General Theorems

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Rolle's Theorem

☺ State Rolle's Theorem

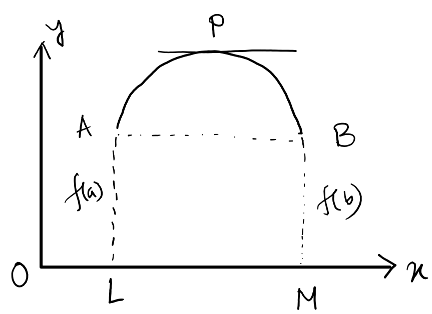
Statement: If a function $f(x)$ is such that,

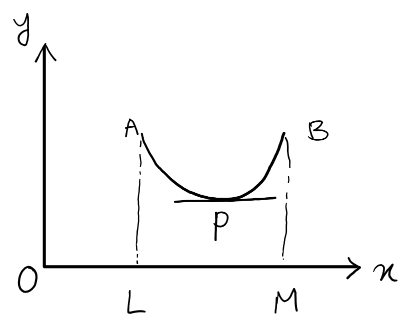
(a) It is continuous in the closed interval $[a,b]$.

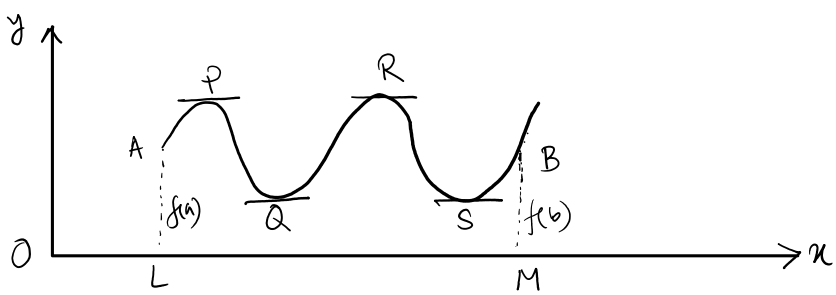
(b) It is derivable in the open interval (a,b).

(c) $f(a)=f(b)$.

then there exists at least one value 'c' in the open interval (a,b) such that $f'(c)=0$.







☺ Geometric Interpretation of Rolle's Theorem

1. As the fu^n $f(x)$ is continuous in the closed interval $[a,b]$, so its graph is a continuous curve from A to B.

2. As the fu^n $f(x)$ is derivable in the open interval (a,b), so the graph of $f(x)$ has a unique tangent at every point between A to B.

3. As $f(a)=f(b)$, so AL and BM, the ordinates of A & B are equal.

It is evident from the figures that there is at least one point P on the curve between A and B, at which the tangent is parallel to x-axis.

Since the tangent at P is parallel to x-axis, so the slope of the tangent at P is zero. If 'c' be the abscissa of P, then $f'(c)=0$, where $a<c<b$.

□ Verify the truth of Rolle's theorem for the function $f(x) = 2x^3+x^2-4x-2$ in the interval $[-\sqrt{2},\sqrt{2}]$.

Solⁿ: The given function is, $f(x)=2x^3+x^2-4x-2$

Since $f(x)$ is a polynomial in x, so $f(x)$ is continuous and diff. in the interval $(-\infty,\infty)$.

$\therefore f(x)$ is continuous in the interval $[-\sqrt{2},\sqrt{2}]$ and differentiable in the interval $(-\sqrt{2},\sqrt{2})$.

Here $f(x)=0$ gives $2x^3-x^2-4x-2=0$

or, $x^2(2x+1)-2(2x+1)=0$

or, $(2x+1)(x^2-2)=0$

$\therefore x=-\frac{1}{2}, \sqrt{2}, -\sqrt{2}$.

i.e. $f(-\frac{1}{2})=0$, $f(\sqrt{2})=0$, $f(-\sqrt{2})=0$

Thus we find that $f(\sqrt{2})=f(-\sqrt{2})$.

