\sigma^a}{2}\langle\varphi\rangle\Big) \equiv \frac{-X^a}{g_1\sqrt{\xi}}
\end{aligned}
$$

$$\frac{\delta G(B_\alpha,\cdots)}{\delta\alpha^b} = -ig_1\sqrt{\xi}\frac{i}{2}\Big(\langle\varphi\rangle^\dagger\frac{\sigma^b}{2}\varphi + \varphi^\dagger\frac{\sigma^b}{2}\langle\varphi\rangle\Big) \equiv \frac{-X^b}{g_2\sqrt{\xi}} \tag{67}$$

$$\frac{\delta G(B_\alpha,\cdots)}{\delta\alpha} = \frac{1}{\sqrt{\xi}}\Big\{g_1^{-1}\partial^2 - ig_1\xi\Big(\frac{1}{2}\langle\varphi\rangle^\dagger\frac{i}{2}\varphi + \frac{1}{2}\varphi^\dagger\frac{i}{2}\langle\varphi\rangle\Big)\Big\} \equiv \frac{-X}{g_1\sqrt{\xi}}$$

where all ∂ acts on rhs as a whole.

To each indept. group parameter is associated a ghost pair: $(c^a,\ \bar{c^a}),\ (c,\ \bar{c})$.

Up to an irrelevant const. in PI, det is reproduced by $\mathcal{L}_{\mathsf{FP}}$ terms in final $\mathcal{L}$:

$$\mathcal{L}_{\mathsf{FP}} = (\bar{c}^a,\ \bar{c})\begin{pmatrix} X^{ab} & X^a \\ X^b & X \end{pmatrix}\begin{pmatrix} c^b \\ c \end{pmatrix} \tag{68}$$

Our final $\mathcal{L}$ is the sum of those in §8.3 and $\mathcal{L}_{\mathsf{g.f.}} + \mathcal{L}_{\mathsf{FP}}$.

To get ready for application, we must express

$A_\mu^a,\ B_\mu$ in terms of $W_\mu^\pm,\ Z_\mu^0,\ A_\mu$

$c^a,\ c$ in terms of $c^\pm,\ c^Z,\ c^A$,

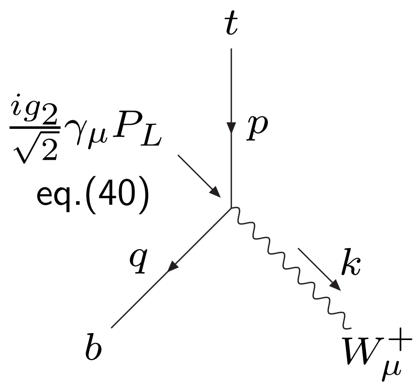
$\bar{c}^a,\ \bar{c}$ in terms of $\bar{c}^\pm,\ \bar{c}^Z,\ \bar{c}^A$.

8.5. Examples of calculations in SM

SM is a quantum mechanically consistent theory that can be tested order by order in perturbation theory against measurements.

Following are a few examples of its prediction verified by experiments.

1. Top quark decay



$m_t \sim 170\mathrm{GeV},\ m_W \sim 80\mathrm{GeV},\ m_b \sim 5\mathrm{GeV}$
t decays dominantly to b quark and W^+ boson.
Naively, we expect the decay rate is of order

$$\Gamma \sim \frac{g_2^2}{4\pi} m_t, \text{ for } m_t \gg m_W \gg m_b$$

Now compute it at the lowest order in pert. theory:

$$i\mathcal{M} \quad = \quad \frac{ig_2}{\sqrt{2}} \underbrace{\bar{u}(q)}_{b} \gamma^\mu P_L \underbrace{u(p)}_{t} \underbrace{\varepsilon^*_\mu(k)}_{W^+} \qquad P_L = \frac{1}{2}(1-\gamma_5)$$

