Small Sample Tests: Test for a Population Mean (t - test)

Aim

To test that the population mean μ be regarded as μ_0, based on a random sample. That is, to investigate the significance of the difference between the sample mean $\bar{X}$ and the assumed population mean μ_0.

Source

A random sample of n observations X_i, $(i = 1, 2, \ldots, n)$ be drawn from a population whose mean μ and variance σ^2 are unknown.

Assumptions

(*i*) The population from which, the sample drawn is Normal distribution.

(*ii*) The population variance σ^2 is unknown. (Since σ^2 is unknown, it is replaced by its unbiased estimate S^2)

Null Hypothesis

H_0 : The sample has been drawn from a population with mean μ be μ_0. That is, there is no significant difference between the sample mean $\bar{X}$ and the assumed population mean μ_0. *i.e.*, $H_0 : \mu = \mu_0$.

Alternative Hypotheses

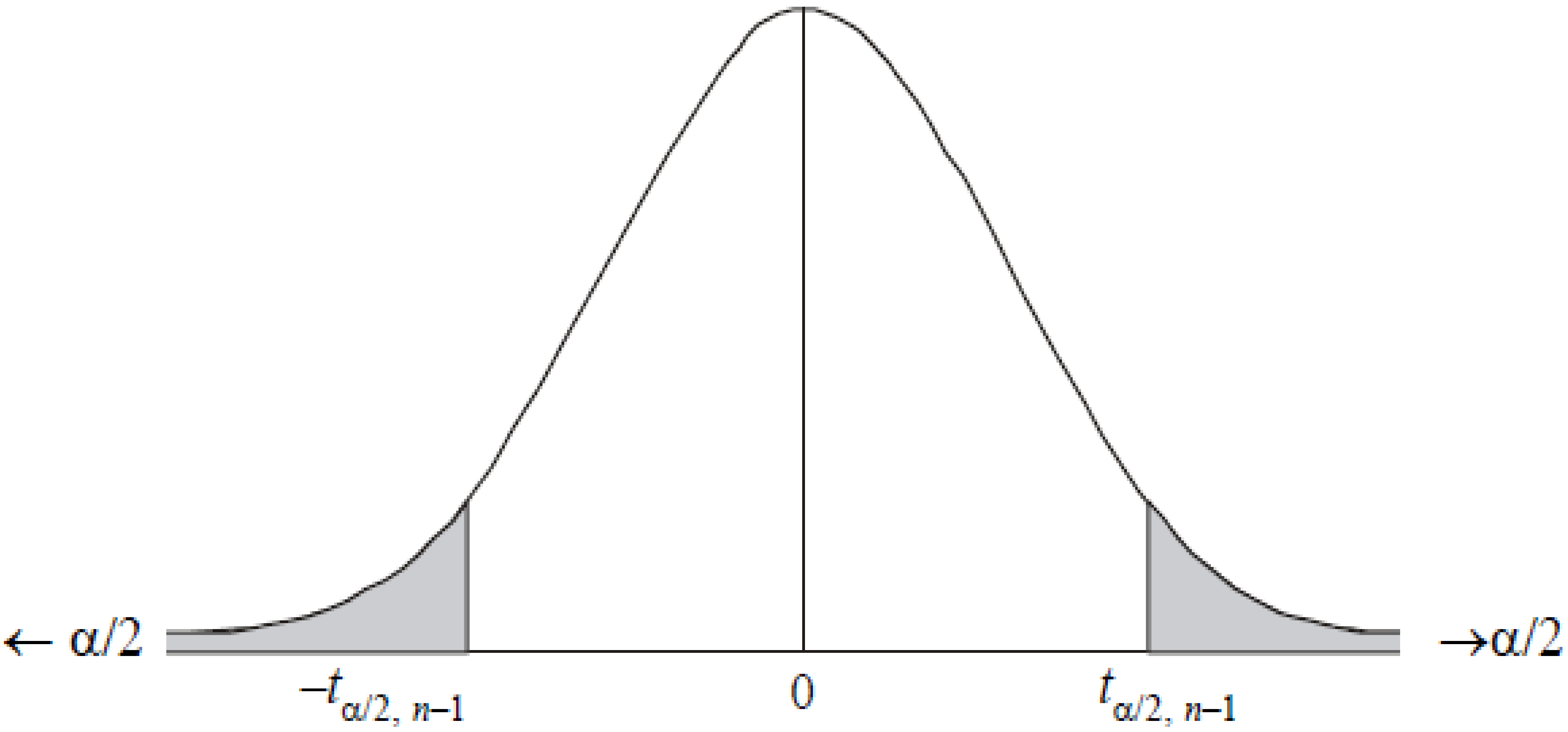
$H_1(1)$: $\mu \neq \mu_0$

$H_1(2)$: $\mu > \mu_0$

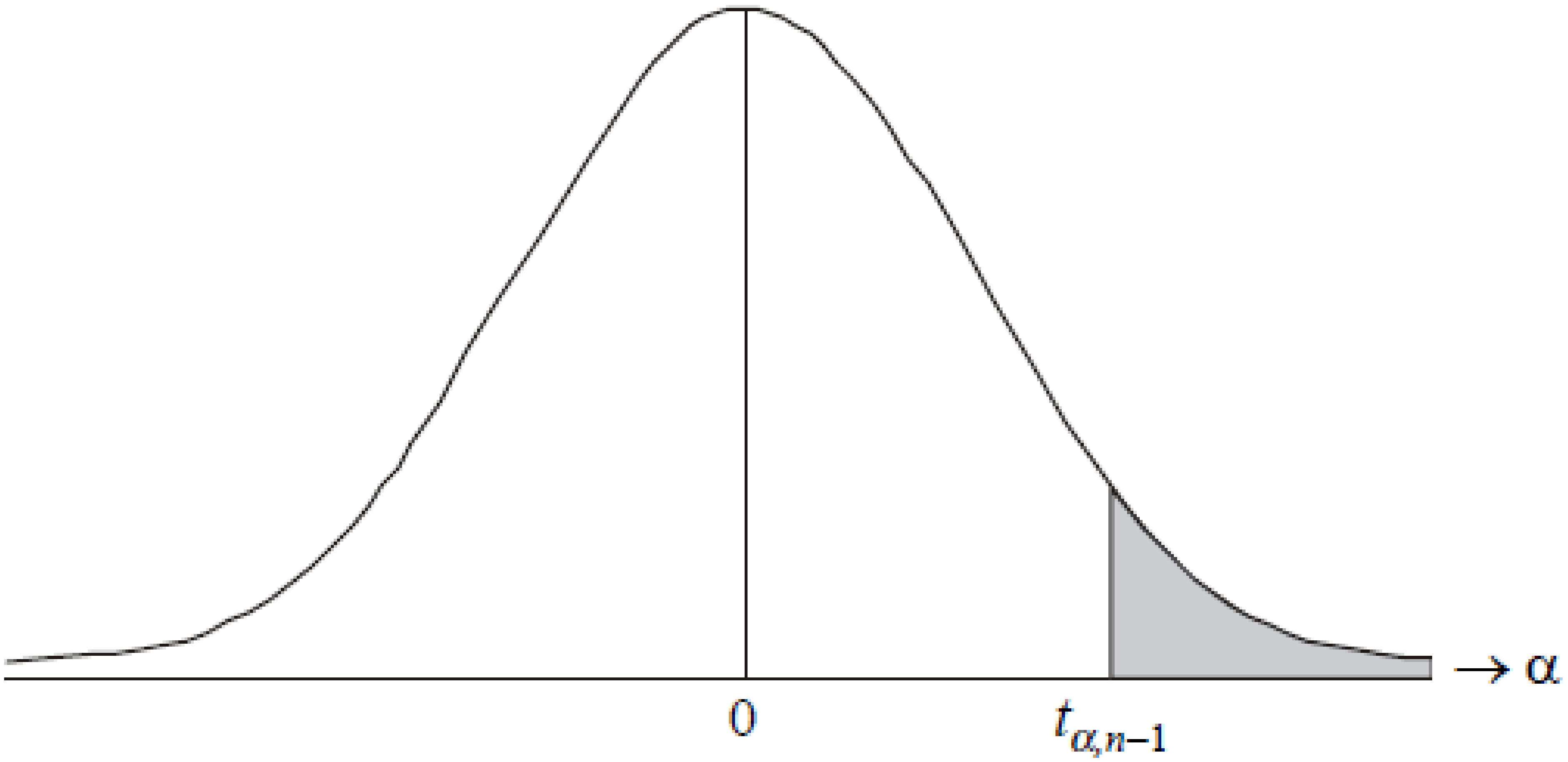
$H_1(3)$: $\mu < \mu_0$

Level of Significance (α) and Critical Region

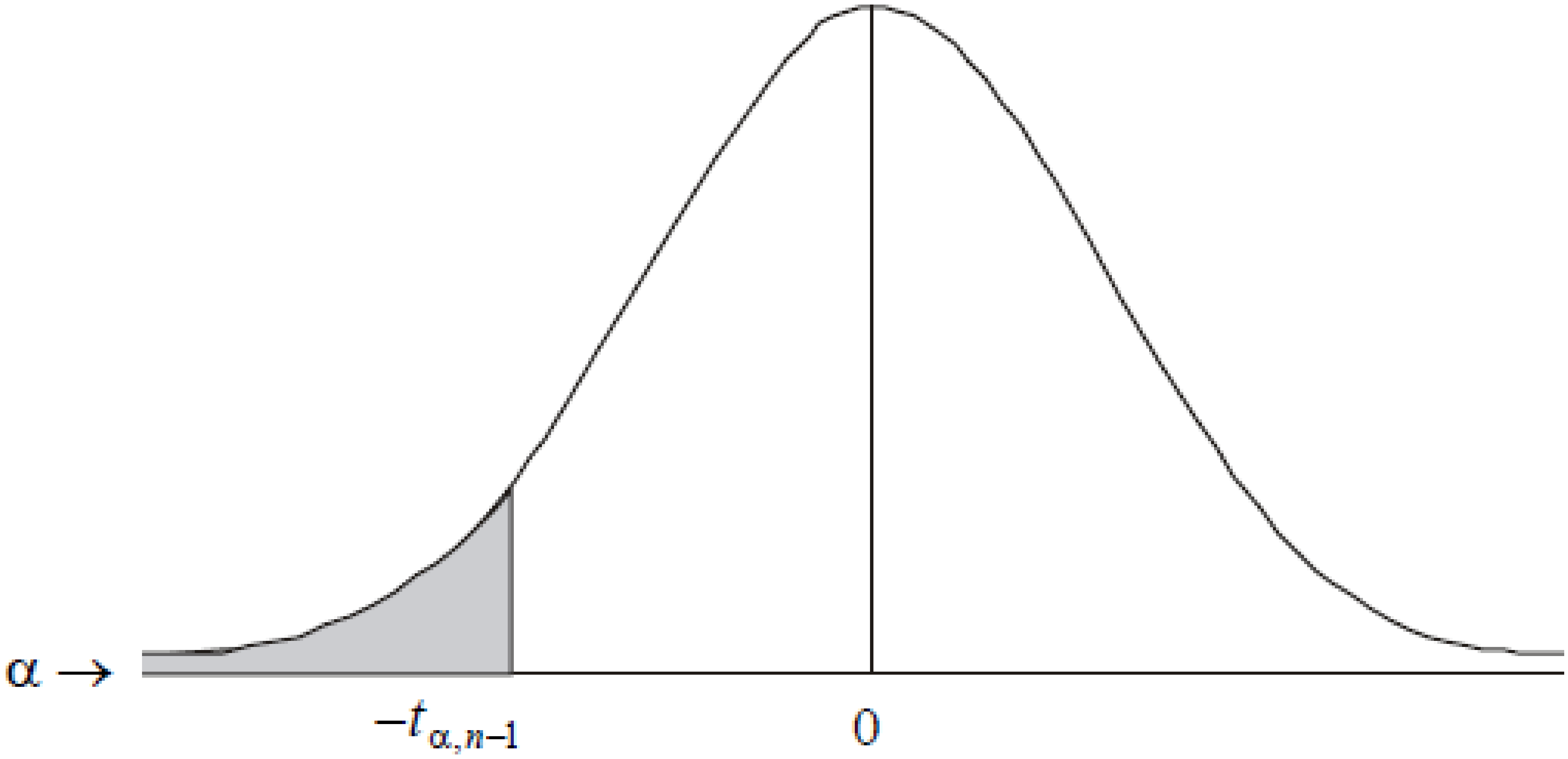
(1) $|t| > t_{\alpha, n-1}$ such that $P\{|t| > t_{\alpha, n-1}\} = \alpha$



(2) $t > t_{\alpha, n-1}$ such that $P\{t > t_{\alpha, n-1}\} = \alpha$



(3) $t < t_{\alpha,n-1}$ such that $P\{t < t_{\alpha,n-1}\} = \alpha$



Test statistic $t = \frac{\bar{X}-\mu}{S/\sqrt{n}}$ ***or*** $t = \frac{\bar{X}-\mu}{s/\sqrt{n-1}}$

Where, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

and $s^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$ $\Longrightarrow$ $\frac{S^2}{n} = \frac{s^2}{n-1}$

The statistic *t*- follows **t** distribution with $(n-1)$ degree of freedom.

Conclusions

1. If $|t| \leq t_{\alpha}$, we conclude that the data do not provide us any evidence against the null hypothesis H_0, and hence it may be accepted at $\alpha\%$ level of significance. Otherwise reject H_0 or accept $H_1(1)$.
2. If $t \leq t_{\alpha}$, we conclude that the data do not provide us any evidence against the null hypothesis H_0, and hence it may be accepted at $\alpha\%$ level of significance. Otherwise reject H_0 or accept $H_1(2)$.
3. If $|t| \leq |t_{\alpha}|$, we conclude that the data do not provide us any evidence against the null hypothesis H_0, and hence it may be accepted at $\alpha\%$ level of significance. Otherwise reject H_0 or accept $H_1(3)$.

