230

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Probability, Statistics and Random Processes

Second Edition

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TESTS OF SIGNIFICANCE FOR SMALL SAMPLES

The tests of significance discussed in the previous section hold good only for large samples, i.e. only when the size of the sample $n \geq 30$. When the sample is small, i.e. $n < 30$, the sampling distributions of many statistics are not normal, even though the parent populations may be normal. Moreover the assumption of near equality of population parameters and the corresponding sample statistics will not be justified for small samples. Consequently we have to develop entirely different tests of significance that are applicable to small samples.

STUDENT'S *t*-DISTRIBUTION

A random variable T is said to follow student's t-distribution or simply t-distribution, if its probability density function is given by

$$f(t) = \frac{1}{\sqrt{v}\,\beta\left(\frac{v}{2}, \frac{1}{2}\right)} \cdot \left(1 + \frac{t^2}{v}\right)^{-(v+1)/2}, \; -\infty < t < \infty.$$

v is called the number of degrees of freedom of the t-distribution.

(***Note***: t-distribution was defined by the mathematician *W.S.D. Gosset* whose pen name is Student.)

Properties of *t*-Distribution

1. The probability curve of the t-distribution is similar to the standard normal curve and is symmetric about $t = 0$, bell-shaped and asymptotic to the t-axis as shown in the Fig. 9.2.
2. For sufficiently large value of n, the t-distribution tends to the standard normal distribution.
3. The mean of the t-distribution is zero.
4. The variance of the t-distribution is $\frac{v}{v-2}$, if $n > 2$ and is greater than 1, but it tends to 1 as $v \to \infty$.

Uses of *t*-Distribution

The t-distribution is used to test the significance of the difference between

1. The mean of a small sample and the mean of the population.
2. The means of two small samples and
3. The coefficient of correlation in the small sample and that in the population, assumed zero.

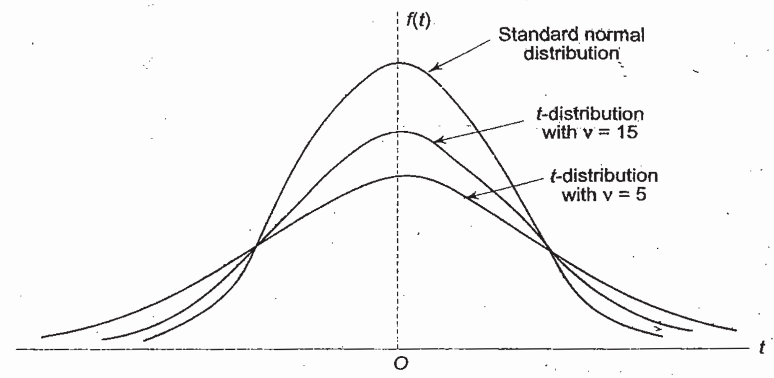

Fig. 9.2

Note on degree of freedom

The number of degrees of freedom, usually denoted by the Greek alphabet v, can be interpreted as the number of useful bits of information generated by a sample of given size for estimating a population parameter. Suppose we wish to find the mean of a sample with observations $x_1, x_2, \ldots, x_n$. We have to use all the 'n' values taken by the variable with full freedom (i.e. without putting any constraint or restriction on them) for computing $\bar{x}$. Hence $\bar{x}$ is said to have n degrees of freedom.

Suppose we wish to further compute the S.D. 's' of this sample using the formula $s^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$. Though we use the n values $x_1 - \bar{x}, x_2 - \bar{x}, \ldots, x_n - \bar{x}$ for this computation, they do not have 'n' degrees of freedom, as they depend on $\bar{x}$ which has been already calculated and fixed. Since there is one restriction regarding the value of $\bar{x}$, 's' is said to have $(n - 1)$ degrees of freedom.

If we compute another statistic of the sample based on $\bar{x}$ and s, then that statistic will be assumed to have $(n - 2)$ degrees of freedom and so on.

Thus the number of independent variates used to compute the test statistic is known as the number of degrees of freedom of that statistic. In general, the number of degrees of freedom is given by $v = n - k$, where n is the number of observations in the sample and k is the number of constraints imposed on them or k is the number of values that have been found out and specified by prior calculations.

Critical Values of *t* and the *t*-Table

The critical value of t at level of significance α and degrees of freedom v is given by $P\{|t| > t_v(\alpha)\} = \alpha$ for two-tailed test, as in the case of normal distribution and large samples and by $P\{t > t_v(\alpha)\} = \alpha$ for the right-tailed test also, as in the case of normal distribution. The critical value of t for a single (right or left) tailed test at LOS 'α' corresponding to v degrees of freedom is the same as that for a two-tailed test at LOS '2α' corresponding to the same degrees of freedom.

Critical values $t_v(\alpha)$ of the t-distribution for two-tailed tests corresponding to a few important levels of significance and a range of values of v have been published by Prof. R.A. Fisher in the form of a table, called the t-table, which is given in the Appendix.

Test 1

Test of significance of the difference between sample mean and population mean.

If $\bar{x}$ is the mean of a sample of size n, drawn from a population $N(\mu, \sigma)$, we have seen that $z = \dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ follows a $N(0, 1)$.

If σ, the S.D. of the population is not known, we have to estimate it using the sample S.D. 's'. From the theory of estimation, it is known that $s\sqrt{\dfrac{n}{n-1}}$ is an unbiased estimate of σ with $(n-1)$ degrees of freedom. When n is large, $\dfrac{n}{n-1} \simeq 1$ and hence s was taken as a satisfactory estimate of σ and hence $z = \dfrac{\bar{x}-\mu}{s/\sqrt{n}}$ was assumed to follow a $N(0, 1)$. But when n is small, we cannot use s as an estimate of σ, since

$$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{x}-\mu}{s\sqrt{\dfrac{n}{n-1}}\cdot\dfrac{1}{\sqrt{n}}} = \frac{\bar{x}-\mu}{s/\sqrt{n-1}}$$

Now $\dfrac{\bar{x}-\mu}{s/\sqrt{n-1}}$ does not follow a normal distribution, but follows a t-distribution with number of degrees of freedom $v = n-1$. Hence $\dfrac{\bar{x}-\mu}{s/\sqrt{n-1}}$ is denoted by t and is taken as the test-statistic.

Sometimes $t = \dfrac{\bar{x}-\mu}{s/\sqrt{n-1}}$ is also taken as $t = \dfrac{\bar{x}-\mu}{S/\sqrt{n}}$,

where $S^2 = \dfrac{1}{n-1}\sum_{r=1}^{n}(x_r-\bar{x})^2$ and is called students 't'. We shall use only $t = \dfrac{\bar{x}-\mu}{s/\sqrt{n-1}}$, where s is the sample S.D.

We get the value of $t_v(\alpha)$ for the L.O.S. α and $v = n-1$ from the t-table.

If the calculated value of t satisfies $|t| < t_v(\alpha)$, the null hypothesis H_0 is accepted at L.O.S. 'α' otherwise, H_0 is rejected at L.O.S. 'α'.

Note *95% confidence interval of m is given by*

$$\left|\frac{\bar{x}-\mu}{s/\sqrt{n-1}}\right| \le t_{0.05}, \quad \textit{since } P\left\{\left|\frac{\bar{x}-\mu}{s/\sqrt{n-1}}\right| \le t_{0.05}\right\} = 0.95$$

i.e. by $\bar{x} - t_{0.05}\dfrac{s}{\sqrt{n-1}} \le m \le \bar{x} + t_{0.05} \times \dfrac{s}{\sqrt{n-1}}$, *where $t_{0.05}$ is the 5 per cent critical value of t for n $(= n-1)$ degrees of freedom for a two-tailed test.*

Test 2

Test of significance of the difference between means of two small samples drawn from the same normal population.

In Test (4) for large samples, the test statistic used to test the significance of the difference between the means of two samples from the same normal population was taken as

$$z = \frac{\bar{x}_1-\bar{x}_2}{\sigma\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}}, \text{ which follows a } N(0, 1) \tag{1}$$

If σ is not known, we may assume that $\sigma \simeq \sqrt{\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2}}$, when n_1 and n_2 are large, where s_1 and s_2 are the sample S.D.'s. This assumption no longer holds good when n_1 and n_2 are small.

In fact, it is known from the theory of estimation, that an estimate of σ is $\sqrt{\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}}$ with (n_1+n_2-2) degrees of freedom, when n_1 and n_2 are small. Using this value of σ in (1), the test statistic becomes

$$\frac{\bar{x}_1-\bar{x}_2}{\sqrt{\left(\dfrac{n_1 s_1^2+n_2 s_2^2}{n_1+n_2-2}\right)\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)}},$$

which does not follow a $N(0, 1)$, but follows a t-distribution with $v = (n_1+n_2-2)$ degrees of freedom. Hence t-test is applied in this case.

Note *1. If $n_1 = n_2 = n$ and if the samples are independent i.e., the observations in the two samples are not at all related, then the test statistic is given by*

$$t = \frac{\bar{x}_1-\bar{x}_2}{\sqrt{\dfrac{s_1^2+s_2^2}{n-1}}} \quad \textit{with} \quad v = 2n-2 \tag{2}$$

2. If $n_1 = n_2 = n$ and if the pairs of values of x_1 and x_2 are associated in some way (or correlated), the formula (2) for t in Note (1) should not be used. In this case, we shall assume that $H_0 : \bar{d}\ (= \bar{x} - \bar{y}) = 0$ and test the significance of the difference between $\bar{d}$ and 0, using the test statistic $t = \dfrac{\bar{d}}{s/\sqrt{n-1}}$ with $\nu = n - 1$, where $d_i = x_i - y_i\ (i = 1, 2, \ldots, n)$, $\bar{d} = \bar{x} - \bar{y}$; and s = S.D. of d's $= \dfrac{1}{n}\sum_{i=1}^{n}\left(d_i - \bar{d}\right)^2$.

SNEDECOR'S F-DISTRIBUTION

A random variable F is said to follow snedecor's F-distribution or simply F-distribution, if its probability density function is given by

$$f(F) = \frac{(\nu_1/\nu_2)^{\nu_1/2}}{\beta\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \cdot \frac{F^{\frac{\nu_1}{2}-1}}{\left(1 + \frac{\nu_1 F}{\nu_2}\right)^{(\nu_1+\nu_2)/2}}, \quad F > 0.$$

Note *(The mathematical variable corresponding to the random variable F is also taken as F.) ν_1 and ν_2 used in $f(F)$ are the degrees of freedom associated with the F-distribution.*

Properties of the F-Distribution

1. The probability curve of the F-distribution is roughly sketched in Fig. 9.3.

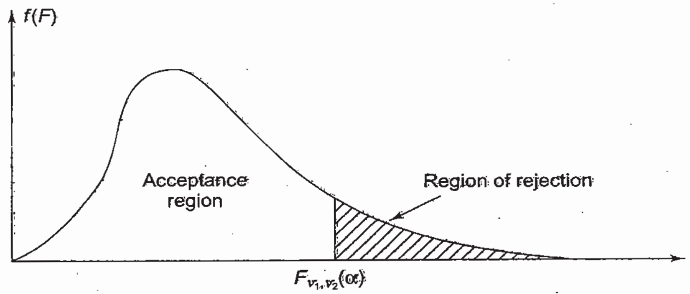

Fig. 9.3

2. The square of the t-variate with n degrees of freedom follows a F-distribution with 1 and n degrees of freedom.
3. The mean of the F-distribution is $\dfrac{\nu_2}{\nu_2 - 2}$ $(\nu_2 > 2)$.

4. The variance of the F-distribution is

$$\frac{2\nu_2^2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)} \quad (\nu_2 > 4).$$

Use of F-Distribution

F-distribution is used to test the equality of the variance of the populations from which two small samples have been drawn.

F-test of significance of the difference between population variances and F-table.

To test the significance of the difference between population variances, we shall first find their estimates, $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ based on the sample variances s_1^2 and s_2^2 and then test their equality. It is known that $\hat{\sigma}_1^2 = \dfrac{n_1 s_1^2}{n_1 - 1}$ with the number of degree of freedom $\nu_1 = n_1 - 1$ and $\hat{\sigma}_2^2 = \dfrac{n_2 s_2^2}{n_2 - 1}$ with the number of degrees of freedom $\nu_2 = n_2 - 1$.

It is also known that $F = \dfrac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}$ follows a F-distribution with ν_1 and ν_2 degrees of freedom. If $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$, then $F = 1$. Hence our aim is to find how far any observed value of F can differ from unity due to fluctuations of sampling.

Snedecor has prepared tables that give, for different values of ν_1 and ν_2, the 5 per cent and 1 per cent critical values of F. An extract from these tables is given in the Appendix. If F denotes the observed (calculated) value and $F_{\nu_1,\nu_2}(\alpha)$ denotes the critical (tabulated) value of F at LOS α, then $P\{F > F_{\nu_1,\nu_2}(\alpha)\} = \alpha$.

Note *F-test is not a two-tailed test and is always a right-tailed test, since F cannot be negative. Thus if $F > F_{\nu_1,\nu_2}(\alpha)$, then the difference between F and 1, i.e. the difference between $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ is significant at LOS 'α'. In other words, the samples may not be regarded as drawn from the same population or from populations with the same variance. If $F < F_{\nu_1,\nu_2}(\alpha)$, the difference is not significant at LOS α.*

1. *We should always make $F > 1$. This is done by taking the larger of the two estimates of σ^2 as σ_1^2 and by assuming that the corresponding degree of freedom as ν_1.*
2. *To test if two small samples have been drawn from the same normal population, it is not enough to test if their means differ significantly or not, because in this test we assumed that the two samples came from the same population or from populations with equal variance. So, before applying the t-test for the significance of the difference of two sample means, we should satisfy ourselves about the equality of the population variances by F-test.*

Worked Example 9(B)

Example 1

Tests made on the breaking strength of 10 pieces of a metal wire gave the results: 578, 572, 570, 568, 572, 570, 570, 572, 596 and 584 kg. Test if the mean breaking strength of the wire can be assumed as 577 kg.

Let us first compute sample mean $\bar{x}$ and sample S.D. 's' and then test if $\bar{x}$ differs significantly from the population mean $\mu = 577$.

We take the assumed mean $A = \dfrac{568 + 596}{2} = 582$

$$di = x_i - A$$

$\therefore$ $x_i = d_i + A$

$\therefore$ $\bar{x} = \frac{1}{n}\sum x_i = \frac{1}{n}\sum d_i + A$

$= \frac{1}{10} \times (-68) + 582 = 575.2$ (see Table 9.7 given below)

Table 9.7

x_i	$d_i = x_i - A$	d_i^2
578	−4	16
572	−10	100
570	−12	144
568	−14	196
572	−10	100
570	−12	144
570	−12	144
572	−10	100
596	14	196
584	2	4
Total	−68	1144

$$s^2 = \frac{1}{n}\sum d_i^2 - \left(\frac{1}{n}\sum d_i\right)^2$$

$$= \frac{1}{10} \times 1144 - \left(\frac{1}{10} \times -68\right)^2 = 68.16$$

$\therefore$ $s = 8.26$

Now $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n-1}} = \dfrac{575.2 - 577}{8.26/\sqrt{9}}$

$= -0.65$

and $v = n - 1 = 9$.

$H_0 : \bar{x} = \mu$ and $H_1 : \bar{x} \neq \mu$.

Let LOS be 5%. Two tailed test is to be used.