Approximations: 1) $V_{\rm CKM} = \mathbf{1}_3$, 2) $m_b = 0$

$$
\begin{aligned}
\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{1}{2}\Big(\frac{g_2}{\sqrt{2}}\Big)^2 \mathrm{Tr}[(\not{p} + m_t)\gamma^\nu P_L \not{q}\gamma^\mu P_L] \sum_\varepsilon \varepsilon^*_\mu(k)\varepsilon_\nu(k) \\
&= \frac{g_2^2}{4}\cdot \mathrm{Tr}[\not{p}\gamma^\nu\not{q}\gamma^\mu P_L + 0]\cdot\Big(- g^{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}\Big) \qquad {\color{red} k^2 = m_W^2} \\
&= \frac{g_2^2}{4}\cdot 2(p^\mu q^\nu + p^\nu q^\mu - g^{\mu\nu} p\cdot q)\cdot\Big(- g^{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}\Big) \\
&= \frac{1}{2} g_2^2 \Big(p\cdot q + \frac{2(k\cdot p)(k\cdot q)}{m_W^2}\Big) \qquad {\color{blue} \begin{array}{l} p = k + q \\ 2k\cdot p = (m_t^2 + m_W^2) \\ 2k\cdot q = 2p\cdot q = (m_t^2 - m_W^2) \end{array}} \\
&= \frac{1}{4} g_2^2 [(m_t^2 - m_W^2) + m_W^{-2}(m_t^2 + m_W^2)(m_t^2 - m_W^2)] \\
&= \frac{1}{4} g_2^2 (m_t^2 - m_W^2)\Big(2 + \frac{m_t^2}{m_W^2}\Big) \\
&= \frac{1}{4} g_2^2 \frac{m_t^4}{m_W^2}\Big(1 - \frac{m_W^2}{m_t^2}\Big)\Big(1 + \frac{2m_W^2}{m_t^2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma &= \frac{1}{2m_t}\int d\Pi_2\, \frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 \\
&= \frac{1}{2m_t}\cdot\frac{g_2^2}{4}\frac{m_t^4}{m_W^2}\left(1-\frac{m_W^2}{m_t^2}\right)\left(1+\frac{2m_W^2}{m_t^2}\right)\cdot\frac{1}{8\pi}\left(1-\frac{m_W^2}{m_t^2}\right) \\
&= \frac{g_2^2 m_t}{64\pi}\frac{m_t^2}{m_W^2}\left(1-\frac{m_W^2}{m_t^2}\right)^2\left(1+\frac{2m_W^2}{m_t^2}\right)
\end{aligned}
\tag{69}
$$

For $m_t \gg m_W \gg m_b$, Γ is larger than naively expected by a factor of m_t^2/m_W^2. Why?

Massive $W_\mu^\pm$ has 3 polarizations : 2 transverse $+$ 1 longitudinal

original gau. boson from eaten-up $G^\pm$

$$
\mathcal{L}^{\psi}_{\text{Yuk.}} = \frac{\sqrt{2}}{v}(m_t\bar{b}_L t_R - m_b\bar{b}_R t_L)G^- + \text{h.c.} + \cdots
$$

$$
\frac{m_t}{v} \text{ in amplitude } \to \frac{m_t^2}{v^2}\left(= \frac{g_2^2 m_t^2}{4m_W^2}\right) \text{ in } \Gamma
$$

2. Polarization asymmetry in Z decay

Eqs. (40,41): Z^0 couples differently to LH/RH fermions of same species:

$$
\begin{aligned}
\mathcal{L}_{Zf\bar{f}} &= \frac{g_2}{c_w} Z^0_\mu \bar{f}\gamma^\mu (T^3 - Q_f s_w^2) f \quad \text{non-vanishing only for LH} \quad \text{for both LH and RH} \\
&= \frac{g_2}{c_w} Z^0_\mu \Big[\bar{f}_L \gamma^\mu (T^3 - Q_f s_w^2) f_L + \bar{f}_R \gamma^\mu (-Q_f s_w^2) f_R \Big]
\end{aligned}
$$

For massless fermions, helicity coincides with the chirality of the field.

For massless anti-fermions, helicity is opposite to the chirality of the field.

For massless fermions & anti-fermions, all kinematical factors are the same for LH/RH helicities and are cancelled in the ratio of left-right asymmetry:

$$
\begin{aligned}
A^f_{LR} &= \frac{\Gamma(Z^0 \to f_L \bar{f}_R) - \Gamma(Z^0 \to f_R \bar{f}_L)}{\cdots \quad + \quad \cdots} \\
&= \frac{[T^3(f_L) - Q_f s_w^2]^2 - [-Q_f s_w^2]^2}{\cdots \quad + \quad \cdots}
\end{aligned}
$$

where subscripts L, R refer to particles' helicities.

In SM, $T^3(f_L)Q_f \geqslant 0$:

$$A_{LR}^f = \frac{\left(\frac{1}{2} - |Q_f| s_w^2\right)^2 - (Q_f s_w^2)^2}{\cdots \quad + \quad \cdots}$$

$$= \begin{cases} \frac{1-4s_w^2}{1-4s_w^2+8s_w^4} \doteq 0.16, \text{ for } f = e, \mu, \tau \\ \frac{9-12s_w^2}{9-12s_w^2+8s_w^4} \doteq 0.94, \text{ for } f = d, s, b \end{cases} \qquad s_w^2 \doteq 0.23$$

[finished in 3 units on Dec 28, 2012 – End of course]