Confidence Interval for Single mean, (*Variance Unknown*)

A $100(1-\alpha)\%$ C.I. for $\boldsymbol{\mu}$ of $N(\mu, \sigma^2)$, when $\boldsymbol{\sigma}$ **– is unknown**

$$P\left\{-t_{\left(\frac{\alpha}{2}, n-1\right)} \leq t \leq t_{\left(\frac{\alpha}{2}, n-1\right)}\right\} = 1-\alpha$$

$$P\left\{-t_{\left(\frac{\alpha}{2}, n-1\right)} \leq \frac{\bar{X}-\mu}{s/\sqrt{n-1}} \leq t_{\left(\frac{\alpha}{2}, n-1\right)}\right\} = 1-\alpha$$

$$P\left\{\bar{X} - t_{\alpha/2, n-1}\, s/\sqrt{n-1} \leq \mu \leq \bar{X} + t_{\alpha/2, n-1}\, s/\sqrt{n-1}\right\} = 1-\alpha$$

$100(1-\alpha)\%$ two-sided confidence interval of $\boldsymbol{\mu}$ are

$$\bar{\boldsymbol{X}} - \boldsymbol{t}_{\boldsymbol{\alpha/2,\,n-1}}\,\boldsymbol{S}/\sqrt{\boldsymbol{n}} \le \boldsymbol{\mu} \le \bar{\boldsymbol{X}} + \boldsymbol{t}_{\boldsymbol{\alpha/2,\,n-1}}\,\boldsymbol{S}/\sqrt{\boldsymbol{n}}$$

$100(1-\alpha)\%$ C.I. for $\boldsymbol{\mu}$ for upper confidence:

$$\boldsymbol{\mu} \le \bar{\boldsymbol{X}} + \boldsymbol{t}_{(\boldsymbol{\alpha},\,\boldsymbol{n-1})}\,\boldsymbol{S}/\sqrt{\boldsymbol{n}}$$

$100(1-\alpha)\%$ C.I. for $\boldsymbol{\mu}$ for lower confidence is :

$$\bar{\boldsymbol{X}} - \boldsymbol{t}_{(\boldsymbol{\alpha},\,\boldsymbol{n-1})}\,\boldsymbol{S}/\sqrt{\boldsymbol{n}} \le \boldsymbol{\mu}$$

Example 1

A sample of 12 students from a school has the following scores in an I.Q. test. 89 87 76 78 79 86 74 83 75 71 76 92. Do this data support that the mean I.Q. mark of the school students is 80? Test at 5% level.

Solution

Aim: To test the mean I.Q. marks of the school students be regarded as 80 or not.

H_0: The mean I.Q. mark of the school students is 80. *i.e.*, H_0: $\mu=80$.

H_1: The mean I.Q. mark of the school students is not 80. *i.e.*, H_1: $\mu \neq 80$.

Level of Significance: $\alpha = 0.05$ and Critical Value: $t_{0.05,11} = 2.20$

Test statistic:

$$t = \frac{\bar{X}-\mu}{s/\sqrt{n-1}}$$

Hence, $\bar{X} = 80.5$ and $s = 6.44$

$$t = \frac{80.5 - 80}{6.44/\sqrt{11}} = 0.2575$$

Conclusion: Since $|\boldsymbol{t}| < \mathbf{2.20}$. We conclude that the data do not provide us any evidence against the null hypothesis $\boldsymbol{H_0}$.

Hence, accept $\boldsymbol{H_0}$ at $\mathbf{5\%}$ level of significance. *i.e.*, the mean **I.Q.** mark of the school students is regarded as **80**.