From the t-table, for $v = 9, t_{5\%} = 2.26$. Since $|t| < t_{5\%}$, the difference between $\bar{x}$ and μ is not significant or H_0 is accepted. $\therefore$ The mean breaking strength of the wire can be assumed as 577 kg at 5% LOS

Example 2

A machinist is expected to make engine parts with axle diameter of 1.75 cm. A random sample of 10 parts shows a mean diameter 1.85 cm. with a S.D. of 0.1 cm. On the basis of this sample, would you say that the work of the machinist is inferior?

$\bar{x} = 1.85$, $s = 0.1$, $n = 10$ and $\mu = 1.75$.

$H_0 : \bar{x} = \mu$; $H_1 : \bar{x} \neq \mu$

Two tailed test is to be used. Let L.O.S. be 5%

$t = \dfrac{\bar{x} - \mu}{s/\sqrt{n-1}} = \dfrac{0.10}{0.1/\sqrt{9}} = 3$ and $v = n - 1 = 9$.

From the t-table, for $v = 9$, $t_{0.05} = 2.26$ and $t_{0.01} = 3.25$.

$\therefore$ $|t| > t_{0.05}$ and $|t| < t_{0.01}$

$\therefore$ H_0 is rejected and H_1 is accepted at 5% level, but H_0 is accepted and H_1 is rejected at 1% level. i.e. At 5% LOS, the work of the machinist can be assumed to be inferior, but at 1% LOS, the work cannot be assumed to be inferior.

Example 3

A certain injection administered to each of 12 patients resulted in the following increases of blood pressure:

5, 2, 8, − 1, 3, 0, 6, − 2, 1, 5, 0, 4.

Can it be concluded that the injection will be, in general, accompanied by an increase in B.P.? The mean of the sample is given by $\bar{x} = \frac{1}{n}\sum x = \frac{31}{12} = 2.58$

The S.D. 's' of the sample is given by

$$s^2 = \frac{1}{n}\sum x^2 - \left(\frac{1}{n}\sum x\right)^2 = \frac{1}{12} \times 185 - (2.58)^2 = 8.76$$

$\therefore$ $s = 2.96$

$H_0 : \bar{x} = \mu$,

where $\mu = 0$, i.e. the injection will not result in increase in B.P.

$H_1 : \bar{x} > \mu$

Right-tailed test is to be used. Let L.O.S. be 5%. Now $t_{5\%}$ for one-tailed test for $(\nu = 11) = t_{10\%}$ for two-tailed test for $(\nu = 11) = 1.80$ (from t-table)

Now $$t = \frac{\bar{x}-\mu}{s/\sqrt{n-1}} = \frac{2.58-0}{2.96/\sqrt{11}} = 2.89$$

We see that $|t| > t_{10\%}$ $(\nu = 11)$

$\therefore$ H_0 is rejected and H_1 is accepted.

i.e. we may conclude that the injection is accompanied by an increase in B.P.

Example 4

The mean lifetime of a sample of 25 bulbs is found as 1550 hours with a S.D. of 120 hours. The company manufacturing the bulbs claims that the average life of their bulbs is 1600 hours. Is the claim acceptable at 5% level of significance?

$$\bar{x} = 1550, \quad s = 120, \quad n = 25 \quad \text{and} \quad \mu = 1600.$$

$$H_0 : \bar{x} = \mu \quad \text{and} \quad H_1 : \bar{x} < \mu.$$

Left-tailed test is to be used. LOS = 5%

Now $$t = \frac{\bar{x}-\mu}{s/\sqrt{n-1}} = \frac{-50\sqrt{24}}{120} = -2.04 \text{ and } \quad \nu = 24$$

$t_{5\%}$ for one-tailed test for $(\nu = 24) = t_{10\%}$ for two-tailed test for $(\nu = 24) = 1.71$.

We see that $|t| > |t_{10\%}|$

$\therefore$ H_0 is rejected and H_1 is accepted at 5% LOS

i.e. The claim of the company cannot be accepted at 5% LOS

Example 5

The heights of ten males of a given locality are found to be 175, 168, 155, 170, 152, 170, 175, 160, 160 and 165 cms. Based on this sample, find the 95% confidence limits for the height of males in that locality.

We shall first find the mean $\bar{x}$ and S.D. 's' of the sample, by taking the assumed mean $A = 165$ (Table 9.8).

$$d_i = x_i - A$$

$$\therefore \quad \bar{x} = A + \bar{d}$$

$$= 165 + \frac{1}{10} \times 0 = 165.$$

$$s^2 = \frac{1}{n}\sum d_i^2 - \left(\frac{1}{n}\sum d_i\right)^2$$

$$= \frac{1}{10} \times 578 = 57.8$$

$$\therefore \quad s = 7.6$$

From the t-table,

$$t_{5\%}\ (\nu = 9) = 2.26.$$

The 95% confidence limits for μ are

$$\left(\bar{x} - 2.26\frac{s}{\sqrt{n-1}},\ \bar{x} + 2.26\frac{s}{\sqrt{n-1}}\right)$$

i.e. $$\left(165 - \frac{2.26 \times 7.6}{\sqrt{9}},\ 165 + \frac{2.26 \times 7.6}{\sqrt{9}}\right)$$

i.e. $$(159.3, 170.7)$$

i.e. the heights of males in the locality are likely to lie within 159.3 cm and 170.7 cm.

Table 9.8

x_i	$d_i = x_i - A$	d_i^2
175	10	100
168	3	9
155	– 10	100
170	5	25
152	– 13	169
170	5	25
175	10	100
160	– 5	25
160	– 5	25
165	0	0
Total	0	578

Example 6

Two independent samples of sizes 8 and 7 contained the following values:

Sample I : 19, 17, 15, 21, 16, 18, 16, 14
Sample II : 15, 14, 15, 19, 15, 18, 16

Is the difference between the sample means significant?

Table 9.9

Sample I			Sample II		
x_1	$d_1 = x_1 - 18$	d_1^2	x_2	$d_2 = x_2 - 16$	d_2^2
19	1	1	15	– 1	1
17	–1	1	14	– 2	4
15	– 3	9	15	– 1	1
21	3	9	19	3	9
16	– 2	4	15	– 1	1
18	0	0	18	2	4
16	– 2	4	16	0	0
14	– 4	16			
Total	– 8	44	Total	0	20

For sample I, $\bar{x}_1 = 18 + \bar{d}_1 = 18 + \frac{1}{8}\sum d_1$

$= 18 + \frac{1}{8} \times (-8) = 17.$

$$s_1^2 = \frac{1}{n_1}\sum d_1^2 - \left(\frac{1}{n_1}\sum d_1\right)^2$$

$$= \frac{1}{8} \times 44 - \left(\frac{1}{8} \times -8\right)^2 = 4.5$$

$\therefore$ $s_1 = 2.12.$

For sample II, $\bar{x}_2 = 16 + \bar{d}_2 = 16 + \frac{1}{7}\sum d_2 = 16.$

$$s_2^2 = \frac{1}{n_2}\sum d_2^2 - \left(\frac{1}{n_2}\sum d_2\right)^2$$

$$= \frac{1}{7} \times 20 - \left(\frac{1}{7} \times 0\right)^2 = 2.857$$

$\therefore$ $s_2 = 1.69$

$H_0 : \bar{x}_1 = \bar{x}_2$ and $H_1 : \bar{x}_1 \neq \bar{x}_2$

Two-tailed test is to be used. Let LOS be 5 %

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{17 - 16}{\sqrt{\left(\frac{8 \times 4\cdot5 + 7 \times 2\cdot857}{13}\right)\left(\frac{1}{8} + \frac{1}{7}\right)}}$$

$= 0.93$

Also $\nu = n_1 + n_2 - 2 = 13.$

From the t-table, $t_{5\%}$ $(\nu = 13) = 2.16$

Since $|t| < t_{5\%}$, H_0 is accepted and H_1 is rejected.

i.e. the two sample means do not differ significantly at 5% LOS

Example 7

Table 9.10 give's the biological values of protein from cow's milk and buffalo's milk at a certain level. Examine if the average values of protein in the two samples significantly differ.

Table 9.10

Cow's milk (x_1):	1.82,	2.02,	1.88,	1.61,	1.81,	1.54
Buffalo's milk (x_2):	2.00,	1.83,	1.86,	2.03,	2.19,	1.88

$n = 6$

$$\bar{x}_1 = \frac{1}{6} \times 10.68 = 1.78$$

$$s_1^2 = \frac{1}{6} \times \Sigma x_1^2 - (\bar{x}_1)^2 = \frac{1}{6} \times 19.167 - (1.78)^2 = 0.0261$$

$$\bar{x}_2 = \frac{1}{6} \times 11.79 = 1.965$$

$$s_2^2 = \frac{1}{6} \times \Sigma x_2^2 - (\bar{x}_2)^2 = \frac{1}{6} \times 23.2599 - (1.965)^2 = 0.0154$$

As the two samples are independent, the test statistic is given by t

$$= \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2 + s_2^2}{n-1}}}$$

with $\nu = 2n - 2$ [Refer to note (2) under Test (2)]

$$t = \frac{1\cdot78 - 1\cdot965}{\sqrt{\frac{0\cdot0261 + 0\cdot0154}{5}}} = \frac{-0\cdot185}{\sqrt{\cdot0083}} = -2.03 \text{ and } \nu = 10.$$

$H_0 : \bar{x}_1 = \bar{x}_2$ and $H_1 : \bar{x}_1 \neq \bar{x}_2$.

Two tailed test is to be used. Let LOS be 5%

From t-table, $t_{5\%}$ $(\nu = 10) = 2.23.$

Since $|t| < t_{5\%}$ $(\nu = 10)$, H_0 is accepted.

i.e. the difference between the mean protein values of the two varieties of milk is not significant at 5% level.

Example 8

Samples of two types of electric bulbs were tested for length of life and the following data were obtained.

	Size	Mean	S.D.
Sample I	8	1234 hours	36 hours
Sample II	7	1036 hours	40 hours

Is the difference in the means sufficient to warrant that type I bulbs are superior to type II bulbs?

$\bar{x}_1 = 1234$, $s_1 = 36$, $n_1 = 8$; $\bar{x}_2 = 1036$, $s_2 = 40$, $n_2 = 7$

$H_0 : \bar{x}_1 = \bar{x}_2$; $H_1 : \bar{x}_1 > \bar{x}_2$.

Right-tailed test is to be used. Let LOS be 5%

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{198}{\sqrt{\left(\frac{21568}{13}\right)\left(\frac{1}{8} + \frac{1}{7}\right)}} = \frac{198}{21\cdot0807}$$

$$= 9.39$$

$$v = n_1 + n_2 - 2 = 13$$

$t_{5\%}$ ($v = 13$) for one-tailed test $= t_{10\%}$ ($v = 13$) for two tailed test = 1.77 (from t-table)

Now $t > t_{10\%}$ ($v = 13$)

$\therefore$ H_0 is rejected and H_1 is accepted

i.e. Type I bulbs may be regarded superior to type II bulbs at 5% LOS.

Example 9

The mean height and the S.D. height of eight randomly chosen soliders are 166.9 cm. and 8.29 cm. respectively. The corresponding values of six randomly chosen sailors are 170.3 cm and 8.50 cm. respectively. Based on this data, can we conclude that soldiers are, in general, shorter than sailors?

$$\bar{x}_1 = 166.9, \quad s_1 = 8.29, \quad n_1 = 8; \quad \bar{x}_2 = 170.3, \quad s_2 = 8.50, \; n_2 = 6.$$

$H_0 : \bar{x}_1 = \bar{x}_2$; $\quad H_1 : \bar{x}_1 < \bar{x}_2$.

Left-tailed test is to be used. Let LOS be 5%.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} = \frac{-3{\cdot}4}{\sqrt{\left(\dfrac{983{\cdot}29}{12}\right)\left(\dfrac{1}{8} + \dfrac{1}{6}\right)}}$$

$$= -0.695$$

$$v = n_1 + n_2 - 2 = 12$$

$t_{5\%}$ ($v = 12$) for one-tailed test $= t_{10\%}$ ($v = 12$) for two tailed test = 1.78 (from t-table)

Now $|t| < t_{10\%}$ ($v = 12$)

$\therefore$ H_0 is accepted and H_1 is rejected.

i.e. based on the given data, we cannot conclude that soldiers are in general, shorter than sailors.

Example 10

The following data relate to the marks obtained by 11 students in two tests, one held at the beginning of a year and the other at the end of the year after intensive coaching. Do the data indicate that the students have benefited by coaching?

Test 1:	19,	23,	16,	24,	17,	18,	20,	18,	21,	19,	20
Test 2:	17,	24,	20,	24,	20,	22,	20,	20,	18,	22,	19

The given data relate to the marks obtained in two tests by the same set of students. Hence the marks in the two tests can be regarded as correlated and so the t-test for paired values should be used.

Let $\qquad d = x_1 - x_2,$

where x_1, x_2 denote the marks in the two tests.

Thus the values of d are 2, – 1 – 4, 0, – 3, – 4, 0, – 2, 3, – 3, 1.

$$\Sigma d = -11 \quad \text{and} \quad \Sigma d^2 = 69$$

$$\therefore \quad \bar{d} = \frac{1}{n}\Sigma d = \frac{1}{11} \times -11 = -1$$

$$s^2 = s_d^2 = \frac{1}{n}\Sigma d^2 - (\bar{d})^2 = \frac{1}{11} \times 69 - (-1)^2 = 5.27$$

$$\therefore \quad s = 2.296$$

$H_0 : \bar{d} = 0$, i.e. the students have not benefited by coaching; $H_1 : \bar{d} < 0$ (i.e. $\bar{x}_1 < \bar{x}_2$).

One-tailed test is to be used. Let LOS be 5%

$$t = \frac{\bar{d}}{s/\sqrt{n-1}} = \frac{-1}{2{\cdot}296/\sqrt{10}} = -1.38 \quad \text{and} \quad v = 10$$

$t_{5\%}$ ($v = 10$) for one-tailed test $= t_{10\%}$ ($v = 10$) for two-tailed test = 1.81 (from t-table).

Now $|t| < t_{10\%}$ ($v = 10$)

$\therefore$ H_0 is accepted and H_1 is rejected.

i.e. there is no significant difference between the two sets of marks.

i.e. the students have not benefitted by coaching.

Example 11

A sample of size 13 gave an estimated population variance of 3.0, while another sample of size 15 gave an estimate of 2.5. Could both samples be from populations with the same variance?

$$n_1 = 13, \quad \hat{\sigma}_1^2 = 3.0 \quad \text{and} \quad v_1 = 12$$

$$n_2 = 15, \quad \hat{\sigma}_2^2 = 2.5 \quad \text{and} \quad v_2 = 14.$$

$H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2$, i.e. The two samples have been drawn from populations with the same variance.

$H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$. Let L.O.S. be 5%

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{3{\cdot}0}{2{\cdot}5} = 1.2$$

$$v_1 = 12 \quad \text{and} \quad v_2 = 14.$$

$F_{5\%}$ ($v_1 = 12$, $v_2 = 14$) = 2.53, from the F-table.

$F < F_{5\%}$. $\therefore$ H_0 is accepted

i.e. the two samples could have come from two normal populations with the same variance.

