Expansion of Functions

Saturday, February 4, 2023 8:20 PM

Maclaurin's Series (or Theorem):

If f(x) be any function of x which can be expanded in powers of x and let the expansion be differentiable term by term any number of times, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

Which is known as Maclaurin's series.

Problem-1: Expand ln(1+x) in ascending powers of x using Maclaurin's theorem.

Ans: let the given fu^n be,

$$f(x) = \ln(1+x).$$

Now differentiating successively w.r.to x we get,

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$f''(x) = -1 + 2x - 3x^2 + \cdots$$

$$f'''(x) = 2 - 6x + \cdots$$

$$\cdots \quad \cdots \quad \cdots$$

Now putting $x=0$ in the above equations we have,

$$f(0) = 0$$

$$f'(0) = 1$$

$$f''(0) = -1$$

$$f'''(0) = 2$$

$$\cdots \quad \cdots$$

$$\cdots \quad \cdots$$

Now by Maclaurin's theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$$\text{or, } \ln(1+x) = 0 + x\cdot 1 + \frac{x^2}{2}(-1) + \frac{x^3}{6}\cdot 2 + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

(Ans)

Problem-2: Expand $sinx$ in terms of Maclaurin's theorem.

Ans: let $f(x) = \sin x$.

Now differentiating successively we get,

$$f'(x) = \cos x = \sin\left(1\cdot\frac{\pi}{2} + x\right)$$

$$f''(x) = -\sin x = \sin\left(2\frac{\pi}{2} + x\right)$$

$$f'''(x) = -\cos x = \sin\left(3\cdot\frac{\pi}{2} + x\right)$$

$$\cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots$$

$$f^n(x) = \sin\left(n\frac{\pi}{2} + x\right).$$

Putting $x=0$ into above equations we get,

$$f(0) = \sin 0 = 0$$

$$f'(0) = \sin\tfrac{1}{2}\pi = 1$$

$$f''(0) = \sin\pi = 0$$

$$f'''(0) = \sin\frac{3\pi}{2} = -1.$$

Q Expand $\tan^{-1}x$ in ascending power of x by Maclaurin's theorem.

Sol^n:

Let,

$$y = f(x) = \tan^{-1}x \qquad \therefore y(0) = 0$$

$$y_1 = f'(x) = \frac{1}{1+x^2} \qquad y_1(0) = 1$$

$$\Rightarrow (1+x^2)y_1 = 1$$

$$\therefore (1+x^2)y_2 + 2xy_1 = 0 \qquad \therefore y_2(0) = 0$$

by leibnitz theorem,

$$(1+x^2)y_{n+2} + n2xy_{n+1} + \frac{n(n-1)}{2}\cdot 2y_n + 2xy_{n+1} + 2ny_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2nxy_{n+1} + (n^2-n)y_n + 2xy_{n+1} + 2ny_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + (n^2+n)y_n = 0$$

Put $x=0$,

$$y_{n+2}(0) = -(n^2+n)y_n(0)$$

Putting $n=1$, $y_3(0) = -2\cdot 1 = -2$

$n=2$, $y_4(0) = 0$

$n=3$, $y_5(0) = 24$

and so on

According Maclaurin's theorem we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$$= x - \frac{2x^3}{3!} + \frac{24x^5}{5!} + \cdots$$

$$\therefore \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

(Ans)

* Expand $e^{a\sin^{-1}x}$ by Maclaurin's theorem and show that, $e^{\theta} = 1 + \sin\theta + \frac{\sin^2\theta}{2!} + \frac{2}{3!}\sin^3\theta + \cdots$

Sol^n: Here, $y = e^{a\sin^{-1}x}$, $y(0) = 1$

$$y_1 = e^{a\sin^{-1}x}\cdot\frac{a}{\sqrt{1-x^2}}, \quad y_1(0) = a.$$

$$\Rightarrow \sqrt{1-x^2}\,y_1 = a\,e^{a\sin^{-1}x}$$

$$\Rightarrow \sqrt{1-x^2}\,y_1 = ay$$

$$\Rightarrow (1-x^2)y_1^2 = a^2y^2$$

Now,

$$(1-x^2)2y_1y_2 - 2xy_1^2 = 2a^2yy_1$$

$$\Rightarrow (1-x^2)y_2 - xy_1 - a^2y = 0$$

put $x=0$, $y_2(0) = a^2$.

Again,

$$(1-x^2)y_3 - 2xy_2 - xy_2 - y_1 - a^2y_1 = 0$$

$$\Rightarrow (1-x^2)y_3 - xy_2 - (1+a^2)y_1 = 0$$

Put $x=0$, $y_3(0) = (1+a^2)a$

Similarly,

$$y_4(0) = a^2(2^2+a^2).$$

Now Maclaurin's theorem,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots$$

$f''(0) = \sin\pi = 0$

$f'''(0) = \sin\frac{3\pi}{2} = -1.$

$\cdots \quad \cdots \quad \cdots$

$f^n(0) = \sin\frac{n\pi}{2}$

Now by Maclaurin's theorem, we have,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

$$\text{or, } \cos x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \cdots + \frac{x^n}{n!}\sin\frac{n\pi}{2} + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!}(-1)^{\frac{n-1}{2}} + \cdots$$

where n is an odd number. Since, $\sin\frac{1}{2}(n\pi) = 0$, if n is even and $(-1)^{(n-1)/2}$, if n is odd. (Ans)

Now Maclaurin's theorem,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) + \cdots$$

$$e^{a\sin^{-1}x} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a(1+a^2)x^3}{3!} + \frac{a^2(2^2+a^2)x^4}{4!} + \cdots$$

Put $a = 1$ and $\sin^{-1}x = \theta \Rightarrow x = \sin\theta$.