P-value: $P\left\{|T| \geq t_{\alpha/2,(n-1)}\right\} = 2\,P\left\{T \geq t_{\alpha/2,(n-1)}\right\}$

$$\Longrightarrow \quad P\{|T| \geq 0.257\}$$

$$= 2\,P\{T \geq 0.257\}$$

p-value $= 0.801$

Comparison: $0.801 > 0.05$

Accept the null hypothesis $\boldsymbol{H_0 : \mu = 140}$ at $\boldsymbol{\alpha = 0.05}$ ***l.o.s.***

Example 2

The average breaking strength of steel rods is specified as 22.25 kg. To test this, a sample of 20 rods was examined. The mean and standard deviations obtained were 21.35 kg and 2.25 respectively. Is the result of the experiment significant at 5% level?

Solution

Aim: To test the average breaking strength of steel rods specified as 22.25 kg is true or not.

H_0: The average breaking strength of steel rods specified as 22.25 kg is true. *i.e.*, $H_0 : \mu = 22.25$.

H_1: The average breaking strength of steel rods specified as 22.25 kg is not true. *i.e.*, $H_1 : \mu \neq 22.25$.

Level of Significance: $\alpha = 0.05$ and Critical Value: $t_{0.05,19} = 2.09$

Test statistic:

$$t = \frac{\bar{X} - \mu}{s/\sqrt{n-1}}$$

$$t = \frac{21.35 - 22.25}{2.25/\sqrt{19}} = -1.7435$$

Conclusion: Since $|\boldsymbol{t}| < \mathbf{2.09}$. We conclude that the data do not provide us any evidence against the null hypothesis $\boldsymbol{H_0}$ and hence it may be accepted $\boldsymbol{H_0}$ at $\mathbf{5\%}$ level of significance.

i.e., the average breaking strength of steel rods is specified as 22.25 kg is true.

P-value: $$P\left\{|T| \geq t_{\alpha/2\,,\,(n-1)}\right\} = 2\,P\left\{T \geq t_{\alpha/2\,,\,(n-1)}\right\}$$

$$\Longrightarrow \quad P\{|T| \geq 1.7435\}$$

$$= 2\,P\{T \geq 1.7435\}$$

p-value $= 0.0955$

Comparison: $0.095 > 0.05$

Accept the null hypothesis $\boldsymbol{H_0 : \mu = 22.25}$ at $\boldsymbol{\alpha = 0.05}$ ***l.o.s.***

Test for Equality of Two Population Means (Population variances are Equal and Unknown)

Aim

To test the null hypothesis of the mean of the two populations are equal, based on two random samples. That is, to investigate the significance of the difference between the two sample means $\overline{X}_1$ and $\overline{X}_2$.

Source

A random sample of n_1 observations X_{1i}, $(i = 1, 2, \ldots, n_1)$ be drawn from a population with unknown mean μ_1 . A random sample of n_2 observations X_{2j}, $(j = 1, 2, \ldots, n_2)$ be drawn from another population with unknown mean μ_2.

Assumptions

(*i*) The populations from which, the two samples drawn, are Normal distributions.

(*ii*) The two Population variances are equal and unknown which is denoted by σ^2 (Since σ^2 is unknown, it is replace by unbiased estimate S^2).

Null Hypothesis

H_0: The two population means μ_1 and μ_2 are equal. That is, there is no significant difference between the two sample means $\overline{X}_1$ and $\overline{X}_2$.

i.e., H_0: $\mu_1 = \mu_2$

Alternative Hypotheses

$H_1(1) : \mu_1 \neq \mu_2$

$H_1(2) : \mu_1 > \mu_2$

$H_1(3) : \mu_1 < \mu_2$

Level of Significance (α) and Critical Region

1. $|t| < t_{\alpha,(n_1+n_2-2)}$ such that $P\{|t| > t_{\alpha,(n_1+n_2-2)}\} = \alpha$
2. $t > t_{\alpha,(n_1+n_2-2)}$ such that $P\{t > t_{\alpha,(n_1+n_2-2)}\} = \alpha$
3. $t < -t_{\alpha,(n_1+n_2-2)}$ such that $P\{t < -t_{\alpha,(n_1+n_2-2)}\} = \alpha$

Critical Values $(t_{\alpha,(n_1+n_2-2)})$ are obtained from Table

Test statistic: Case-1

$$t = \frac{(\overline{X_1} - \overline{X_2}) - (\mu_1 - \mu_2)}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Where, $\overline{X_1} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_{1i}$, $\overline{X_2} = \frac{1}{n_2}\sum_{j=1}^{n_2} X_{2j}$

And $$S^2 = \frac{\sum_{i=1}^{n_1}(X_{i1} - \overline{X_1})^2 + \sum_{j=1}^{n_2}(X_{j2} - \overline{X_2})^2}{n_1 + n_2 - 2}$$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$$

The statistic *t*- follows t distribution with $(\boldsymbol{n_1 + n_2 - 2})$ degree of freedom.

Test statistic: Case-2

$$t = \frac{(\overline{X_1} - \overline{X_2}) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1 - 1} + \frac{s_2^2}{n_2 - 1}}}$$

The statistic *t*- follows t distribution with $(\boldsymbol{n_1 + n_2 - 2})$ degree of freedom.

Example 3

The gain in weight of two random samples of chicks on two different diets A and B are given below. Examine whether the difference in mean increases in weight is significant.

Diet A: 2.5 2.25 2.35 2.60 2.10 2.45 2.5 2.1 2.2

Diet B: 2.45 2.50 2.60 2.77 2.60 2.55 2.65 2.75 2.45 2.50

Solution

Aim: To test the mean increases in weights by diet-A (μ_1) and diet-B (μ_2) are equal or not.

H_0 : The mean increases in weights by both diets are equal. *i.e.*, $H_0 : \mu_1 = \mu_2$

H_1 : The mean increases in weights by both diets are not equal. *i.e.*, $H_1 : \mu_1 \neq \mu_2$

Level of significance: $\alpha = 0.05$(say) and *Critical value*: $t_{0.05}$ for 17 d.f = 2.11

Test Statistic:

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \quad (\text{Under } H_0 : \mu_1 = \mu_2)$$

$$= \frac{(2.34 - 2.58)}{0.16\sqrt{\frac{1}{9} + \frac{1}{10}}} = -2.25$$

Conclusion: Since $|t| > t_\alpha$, we conclude that the data provide us evidence against the null hypothesis H_0 and in favor of H_1. Hence, H_1 is accepted at 5% level of significance. That is, the mean increase in weights by two diets A and B are not equal.

Example 4

A researcher is interested to know whether the performance in a public examination by students of schools from Tsunami affected area compared with other students is poor or not. A random sample of 10 students from coastal area schools is selected whose marks are given below. 68 72 64 65 56 72 64 56 60 73. Another sample of 8 students from non-coastal area schools has the following marks 76 78 68 72 83 85 88 78. Test at 1% level of the hypothesis.

Solution

Aim: To test the performance in a public examination by students of schools from Tsunami affected area compared with other students is equal or less.

H_0: The performance in a public examination by students of schools from Tsunami affected area (μ_1) compared with other students (μ_2) is equal. *i.e.*, H_0: $\mu_1 = \mu_2$

H_1: The performance in a public examination by students of schools from Tsunami affected area is less than that of other students. *i.e.*, H_1: $\mu_1 < \mu_2$

Level of Significance: $\alpha = 0.01$ and *Critical value*: $t_{0.01}$ for 16 d.f = – 2.58

Test Statistic:
$$t = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \quad \text{(Under } H_0 : \mu_1 = \mu_2\text{)}$$

$$= \frac{(65 - 78.5)}{6.88\sqrt{\frac{1}{10} + \frac{1}{8}}} = -4.13$$

Conclusion: Since $|t| > |t_\alpha|$, we conclude that the data provide us evidence against the null hypothesis H_0 and in favor of H_1. Hence, H_1 is accepted at 1% level of significance. That is, the performance in a public examination by students of schools from Tsunami affected area is less than that of other students.

Example 5

To test the claim that the resistance of electric wire can be reduced by at least 0.05 ohm by alloying, 25 values obtained for each alloyed wire and standard wire produced the following results:

	Mean	Standard Deviation
Alloyed wire	0.083 ohm	0.003 ohm
Standard wire	0.136 ohm	0.002 ohm

Test at 5% level of significance whether or not the claim is substantiated.

t - test for Paired Observations

Aim

To test the treatment applied is effective or not, based on a random sample. That is, to investigate the significance of the difference between before and after the treatment in the sample.

Source

Let X_i, $(i = 1, 2, \ldots, n)$ be the observations made initially from n individuals as a random sample of size n. A treatment is applied to the above individuals and observations are made after the treatment and are denoted by Y_i, $(i = 1, 2, \ldots, n)$. That is, (X_i, Y_i) denotes the pair of observations obtained from the i^{th} individual, before and after the treatment applied. Let μ_X is unknown population mean before the treatment and μ_Y is the unknown population mean after the treatment.