Example 12

Two samples of sizes nine and eight gave the sums of squares of deviations from their respective means equal to 160 and 91 respectively. Can they be regarded as drawn from the same normal population?

$$n_1 = 9, \quad \Sigma(x_i - \bar{x})^2 = 160, \quad \text{i.e. } n_1 s_1^2 = 160$$

$$n_2 = 8, \quad \Sigma(y_i - \bar{y})^2 = 91, \quad \text{i.e. } n_2 s_2^2 = 91$$

$$\hat{\sigma}_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{1}{8} \times 160 = 20\,; \quad \hat{\sigma}_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{1}{7} \times 91 = 13$$

Since $\hat{\sigma}_1^2 > \hat{\sigma}_2^2$, $\nu_1 = n_1 - 1 = 8$ and $\nu_2 = n_2 - 1 = 7$

$$H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2 \quad \text{and} \quad H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2.$$

Let the LOS be 5%

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{20}{13} = 1.54$$

$F_{5\%}(\nu_1 = 8, \nu_2 = 7) = 3.73$, from the F-table.
Since $F < F_{5\%}$, H_0 is accepted.

i.e. the two samples could have come from two normal populations with the same variance.

We cannot say that the samples have come from the same population, as we are unable to test if the means of the samples differ significantly or not.

Example 13

Two independent samples of eight and seven items respectively had the following values of the variable.

Sample 1 :	9,	11,	13,	11,	15,	9,	12,	14
Sample 2 :	10,	12,	10,	14,	9,	8,	10	

Do the two estimates of population variance differ significantly at 5% level of significance?

For the first sample, $\Sigma x_1 = 94$ and $\Sigma x_1^2 = 1138$

$$\therefore \quad s_1^2 = \frac{1}{n_1}\sum x_1^2 - \left(\frac{1}{n_1}\sum x_1\right)^2$$

$$= \frac{1}{8} \times 1138 - \left(\frac{1}{8} \times 94\right)^2 = 4.19$$

For the second sample, $\Sigma x_2 = 73$ and $\Sigma x_2^2 = 785$

$$\therefore \quad s_2^2 = \frac{1}{n_2}\sum x_2^2 - \left(\frac{1}{n_2}\sum x_2\right)^2$$

$$= \frac{1}{7} \times 785 - \left(\frac{1}{7} \times 73\right)^2 = 3.39$$

$$\hat{\sigma}_1^2 = \frac{n_1}{n_1 - 1} s_1^2 = 4.79 \quad \text{and} \quad \hat{\sigma}_2^2 = \frac{n_2}{n_2 - 1} s_2^2 = 3.96$$

since $\hat{\sigma}_1^2 > \hat{\sigma}_2^2$, $\nu_1 = 7$ and $\nu_2 = 6$

$$H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2 \quad \text{and} \quad H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$$

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{4 \cdot 79}{3 \cdot 96} = 1.21$$

$F_{5\%}(\nu_1 = 7, \nu_2 = 6) = 4.21$, from the F-table. Since $F < F_{5\%}$, H_0 is accepted.
i.e. $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ do not differ significantly at 5% level of significance.

Example 14

Two random samples gave the following data:

	Size	Mean	Variance
Sample I	8	9.6	1.2
Sample II	11	16.5	2.5

Can we conclude that the two samples have been drawn from the same normal population?

Refer to Note (2) under F-test. To conclude that the two samples have been drawn from the same population, we have to check first that the variances of the populations do not differ significantly and then check that the sample means (and hence the population means) do not differ significantly.

$$\hat{\sigma}_1^2 = \frac{8 \times 1 \cdot 2}{7} = 1.37\,; \quad \hat{\sigma}_2^2 = \frac{11 \times 2 \cdot 5}{10} = 2.75$$

$$F = \frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2} = \frac{2 \cdot 75}{1 \cdot 37} = 2.007 \text{ with degrees of freedom 10 and 7.}$$

From the F-table, $F_{5\%}(10, 7) = 3.64$

If $\quad H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2$ and $H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$,

H_0 is accepted, since $F < F_{5\%}$
i.e. the variances of the populations from which samples are drawn may be regarded as equal.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} = \frac{-6 \cdot 9}{\sqrt{\left(\dfrac{9 \cdot 6 + 27 \cdot 5}{17}\right)\left(\dfrac{1}{8} + \dfrac{1}{11}\right)}}$$

$$= -\frac{6 \cdot 9}{0 \cdot 6864} = -10.05$$

and $\qquad v = n_1 + n_2 - 2 = 17.$

$t_{5\%}\ (v = 17) = 2.11$, from the t-table.

If $H_0 : \bar{x}_1 = \bar{x}_2$ and $H_1 : \bar{x}_1 \neq \bar{x}_2$, H_0 is rejected, since $|t| > t_{5\%}$.

i.e. the means of two samples (and so the populations) differ significantly.

$\therefore$ The two samples could not have been drawn from the same normal population.

Example 15

The nicotine contents in two random samples of tobacco are given below.

Sample I :	21	24	25	26	27	
Sample II :	22	27	28	30	31	36.

Can you say that the two samples came from the same population?

$$\bar{x}_1 = \text{Mean of sample I} = \frac{123}{5} = 24.6$$

$$\bar{x}_2 = \text{Mean of sample II} = \frac{174}{6} = 29.0$$

$$s_1^2 = \text{Variance of sample I} = \frac{1}{5}\,\Sigma\,(x_i - 24.6)^2 = 4.24$$

$$s_2^2 = \text{Variance of sample II} = \frac{1}{6}\,\Sigma\,(x_i - 29.0)^2 = 18.0$$

$$\hat{\sigma}_1^2 = \frac{5}{4} \times 4.24 = 5.30 \text{ and } v = 4;\ \hat{\sigma}_2^2 = \frac{6}{5} \times 18.0 = 21.60 \text{ and } v = 5$$

$$H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2;\quad H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$$

$$F = \frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2} = \frac{21{\cdot}60}{5{\cdot}30} = 4.07$$

$F_{5\%}\ (5, 4) = 6.26.$

Since $\quad F < F_{5\%}$, H_0 is accepted.

$\therefore$ The variances of the two populations can be regarded as equal.

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} = \frac{-4{\cdot}4}{\sqrt{\left(\dfrac{21{\cdot}2 + 108{\cdot}0}{9}\right)\left(\dfrac{1}{5} + \dfrac{1}{6}\right)}}$$

$$= \frac{-4{\cdot}4}{2{\cdot}2943} = -1.92$$

and $\qquad v = 9.$

From t-table, $F_{5\%}\ (v = 9) = 2.26.$

If $\quad H_0 : \bar{x}_1 = \bar{x}_2$ and $H_1 : \bar{x}_1 \neq \bar{x}_2$, H_0 is accepted

since $|t| < F_{5\%}$.

That is, the means of two samples (and hence the populations) do not differ significantly. Therefore, the two samples could have been drawn from the same normal population.

Exercise 9(B)

Part A

(Short Answer Questions)

1. Write down the probability density of student's t-distribution.
2. State the important properties of the t-distribution.
3. Give any two uses of t-distribution.
4. What do you mean by degrees of freedom?
5. How will you get the critical value of t for a single-tailed test at level of sign-ificance a?
6. What is the test statistic used to test the significance of the difference between small sample mean and population mean?
7. Give the 95% confidence interval of the population mean in terms of the mean and S.D. of a small sample.
8. What is the test statistic used to test the significance of the difference between the means of two small samples?
9. Give an estimate of the population variance in terms of variances of two small samples. What is the associated number of degrees of freedom?
10. What is the test statistic used to test the significance of the difference between the means of two small samples of the same size? What is the associated number of degree of freedom?
11. What is the test statistic used to test the significance of the difference between the means of two small samples of the same size, when the sample items are correlated?
12. Write down the probability density function of the F-distribution.
13. State the important properties of the F-distribution.
14. What is the use of F-distribution?
15. Why is the F-distribution associated with two numbers of degrees of freedom?

Part B

16. A random sample of ten boys had the following I.Q.'s: 70, 120, 110, 101, 88, 83, 95, 98, 107, 100. Does the data support the assumption of a population mean I.Q. of 100? Find a reasonable range in which most of the mean I.Q. values of samples of ten boys lie.
17. A random sample of 16 values from a normal population showed a mean of 103.75 cm. and the sum of the squares of deviations from this mean is equal to 843.75 square cms. Show that the assumption of a mean of 108.75 cm for the population is not reasonable. Obtain 95% and 99% fiducial limits for the same.

$$f(\chi^2) = \frac{1}{2^{\nu/2}\sqrt{\left(\frac{\nu}{2}\right)}} \cdot (\chi^2)^{\nu/2-1} \, e^{-\chi^2/2}$$

$0 < \chi^2 < \infty$, where ν is the number of degrees of freedom.

Properties of χ^2-Distribution

1. A rough sketch of the probability curve of the χ^2-distribution for $\nu = 3$ and $\nu = 6$ is given in Fig. 9.4.
2. As ν becomes smaller and smaller, the curve is skewed more and more to the right. As ν increases, the curve becomes more and more symmetrical.
3. The mean and variance of the χ^2-distribution are ν and 2ν respectively.

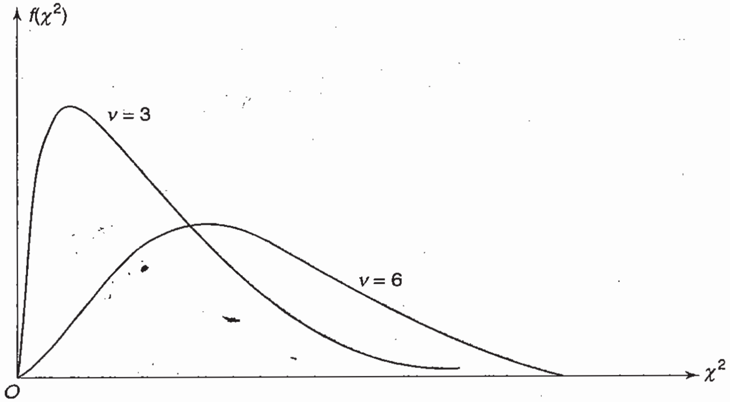

Fig. 9.4

4. As n tends to ∞, the χ^2-distribution becomes a normal distribution.

Uses of χ^2-Distribution

1. χ^2-distribution is used to test the goodness of fit. i.e., it is used to judge whether a given sample may be reasonably regarded as *a* simple sample from a certain hypothetical population.
2. It is used to test the independence of attributes. i.e. If a population is known to have two attributes (or traits), then χ^2-distribution is used to test whether the two attributes are associated or independent, based on a sample drawn from the population.

χ^2-Test of Goodness of Fit

On the basis of the hypothesis assumed about the population, we find the expected frequencies $E_i (i = 1, 2, \ldots, n)$, corresponding to the observed frequencies

$O_i (i = 1, 2, \ldots, n)$ such that $\sum E_i = \sum O_i$. It is known that $\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$

follows approximately a χ^2-distribution with degrees of freedom equal to the number of independent frequencies. In order to test the goodness of fit, we have to determine how far the differences between O_i and E_i can be attributed to fluctuations of sampling and when we can assert that the differences are large enough to conclude that the sample is not a simple sample from the hypothetical population. In other words, we have to determine how large a value of χ^2 we can get so as to assume that the sample is a simple sample from the hypothetical population.

The critical value of χ^2 for ν degrees of freedom at α level of significance, denoted by $\chi^2_\nu(\alpha)$ is given by

$$P[\chi^2 > \chi^2_\nu(\alpha)] = \alpha.$$

Critical values of the χ^2-distribution corresponding to a few important levels of significance and a range of values of ν are available in the form of a table called χ^2-table, which is given in the Appendix.

If the calculated $\chi^2 < \chi^2_\nu(\alpha)$, we will accept the null hypothesis H_0 which assumes that the given sample is one drawn from the hypothetical population, i.e. we will conclude that the difference between the observed and expected frequencies is not significant at α % LOS If $\chi^2 > \chi^2_\nu(\alpha)$, we will reject H_0 and conclude that the difference is significant.

Conditions for the Validity of χ^2-Test

1. The number of observations N in the sample must be reasonably large, say ≥ 50.
2. Individual frequencies must not be too small, i.e. $O_i \geq 10$. In case $O_i < 10$, it is combined with the neighbouring frequencies, so that the combined frequency is ≥ 10.
3. The number of classes n must be neither too small nor too large i.e., $4 \leq n \leq 16$.

χ^2-Test of Independence of Attributes

If the population is known to have two major attributes A and B, then A can be divided into m categories $A_1, A_2, \ldots, A_m$ and B can be divided into n categories $B_1, B_2, \ldots, B_n$. Accordingly the members of the population and hence those of the sample can be divided into mn classes. In this case, the sample data may be presented in the form of a matrix containing m rows and n columns and hence mn cells and showing the observed frequencies O_{ij}, in the various cells, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots n$. O_{ij} means the number of observed frequencies possessing the attributes A_i and B_j. The matrix or tabular form of the sample data, called an $(m \times n)$ *contingency table* is given below:

Table 9.15

A\B	B_1	B_2	-	B_j	-	B_n	Row Total
A_1	O_{11}	O_{12}	-	O_{1j}	-	O_{1n}	O_{1*}
A_2	O_{21}	O_{22}	-	O_{2j}	-	O_{2n}	O_{2*}
:	-	-	-	-	-	-	-
A_i	O_{i1}	O_{i2}	-	O_{ij}	-	O_{in}	O_{i*}
:	-	-	-	-	-	-	-
A_m	O_{m1}	O_{m2}		O_{mj}	-	O_{mn}	O_{m*}
Column Total	O_{*1}	O_{*2}	-	O_{*j}	-	O_{*n}	N

Now, based on the null hypothesis H_0 i.e. the assumption that the two attributes A and B are independent, we compute the expected frequencies E_{ij} for various cells, using the following formula $E_{ij} = \frac{O_{i*} \cdot O_{*j}}{N}$, $i = 1, 2, \ldots\, m$; and $j = 1, 2, \ldots\, n$

i.e.
$$E_{ij} = \left\{ \frac{(\text{Total of observed frequencies in the } i^{\text{th}} \text{ row}) \times (\text{total of observed frequencies in the } j^{\text{th}} \text{ column})}{\text{Total of all cell frequecies}} \right\}$$

Then we compute $\chi^2 = \sum_{i=1}^{m} \sum_{J=1}^{n} \left\{ \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \right\}$

The number of degrees of freedom for this χ^2 computed from the $(m \times n)$ contingency table is $v = (m-1)(n-1)$.

If $\chi^2 < \chi^2_v(\alpha)$, H_0 is accepted at $\alpha\,\%$ LOS i.e. the attributes A and B are independent.

If $\chi^2 > \chi^2_v(\alpha)$, H_0 is rejected at $\alpha\,\%$ LOS i.e. A and B are not independent.

Worked Example 9(C)

Example 1

The following table shows the distribution of digits in the numbers chosen at random from a telephone directory:

Table 9.16

Digit:	0	1	2	3	4	5	6	7	8	9	Total
Frequency:	1026,	1107,	997,	966,	1075	933,	1107,	972,	964,	853	10,000

Test whether the digits may be taken to occur equally frequently in the directory.

H_0 : The digits occur equally frequently, i.e. they follow a uniform distribution.

Based on H_0, we compute the expected frequencies.

The total number of digits = 10,000.

If the digits occur uniformly, then each digit will occur

$$\frac{10{,}000}{10} = 1000 \text{ times}$$

$$O_i : 1026, 1107, \ldots\ldots, 853$$

$$E_i : 1000, 1000, \ldots\ldots, 1000$$

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

$$= \frac{1}{1000}\{(26)^2 + (107)^2 + (-3)^2 + (-34)^2 + (75)^2 + (-67)^2 + (107)^2 + (-28)^2 + (-36)^2 + (-147)^2\}$$

$$= 58.542$$

Since ΣE_i was taken equal to ΣO_i (i.e. an information from the sample), $v = n - 1 = 10 - 1 = 9$. From the χ^2–table,

$$\chi^2_{5\%}\ (n = 9) = 16.919$$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected i.e. the digits do not occur uniformly in the directory.