Hence $f(x)$ satisfies all the three conditions of Rolle's theorem. So there exist at least one number 'c' in $(-\sqrt{2},\sqrt{2})$ such that $f'(c)=0$. i.e., $6c^2+2c-4=0$

or, $3c^2+c-2=0$

or, $(c+1)(3c-2)=0$

$\therefore c=-1, \frac{2}{3} \in (-\sqrt{2},\sqrt{2})$

Hence the Rolle's theorem is verified.

Q. Verify the truth of Rolle's theorem for the function $f(x) = x^2 - 3x + 2$ in the interval $[1, 2]$.

Soln: The given funⁿ is, $f(x) = x^2 - 3x + 2$.

Since $f(x)$ is a polynomial in x, so $f(x)$ is continuous and differentiable in the interval $(-\infty, \infty)$.

$\therefore$ $f(x)$ is continuous in the interval $[1, 2]$ and differentiable in the interval $(1, 2)$.

Here, $f(x) = 0$ gives

$$x^2 - 3x + 2 = 0$$

or, $x^2 - 2x - x + 2 = 0$

or, $x(x-2) - 1(x-2) = 0$

or, $(x-2)(x-1) = 0$

$\therefore x = 2, 1$

i.e., $f(1) = 0$, $f(2) = 0$.

Thus we find that $f(1) = f(2)$.

Hence $f(x)$ satisfies all the three conditions of Rolle's theorem. So there exist at least one number 'c' in $(1, 2)$ such that

$$f'(c) = 0$$

i.e. $2c - 3 = 0$

$\therefore c = \frac{3}{2} \in (1, 2)$.

Hence the Rolle's theorem is verified. Ans

Q. Verify for $f(x) = (x-3)(x-5)$ in $[3, 5]$.

Soln: The given function is, $f(x) = (x-3)(x-5) = x^2 - 8x + 15$.

Since $f(x)$ is a polynomial in x, so $f(x)$ is continuous and differentiable in the interval $(-\infty, \infty)$.

$\therefore$ $f(x)$ is continuous in the interval $[3, 5]$ and differentiable in the interval $(3, 5)$.

Here $f(x)=0$ gives $x^2-8x+15=0$.

$$\text{or, } (x-5)(x-3)=0$$

$$\therefore x=3,5$$

Hence $f(x)$ satisfies all three conditions of Rolle's theorem. So there exists at least one number 'c' such that

$$f'(c)=0$$

$$\text{i.e., } 2c-8=0$$

$$\text{or, } c=4 \in (3,5).$$

Hence the Rolle's theorem is verified. (Ans)

Q Verify $f(x) = 3x^3+7x^2-11x-15$ in $[-3, \frac{5}{3}]$.

Soln: Given,

$$f(x) = 3x^3+7x^2-11x-15$$

Since $f(x)$ is a polynomial so it is continuous and diff. in the interval $(-\infty, \infty)$.

Here, $f(x)=0$ gives $3x^3+7x^2-11x-15=0$

$$\text{or, } 3x^2(x+1)+4x(x+1)-15(x+1)=0$$

$$\text{or, } (x+1)(3x^2+4x-15)=0$$

$$\text{or, } (x+1)(x+3)(3x-5)=0$$

$$\therefore x=-1,-3,\frac{5}{3}$$

i.e., $f(-1)=0$, $f(-3)=0$ and $f(5/3)=0$

$\therefore f(x)$ is continuous in the interval $[-3, 5/3]$ and diff. in the interval $(-3, 5/3)$.

Hence $f(x)$ satisfies all the three conditions of Rolle's theorem. So there exists at least one number 'c' such that

$$f'(c)=0$$

$$\text{i.e., } 9c^2+14c-11=0$$

or, $c = \frac{-7 \pm 2\sqrt{37}}{9}$

$\therefore c = \frac{-7-2\sqrt{37}}{9}, \frac{-7+2\sqrt{37}}{9} \in (-3, 5/3)$.

Hence the Rolle's theorem is verified. (Ans)

Lagrange's Mean Value Theorem

□ State Mean Value Theorem: If a function $f(x)$ is such that

(a) It is continuous in the closed interval $[a,b]$.

(b) It is differentiable in the open interval (a,b).

then there exists at least one 'c' of x in the open interval (a,b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$ i.e. $f(b)-f(a) = (b-a) f'(c)$.