Assumptions

(*i*) The observations for the two samples must be obtained in pair.

(*ii*) The population from which, the sample drawn is normal.

Null Hypothesis

H_0: The treatment applied, is ineffective. That is, there is no significant difference between before and after the treatment applied.

i.e., H_0: $\mu_d = \mu_X - \mu_Y = 0$.

Alternative Hypotheses

$H_1(1) : \mu_d \neq 0$

$H_1(2) : \mu_d > 0$

$H_1(3) : \mu_d < 0$

Test Statistic

$$t = \frac{\overline{d} - \mu_d}{S_d / \sqrt{n}} \quad (\text{Under } H_0 : \mu_d = 0)$$

$$\overline{d} = \frac{\sum_{i=1}^{n} d_i}{n}, \; d_i = X_i - Y_i, \;\; S_d^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(d_i - \overline{d}\right)^2$$

The statistic t follows t distribution with $(n–1)$ degrees of freedom.

Example 6

A health spa has advertised a weight-reducing program and has claimed that the average participant in the program loses more than 5 kgs. A random sample of 10 participants has the following weights before and after the program. Test his claim at 5% level of significance.

Solution

Weights before: 80 78 75 86 90 87 95 78 86 90

Weights after: 76 75 70 80 84 83 91 72 83 83

Aim: To test the claim of health spa on average weight reduction is five kgs or more.

H_0: The average weight reduction is only 5 kgs. *i.e.*, $H_0: \mu_d = \mu_x - \mu_y = 5$

H_1: The average weight reduction is more than 5 kgs. *i.e.*, $H_1: \mu_d > 5$.

Level of Significance: $\alpha = 0.05$ and *Critical value*: $t_{0.05,9} = 1.83$

Test statistic:

$$t = \frac{\bar{d} - \mu_d}{S/\sqrt{n}}$$

$$t = \frac{4.8 - 5}{1.3984/\sqrt{10}} = -0.4523$$

Conclusion: Since $|\boldsymbol{t}| < 1.83$. We conclude that the data do not provide us any evidence against the null hypothesis $\boldsymbol{H_0}$ and hence it may be accepted at $\mathbf{5\%}$ level of significance.

Example

Twenty-two volunteers at a cold research institute caught a cold after having been exposed to various cold viruses. A random selection of 10 of these volunteers was given tablets containing 1 gram of vitamin C. These tablets were taken four times a day. The control group consisting of the other 12 volunteers was given placebo tablets that looked and tasted exactly the same as the vitamin C tablets. This was continued for each volunteer until a doctor, who did not know if the volunteer was receiving the vitamin C or the placebo tablets, decided that the volunteer was no longer suffering from the cold.

The length of time the cold lasted was then recorded.

Vitamin C	5.5	6.0	7.0	6.0	7.5	6.0	7.5	5.5	7.0	6.5		
Placebo tablet	6.5	6.0	8.5	7.0	6.5	8.0	7.5	6.5	7.5	6.0	8.5	7.0

Do the data listed prove that taking 4 grams daily of vitamin C reduces the mean length of time a cold lasts? At what level of significance?

Example

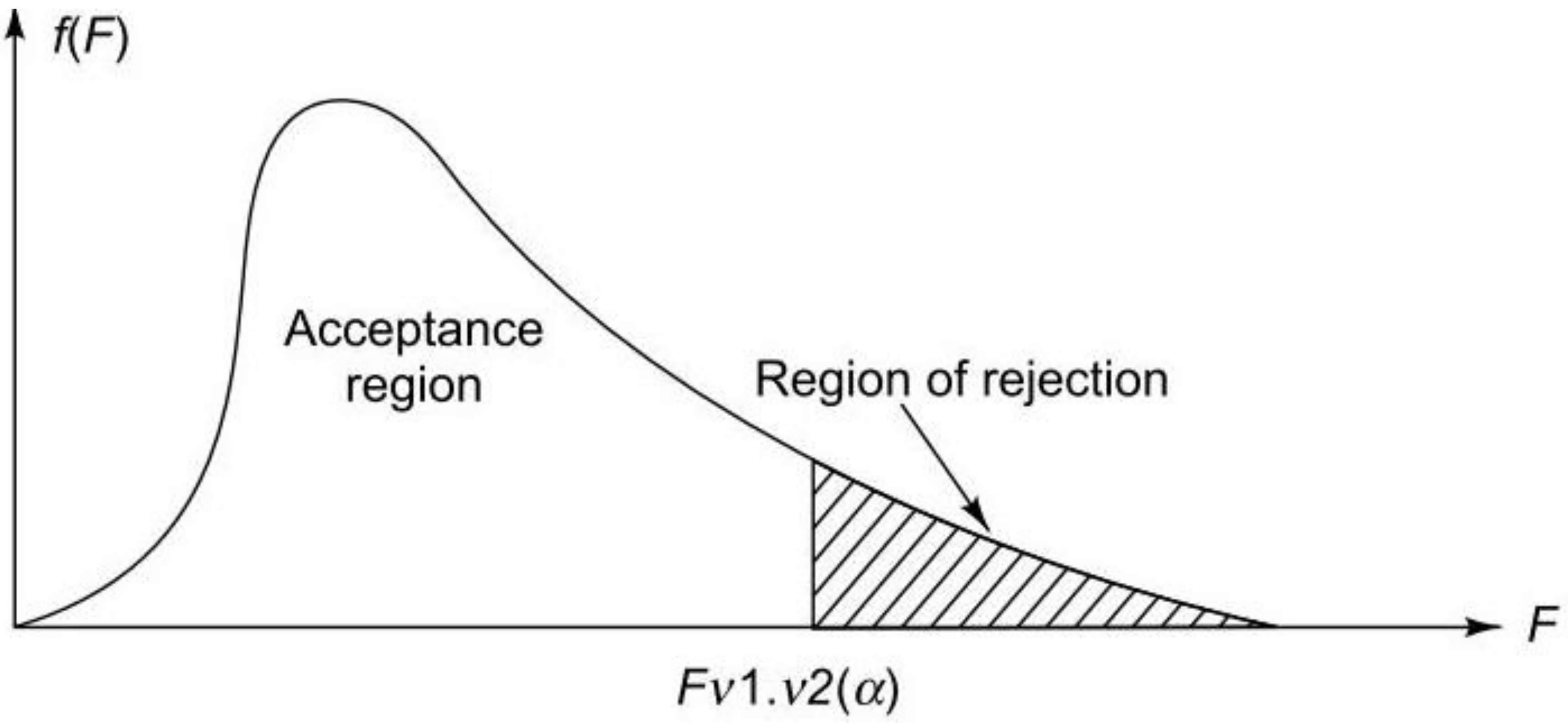
An industrial safety program was recently instituted in the computer chip industry. The average weekly loss (averaged over 1 month) in man-hours due to accidents in 10 similar plants both before and after the program are as follows:

Plant	1	2	3	4	5	6	7	8	9	10
Before	30.5	18.5	24.5	32	16	15	23.5	25.5	28	18
After	23	21	22	28.5	14.5	15.5	24.5	21	23.5	16.5

Determine, at the 5 percent level of significance, whether the safety program has been proven to be effective.

F-distribution

1. The probability curve of the F-distribution is roughly sketched



2. F-test of significance of the difference between population variances and F-table.

To test the significance of the difference between population variances, we shall first find their estimates, $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ based on the sample variances s_1^2 and s_1^2 and then test their equality.

It is known that ,

$$\hat{\sigma}_1^2 = \frac{n_1 s_1^2}{n_1 - 1} \text{ with the number of degrees of freedom } v_1 = n_1 - 1$$

$$\hat{\sigma}_2^2 = \frac{n_2 s_2^2}{n_2 - 1} \text{ with the number of degrees of freedom } v_2 = n_2 - 1$$

Test for Equality of Two Population Variances

Aim

To test the variances of the two populations are equal, based on two random samples. That is, to investigate the significance of the difference between the two sample variances.

Source

Let X_{1i}, (i = 1, 2,…, n_1) be a random sample of n_1 observations drawn from a population with unknown variance σ_1^2. Let Y_{2j} (j = 1, 2,…, n_2) be a random sample of n_2 observations drawn from another population with unknown variance σ_2^2.

Assumption

The populations from which, the samples drawn are normal distributions.

Null Hypothesis

H_0: The two population variances σ_1^2 and σ_2^2 are equal. That is, there is no significant difference between the two, sample variances s_1^2 and s_2^2. *i.e.*, H_0: $\sigma_1^2 = \sigma_2^2$.