Example 2

Table 9.17 gives the number of air-craft accidents that occurred during the various days of a week. Test whether the accidents are uniformly distributed over the week.

Table 9.17

Day:	Mon	Tues	Wed	Thu	Fri	Sat
No. of accidents:	15	19	13	12	16	15

H_0 : Accidents occur uniformly over the week.

Total number of accidents = 90

Based on H_0, the expected number of accidents on any day $= \frac{90}{6} = 15$.

O_i	:	15	19	13	12	16	15
E_i	:	15	15	15	15	15	15

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{1}{15}(0 + 16 + 4 + 9 + 1 + 0) = 2.$$

Since $\sum E_i = \sum O_i, \quad v = 6 - 1 = 5$

From the χ^2-table, $\chi^2_{5\%}(v = 5) = 11.07$.

Since $\chi^2 < \chi^2_{5\%}$, H_0 is accepted.

i.e. accidents may be regarded to occur uniformly over the week.

Example 3

Table 9.18 shows defective articles produced by four machines:

Table 9.18

Machine:	A	B	C	D
Production time:	1 hour	1 hour	2 hours	3 hours
No. of detectives:	12	30	63	98

Do the figures indicate a significant difference in the performance of the machines?

H_0 : Production rates of the machines are the same.

Total number of detectives = 203.

Based on H_0, the expected numbers of defectives produced by the machines are

$$E_i : \quad \frac{1}{7} \times 203, \quad \frac{1}{7} \times 203, \quad \frac{2}{7} \times 203, \quad \frac{3}{7} \times 203$$

i.e. E_i : 29, 29, 58, 87

O_i : 12, 30, 63, 98

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{17^2}{29} + \frac{1^2}{29} + \frac{5^2}{58} + \frac{11^2}{87} = 11.82$$

Since $\sum E_i = \sum O_i, v = 4 - 1 = 3$

From the χ^2-table, $\chi^2_{5\%}(v = 3) = 7.815$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

i.e. There is significant difference in the performance of machines.

Example 4

The following data represents the monthly sales (in Rs) of a certain retail stores in a leap year. Examine if there is any seasonality in the sales.

6100, 5600, 6350, 6050, 6250, 6200, 6300, 6250, 5800, 6000, 6150 and 6150.

H_0 : There is no seasonability in the sales, i.e. the daily sales are uniform throughout the year or the daily sales follow a uniform distribution.

Based on H_0, we compute the expected frequencies.

The total sales in the year = Rs. 73,200.

If the daily sales are uniform, then the sales on each day

$$= \frac{73{,}200}{366} = \text{Rs } 200$$

O_i : 6100, 5600, 6350, 6050, 6250, 6200, 6300, 6250, 5800, 6000, 6150, 6150.

Assuming that the months are taken in the usual calendar order, namely, January, February etc. the expected monthly sales are:

E_i : 6200, 5800, 6200, 6000, 6200, 6000, 6200, 6200, 6000, 6200, 6000, 6200

Then
$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$
$$= \frac{(-100)^2}{6200} + \frac{(-200)^2}{5800} + \ldots + \frac{(-50)^2}{6200} = 38.913$$

Since $\sum E_i$ was found as $\sum O_i$ from the sample, $v = n - 1 = 12 - 1 = 11$.

From the χ^2-table $\chi^2_{5\%}(v = 11) = 19.675$.

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected, i.e. the daily sales are not uniform throughout the year.

Example 5

Theory predicts that the proportion of beans in four groups A, B, C, D should be 9 : 3 : 3 : 1. In an experiment among 1600 beans, the numbers in the four groups were 882, 313, 287 and 118. Does the experiment support the theory?

H_0 : The experiment supports the theory, i.e. the numbers of beans in the four groups are in the ratio 9 : 3 : 3 : 1

Based on H_0, the expected numbers of beans in the four groups are as follows

$$E_i : \quad \frac{9}{16} \times 1600, \quad \frac{3}{16} \times 1600, \quad \frac{3}{16} \times 1600, \quad \frac{1}{16} \times 1600$$

i.e. E_i : 900, 300, 300, 100

O_i : 882, 313, 287, 118

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{18^2}{900} + \frac{13^2}{300} + \frac{13^2}{300} + \frac{18^2}{100} = 4.73$$

Since $\sum E_i = \sum O_i, v = 4 - 1 = 3$

From the χ^2-table, $\chi^2_{5\%}(v = 3) = 7.82$

Since $\chi^2 < \chi^2_{5\%}$, H_0 is accepted.

i.e. the experimental data support the theory.

Example 6

A survey of 320 families with five children each revealed the following distribution:

Table 9.19

No. of boys :	0	1	2	3	4	5
No. of girls :	5	4	3	2	1	0
No. of families:	12	40	88	110	56	14

Is this result consistent with the hypothesis that male and female births are equally probable?

H_0 : Male and female births are equally probable, i.e. P (male birth) $= p = 1/2$ and P (female birth) $= q = 1/2$.

Based on H_0, the probability that a family of 5 children has r male children

$$= 5C_r \left(\frac{1}{2}\right)^5 \qquad \text{(by binomial law)}$$

$\therefore$ Expected number of families having r male children $= 320 \times 5\,C_r \times \dfrac{1}{2^5}$

$$= 10 \times 5\,C_r.$$

Thus	E_i :	10	50	100	100	50	10
and	O_i :	12	40	88	110	56	14

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{2^2}{10} + \frac{10^2}{50} + \frac{12^2}{100} + \frac{10^2}{100} + \frac{6^2}{50} + \frac{4^2}{10}$$

$$= 7.16$$

We have used the sample data to get $\sum E_i$ only. The values of p and q have not been found by using the sample data.

$\therefore$ $\quad v = n - 1 = 6 - 1 = 5$ and $\chi^2_{5\%}\,(v = 5) = 11.07$

Since $\quad \chi^2 < \chi^2_{5\%}$, H_0 is accepted.

i.e. male and female births are equally probable.

Example 7

Twelve dice were thrown 4096 times and a throw of six was considered a success. The observed frequencies were as given below.

No. of successes:	0	1	2	3	4	5	6	7 and over
Frequency:	447	1145	1180	796	380	115	25	8

Test whether the dice were unbiased.

H_0 : All the dice were unbiased. i.e. P (getting 6) $= p = \dfrac{1}{6}$ $\therefore q = \dfrac{5}{6}$.

Based on H_0, the probability of getting exactly 'r' successes $= 12\,C_r\, p^r q^{12-r}$ $(r = 0, 1, 2, ..., 12)$

$\therefore$ Expected number of times in which 'r' successes are obtained

$$= 4096 \times 12\,C_r \left(\frac{1}{6}\right)^r \cdot \left(\frac{5}{6}\right)^{12-r}$$

$$= 4096 \times 12\,C_r \times \frac{5^{12-r}}{6^{12}} \quad (r = 0, 1, 2, ..., 12)$$

i.e.

$$E_0 = N\,(0 \text{ success}) = N\,(r = 0) = 459.39$$
$$E_1 = N\,(r = 1) = 1102.54$$
$$E_2 = N\,(r = 2) = 1212.80$$
$$E_3 = N\,(r = 3) = 808.53$$
$$E_4 = N\,(r = 4) = 363.84$$
$$E_5 = N\,(r = 5) = 116.43$$
$$E_6 = N\,(r = 6) = 27.17$$
$$E_7 = N\,(r \geq 7) = 5.30$$

Converting E_i's into whole numbers subject to the condition that $\sum E_i = 4096$, we get

E_i :	459,	1103,	1213,	809,	364,	116,	27,	5
O_i :	447,	1145,	1180,	796,	380,	115,	25,	8,

Since E and O corresponding to the last class i.e. 5 and 8 are less than 10, we combine the last two classes and consider as a single class.

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{12^2}{459} + \frac{42^2}{1103} + \frac{33^2}{1213} + \frac{13^2}{809} + \frac{16^2}{364} + \frac{1^2}{116} + \frac{1^2}{32}$$

$$= 3.76$$

$v = n - 1$, since only $\sum E_i$ has been found using the sample data.

$= 7 - 1$ [n must be taken as the number of classes after combination of end classes, if any]

$= 6$

and $\chi^2_{5\%}\,(v = 6) = 12.59$, from the χ^2–table. Since $\chi^2 < \chi^2_{5\%}$, H_0 is accepted, i.e. the dice were unbiased.

Example 8

Fit a binomial distribution for the following data and also test the goodness of fit.

x :	0	1	2	3	4	5	6	Total
f:	5	18	28	12	7	6	4	80

To find the binomial distribution $N\,(q + p)^n$, which fits the given data, we require p.

We know that the mean of the binomial distribution is np, from which we can find p. Now the mean of the given distribution is found out and is equated to np.

x :	0	1	2	3	4	5	6	Total
f:	5	18	28	12	7	6	4	80
fx :	0	18	56	36	28	30	24	192

$$\bar{x} = \frac{\sum f x}{\sum f} = \frac{192}{80} = 2.4$$

i.e. $np = 2.4$ or $6p = 2.4$, since the maximum value taken by x is n.

$\therefore$ $p = 0.4$ and hence $q = 0.6$

$\therefore$ The expected frequencies are given by the successive terms in the expansion of $80\,(0.6 + 0.4)^6$.

Thus E_i: 3.73, 14.93, 24.88, 22.12, 11.06, 2.95, 0.33

Converting the E_i's into whole number such that $\sum E_i = \sum O_i = 80$, we get

E_i: 4 15 25 22 11 3 0

Let us now proceed to test the goodness of binomial fit.

O_i: 5 18 28 12 7 6 4

The first class is combined with the second and the last two classes are combined with the last but second class in order to make the expected frequency in each class greater than or equal to 10. Thus, after regrouping, we have,

E_i:	19	25	22	14
O_i:	23	28	12	17

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{4^2}{19} + \frac{3^2}{25} + \frac{10^2}{22} + \frac{3^2}{14} = 6.39$$

We have used the given sample to find

$\sum E_i \; (= \sum O_i)$ and p through its mean.

Hence $\nu = n - k$

$= 4 - 2 = 2$

$\chi^2_{5\%}\,(\nu = 2) = 5.99$, from the χ^2-table.

Since $\chi^2 > \chi^2_{5\%}$, H_0, which assumes that the given distribution is approximately a binomial distribution, is rejected. i.e. the binomial fit for the given distribution is not satisfactory.

Example 9

Fit a Poisson distribution for the following distribution and also test the goodness of fit.

x:	0	1	2	3	4	5	Total
f:	142	156	69	27	5	1	400

To find the Poisson distribution whose probability law is

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}, \; r = 0, 1, 2, \ldots,$$

we require λ, which is the mean of the Poisson distribution.

We find the mean of the given distribution and assume it as λ.

x:	0	1	2	3	4	5	Total
f:	142	156	69	27	5	1	400
fx:	0	156	138	81	20	5	400

$$\bar{x} = \frac{\sum f x}{\sum f} = \frac{400}{400} = 1 = \lambda.$$

The expected frequencies are given by

$$\frac{N \cdot e^{-\lambda}\lambda^r}{r!} \text{ or } \frac{400 \times e^{-1}}{r!}, \; r = 0, 1, 2, \ldots \infty$$

Thus

E_i: 147.15, 147.15, 73.58, 24.53, 6.13, 1.23

The values of E_i are very small for $i = 6, 7, \ldots$ and hence neglected.

Converting the values of E_i's into whole numbers such that $\sum E_i = 400$, we get

E_i: 147, 147, 74, 25, 6, 1

Let us now proceed to test the goodness of Poisson fit.

O_i: 142, 156, 69, 27, 5, 1

The last three classes are combined into one, so that the expected frequency in that class may be ≥ 10. Thus, after regrouping, we have

O_i:	142	156	69	33
E_i:	147	147	74	32

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{5^2}{147} + \frac{9^2}{147} + \frac{5^2}{74} + \frac{1^2}{32}$$

$= 1.09$

We have used the sample data to find $\sum E_i$ and λ. Hence

$n = n - k = 4 - 2 = 2$

From the χ^2-table, $\chi^2_{5\%}\,(\nu = 2) = 5.99$.

Since $\chi^2 < \chi^2_{5\%}$, H_0, which assumes that the given distribution is nearly Poisson, is accepted.

i.e. the Poisson fit for the given distribution is satisfactory.

Example 10

Test the normality of the following distribution by using χ^2-test of goodness of fit

x:	125,	135,	145,	155,	165,	175,	185,	195,	205	Total
f:	1,	1,	14,	22,	25,	19,	13,	3,	2	100

Let us first fit a normal distribution to the given data and then test the goodness of fit.

To fit a normal distribution and hence find the expected frequencies, we require the density function of the normal distribution which involves the mean and S.D. Let us now compute the mean $\bar{x}$ and S.D. 's' of the sample distribution and assume them as μ and σ.

Table 9.20

x	f	$d = \frac{x-165}{10}$	fd	fd^2
125	1	−4	−4	16
135	1	−3	−3	9
145	14	−2	−28	56
155	22	−1	−22	22
165	25	0	0	0
175	19	1	19	19
185	13	2	26	52
195	3	3	9	27
205	2	4	8	32
Total	100	–	5	233

$$\bar{x} = A + \frac{c}{N}\sum fd = 165 + \frac{10}{100} \times 5 = 165.5$$

$$s^2 = c^2\left\{\frac{1}{N}\sum fd^2 - \left(\frac{1}{N}\sum fd\right)^2\right\} = 10^2\,(2.33 - 0.0025)$$

$$= 232.75$$

$\therefore \quad s = 15.26$

$\therefore$ The density function of the normal distribution which fits the given distribution is $f(x) = \frac{1}{15 \cdot 26\sqrt{2\pi}}\, e^{-(x-165 \cdot 5)^2/465 \cdot 5}$.

To find the expected frequency corresponding to a given x, we find $y = f(x)$ and multiply y by the class-width and then by the total frequency N.

Note $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$. *If we put* $\frac{x-\mu}{\sigma} = z$, *then* $y = \frac{1}{\sigma}\left\{\frac{1}{\sqrt{2\pi}} e^{-z^2/2}\right\} = \frac{\phi(z)}{\sigma}$, *where* $\phi(z)$ *is density function of the standard normal distribution. Values of* $f(z)$ *are got from the normal table given in the Appendix.*

Table 9.21

x	$y = \frac{x-165 \cdot 5}{15 \cdot 26}$	$\phi(z)$	$\frac{c \cdot \phi(z)}{\sigma} = \frac{10\phi(z)}{15 \cdot 26}$	*Expected frequency* $= N\phi(z)/\sigma$
125	−2.65	.0119	.0078	0.78
135	−2.00	.0540	.0354	3.54
145	−1.34	.1626	.1066	10.66
155	−0.69	.3144	.2060	20.60
165	−0.03	.3988	.2613	26.13
175	0.62	.3292	.2157	21.57
185	1.28	.1758	.1152	11.52
195	1.93	.0620	.0406	4.06
205	2.59	.0139	.0091	0.91

Converting the expected frequencies as whole numbers such that $\sum E_i = 100$, we get

E_i : 1, 3, 11, 21, 26, 22, 11, 4, 1

Let us now proceed to test the goodness of normal fit.

Combining the end classes so as to make the individual frequencies greater than 10,

E_i : 15, 21, 26, 22, 16
O_i : 16, 22, 25, 19, 18

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{1^2}{15} + \frac{1^2}{21} + \frac{1^2}{26} + \frac{3^2}{22} + \frac{2^2}{16}$$

$$= 0.82$$

We have used the sample data to find $\sum E_i$, μ and σ. Hence $v = n - k = 5 - 3 = 2$.