Thus, $e^\theta = 1 + \sin\theta + \frac{\sin^2\theta}{2!} + \frac{2\sin^3\theta}{3!} + \frac{5\sin^4\theta}{4!} + \cdots$ (Ans)

* when does the Maclaurin's series fail?

Ans: Maclaurin's series fails to expand a function $f(x)$ if,

(i) any of $f(x), f'(x), \dots, f^n(x)$ is infinite or does not exists.

(ii) any of $f(x), f'(x), \dots, f^{n-1}(x)$ is discontinuous.

(iii) $\lim_{n\to\infty} R_n \neq 0$; i.e. The series does not converge.

* Maclaurin's theorem is known as the Taylor's theorem for the function about $x = 0$.

Taylor's Theorem

Statement: let $f(x+h)$ be a function of h which can be expanded in powers of h, and let the expansion be differentiable any number of times with respect to h, then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^n(x) + \cdots$$

* If we put $x = 0$ and $h = x$ in the above thm we get Maclaurin's thm.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots$$

Other forms of Taylor series:

Putting $x = a$,

1. $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^n(a) + \cdots$

Putting $x = h$, $h = a$,

2. $f(h+a) = f(h) + af'(h) + \frac{a^2}{2!}f''(h) + \cdots + \frac{a^n}{n!}f^n(h) + \cdots$

Putting $h = (x-a)$

3. $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(h) + \cdots$

Taylor's Theorem with Lagrange's form of Remainder:

Problem-1: Express the polynomial $2x^3 + 7x^2 + x - 1$ in powers of (x-2).

Ans: Here, $f(x) = 2x^3 + 7x^2 + x - 1$

$\therefore f'(x) = 6x^2 + 14x + 1$

$f''(x) = 12x + 14$

$f'''(x) = 12$

$f^{iv}(x) = 0.$

Now by the Taylor's theorem we have,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots \quad \text{—— (1)}$$

Here $a=2$, as we are to expand in powers of $x-2$.

$\therefore$ putting $x=2$ in $f(x)$, $f'(x)$ etc. we get

$$f(2) = 2(2)^3+7(2)^2+2-1 = 45$$
$$f'(2) = 6(2)^2+14(2)+1 = 53.$$
$$f''(2) = 12(2)+14 = 38$$
$$f'''(2) = 12, \quad f^{iv}(2) = 0.$$

$$f(x) = f(2)+(x-2)f'(2)+\frac{(x-2)^2}{2!}f''(2)+\frac{(x-2)^3}{3!}f'''(2)$$
$$= 45+(x-2)53+\frac{(x-2)^2}{2!}38+\frac{(x-2)^3}{3!}12$$

$\therefore$ From (1) substituting $a=2$ and these values, we get

$$2x^3+7x^2+x-1 = 45+(x-2)53+\frac{(x-2)^2}{2!}(38)+\frac{(x-2)^3}{3!}(12)$$
$$= 45+53(x-2)+19(x-2)^2+2(x-2)^3.$$ (Ans)

Q Expand e^x in powers of $(x-1)$.

Soln: Here, $f(x) = e^x = e^{x-1+1}$

Here we are to use the following form of Taylor's theorem

$$f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\cdots \quad \text{——(1)}$$

Here, $f(x) = e^x$.

$$f'(x) = e^x; \ f''(x) = e^x; \ f'''(x) = e^x \text{ etc.}$$

putting $x=1$ we get,

$$f(1) = e; \ f'(1) = e; \ f''(1) = e; \ f'''(1) = e \text{ etc.}$$

$\therefore$ From (1), putting $a=1$, we get

$$f(x) = f(1)+(x-1)f'(1)+\frac{(x-1)^2}{2!}f''(1)+\frac{(x-1)^3}{3!}f'''(1)+\cdots$$

or, $$e^x = e+(x-1)e+\frac{(x-1)^2}{2!}e+\frac{(x-1)^3}{3!}e+\cdots$$
$$= e\left\{1+(x-1)+\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}+\cdots\right\}.$$ (Ans)

* Expand $\ln x$ in powers of $(x-1)$.

Soln: let $f(x) = \ln x$

Now, $$f(x) = f(1)+(x-1)f'(1)+\frac{(x-1)^2}{2!}f''(1)$$
$$+\frac{(x-1)^3}{3!}f'''(1)+\cdots \quad \text{——(1)}$$

As $f(x) = \ln x \quad \therefore f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} \qquad f'(1) = 1.$

$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1.$

$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = 2$

$\cdots \qquad \cdots \qquad \cdots$

Now from (i) we have,

Now from (i) we have,

$$\ln x = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

$$\therefore \ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

(Ans)

* Use Taylor's series to expand $5x^2+7x+3$ in powers of $(x-1)$.

Sol^n: let $f(x) = 5x^2+7x+3$

$$\therefore f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots \quad \text{(1)}$$

As $f(x) = 5x^2+7x+3$ $\quad \therefore f(1) = 15$

$f'(x) = 10x+7$ $\quad f'(1) = 17$

$f''(x) = 10$ $\quad f''(1) = 10$

Now from (i) we have,

$$5x^2+7x+3 = 15 + 17(x-1) + 10\frac{(x-1)^2}{2!}$$

$$= 15 + 17(x-1) + 5(x-1)^2.$$

(Ans)