Alternative Hypotheses

$H_1(1) : \sigma_1^2 \neq \sigma_2^2$

$H_1(2) : \sigma_1^2 > \sigma_2^2$

$H_1(3) : \sigma_1^2 < \sigma_2^2$

Level of Significance (α) and Critical Values (F_α)

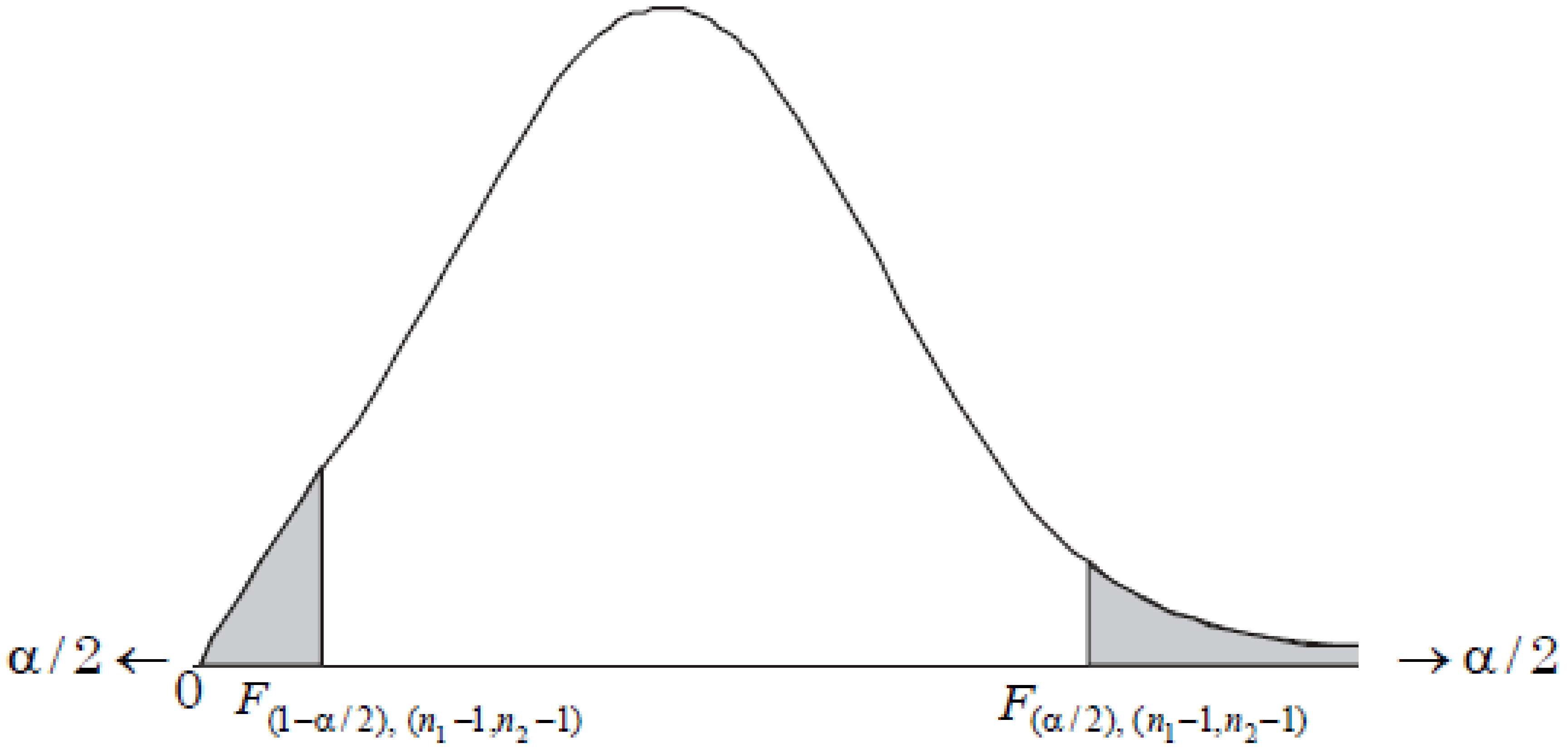
The critical values of F for right tailed test are available in Table 4. That is, the critical region is determined by the right tail areas. Thus the significant value $F_{\alpha,\,(n_1-1,\,n_2-1)}$ at level of significance α and (n_1-1, n_2-1) degrees of freedom is determined by $P\{F > F_{\alpha,\,(n_1-1,\,n_2-1)}\} = \alpha$. The critical values of F for left tailed test is $F < F_{(1-\alpha),\,(n_1-1,n_2-1)}$ and for two tailed test is $F > F_{(\alpha/2),(n_1-1,n_2-1)}$ and $F < F_{(1-\alpha/2),(n_1-1,n_2-1)}$. We have the following reciprocal relation between the upper and lower α significant points of F-distribution:

$$F_\alpha(n_1, n_2) = \frac{1}{F_{1-\alpha}(n_2, n_1)} \Rightarrow F_\alpha(n_1, n_2) \times F_{1-\alpha}(n_2, n_1) = 1.$$

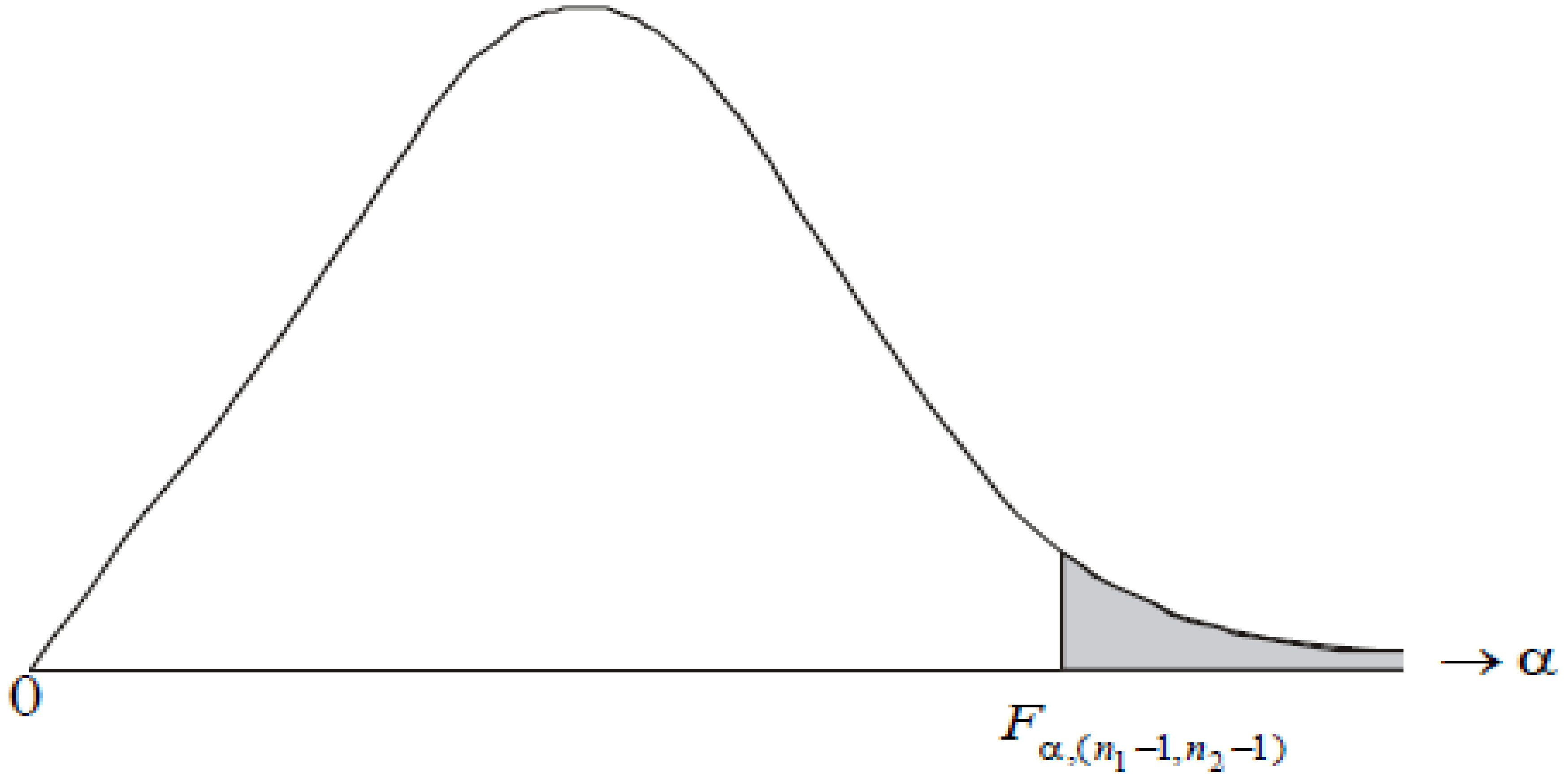
Critical Regions

1. $F > F_{(\alpha/2),(n_1-1,n_2-1)}$ and $F < F_{(1-\alpha/2),(n_1-1,n_2-1)}$ such that