From the χ^2-table, $\chi^2_{5\%}\,(v = 2) = 5.99$.

Since $\chi^2 < \chi^2_{5\%}$, H_0, which assumes that the given distribution is nearly normal, is accepted.

i.e. the normal fit for the given distribution is satisfactory.

Example 11

The following data are collected on two characters (Table 9.22).

Table 9.22

	Smokers	*Non-smokers*
Literates :	83	57
Illiterates :	45	68

Based on this, can you say that there is no relation between smoking and literacy?
H_0: Literacy and smoking habit are independent

Table 9.23

	Smokers	Non-smokers	Total
Literates	83	57	140
Illiterates	45	68	113
Total	128	125	253

Table 9.24

O	E	E (rounded)	$(O-E)^2/E$
83	$\frac{128 \times 140}{253} = 70.83$	71	$122/71 = 2.03$
57	$\frac{125 \times 140}{253} = 69.17$	69	$122/69 = 2.09$
45	$\frac{128 \times 113}{253} = 57.17$	57	$122/57 = 2.53$
68	$\frac{125 \times 113}{253} = 55.83$	56	$122/56 = 2.57$
			$\chi 2 = 9.22$

$$v = (m-1)(n-1)$$
$$= (2-1)(2-1) = 1.$$

From the χ^2-table, $\chi^2_{5\%}\ (v = 1) = 3.84$
Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.
i.e. there is some association between literacy and smoking.

Example 12

Prove that the value of χ^2 for the 2×2 contingency table $\begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$ is given by

$$\chi^2 = \frac{N(ad - bc)^2}{(a+b)(c+d)(a+c)(b+d)}, \text{ where } N = a+b+c+d.$$

Hence compute χ^2 for the 2×2 contingency table given in Example 11.

The value of E corresponding to the cell in which $O = a$ is given by

$$E = \frac{(a+b)(a+c)}{(a+b+c+d)}.$$

$\therefore$ The value of χ^2 corresponding to this cell is given by

$$\chi^2 = \left\{ a - \frac{(a+b)(a+c)}{a+b+c+d} \right\}^2 \div \frac{(a+b)(a+c)}{(a+b+c+d)}$$

$$= \frac{\{a(a+b+c+d) - (a+b)(a+c)\}^2}{N(a+b)(a+c)}$$

$$= \frac{(ad - bc)^2}{N(a+b)(a+c)}$$

Similarly the value of χ^2 are found out for the other three cells.

$\therefore \chi^2$ for the table

$$= \frac{(ad-bc)^2}{N} \left\{ \frac{1}{(a+b)(a+c)} + \frac{1}{(a+b)(b+d)} + \frac{1}{(a+c)(c+d)} + \frac{1}{(b+d)(c+d)} \right\}$$

$$= \frac{(ad-bc)^2}{N(a+b)(c+d)(a+c)(b+d)} \{(b+d)(c+d) + (a+c)(c+d) + (a+b)(b+d) + (a+b)(a+c)\}$$

$$= \frac{(ad-bc)^2 \left\{ \sum a^2 + 2\sum ab \right\}}{N(a+b)(c+d)(a+c)(b+d)} = \frac{(ad-bc)^2 (a+b+c+d)^2}{N(a+b)(c+d)(a+c)(b+d)}$$

$$= \frac{N(ad-bc)^2}{(a+b)(c+d)(a+c)(b+d)} \qquad (1)$$

Using (1) for the contingency table in Example 11,

we get $$\chi^2 = \frac{253(83 \times 68 - 45 \times 57)^2}{140 \times 113 \times 128 \times 125} = 9.48.$$

Example 13

Two batches each of 12 animals are taken for test of inoculation. One batch was inoculated and the other batch was not inoculated. The numbers of dead and surviving animals are given in Table 9.25 in both cases. Can the inoculation be regarded as effective against the disease? Make Yate's correction for continuity of χ^2.

Table 9.25

	Dead	Survived	Total
Inoculated	2	10	12
Not inoculated	8	4	12
Total	10	14	24

Note on Yate's correction for continuity of χ^2.

The χ^2-table was prepared using the theoretical χ^2-distribution which is continuous, whereas the approximate values of χ^2 that we are using are discrete. To rectify this defect, Yates has shown that, when

$$\chi^2 = \sum \left[\frac{\left\{ \left| 0_i - E_i \right| - \frac{1}{2} \right\}^2}{E_i} \right]$$

is used, the χ^2-approximation is improved. Yate's correction is used only when $v = 1$ and hence for a 2×2 contingency table. It is used only when some cell frequency is small, i.e. less than 5.)

In the present problem, two cell frequencies are less than 5 each. Hence we apply Yate's correction (Table 9.26).

Table 9.26

O	E	$\lvert O - E \rvert - 0.5$	$\{\lvert O - E \rvert - 0.5\}^2/E$
2	$\frac{12 \times 10}{24} = 5$	2.5	6.25/5 = 1.25
10	$\frac{12 \times 14}{24} = 7$	2.5	6.25/7 = 0.89
8	$\frac{12 \times 10}{24} = 5$	2.5	6.25/5 = 1.25
4	$\frac{12 \times 14}{24} = 7$	2.5	6.25/7 = 0.89
	$v = (2-1)(2-1) = 1$		$\chi 2 = 4.28$

From the χ^2-table, $\chi^2_{5\%}$ $(v = 1) = 3.84$

If H_0 : Inoculation and effect on the diseases are independent, then H_0 is rejected as $\chi^2 > \chi^2_{5\%}$ i.e. Inoculation can be regarded as effective against the disease.

Note *Even if Yate's correction is not made, we would have arrived at the same conclusion.*

Example 14

A total number of 3759 individuals were interviewed in a public opinion survey on a political proposal. Of them, 1872 were men and the rest women. 2257 individuals were in favour of the proposal and 917 were opposed to it. 243 men were undecided and 442 women were opposed to the proposal. Do you justify or contradict the hypothesis that there is no association between sex and attitude?

A careful analysis of the problem results in the following contingency (Table 9.27).

Table 9.27

	Favoured	*Opposed*	*Undecided*	*Total*
Men	1154	475	243	1872
Women	1103	442	342	1887
Total	2257	917	585	3759

H_0 : Sex and attitude are independent, i.e. there is no association between sex and attitude.

Table 9.28

O	E (rounded E)	$(O-E)^2/E$
1154	$\frac{1872 \times 2257}{3759} \simeq 1124$	302/1124 = 0.80
475	$\frac{1872 \times 917}{3759} \simeq 457$	182/457 = 0.71
243	$\frac{1872 \times 585}{3759} \simeq 291$	482/291 = 7.92
1103	$\frac{1887 \times 2257}{3759} \simeq 1133$	302/1133 = 0.79
442	$\frac{1887 \times 917}{3759} \simeq 460$	182/460 = 0.70
342	$\frac{1887 \times 585}{3759} \simeq 294$	482/294 = 7.84
	$v = (3-1)(2-1) = 2$	$\chi^2 = 18.76$

From the χ^2-table, $\chi^2_{5\%}$ $(v = 2) = 5.99$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

That is, sex and attitude are not independent i.e. there is some association between sex and attitude.

Example 15

The following table gives for a sample of married women, the level of education and the marriage adjustment score:

Table 9.29

Level of education	*Marriage adjustment*				
	Very low	*low*	*high*	*very high*	*Total*
College	24	97	62	58	241
High school	22	28	30	41	121
Middle school	32	10	11	20	73
Total	78	135	103	119	435

Can you conclude from the above data that the higher the level of education, the greater is the degree of adjustment in marriage? H_0 : There is no relation between the level of education and adjustment in marriage.

$$v = (4-1)(3-1) = 6$$

$$\chi^2_{5\%}(v = 6) = 12.59$$

Table 9.30

O	E (rounded)	$(O-E)^2/E$
24	43	$19^2/43 = 8.40$
97	75	$22^2/75 = 6.45$
62	57	$5^2/57 = 0.44$
58	66	$8^2/66 = 0.97$
22	22	$0^2/22 = 0.00$
28	37	$9^2/37 = 2.19$
30	29	$1^2/29 = 0.03$
41	33	$8^2/33 = 1.94$
32	13	$19^2/13 = 27.77$
10	23	$13^2/23 = 7.35$
11	17	$6^2/17 = 2.12$
20	20	$0^2/20 = 0.00$
$\chi^2_{5\%}(v = 6) = 12.59$		$\chi^2 = 57.66$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

That is, the level of education and adjustment in marriage are associated.

Thus we may conclude that the higher the level of education, the greater is the degree of adjustment in marriage.

Exercise 9(C)

Part A

(Short Answer Questions)

1. Define Chi-square distribution.
2. Write down the probability density function of the χ^2-distribution.
3. State the important properties of χ^2-distribution.
4. Give two uses of χ^2-distribution.
5. What is χ^2-test of goodness of fit?
6. State the conditions under which χ^2-test of goodness of fit is valid.
7. What is χ^2-test of independence of attributes?
8. What is contingency table?
9. Write down the value of χ^2 for a 2×2 contingency table with cell frequencies a, b, c and d.
10. What is Yate's correction for continuity of χ^2?

Part B

11. In 250 digits from the lottery numbers, the frequencies of the digits were as follows:

Digit:	0	1	2	3	4	5	6	7	8	9
Frequency:	23	25	20	23	23	22	29	25	33	27

Test the hypothesis that the digits were randomly drawn.

12. The following table gives the number of fatal road accidents that occurred during the seven days of the week. Find whether the accidents are uniformly distributed over the week.

Day :	Sun	Mon	Tues	Wed	Thu	Fri	Sat
Number :	8	14	16	12	11	14	9

13. In 120 throws of a single die, the following distribution of faces are obtained:

Face :	1	2	3	4	5	6
Frequency :	30	25	18	10	22	15

Do these results support the equal probability hypothesis?

14. The number of demands for a particular spare part in a shop was found to vary from day to day. In a sample study, the following information was obtained:

Day :	Mon	Tues	Wed	Thu	Fri	Sat
No. of demands :	124	125	110	120	126	115

Test the hypothesis that the number of parts demanded does not depend on the day of the week.

15. According to genetic theory, children having one parent of blood type M and the other of blood type N will always be one of the three types-M, MN and N and the average proportions of these types will be 1 : 2 : 1. Out of 300 children, having one M parent and one N parent, 30 per cent were found to be of type M, 45 per cent of type MN and the remaining of type N. Test the genetic theory by χ^2-test.

16. 5 coins are tossed 256 times. The number of heads observed is given below. Examine if the coins are true.

No. of heads :	0	1	2	3	4	5
Frequency :	5	35	75	84	45	12

17. 5 dice were thrown 243 times and the numbers of times 1 or 2 was thrown (x) are given below:

x :	0	1	2	3	4	5
Frequency :	30	75	76	47	13	2

Examine if the dice were unbiased.

18. Fit a binomial distribution for the following data and also test the goodness of fit.

x :	0	1	2	3	4
f :	5	29	36	25	5

19. Fit a binomial distribution for the following data and also test the goodness of fit

x:	0	1	2	3	4	5	6	7	8	9
f:	3	8	11	15	16	14	12	11	9	1

20. Fit a Poisson distribution for the following distribution and also test the goodness of fit.

x:	0	1	2	3	4	5	6	7
f:	314	335	204	86	29	9	3	0

21. Fit a Poisson distribution for the following distribution and also test the goodness of fit.

x:	0	1	2	3	4
f:	123	59	14	3	1

22. The figures given below are (a) the observed frequencies of a distribution and (b) the expected frequencies of the normal distribution having the same mean, S.D. and total frequency as in (a).
    (a) 1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1
    (b) 2, 15, 66, 210, 484, 799, 943, 799, 484, 210, 66, 15, 2

    Do you think that the normal distribution provides a good fit to the data?
23. Fit a normal distribution to the following data and find also the goodness of fit.

x:	4	6	8	10	12	14	16	18	20	22	24
f:	1	7	15	22	35	43	38	20	13	5	1

24. In an epidemic of certain disease, 92 children contacted the disease. Of these 41 received no treatment and of these 10 showed after effects. Of the remainder who did receive the treatment, 17 showed after effects. Test the hypothesis that the treatment was not effective.
25. Out of 1660 candidates who appeared for a competitive examination, 422 were successful. Out of these, 256 had attended a coaching class and 150 of them came out successful. Examine whether coaching was effective as regards the success in the examination.
26. In a pre-poll survey, out of 1000 rural voters, 620 favoured A and the rest B. Out of 1000 urban voters, 450 favoured B and the rest A. Examine if the nature of the area is related to voting preference.
27. The following information was obtained in a sample of 40 small general shops:

Table 9.31

	Shops in urban areas	*Shops in rural areas*
Owned by men	17	18
Owned by women	3	12

Can it be said that there are more women owners in rural areas than in urban areas? Use Yate's correction for continuity.

28. A certain drug is claimed to be effective in curing cold. In an experiment on 500 persons with cold, half of them were given the drug and half of them were given the sugar pills. The patients' reaction to the treatment are recorded in the following table.

Table 9.32

	Helped	*Harmed*	*No effect*
Drug	150	30	70
Sugar pills	130	40	80

On the basis of this data, can it be concluded that the drug and sugar pills differ significantly in curing cold?

29. A survey of radio listeners' preference for two types of music under various age groups gave the following information.

Table 9.33

Type of music	*Age group*		
	19-25	26-35	Above 36
Carnatic music :	80	60	90
Film music :	210	325	44
Indifferent :	16	45	132

Is preference for type of music influenced by age?

30. The table given below shows the results of a survey in which 250 respondents were classified according to levels of education and attitude towards students' agitation in a certain town. Test whether the two criteria of classification are independent.

Table 9.34

	Attitude		
Education	*Against*	*Neutral*	*For*
Middle school:	40	25	5
High school:	40	20	5
College:	30	15	30
Postgraduate:	15	15	10

ANSWERS

Exercise 9 (A)

29. $z = 2.83$; significant
30. $z = 4.5$; the coin is not fair
31. $z = 1.79$; claim cannot be supported.

32. No, since z (= 1.40) < z_α (= 1.645).
34. No ; (16.7, 27.0)
35. difference due to sampling fluctuations
36. $z = 0.725$; not significant 37. $z = 6.49$; Yes; Yes.
38. Yes, since z (= 1.80) > $z_{.05}$ (= 1.645)
39. $z = 2.56$; the difference cannot be hidden
40. z (= .104) < $z_{.05}$ (= 1.645) ; the machine has improved
41. z (= 3.17) > $z_{.05}$ (= 1.96) ; difference significant.
42. $|z| = 1.02$; the claim is valid
43. $z = 3.12$; No
44. No, since $z = 5$; 96.8
45. Claim cannot be true as $z = 1.89$ and $z_{5\%} = 1.645$
46. Yes, since z (= 1.6) < $z_{5\%}$ (= 1.645)
47. 41 48. (57.2, 58.8) and (57.0, 59.0)
49. No, since $z = 2.58$ 50. No, since $z = 8.82$
51. No, since $z = 1.32$
52. Yes, since z (= 4.78) > $z_{1\%}$ (= 2.33)
53. Yes, at 5% level, since $|z|$ (= 1.937) > $z_{5\%}$ (= 1.645) and No, at 1% level, since $|z|$ (= 1.937) < $|z_\alpha|$ (= 2.33)
54. (1.98, 6.02)
55. $H_0 : \mu_1 - \mu_2 = 35$ accepted, as $z = 1.90$
57. Yes, as $z = 1.71$
58. Yes, as $|z|$ (= 2.5) > $z_{1\%}$ (= 2.33)
59. Yes, as $z = 1.70$ 60. No, as $z = 3.61$.