□ Justify the validity of mean value theorem for the fun $f(x) = 3+2x-x^2$ in the interval $(0,1)$.

Soln: Given function is, $f(x) = 3+2x-x^2$ —— (1)

Here $f(x)$ is a polynomial, so $f(x)$ is continuous and differentiable in the interval $(-\infty, \infty)$.

Hence it is continuous in $[0,1]$ and differentiable in $(0,1)$.

But $f(a) = f(0) = 3+2\cdot 0 - 0^2 = 3$

and $f(b) = f(1) = 3+2\cdot 1 - 1^2 = 4$.

And $f'(x) = 2-2x$ $\therefore f'(c) = 2-2c$.

So by the mean value theorem we have,

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{4-3}{1-0} = 1.$$

or, $2-2c = 1$

or, $c = \frac{1}{2} \in (0,1)$.

So $c=\frac{1}{2}$ is the number whose existence is guaranteed by the mean value theorem.

Hence the MVT is valid for the given function. (Ans)

□ $f(x) = x^2-3x+2$, $[-2,3]$.

Soln: Given function is, $f(x)=x^2-3x+2$

Since $f(x)$ is a polynomial, so $f(x)$ is continuous and differentiable in the interval $(-\infty,\infty)$.

Hence it is continuous in the interval $[-2,3]$ and diff. in the interval $(-2,3)$. Thus the hypothesis of the mean value thm is satisfied with $a=-2$ and $b=3$.

But $f(a) = f(-2) = (-2)^3 - 3(-2) + 2 = 12$

and $f(b) = f(3) = 3^2 - 3\cdot3 + 2 = 2$.

Also $f'(x) = 2x-3$

$\therefore f'(c) = 2c-3$

So by the mean value theorem we have

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or, $$2c-3 = \frac{2-12}{3-(-2)} = \frac{-10}{5} = -2$$

or, $2c = -2+3 = 1$

$\therefore c = \frac{1}{2} \in (-2,3)$.

So $c=\frac{1}{2}$ is the number whose existence is guaranteed by the MVT.

Hence the MVT is valid for the given function.

□ $f(x) = (x-1)(x-2)(x-3)$, $[0,4]$.

Ans: $c = 1.15, .85$.

□ $f(x) = 2x-x^2$, $[0,1]$. Ans: $c=\frac{1}{2}$.

Ascertain the validity of lagrange's MVT for $f(x) = x(x-1)(x-2)$ on the interval $[0,1]$.

$f(x) = x(x-1)(x-2)$, $[0, \frac{1}{2}]$

Solⁿ: Here, $f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$.

$$f(a) = f(0) = 0$$

$$f(b) = f(\tfrac{1}{2}) = (\tfrac{1}{2})^3 - 3(\tfrac{1}{2})^2 + 2 \cdot \tfrac{1}{2} = \tfrac{1}{8} - 3 \cdot \tfrac{1}{4} + 1 = \tfrac{3}{8}.$$

$$f'(x) = 3x^2 - 6x + 2$$

$$\therefore f'(c) = 3c^2 - 6c + 2.$$

Thus, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

or, $3c^2 - 6c + 2 = \dfrac{\frac{3}{8} - 0}{\frac{1}{2} - 0} = \frac{3}{8} \times \frac{2}{1} = \frac{3}{4}$.

or, $12c^2 - 24c + 8 - 3 = 0$

or, $12c^2 - 24c + 5 = 0$

$$\therefore c = \frac{24 \pm \sqrt{24^2 - 4 \cdot 12 \cdot 5}}{2 \cdot 12} = 1.76,\ 0.24 \Rightarrow c = 0.24 \in (0, \tfrac{1}{2})$$

Thus the MVT is verified in the interval $[0, \frac{1}{2}]$ for the given funⁿ.

(Ans)

* $f(x) = x - x^3$, $[-2, 1]$ Ans: $c = \pm 1$.
* $f(x) = x^3 - x^2 - 4x + 4$, $[-2, 1]$ Ans: $c = 1.53, -0.87$.
* $f(x) = x^3 + x - 4$, $[-1, 2]$ Ans: $c = \pm 1$.