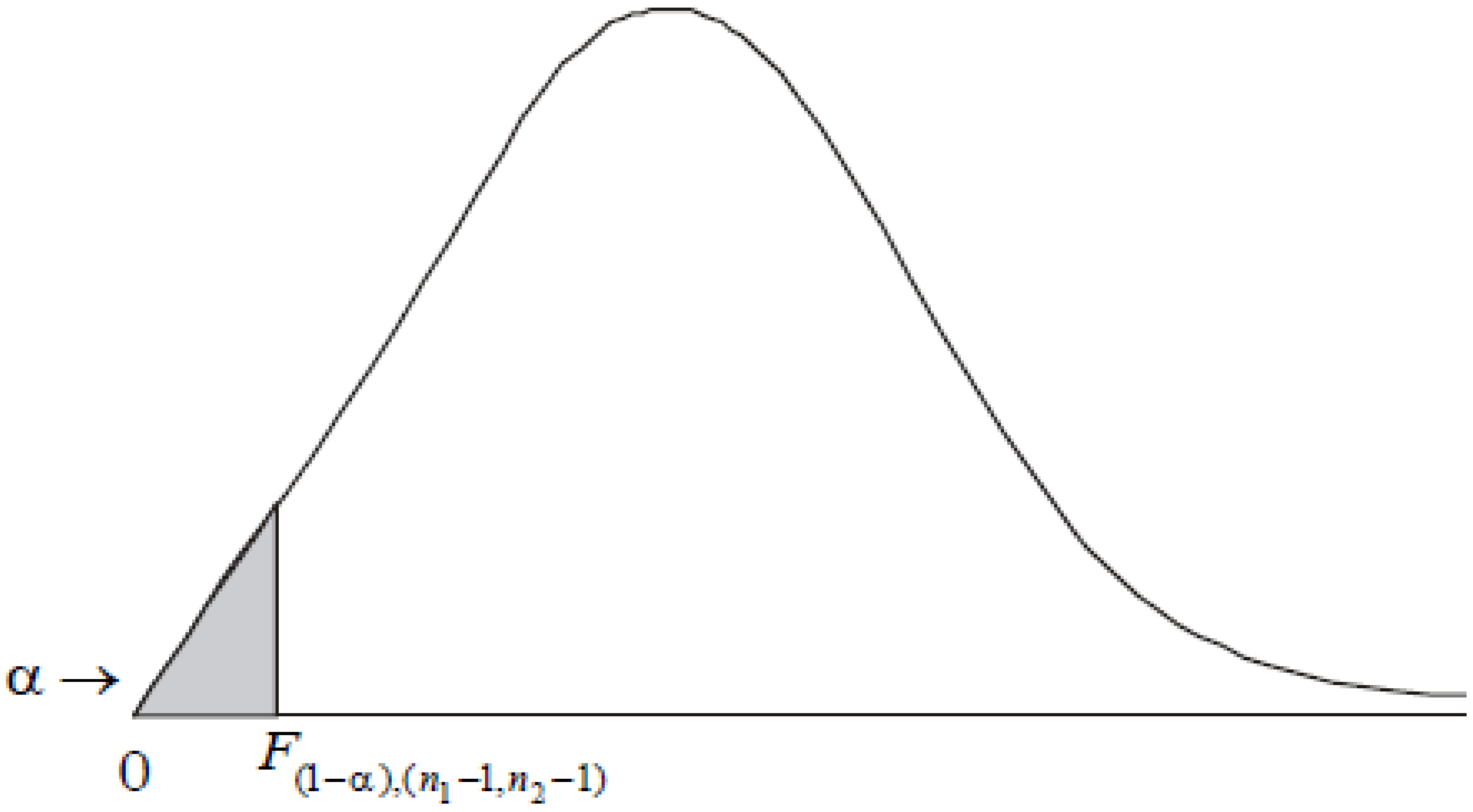
$$P\{F > F_{(\alpha/2),(n_1-1,n_2-1)}\} + P\{F < F_{(1-\alpha/2),(n_1-1,n_2-1)}\} = \alpha$$



2. $F > F_{\alpha,(n_1-1,n_2-1)}$ such that $P\{F > F_{\alpha,(n_1-1,n_2-1)}\} = \alpha$.



3. $F < F_{(1-\alpha),(n_1-1,n_2-1)}$ such that $P\{F < F_{(1-\alpha),(n_1-1,n_2-1)}\} = \alpha$



Test statistic

$$F = \frac{S_1^2}{S_2^2}$$

Where, $\overline{X_1} = \frac{1}{n_1} \sum_{i=1}^{n_1} X_{1i}$, $\overline{X_2} = \frac{1}{n_2} \sum_{j=1}^{n_2} X_{2j}$

And
$$S_1^2 = \frac{\sum_{i=1}^{n_1}(X_{i1}-\overline{X_1})^2}{n_1-1} = \left(\frac{n_1}{n_1-1}\right) s_1^2$$
$$S_2^2 = \frac{\sum_{j=1}^{n_2}(X_{j2}-\overline{X_2})^2}{n_2-1} = \left(\frac{n_2}{n_2-1}\right) s_2^2$$

The statistic F - follows F distribution with $(n_1 - 1, n_2-1)$ degree of freedom.

Conclusions

1. If $F_{(1-\alpha/2),(n_1-1,n_2-1)} \le F \le F_{(\alpha/2),(n_1-1,n_2-1)}$, we conclude that the data do not provide us any evidence against the null hypothesis H_0, and hence it may be accepted at α% level of significance. Otherwise reject H_0 or accept $H_1(1)$.

2. If $F \le F_{(\alpha),(n_1-1,n_2-1)}$, we conclude that the data do not provide us any evidence against the null hypothesis H_0, and hence it may be accepted at α% level of significance. Otherwise reject H_0 or accept $H_1(2)$.

3. If $F \ge F_{(1-\alpha),\ (n_1-1,n_2-1)}$, we conclude that the data do not provide us any evidence against the null hypothesis H_0, and hence it may be accepted at α% level of significance. Otherwise reject H_0 or accept $H_1(3)$.

P-value

$$\frac{S_1^2}{S_2^2} \sim F_{[(n_1-1)\ ,(n_2-1)]}$$

$$\mathrm{P}\left\{F_{\left[\left(1-\frac{\alpha}{2}\right),(n_1-1)\ ,(n_2-1)\right]} \leq \frac{S_1^2}{S_2^2} \leq F_{\left[\left(\frac{\alpha}{2}\right),(n_1-1)\ ,(n_2-1)\right]}\right\} = \ 1-\alpha$$

p-value = $2 * \ \min(P\{F_{n_1-1,\, n_2-1} < \ v\}, 1 \ - \ P\{F_{n_1-1,\, n_2-1} \ < \ v\})$

Example 7

A quality control supervisor for an automobile manufacturer is concerned with uniformity in the number of defects in cars coming off the assembly line. If one assembly line has significantly more variability in the number of defects, then changes have to be made. The supervisor has obtained the following data.

	Number of Defects	
	Assembly Line-A	*Assembly Line-B*
Mean	12	14
Variance	20	13
Sample size	16	20

Does assembly line A have significantly more variability in the number of defects? Test at 5% level of significance.

Solution

Aim: To test the assembly line A have significantly more variability than assembly line B in the number of defects or not.

H_0: There is no significant difference in variability between assembly line A and assembly line B in the number of defects. *i.e.*, $H_0: \sigma_1^2 = \sigma_2^2$.

H_1: The assembly line A has significantly more variability than assembly line B in the number of defects. *i.e.*, $H_1: \sigma_1^2 > \sigma_2^2$.

Level of Significance: $\alpha = 0.05$ and *Critical value*: $F_{0.05,\,(16\text{-}1,\ 20-1)} = 2.23$

Test statistic $$F = \frac{S_1^2}{S_2^2}$$

$$F = \frac{\left(\frac{n_1}{n_1-1}\right) s_1^2}{\left(\frac{n_2}{n_2-1}\right) s_2^2} = \frac{\left(\frac{16}{15}\right) 20}{\left(\frac{20}{19}\right) 13} = 1.56$$

***Conclusion*:** Since $|\boldsymbol{F}| < \mathbf{2.23}$, we conclude that the data do not provide us any evidence against the null hypothesis $\boldsymbol{H_0}$ and hence it is be accepted at $\mathbf{5\%}$ level of significance.

i.e., there is no significance difference in variability between assembly line A and assembly lien B in the number of defects.