Exercise 9 (B)

16. $|t| = 0.62$; Yes ; $83.66 < \mu < 110.74$
17. $|t| = 2.67$; (99.76, 107.74) and (98.22, 109.29)
18. $t = 2.5$; the campaign was successful
19. $|t| = 1.44$; $t_{10\%} = 1.90$; $\bar{x}$ is not less than μ.
20. $|t| = 0.26$; machine is reliable
21. (11.82, 12.38); (11.69, 12.51)
22. $t = -1.067$; No 23. $|t| = 0.424$; No
24. $t = 4.33$; Yes 25. $|t| = 1.85$; Not significant
26. $t = 4.0$; coaching was effective 27. $F = 1.057$; Not significant
28. The populations have the same variance
29. No, though the difference between variances is not significant, the difference between the mean is significant.
30. Yes, as the differences between the means and between the variance are not significant.

Exercise 9 (C)

11. $\chi^2 = 5.2$; $v = 9$; digits randomly drawn.
12. $\chi^2 = 4.17$; $v = 6$; accidents occur uniformly.

13. $\chi^2 = 12.9$; $v = 5$; equal probability hypothesis is refuted.
14. $\chi^2 = 1.68$; $v = 5$; the demand does not depend on the day of the week
15. $\chi^2 = 4.5$; $v = 2$; genetic theory may be correct
16. $\chi^2 = 3.54$; $v = 3$; coins are true.
17. $\chi^2 = 2.76$; $v = 4$; dice are unbiased
18. E_i : 7, 26, 37, 24, 6 ; $\chi^2 = 0.06$; $v = 1$; binomial fit is good. (E_i: 1, 5, 11, 18, 21, 19, 13, 8, 3, 1)
19. $\chi^2 = 11.30$; $v = 4$; binomial fit is not satisfactory;
20. E_i : 301, 362, 217, 87, 26, 6, 1; $\chi^2 = 5.40$; $v = 4$; Poisson fit is good.
21. E_i : 121, 61, 15, 3, 0 ; $\chi^2 = 0.99$; $v = 1$; Poisson fit is good.
22. $x^2 = 3.84$; $v = 8$; Normal fit is good.
23. E_i : 2, 5, 13, 25, 37, 42, 36, 23, 12, 4, 1, ; $\chi^2 = 1.68$; $v = 4$; Normal fit is good.
24. $\chi^2 = 0.85$; $v = 1$; No association between treatment and after-effect.
25. $\chi^2 = 176.12$; $v = 1$; coaching was effective.
26. $\chi^2 = 10.09$; $v = 1$; some relation between area and voting preference
27. $\chi^2 = 2.48$; $v = 1$; No, as there is no relation between area and sex of ownership
28. $\chi^2 = 3.52$; $v = 2$; do not differ significantly
29. $\chi^2 = 373.40$; $v = 4$; preference for type of music influenced by age.
30. $\chi^2 = 35.42$; $v = 6$; the two criteria are not independent.

Chapter 10

Design of Experiments

By 'experiment', we mean collection of data (which usually consist of a series of measurement of some feature of an object) for a scientific investigation, according to certain specified sampling procedures. Statistics provides not only the principles and the basis for the proper planning of the experiments but also the methods for proper interpretation of the results of the experiment.

In the beginning, the study of the design of experiments was associated only with agricultural experimentation. The need to save time and money has led to a study of ways to obtain maximum information with the minimum cost and labour. Such motivations resulted in the subsequent acceptance and wide use of the design of experiments and the related analysis of variance techniques in all fields of scientific experimentation. In this chapter we consider some aspects of experimental design briefly and analysis of data from such experiments using analysis of variance techniques.

AIM OF THE DESIGN OF EXPERIMENTS

A statistical experiment in any field is performed to verify a particular hypothesis. For example, an agricultural experiment may be performed to verify the claim that a particular manure has got the effect of increasing the yield of paddy. Here the quantity of the manure used and the amount of yield are the two variables involved directly. They are called *experimental variables*. Apart from these two, there are other variables such as the fertility of the soil, the quality of the seed used and the amount of rainfall, which also affect the yield of paddy. Such variables are called *extraneous variables*. The main aim of the design of experiments is to control the extraneous variables and hence to minimise the experimental error so that the results of the experiments could be attributed only to the experimental variables.

Basic Principles of Experimental Design

In order to achieve the objective mentioned above, the following three principles are adopted while designing the experiments– (1) randomisation, (2) replication and (3) local control.

1. Randomisation

As it is not possible to eliminate completely the contribution of extraneous variables to the value of the response variable (the amount of yield of paddy), we try to control it by randomisation. The group of experimental units (plots of the same size) in which the manure is used is called the *experimental group* and the other group of plots in which the manure is not used and which will provide a basis for comparison is called the *control group*. If any information regarding the extraneous variables and the nature and magnitude of their effect on the response variable in question is not available, we resort to randomisation. That is, we select the plots for *the* experimental and control groups in a random manner, which provides the most effective way of eliminating any unknown bias in the experiment.

2. Replication

In a comparative experiment, in which the effects of different manures on the yield are studied, each manure is used in more than one plot. In other words, we resort to replication which means repetition. It is essential to carry out more than one test on each manure in order to estimate the amount of the experimental error and hence to get some idea of the precision of the estimates of the manure effects.

3. Local Control

To provide adequate control of extraneous variables, another essential principle used in the experimental design is the local control. This includes techniques such as grouping, blocking and balancing of the experimental units used in the experimental design. By *grouping*, we mean combining sets of homogeneous plots into groups, so that different manures may be used in different groups. The number of plots in different groups need not necessarily be the same. By *blocking*, we mean assigning the same number of plots in different blocks. The plots in the same block may be assumed to be relatively homogeneous. We use as many manures as the number of plots in a block in a random manner. By *balancing*, we mean adjusting the procedures of grouping, blocking and assigning the manures in such a manner that a balanced configuration is obtained.

SOME BASIC DESIGNS OF EXPERIMENT

1. Completely Randomised Design (C.R.D.)

Let us suppose that we wish to compare 'h' treatments (use of 'h' different manures) and there are n plots available for the experiment.

Let the ith treatment be replicated (repeated) n_i times, so that $n_1 + n_2 + \dots + n_h = n$.

The plots to which the different treatments are to be given are found by the following randomisation principle. The plots are numbered from 1 to n serially. n identical cards are taken, numbered from 1 to n and shuffled thoroughly. The numbers on the first n_1 cards drawn randomly give the numbers of the plots to which the first treatment is to be given. The numbers on the next n_2 cards drawn at random give the numbers of the plots to which the second treatment is to be given and so on. This design is called a completely randomised design, which is used when the plots are homogeneous or the pattern of heterogeneity of the plots is unknown.

Analysis of Variance (ANOVA)

The analysis of variance is a widely used technique developed by R.A. Fisher. It enables us to divide the total variation (represented by variance) in a group into parts which are ascribable to different factors and a residual random variation which could not be accounted for by any of these factors. The variation due to any specific factor is compared with the residual variation for significance by applying the F-test, with which the reader is familiar. The details of the procedure will be explained in the sequel.

Analysis of Variance for One Factor of Classification

Let a sample of N values of a given random variable X (representing the yield of paddy) be subdivided into 'h' classes according to some factor of classification (different manures).

We wish to test the null hypothesis that the factor of classification has no effect on the variable, viz., there is no difference between various classes, viz., the classes are homogeneous. Let x_{ij} be the value of the j^{th} member of the i^{th} class, which contains n_i members. Let the general mean of all the N values be $\bar{x}$ and the mean of n_i values in the i^{th} class be $\bar{x}_i$.

Now
$$\begin{aligned}
\sum_i \sum_j (x_{ij} - \bar{x})^2 &= \sum_i \sum_j \{(x_{ij} - \bar{x}_i) + (x_i - \bar{x})\}^2 \\
&= \sum_i \sum_j (x_{ij} - \bar{x}_i)^2 + \sum_i \sum_j (x_i - \bar{x})^2 \\
&\quad + 2 \sum_i \sum_j (x_{ij} - \bar{x}_i)(x_i - \bar{x}) \\
&= \sum_i \sum_j (x_{ij} - \bar{x}_i)^2 + \sum_i \sum_{j=1}^{n_i} (x_i - \bar{x})^2 \\
&\quad + 2 \sum_i (x_i - \bar{x}) \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i) \\
&= \sum_i \sum_j (x_{ij} - \bar{x}_i)^2 + \sum_i n_i (x_i - \bar{x})^2
\end{aligned}$$

$$\left[\because \sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i) = \text{sum of the deviations of } n_i \text{ values of the } x_{ij} \text{ in the } i^{th} \text{ class from their mean} = \bar{x}_i = 0\right]$$

i.e., $Q = Q_2 + Q_1$, say, where

$Q_1 = \sum_i n_i(\bar{x}_i - \bar{x})^2$ = sum of the squared deviations of class means from the general mean (variation between classes)

$Q_2 = \sum_i \sum_j (x_{ij} - \bar{x}_i)^2$ = sum of the squared deviations of variates from the corresponding class means (variation within classes) and Q = total variation.

Since $Q_2 = Q - Q_1$, viz., the variation Q_2 within classes is got after removing the variation Q_1 between classes from the total variation Q, Q_2 is the residual variation.

If s^2 is the variance of a sample of size n drawn from a population with variance σ^2, then it is known from the theory of estimation that $\left(\frac{ns^2}{n-1}\right)$ is an unbiased estimate of σ^2.

i.e., $$E\left(\frac{ns^2}{n-1}\right) = \sigma^2.$$

Since the items in the i^{th} class with variance $\frac{1}{n_i}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2$ may be considered as a sample of size n_i drawn from a population with variance σ^2,

$$E\left\{\frac{n_i}{n_i-1}\cdot\frac{1}{n_i}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2\right\} = \sigma^2$$

i.e., $$E\left[\sum_j (x_{ij}-\bar{x}_i)^2\right] = (n_i - 1)\,\sigma^2$$

$\therefore$ $$E\left[\sum_i \sum_j (x_{ij}-\bar{x}_i)^2\right] = \sum_{i=1}^{h}(n_i - 1)\,\sigma^2$$

i.e., $$E(Q_2) = (N-h)\,\sigma^2 \text{ or } E\left\{\frac{Q_2}{N-h}\right\} = \sigma^2$$

i.e., $\frac{Q_2}{N-h}$ is an unbiased estimate of σ^2 with $(N-h)$ degrees of freedom.

Now if we consider the entire group of N items with variance $\frac{1}{N}\sum_i\sum_j(x_{ij}-\bar{x})^2$ as a sample of size N drawn from the same population,

$$E\left\{\frac{N}{N-1}\frac{1}{N}\sum_i\sum_j(x_{ij}-\bar{x})^2\right\} = \sigma^2$$

i.e., $$E\left(\frac{Q}{N-1}\right) = \sigma^2$$

i.e., $\frac{Q}{N-1}$ is an unbiased estimate of σ^2 with $(N-1)$ degrees of freedom.

Now $$Q_1 = Q - Q_2$$

$\therefore$ $$\begin{aligned} E(Q_1) &= E(Q) - E(Q_2) \\ &= (N-1)\sigma^2 - (N-h)\sigma^2 \\ &= (h-1)\,\sigma^2 \text{ or } E\left(\frac{Q_1}{h-1}\right) = \sigma^2 \end{aligned}$$

i.e., $\frac{Q_1}{N-1}$ is also an unbiased estimate of σ^2 with $(h-1)$ degrees of freedom.

If we assume that the sampled population is normal, then the estimates $\frac{Q_1}{h-1}$ and $\frac{Q_2}{N-h}$ are independent and hence the ratio $\frac{Q_1/(h-1)}{Q_2/(N-h)}$ follows a F-distribution with $(h-1, N-h)$ degrees of freedom or the ratio $\frac{Q_2/(N-h)}{Q_1/(h-1)}$ follows a F-distribution with $(N-h, h-1)$ degrees of freedom. Choosing the ratio which is greater than one, we employ the F-test.

If the calculated value of $F < F_{5\%}$, the null hypothesis is accepted, viz., different treatments do not contribute significantly different yields.

These results are displayed in the form of a table, called the ANOVA table, as given below:

Table 10.1 *ANOVA table for one factor of classification*

Source of variation (S.V.)	*Sum of squares (S.S.)*	*Degree of freedom (d.f.)*	*Mean square (M.S.)*	*Variance ratio (F)*
Between classes	Q_1	$h-1$	$Q_1/(h-1)$	$\dfrac{Q_1/(h-1)}{Q_2/(N-h)}$ (OR)
Within classes	Q_2	$N-h$	$Q_2/(N-h)$	$\dfrac{Q_2/(N-h)}{Q_1/(h-1)}$
Total	Q	$N-1$	–	–

Note *For calculating Q, Q_1, Q_2, the following computational formulas may be used:*

$$Q = N\left\{\frac{1}{N}\sum\sum x_{ij}^2 - \bar{x}^2\right\}$$

$$= N\left\{\frac{1}{N}\sum\sum x_{ij}^2 - \left(\frac{1}{N}\sum\sum x_{ij}\right)^2\right\}$$

$$= \sum\sum x_{ij}^2 - \frac{T^2}{N}, \text{ where } T = \sum\sum x_{ij}$$

Similarly, for the ith class,

$$\sum_j (x_{ij} - \bar{x}_i)^2 = \sum_j x_{ij}^2 - \frac{T_i^2}{n_i}, \text{ where } T_i = \sum_j x_{ij}.$$

$$\therefore \quad Q_2 = \sum_i\sum_j (x_{ij} - \bar{x}_i)^2 = \sum_i\sum_j x_{ij}^2 - \sum_i \frac{T_i^2}{n_i}$$

Hence $$Q_1 = Q - Q_2$$

$$= \sum_i \frac{T_i^2}{n_i} - \frac{T^2}{N}$$

2. Randomised Block Design (R.B.D.)

Let us consider an agricultural experiment using which we wish to test the effect of 'k' fertilizing treatments on the yield of a crop. We assume that we know some information about the soil fertility of the plots. Then we divide the plots into 'h' blocks, according to the soil fertility, each block containing 'k' plots. Thus the plots in each block will be of homogeneous fertility as far as possible.

Within each block, the 'k' treatments are given to the 'k' plots in a perfectly random manner, such that each treatment occurs only once in any block. But the same 'k' treatments are repeated from block to block. This design is called Randomised Block Design.

Analysis of Variance for two Factors of Classification

Let the N variate values $\{x_{ij}\}$ (representing the yield of paddy) be classified according to two factors. Let there be 'h' rows (blocks) representing one factor of classification (soil fertility) and 'k' columns representing the other factor (treatment), so that $N = hk$.

We wish to test the null hypothesis that the rows and columns are homogeneous viz., there is no difference in the yields of paddy between the various rows and between the various columns.

Let x_{ij} be the variate value in the i^{th} row and j^{th} column.

Let $\bar{x}$ be the general mean of all the N values, $\bar{x}_{i*}$ be the mean of the k values in the i^{th} row and $\bar{x}_{*j}$ be the mean of the h values in the j^{th} column.