Example 8

An insurance company is interested in the length of hospital-stays for various illnesses. The company has selected 15 patients from hospital A and 10 from hospital B who were treated for the same ailment. The amount of time spent in hospital A had an average of 2.6 days with a standard deviation of 0.8 day. The treatment time in hospital B averaged 2.2 days with a standard deviation of 1.2 day. Do patients in hospital A have significantly less variability in their recovery time? Test at 1% level of significance.

Solution

Aim: To test the patients in hospital A, have significantly less variability than the patients do in hospital B, in their recovery time.

H_0: There is no significant difference in recovery time in variability between the patients in hospital A and hospital B. *i.e.*, $H_0: \sigma_1^2 = \sigma_2^2$.

H_1: The patients in hospital A, have significantly less variability than the patients do in hospital B, in their recovery time.

i.e., $H_1: \sigma_1^2 < \sigma_2^2 \Rightarrow H_1: \sigma_2^2 > \sigma_1^2$.

Level of Significance: $\alpha = 0.01$ and *Critical value*: $F_{0.01,\,(10-1,\,15-1)} = 4.03$.

The moisture content in 10 samples of wheat determined by a research laboratory and 11 samples of same wheat by a government laboratory was found to be as presented below:

Res. Lab	7.7	9.4	6.6	5.5	8.1	5.9	7.9	6.9	9.7	7.4	
Govt. Lab.	7.5	9.1	6.8	8.0	6.4	7.4	6.5	9.6	9.3	7.7	8.5

Test the hypothesis that there is no difference in the determinations made by two laboratories with regard to average moisture content. [$t = -0.6954$]

Example

Following data give the price per kg of a standard commodity in tow cites. The prices were enquired from a sample of 12 shops in city A and 14 shops in city B. Test whether the variability in prices in cities A and B is same ? [F =1.24]

City A	60.5	71.3	65.3	67.8	61.4	68.3	66.0	64.5	63.3	62.8	69.0	70.5		
City B	61.4	58.4	56.6	58.3	54.5	59.2	55.5	53.2	60.0	51.8	53.6	52.5	57.4	52.0

5. Eleven students were given a test in Statistics. There were given a month's tuition and a second test was held at the end of it. Do the marks give evidence that the students have benefited by the extra coaching? Test the hypothesis at 5% and 1% level of significance. Also estimate the 95% confidence limits for paired observations.

$[t = 1.483]$

Marks in 1^{st} Test	23	20	19	21	18	20	18	17	23	16	19
Marks in 2^{nd} Test	24	19	22	18	20	22	20	20	23	20	18

Test for Goodness of Fit

Aim

To test that, the observed frequencies are good for fit with the theoretical frequencies. That is, to investigate the significance of the difference between the observed frequencies and the expected frequencies, arranged in K classes.

Source

Let O_i, $(i = 1, 2, \ldots, K)$ is a set of observed frequencies on K classes based on any experiment and E_i $(i = 1, 2, \ldots, K)$ is the corresponding set of expected (theoretical or hypothetical) frequencies.

Assumptions

(*i*) The observed frequencies in the K classes should be independent.

(*ii*) $$\sum_{i=1}^{K} O_i = \sum_{i=1}^{K} E_i = N.$$

(*iii*) The total frequency, N should be sufficiently large (*i.e.*, $N > 50$).

(*iv*) Each expected frequency in the K classes should be at least 5.

Null Hypothesis

$\boldsymbol{H_0}$: The observed frequencies are good for fit with the theoretical frequencies. That is, there is no significant difference between the observed frequencies and the expected frequencies, arranged in *K* classes.

Alternative Hypothesis

$\boldsymbol{H_1}$: The observed frequencies are not good for fit with the theoretical frequencies. That is, there is a significant difference between the observed frequencies and the expected frequencies, arranged in *K* classes.

Level of Significance$(\boldsymbol{\alpha})$ and critical region

$$\chi^{\mathbf{2}} > \chi^{\mathbf{2}}_{\boldsymbol{\alpha},(\boldsymbol{k}-\mathbf{1})} \quad \text{such that} \qquad \boldsymbol{P}\left\{\chi^{\mathbf{2}} > \chi^{\mathbf{2}}_{\boldsymbol{\alpha},(\boldsymbol{k}-\mathbf{1})}\right\} = \boldsymbol{\alpha}$$

Test Statistic:

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

The statistic χ^2 follows χ^2 distribution with $(k-1)$ degree of freedom.

Conclusion: If $\chi^2 \leq {\chi^2}_{(\alpha,\, k-1)}$, we conclude that the data do not provide any evidence against the null hypothesis $\boldsymbol{H_0}$ and hence it may be accepted at $\alpha\%$ level of significance. Otherwise reject $\boldsymbol{H_0}$ and accept $\boldsymbol{H_1}$.

If the population follows the hypothesized distribution, χ^2_{cal} has, approximately,

a chi-square distribution with $(\boldsymbol{k-p-1})$ degrees of freedom, where $\boldsymbol{p}-$ represents the number of parameters of the hypothesized distribution estimated by sample statistics. This approximation improves as $\boldsymbol{n}$ increases.

Important Note: The test procedure concerns the magnitude of the expected frequencies. If these expected frequencies are too small, the test statistic will not reflect the departure of observed from expected, but only the small magnitude of the expected frequencies.

There is no general agreement regarding the minimum value of expected frequencies, but values of 3, 4, and 5 are widely used as minimal. Some writers suggest that an expected frequency could be as small as 1 or 2, so long as most of them exceed 5. Should an expected frequency be too small, it can be combined with the expected frequency in an adjacent class interval. The corresponding observed frequencies would then also be combined, and $\boldsymbol{k}$ would be reduced by $\boldsymbol{1}$. Class intervals are not required to be of equal width.

Example 9

The sales of milk from a milk booth are varying from day-to-day. A sample of one-week sales (Number of Liters) is observed as follows.

Day:	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Sales:	154	145	152	140	135	165	173

Examine whether the sales of milk are same over the entire week at 1% level of significance.

Solution

Aim: To test the sales of milk is same over the entire week or not.

$\boldsymbol{H_0}$: The sale of milk is same over the entire week.

$\boldsymbol{H_1}$: The sale of milk is not same over the entire week.

Level of Significance: $\boldsymbol{\alpha} = 0.01$

Critical value: $\boldsymbol{\chi^2}_{\mathbf{0.01,\,6}} = 16.812$

Day	*Frequency*		$(O_i - E_i)^2$	$\dfrac{(O_i - E_i)^2}{E_i}$
	Observed (O_i)	*Expected* (E_i)		
Monday	154	152	4	0.0263
Tuesday	145	152	49	0.3224
Wednesday	152	152	0	0.0000
Thursday	140	152	144	0.9474
Friday	135	152	289	1.9013
Saturday	165	152	169	1.1118
Sunday	173	152	441	2.9013
1064	1064		7.2105	

Test Statistic:

$$\chi^2 = \sum_{i=1}^{7} \frac{(O_i - E_i)^2}{E_i} = 7.2105$$

Conclusion: If $\chi^2 \leq \chi^2{}_{(0.01,\,6)}$, we conclude that the data do not provide any evidence against the null hypothesis $\boldsymbol{H_0}$ and hence $\boldsymbol{H_0}$ is accepted at 1% level of significance.

i.e., The sale of milk is same over the entire week

Example

The number of defects in printed circuit boards is hypothesized to follow a Poisson distribution.

A random sample of **60** printed boards has been collected and the following number of defects observed.

Number of defects	**Observed Frequency**
0	**32**
1	**15**
2	**9**
3	**4**

Does the assumption of the Poisson distribution seem appropriate as a probability model for this data? Or Perform a goodness-of-fit procedure with $\boldsymbol{\alpha = 0.05}$.

Example

Fit a binomial distribution to the following data:

X	0	1	2	3	4
Frequency	8	46	55	40	11

Perform a goodness-of-fit procedure with $\boldsymbol{\alpha = 0.05}$.

Example

A computer program is written to generate random numbers, $\boldsymbol{X}$, **uniformly** in the interval $\mathbf{0 \leq X \leq 10}$. From 250 consecutive values, the following data are obtained:

$\boldsymbol{X}$	0 – 1.99	2 – 3.99	4 – 5.99	6 – 7.99	8 – 9.99
Frequency	38	55	54	41	62

Do these data offer any evidence that the program is not properly written?

Example

Fit a binomial distribution to the following data:

X	0	1	2	3	4
Frequency	8	46	55	40	11

Perform a goodness-of-fit procedure with 5% *l.o.s*.

Example

An experiment consists of tossing a coin until the first head shows up. One hundred repetitions of this experiment are performed. The frequency distribution of the number of trials required for the first head is as follows:

No. of trails	0	1	2	3	4
Frequency	40	32	15	7	6

Can we conclude that the coin is fair?

Test for Independence of Attributes

Aim

To test the given two attributes are independent, based on the observed frequencies, obtained from any sample survey.