Now $$x_{ij} - \bar{x} = (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} + \bar{x}) + (\bar{x}_{i*} - \bar{x}) + (\bar{x}_{*j} - \bar{x})$$

$$\therefore \quad \sum\sum (x_{ij} - \bar{x})^2 = \sum\sum (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} + \bar{x})^2 + \sum\sum (\bar{x}_{i*} - \bar{x})^2$$
$$+ \sum\sum (\bar{x}_{*j} - \bar{x})^2 + 2\sum\sum (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} + \bar{x})(\bar{x}_{i*} - \bar{x})$$
$$+ 2\sum\sum (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} + \bar{x})(\bar{x}_{*j} - \bar{x})$$
$$+ 2\sum\sum (\bar{x}_{i*} - \bar{x})(\bar{x}_{*j} - \bar{x}) \qquad (1)$$

Now, the fourth member in the R.H.S. of (1)

$$= 2\sum_i (\bar{x}_{i*} - \bar{x}) \sum_{j=1}^{k} (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} + \bar{x})$$

$$= 2\sum_i (\bar{x}_{i*} - \bar{x})(k\,\bar{x}_{i*} - k\,\bar{x}_{i*} - k\,\bar{x} + k\,\bar{x})$$

$$= 0$$

Similarly, the last two members in the R.H.S. of (1) also become zero each.

Also $$\sum_i\sum_j (\bar{x}_{i*} - \bar{x})^2 = k\sum_i (\bar{x}_{i*} - \bar{x})^2 = Q_1, \text{ say}$$

$$\sum_i\sum_j (\bar{x}_{*j} - \bar{x})^2 = h\sum_j (\bar{x}_{*j} - \bar{x})^2 = Q_2, \text{ say}$$

Let $$Q = \Sigma\Sigma(x_{ij} - \bar{x})^2 \text{ and}$$
$$Q_3 = \Sigma\Sigma(x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} + \bar{x})^2$$

Using all these in (1), we get

$Q = Q_1 + Q_2 + Q_3$, where

Q = total variation.
Q_1 = sum of the squares due to the variations in the rows,
Q_2 = that in the columns and
Q_3 = that due to the residual variations.

Proceeding as in one factor of classification, we can prove that $\frac{Q_1}{h-1}, \frac{Q_2}{k-1}$, $\frac{Q_3}{(h-1)(k-1)}$ and $\frac{Q}{hk-1}$ are unbiased estimates of the population variance σ^2 with degrees of freedom $h-1, k-1, (h-1)(k-1)$ and $(hk-1)$ respectively. If the sampled population is assumed normal, all these estimates are independent.

$\therefore \frac{Q_1/(h-1)}{Q_3/(h-1)(k-1)}$ follows a F-distribution with $\{h-1, (h-1)(k-1)\}$ degrees of freedom and $\frac{Q_2/(k-1)}{Q_3/(h-1)(k-1)}$ follows a F-distribution with $\{k-1, (h-1)(k-1)\}$ degrees of freedom. Then the F-tests are applied as usual and the significance of difference between rows and between columns is analysed.

Table 10.2 ***The ANOVA table for the two factors of classifications***

S.V.	*S.S.*	*d.f.*	*M.S.*	*F*
Between rows	Q_1	$h-1$	$Q_1/(h-1)$	$\left[\frac{Q_1/(h-1)}{Q_3/(h-1)(k-1)}\right]^{\pm 1}$
Between columns	Q_2	$k-1$	$Q_2/(k-1)$	$\left[\frac{Q_2/(k-1)}{Q_3/(h-1)(k-1)}\right]^{\pm 1}$
Residual	Q_3	$(h-1)(k-1)$	$Q_3/(h-1)(k-1)$	–
Total	Q	$hk-1$	–	–

Note *The following working formulas that can be easily derived may be used to compute Q, Q_1, Q_2 and Q_3:*

1. $Q = \sum\sum x_{ij}^2 - \frac{T^2}{N}$, *where* $T = \sum\sum x_{ij}$
2. $Q_1 = \frac{1}{k}\sum T_i^2 - \frac{T^2}{N}$, *where* $T_i = \sum_{j=1}^{k} x_{ij}$
3. $Q_2 = \frac{1}{h}\sum T_j^2 - \frac{T^2}{N}$, *where* $T_j = \sum_{i=1}^{h} x_{ij}$
4. $Q_3 = Q - Q_1 - Q_2$

It may be verified that $\sum_i T_i = \sum_j T_j = T$.

3. Latin Square Design (L.S.D.)

We consider an agricultural experiment, in which n^2 plots are taken and arranged in the form of an $n \times n$ square, such that the plots in each row will be homogeneous as far as possible with respect to one factor of classification, say, soil fertility and plots in each column will be homogeneous as far as possible with respect to another factor of classification, say, seed quality.

Then n treatments are given to these plots such that each treatment occurs only once in each row and only once in each column. The various possible arrangements obtained in this manner are known as Latin squares of order n. This design of experiment is called the Latin Square Design.

Analysis of Variance for Three Factors of Classifications

Let the $N (= n^2)$ variate values $\{x_{ij}\}$, representing the yield of paddy, be classified according to three factors. Let the rows, columns and letters stand for the three factors, say soil fertility, seed quality and treatment respectively.

We wish to test the null hypothesis that the rows, columns and letters are homogeneous. viz., there is no difference in the yield of paddy between the rows (due to soil fertility), between the columns (due to seed quality) and between the letters (due to the treatments).

Let x_{ij} be the variate value corresponding to the i^{th} row, j^{th} column and k^{th} latter.

Let $\bar{x} = \frac{1}{n^2}\sum\sum x_{ij}$, $\bar{x}_{i*} = \frac{1}{n}\sum_j x_{ij}$, $\bar{x}_{*j} = \frac{1}{n}\sum_i x_{ij}$ and $\bar{x}_k$ be the mean of the values of x_{ij} corresponding to the k^{th} treatment.

Now $x_{ij} - \bar{x} = (\bar{x}_{i*} - \bar{x}) + (\bar{x}_{*j} - \bar{x}) + (\bar{x}_k - \bar{x}) + (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} - \bar{x}_k + 2\bar{x})$

$$\therefore \sum\sum (x_{ij} - \bar{x})^2 = n\sum_i (\bar{x}_{i*} - \bar{x})^2 + n\sum_j (\bar{x}_{*j} - \bar{x})^2 + n\sum_k (\bar{x}_k - \bar{x})^2 + \sum_i\sum_j (x_{ij} - \bar{x}_{i*} - \bar{x}_{*j} - \bar{x}_k + 2\bar{x})^2$$

($\because$ all the product terms vanish as in two factor classification)

i.e.,
$$Q = Q_1 + Q_2 + Q_3 + Q_4$$

As before we can prove that $\frac{Q_1}{n-1}, \frac{Q_2}{n-1}, \frac{Q_3}{n-1}, \frac{Q_4}{(n-1)(n-2)}$ and $\frac{Q}{n^2-1}$ are unbiased estimates of the population variance σ^2 with degrees of freedom $n-1, n-1, n-1, (n-1)(n-2)$ and (n^2-1) respectively.

If the sampled population is assumed normal, all these estimates are independent.

$\therefore$ Each of $\dfrac{Q_1/(n-1)}{Q_4/(n-1)(n-2)}$, $\dfrac{Q_2/(n-1)}{Q_4/(n-1)(n-2)}$ and $\dfrac{Q_3/(n-1)}{Q_4/(n-1)(n-2)}$

follows a F-distribution with $\{(n-1), (n-1)(n-2)\}$ degrees of freedom.

Then the F-tests are applied as usual and the significance of differences between rows, columns and treatments is analysed.

Table 10.3 ***The ANOVA table for three factors of classification***

S.V.	S.S.	d.f.	M.S.	F
Between rows	Q_1	$n-1$	$Q_1/(n-1) = M_1$	$\left(\dfrac{M_1}{M_4}\right)^{\pm 1}$
Between columns	Q_2	$n-1$	$Q_2/(n-1) = M_2$	$\left(\dfrac{M_2}{M_4}\right)^{\pm 1}$
Between letters	Q_3	$n-1$	$Q_3/(n-1) = M_3$	$\left(\dfrac{M_3}{M_4}\right)^{\pm 1}$
Residual	Q_4	$(n-1)(n-2)$	$Q_4/(n-1)(n-2) = M_4$	–
Total	Q	n^2-1	–	–

Note *The following working formulas may be used to compute the Q's:*

1. $Q = \sum\sum x_{ij}^2 - \dfrac{T^2}{n^2}$, *where* $T = \sum\sum x_{ij}$

2. $Q_1 = \dfrac{1}{n}\sum T_i^2 - \dfrac{T^2}{n^2}$, *where* $T_i = \sum_{j=1}^{n} x_{ij}$

3. $Q_2 = \dfrac{1}{n}\sum T_j^2 - \dfrac{T^2}{n^2}$, *where* $T_j = \sum_{i=1}^{n} x_{ij}$

4. $Q_3 = \dfrac{1}{n}\sum T_k^2 - \dfrac{T^2}{n^2}$, *where* T_k *is the sum of all* x_{ij}*s receiving the* k^{th} *treatment.*

5. $Q_4 = Q - Q_1 - Q_2 - Q_3$.

Also $T = \sum_i T_i = \sum_j T_j = \sum_k T_k$

COMPARISON OF RBD AND LSD

1. The number of replications of each treatment is equal to the number of treatments in LSD, whereas there is no such restrictions on treatments and replication in RBD.
2. LSD can be performed on a square field, while RBD can be performed either on a square field or a rectangular field.
3. LSD is known to be suitable for the case when the number of treatments is between 5 and 12, whereas RBD can be used for any number of treatments.
4. The main advantage of LSD is that it controls the effect of two extraneous variables, whereas RBD controls the effect of only one extraneous variable. Hence the experimental error is reduced to a larger extent in LSD than in RBD.

Note on Simplification of Computational Work

The variance of a set of values is independent of the origin and so a shift of origin does not affect the variance calculations. Hence in analysis of variance problems, we can subtract a convenient number from the original values and work out the problems with the new values obtained. Also since we are concerned with the variance ratios, change of scale also may be introduced without affecting the values of F.

Worked Example 10

Example 1

A completely randomised design experiment with 10 plots and 3 treatments gave the following results:

Plot No.	:	1	2	3	4	5	6	7	8	9	10
Treatment	:	A	B	C	A	C	C	A	B	A	B
Yield	:	5	4	3	7	5	1	3	4	1	7

Analyse the results for treatment effects.

Rearranging the data according to the treatments, we have the following table:

Treatment	*Yield from plots* (x_{ij})				T_i	T_i^2	n_i	$\dfrac{T_i^2}{n_i}$
A	5	7	3	1	16	256	4	64
B	4	4	7	–	15	225	3	75
C	3	5	1	–	9	81	3	27
				Total	$T = 40$	–	$N = 10$	166

$$\sum\sum x_{ij}^2 = (25 + 49 + 9 + 1) + (16 + 16 + 49) + (9 + 25 + 1)$$
$$= 84 + 81 + 35 = 200$$

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 200 - \frac{40^2}{10} = 200 - 160 = 40$$

$$Q_1 = \sum \frac{T_i^2}{n_i} - \frac{T^2}{N} = 166 - 160 = 6$$

$$\therefore \quad Q_2 = Q - Q_1 = 40 - 6 = 34$$

ANOVA table

S.V.	*S.S.*	*d.f.*	*M.S.*	F_0
Between classes (treatments)	$Q_1 = 6$	$h - 1 = 2$	3.0	$\frac{4.86}{3.0}$
Within classes	$Q_2 = 34$	$N - h = 7$	4.86	$= 1.62$
Total	$Q = 40$	$N - 1 = 9$	–	–

From the F-table, $F_{5\%}$ $(v_1 = 7, v_2 = 2) = 19.35$
We note that $F_0 < F_{5\%}$
Let H_0 : The treatments do not differ significantly.
$\therefore$ The null hypothesis is accepted.
i.e., the treatments are not significantly different.

Example 2

The following table shows the lives in hours of four brands of electric lamps:

Brand
A : 1610, 1610, 1650, 1680, 1700, 1720, 1800
B : 1580, 1640, 1640, 1700, 1750
C : 1460, 1550, 1600, 1620, 1640, 1660, 1740, 1820
D : 1510, 1520, 1530, 1570, 1600, 1680

Perform an analysis of variance and test the homogeneity of the mean lives of the four brands of lamps.

We subtract 1640 (= the average of the extreme values) from the given values and work out with the new values of x_{ij}

Brand	*Lives of lamps* (x_{ij})								*Ti*	n_i	$\frac{T_i^2}{n_i}$
A	– 30	– 30	10	40	60	80	160	–	290	7	12014
B	– 60	0	0	60	110	–	–	–	110	5	2420
C	– 180	– 90	– 40	– 20	0	20	100	180	– 30	8	113
D	– 130	– 120	– 110	– 70	– 40	40	–	–	– 430	6	30817
								Total	– 60	26	45364

$$\sum\sum x_{ij}^2 = (900 + 900 + 100 + 1600 + 3600 + 6400 + 25600)$$
$$+ (3600 + 0 + 0 + 3600 + 12100)$$
$$+ (32400 + 8100 + 1600 + 400 + 0 + 400 + 10000 + 32400)$$
$$+ (16900 + 14400 + 12100 + 4900 + 1600 + 1600)$$
$$= 39100 + 19300 + 85300 + 51500 = 195200$$

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 1,95,200 - 138 = 1,95,062$$

$$Q_1 = \sum \frac{T_i^2}{n^i} - \frac{T^2}{N} = 45,364 - 138 = 45,226$$

$$Q_2 = Q - Q_1 = 1,95,062 - 45,226 = 1,49,836$$

ANOVA table

S.V.	*S.S.*	*d.f.*	*M.S.*	F_0
Between brands	$Q_1 = 45,226$	$h - 1 = 3$	15,075	$\frac{15,075}{6,811}$
Within brands	$Q_2 = 1,49,836$	$N - h = 22$	6,811	$= 2.21$
Total	$Q = 1,95,062$	$N - 1 = 25$	–	–

From the F-tables, $F_{5\%}$ $(v_1 = 3, v_2 = 22) = 3.06$

$$F_0 < F_{5\%}$$

Hence the null hypothesis H_0, namely, the means of the lives of the four brands are homogeneous, is accepted viz., the lives of the four brands of lamps do not differ significantly.

Note *We could have used a change of scale also. viz., we could have made the change*

New $x_{ij} = \frac{old\ x_{ij} - 1640}{10}$ *and simplified the numerical work still further*

Example 3

A car rental agency, which uses 5 different brands of tyres in the process of deciding the brand of tyre to purchase as standard equipment for its fleet, finds that each of 5 tyres of each brand last the following number of kilometres (in thousands):

Tyre brands

A	B	C	D	E
36	46	35	45	41
37	39	42	36	39
42	35	37	39	37
38	37	43	35	35
47	43	38	32	38

Test the hypothesis that the five tyre brands have almost the same average life.

We shift the origin to 40 and work out with the new values of x_{ij}.

Tyre brand	x_{ij}					T_i	n_i	$\frac{T_i^2}{n_i}$	$\sum_{j=1}^{5} x_{ij}^2$
A	−4	−3	2	−2	7	0	5	0	82
B	6	−1	−5	−3	3	0	5	0	80
C	−5	2	−3	3	−2	−5	5	5	51
D	5	−4	−1	−5	−8	−13	5	33.8	131
E	1	−1	−3	−5	−2	−10	5	20	40
			Total			−28	25	58.8	384

$$T = \sum_i T_i = -28; \sum\sum x_{ij}^2 = \sum_i \left(\sum_j x_{ij}^2\right) = 384$$

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 384 - \frac{(-28)^2}{25} = 352.64$$

$$Q_1 = \sum \frac{T_i^2}{n_i} - \frac{T^2}{N} = 58.8 - 31.36 = 27.44$$

$$Q_2 = Q - Q_1 = 352.64 - 27.44 = 325.20$$

ANOVA table   Mean Square

S.V.	S.S.	d.f.	M.S.	F_0
Between tyre brands	$Q_1 = 27.44$	$h - 1 = 4$	6.86	$\frac{16.26}{6.86}$
Within tyre brands	$Q_2 = 325.20$	$N - h = 20$	16.26	$= 2.37$
Total	$Q = 352.64$	$N - 1 = 24$	–	–

From the F-tables, $F_{5\%}\ (\nu_1 = 20, \nu_2 = 4) = 5.80$

$$F_0 < F_{5\%}$$

Hence $\mathcal{H}_0$: the five tyre brands have almost the same average life) is accepted viz., the five tyre brands do not differ significantly in their lives.