Source

A random sample of N observed frequencies be classified into m classes by attribute-A and n classes by attribute-B. The above observed frequencies can be expressed in the following table known as $m \times n$ contingency table.

		Attribute-B 1	2	…	j	…	n	*Total*
	1	O_{11}	O_{12}	…	O_{1j}	…	O_{1n}	$O_{1\cdot}$
	2	O_{21}	O_{22}	…	O_{2j}	…	O_{2n}	$O_{2\cdot}$
	…	…	…	…	…	…	…	…
	…	…	…	…	…	…	…	…
	…	…	…	…	…	…	…	…
Attribute A	i	O_{i1}	O_{i2}	…	O_{ij}	…	O_{in}	$O_{i\cdot}$
	…	…	…	…	…	…	…	…
	…	…	…	…	…	…	…	…
	…	…	…	…	…	…	…	…
	m	O_{m1}	O_{m2}	…	O_{mj}	…	O_{mn}	$O_{m\cdot}$
	Total	$O_{\cdot 1}$	$O_{\cdot 2}$	…	$O_{\cdot j}$	…	$O_{\cdot n}$	N

Assumptions

(*i*) The sample size N, should be sufficiently large.

(*ii*) Each cell frequencies O_{ij} should be independent.

(*iii*) Each cell frequencies O_{ij} should be at least 5.

Null Hypothesis H_0

The two attributes are independent.

Alternative Hypothesis H_1

The two attributes are dependent.

Level of Significance (α) and Critical Region

$$\chi^2 > \chi^2_{\alpha,(m-1)\times(n-1)} \text{ such that } P\{\chi^2 > \chi^2_{\alpha,(m-1)\times(n-1)}\} = \alpha$$

Test Statistic:

$$\chi^2 = \sum_{i=1}^{m} \sum_{J=1}^{n} \left[\frac{(O_{ij} - E_{ij})^2}{E_{ij}}\right]$$

$$E_{ij} = \frac{O_{i.} \times O_{.j}}{N}$$

The statistic χ^2 follows χ^2-distribution with $[(m-1) \times (n \times 1)]$ degree of freedom.

Conclusion

If $\chi^2 \le \chi^2_{\alpha,(m-1)\times(n-1)}$, we conclude that the data do not provide us any evidence against the null hypothesis H_0, and hence it may be accepted at $\alpha\%$ level of significance. Otherwise reject H_0 or accept H_1.

Example 2

A company offers three bonus plans to its employees to give their preference to only one of the three plans. A sample of employees was selected and their opinion was obtained. The information gathered from them in accordance to the category of employees is presented in the following contingency table.

Category of employees	No. of employees favouring Plan A	Plan B	Plan C	Total
Factory workers	75	55	20	150
Clerical Staff	50	42	28	120
Technical Staff	26	18	16	60
Executives	19	25	16	60
Total	170	140	80	390

Can it be believed that the choice of bonus plan is independent of the category of employees.

Can it be believed that the choice of bonus plan is independent of the category of employees ?

Solution: $\boldsymbol{H_0}$: Employees category and bonus plans are independent.

$\boldsymbol{H_1}$: Employees category and bonus plans are not independent.

$$E(75) = \frac{row\ total \times coloumn\ total}{Grand\ total} = \frac{150 \times 170}{390} = 65.38$$

$$E(55) = \frac{150 \times 140}{390} = 53.85$$

$$E(20) = \frac{150 \times 80}{390} = 50.76$$

$$E(50) = \frac{120 \times 170}{390} = 52.30$$

$$E(42) = \frac{120 \times 140}{390} = 43.08$$

$$E(28) = \frac{120 \times 80}{390} = 24.62$$

$$E(26) = \frac{60 \times 170}{390} = 26.15$$

$$E(18) = \frac{60 \times 140}{390} = 21.54$$

$$E(16) = \frac{60 \times 80}{390} = 12.31$$

$$E(19) = \frac{60 \times 170}{390} = 26.15$$

$$E(25) = \frac{60 \times 140}{390} = 21.54$$

$$E(16) = \frac{60 \times 80}{390} = 12.31$$

$$\chi^2 = \sum_{i=1}^{4} \sum_{J=1}^{3} \left[\frac{(O_{ij} - E_{ij})^2}{E_{ij}} \right]$$

$$= \frac{(75 - 65.38)^2}{65.38} + \frac{(55 - 53.85)^2}{53.85} + \cdots + \frac{(16 - 12.31)^2}{12.31} = 11.84$$

Degree of freedom for $\chi^2 = (4 - 1)(3 - 1) = 6$

Critical value of χ^2 for 5% level of significance at 6 *d.f.* = 12.59

$$\therefore \chi^2{}_{cal} < \chi^2{}_{6,\,5\%}$$

Accept the null hypothesis $\boldsymbol{H_0}$

Conclusion: Preference of bonus scheme is independent of category of employees.

Example 3

In an experiment on pea breeding, Mendal obtained the following frequencies of seeds from 560 seeds: 312 rounded and yellow (RY), 104 wrinkled and yellow (WY); 112 round and green (RG), 32 wrinkled and green (WG). Theory predicts that the frequencies should be in the proportion 9:3:3:1 respectively. Set up the hypothesis and test it for 1% level.

Solution

Aim: To test the observed frequencies of the pea breeding in the ratio 9:3:3:1.

$\boldsymbol{H_0}$: The observed frequencies of the pea breeding are in the ratio 9:3:3:1.

$\boldsymbol{H_1}$: The observed frequencies of the pea breeding are not in the ratio 9:3:3:1.

Level of Significance: $\boldsymbol{\alpha} = 0.01$

Critical value: $\boldsymbol{\chi^2}_{\mathbf{0.01,\,3}} = 11.345$

Seed type	*Frequency*		$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
	Observed (O_i)	*Expected* (E_i)		
RY	312	315	9	0.0286
WY	104	105	1	0.0095
RG	112	105	49	0.4667
WG	32	35	9	0.2571
	560	560		0.7619

Test Statistic: $\chi^2 = \sum_{i=1}^{K} \left(\frac{O_i - E_i}{E_i} \right)^2 = 0.7619$

Conclusion: Since $\chi^2 < \chi^2_{\alpha,(K-1)}$, we conclude that the data do not provide us any evidence against the null hypothesis H_0. Hence, H_0 is accepted at 1% level of significance. That is, the observed frequencies of the pea breeding are in the ratio 9:3:3:1.

1. The mean height of 50 male students who showed above average participation in college athletics was 68.2 inches with a standard deviation of 2.5 inches, while 50 male students who showed no interest in such participation had a mean height of 67.5 inches with a standard deviation of 2.8 inches.

   (i) Test the hypothesis that male students who participate in college athletics are taller than other male students at 5% and 10% level of significance .

   (ii) Find the 90% confidence limits for difference mean height of two types of athletics.

2. In a year there are 956 births in a town A, of which 52.5% were males, while in towns A and B combined, this proportion in a total of 1406 births was 0.496. Is there any significant difference in the proportion of male births in the two towns?

3. The moisture content in 10 samples of wheat determined by a research laboratory and 11 samples of same wheat by a government laboratory was found to be as presented below:

Res. Lab	7.7	9.4	6.6	5.5	8.1	5.9	7.9	6.9	9.7	7.4	
Govt. Lab.	7.5	9.1	6.8	8.0	6.4	7.4	6.5	9.6	9.3	7.7	8.5

Test the hypothesis that there is no difference in the determinations made by two laboratories with regard to average moisture content. [t = -0.6954]

4. Following data give the price per kg of a standard commodity in tow cites. The prices were enquired from a sample of 12 shops in city A and 14 shops in city B. Test whether the variability in prices in cities A and B is same ? [F =1.24]

City A	60.5	71.3	65.3	67.8	61.4	68.3	66.0	64.5	63.3	62.8	69.0	70.5		
City B	61.4	58.4	56.6	58.3	54.5	59.2	55.5	53.2	60.0	51.8	53.6	52.5	57.4	52.0

5. Eleven students were given a test in Statistics. There were given a month's tuition and a second test was held at the end of it. Do the marks give evidence that the students have benefited by the extra coaching? Test the hypothesis at 5% and 1% level of significance. Also estimate the 95% confidence limits for paired observations.

$[t = 1.483]$

Marks in 1st Test	23	20	19	21	18	20	18	17	23	16	19
Marks in 2nd Test	24	19	22	18	20	22	20	20	23	20	18

6. A psychometric test was conducted among 150 children of three types, to test their ability. They were rated in three categories- good, poor and bad. Their frequencies in different cells were as displayed below.

Category	Types of children		
	Mentally retarted	Educationally handicapped	Physically handicapped
Good	6	14	34
Poor	13	28	10
Bad	31	8	6

Test whether there is any relationship between type of deficiency in children and their performance. [$\chi^2 = 59.786$]