Example 4

In order to determine whether there is significant difference in the durability of 3 makes of computers, samples of size 5 are selected from each make and the frequency of repair during the first year of purchase is observed. The results are as follows :

Makes

A	B	C
5	8	7
6	10	3
8	11	5
9	12	4
7	4	1

In view of the above data, what conclusion can you draw?

Make	x_{ij}					T_i	n_i	T_i^2 / n_i	$\sum_j x_{ij}^2$
A	5	6	8	9	7	35	5	245	255
B	8	10	11	12	4	45	5	405	445
C	7	3	5	4	1	20	5	80	100
					Total	100	15	730	800

$$T = \sum T_i = 100 ; \sum\sum x_{ij}^2 = 800; N = \sum n_i = 15$$

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 800 - \frac{100^2}{15} = 133.33$$

$$Q_1 = \sum \frac{T_i^2}{n_i} - \frac{T^2}{N} = 730 - 666.67 = 63.33$$

$$Q_2 = Q - Q_1 = 70$$

ANOVA table

S.V.	S.S.	d.f.	M.S.	F_0
Between makes	$Q_1 = 63.33$	$h - 1 = 2$	31.67	$\frac{31.67}{5.83}$
Within makes	$Q_2 = 70$	$N - h = 12$	5.83	5.43
Total	$Q = 133.33$	$N - 1 = 14$	–	–

From the F-tables, $F_{5\%}$ $(v_1 = 2, v_2 = 12) = 3.88$

$$F_0 > F_{5\%}$$

Hence the null hypothesis (H_0: the 3 makes of computers do not differ in the durability) is rejected.
viz., there is significant difference in the durability of the 3 makes of computers.

Example 5

Three varieties of a crop are tested in a randomised block design with four replications, the layout being as given below: The yields are given in kilograms. Analyse for significance

C48	A51	B52	A49
A47	B49	C52	C51
B49	C53	A49	B50

Rewriting the data such that the rows represent the blocks and the columns represent the varieties of the crop (as assumed in the discussion of analysis of variance for two factors of classification), we have the following table:

Crops

Blocks	A	B	C
1	47	49	48
2	51	49	53
3	49	52	52
4	49	50	51

We shift the origin to 50 and work out with the new values of x_{ij}.

Crops

Blocks	A	B	C	T_i	T^2_i / k	$\sum_j x^2_{ij}$
1	−3	−1	−2	−6	36/3 = 12	14
2	1	−1	3	3	9/3 = 3	11
3	−1	2	2	3	9/3 = 3	9
4	−1	0	1	0	0/3 = 0	2
T_j	−4	0	4	$T = 0$	$\sum \frac{T_i^2}{k} = 18$	36
T^2_j / h	$\frac{16}{4} = 4$	$\frac{0}{4} = 0$	$\frac{16}{4} = 4$	$\sum \frac{T_j^2}{h} = 8$		
$\sum_i x^2_{ij}$	12	6	18	36		

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 36 - \frac{0^2}{12} = 36$$

$$Q_1 = \frac{1}{k}\sum T_i^2 - \frac{T^2}{N} = 18 - 0 = 18$$

$$Q_2 = \frac{1}{h}\sum T_j^2 - \frac{T^2}{N} = 8 - 0 = 8$$

$$Q_3 = Q - Q_1 - Q_2 = 36 - 18 - 8 = 10$$

ANOVA table

S.V.	S.S.	d.f.	M.S.	F_0
Between rows (blocks)	$Q_1 = 18$	$h - 1 = 3$	6	$\frac{6}{1.67} = 3.6$
Between columns (crops)	$Q_2 = 8$	$k - 1 = 2$	4	$\frac{4}{1.67} = 2.4$
Residual	$Q_3 = 10$	$(h-1)(k-1) = 6$	1.67	–
Total	$Q = 36$	$hk - 1 = 11$	–	–

From F-tables, $F_{5\%}$ $(v_1 = 3, v_2 = 6) = 4.76$ and $F_{5\%}$ $(v_1 = 2, v_2 = 6) = 5.14$
Considering the difference between rows, we see that F_0 $(= 3.6) < F_{5\%}$ $(= 4.76)$
Hence the difference between the rows is not significant. (H_0 is accepted) viz., the blocks do not differ significantly with respect to the yield.

Considering the difference between columns, we see that F_0 $(= 2.4) < F_{5\%}$ $(= 5.14)$

Hence the difference between the columns is not significant. (H_0 is accepted) viz., the varieties of crop do not differ significantly with respect to the yield.

Example 6

Five breeds of cattle B_1, B_2, B_3, B_4, B_5 were fed on four different rations R_1, R_2, R_3, R_4. Gains in weight in kg. over a given period were recorded and given below:

	B_1	B_2	B_3	B_4	B_5
R_1	1.9	2.2	2.6	1.8	2.1
R_2	2.5	1.9	2.3	2.6	2.2
R_3	1.7	1.9	2.2	2.0	2.1
R_4	2.1	1.8	2.5	2.3	2.4

Is there a significant difference between (i) breeds and (ii) rations?

We effect the change of origin and scale using $y_{ij} = \frac{x_{ij} - 2}{1/10} = 10(x_{ij} - 2)$ and work out with y_{ij} values.

	B_1	B_2	B_3	B_4	B_5	T_i	$\frac{T_i^2}{k}$	$\sum_j y_{ij}^2$
R_1	−1	2	6	−2	1	6	7.2	46
R_2	5	−1	3	6	2	15	45.0	75
R_3	−3	−1	2	0	1	−1	0.2	15
R_4	1	−2	5	3	4	11	24.2	55
T_j	2	−2	16	7	8	$T = 31$	$\sum \frac{T_i^2}{k} = 76.6$	191
T_j^2/h	1	1	64	12.25	16	$\sum T_j^2/h = 94.25$		
$\sum_i y_{ij}^2$	36	10	74	49	22	191		

$$Q = \sum\sum y_{ij}^2 - \frac{T^2}{N} = 191 - \frac{(31)^2}{20} = 142.95$$

$$Q_1 = \frac{1}{k}\sum T_i^2 - \frac{T^2}{N} = 76.6 - 48.05 = 28.55$$

$$Q_2 = \frac{1}{h}\sum T_j^2 - \frac{T^2}{N} = 94.25 - 48.05 = 46.20$$

$$Q_3 = Q - Q_1 - Q_2 = 142.95 - (28.55 + 46.20) = 68.20$$

ANOVA table

S.V.	S.S.	d.f.	M.S.	F_0
Between rows (rations)	$Q_1 = 28.55$	$h - 1 = 3$	9.52	9.52/5.68 = 1.68
Between Cols. (breeds)	$Q_2 = 46.20$	$K - 1 = 4$	11.55	11.55/5.68 = 2.03
Residual	$Q_3 = 68.20$	$(h-1)(k-1) = 12$	5.68	–
Total	$Q = 142.95$	$hk - 1 = 19$	–	–

From the F-tables, $F_{5\%}(\nu_1 = 3, \nu_2 = 12) = 3.49$ and $F_{5\%}(\nu_1 = 4, \nu_2 = 12) = 3.26$
With respect to the rows, $F_0 (= 1.68) < F_{5\%} (= 3.49)$
With respect to the columns, $F_0 (= 2.03) < F_{5\%} (= 3.26)$

Hence the difference between the rations and that between the breeds are not significant.

Example 7

The following data represent the number of units of production per day turned out by 5 different workers using 4 different types of machines:

		Machine Type			
		A	B	C	D
Workers:	1	44	38	47	36
	2	46	40	52	43
	3	34	36	44	32
	4	43	38	46	33
	5	38	42	49	39

(a) Test whether the five men differ with respect to mean productivity.

(b) Test whether the mean productivity is the same for the four different machine types.

We subtract 40 from the given values and work out with new values of x_{ij}.

Worker	*A* (Machine Type)	*B*	*C*	*D*	T_i	T_i^2/k	$\sum_j x_{ij}^2$
1	4	−2	7	−4	5	6.25	85
2	6	0	12	3	21	110.25	189
3	−6	−4	4	−8	−14	49.00	132
4	3	−2	6	−7	0	0	98
5	−2	2	9	−1	8	16.00	90
T_j	5	−6	38	−17	$T = 20$	$\sum \frac{T_j^2}{k} = 181.5$	594
T_j^2/h	5	7.2	288.8	57.8	$\sum T_j^2/h = 358.8$		
$\sum_i x_{ij}^2$	101	28	326	139	594		

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 594 - \frac{400}{20} = 574$$

$$Q_1 = \sum \frac{T_i^2}{k} - \frac{T^2}{N} = 181.5 - 20 = 161.5$$

$$Q_2 = \sum \frac{T_j^2}{h} - \frac{T^2}{N} = 358.8 - 20 = 338.8$$

$$Q_3 = Q - Q_1 - Q_2 = 574 - (161.5 + 338.8) = 73.7$$

ANOVA table

S.V.	S.S.	d.f.	M.S.	F_0
Between rows (workers)	$Q_1 = 161.5$	$h - 1 = 4$	40.375	$\frac{40.375}{6.142} = 6.57$
Between Cols. (machines)	$Q_2 = 338.8$	$k - 1 = 3$	112.933	$\frac{112.933}{6.142} = 18.39$
Residual	$Q_3 = 73.7$	$(h-1)(k-1) = 12$	6.142	–
Total	$Q = 574$	$hk - 1 = 19$	–	–

From the F-tables, $F_{5\%}$ $(v_1 = 4, v_2 = 12) = 3.26$
and $F_{5\%}$ $(v_1 = 3, v_2 = 12) = 3.49$
With respect to the rows, $F_0(= 6.57) > F_{5\%}$ $(= 3.26)$
With respect to the columns, $F_0(= 18.39) > F_{5\%}(= 3.49)$

Hence the 5 workers differ significantly and the 4 machine types also differ significantly with respect to mean productivity.

Example 8

Four doctors each test four treatments for a certain disease and observe the number of days each patient takes to recover. The results are as follows (recovery time in days)

	Treatment			
Doctor	1	2	3	4
A	10	14	19	20
B	11	15	17	21
C	9	12	16	19
D	8	13	17	20

Discuss the difference between (a) doctors and (b) treatments.
We subtracted 15 from the given values and work out with the new values of x_{ij}.

Doctor	Treatment 1	2	3	4	T_i	$\frac{T_i^2}{k}$	$\sum_j x_{ij}^2$
A	–5	–1	4	5	3	2.25	67
B	–4	0	2	6	4	4.00	56
C	–6	–3	1	4	–4	4.00	62
D	–7	–2	2	5	–2	1.00	82
T_j	–22	–6	9	20	$T = 1$	$\sum \frac{T_j^2}{k} = 11.25$	267
T_j^2 / h	121	9	20.25	100	$\sum T_j^2 / h = 250.25$		
$\sum_i x_{ij}^2$	126	14	25	102	267		

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 267 - \frac{1}{16} = 266.94$$

$$Q_1 = \sum \frac{T_i^2}{k} - \frac{T^2}{N} = 11.25 - 0.0625 = 11.19$$

$$Q_2 = \sum \frac{T_j^2}{h} - \frac{T^2}{N} = 250.25 - 0.0625 - 250.19$$

$$Q_3 = Q - Q_1 - Q_2 = 266.94 - 261.38 = 5.56$$

ANOVA table

S.V.	S.S.	d.f.	M.S.	F_0
Between rows (doctors)	$Q_1 = 11.19$	$h - 1 = 3$	3.73	$\frac{3.73}{0.62} = 6.02$
Between cols. (treatments)	$Q_2 = 250.19$	$k - 1 = 3$	83.40	$\frac{83.40}{0.62} = 134.52$
Residual	$Q_3 = 5.56$	$(h-1)(k-1) = 9$	0.62	–
Total	$Q = 266.94$	$hk - 1 = 15$	–	–

From the F-tables, $F_{5\%}$ $(v_1 = 3, v_2 = 9) = 3.86$
Since $F_0 > F_{5\%}$ with respect to rows and columns, the difference between the doctors is significant and that between the treatments is highly significant.

Example 9

The following data resulted from an experiment to compare three burners B_1, B_2 and B_3. A Latin square design was used as the tests were made on 3 engines and were spread over 3 days.

	Engine 1	Engine 2	Engine 3
Day 1	$B_1 - 16$	$B_2 - 17$	$B_3 - 20$
Day 2	$B_2 - 16$	$B_3 - 21$	$B_1 - 15$
Day 3	$B_3 - 15$	$B_1 - 12$	$B_2 - 13$

Test the hypothesis that there is no difference between the burners.
We subtract 16 from the given values and work out with new values of x_{ij}.

	E_1	E_2	E_3	T_i	$\frac{T_i^2}{n}$	$\sum_j x_{ij}^2$
D_1	$0(B_1)$	$1(B_2)$	$4(B_3)$	5	8.33	17
D_2	$0(B_2)$	$5(B_3)$	$-1(B_1)$	4	5.33	26
D_3	$-1(B_3)$	$-4(B_1)$	$-3(B_2)$	-8	21.33	26
T_j	-1	2	0	$T = 1$	$\sum T_i^2 / n = 35$	69
T_j^2 / n	0.33	1.33	0	$\sum T_j^2 / n = 1.66$		
$\sum_i x_{ij}^2$	1	42	26	69		

Rearranging the data values according to the burners, we have

Burner	D_1	x_k		T_k	T_k^2 / n
B_1	0	-1	-4	-5	8.33
B_2	1	0	-3	-2	1.33
B_3	4	5	-1	8	21.33
		Total		$T = 1$	$\sum \frac{T_k^2}{n} = 31$

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 69 - \frac{1}{9} = 68.89$$

$$Q_1 = \frac{1}{n}\sum T_i^2 - \frac{T^2}{N} = 35 - \frac{1}{9} = 34.89$$

$$Q_2 = \frac{1}{n}\sum T_j^2 - \frac{T^2}{N} = 1.67 - \frac{1}{9} = 1.56$$

$$Q_3 = \frac{1}{n}\sum T_k^2 - \frac{T^2}{N} = 31 - \frac{1}{9} = 30.89$$

$$Q_4 = Q - Q_1 - Q_2 - Q_3 = 1.55$$

ANOVA table

S.V.	S.S.	d.f.	M.S.	F_0
Between rows (days)	$Q_1 = 34.89$	$n - 1 = 2$	17.445	$\frac{17.445}{0.775} = 22.51$
Between Cols. (engines)	$Q_2 = 1.56$	$n - 1 = 2$	0.780	$\frac{0.780}{0.775} = 1.01$
Between letters (burners)	$Q_3 = 30.89$	$n - 1 = 2$	15.445	$\frac{15.445}{0.775} = 19.93$
Residual	$Q_4 = 1.55$	$(-1)(n-2) = 2$	0.775	–
Total	$Q = 68.89$	$n^2 - 1 = 8$	–	–

From the F-tables, $F_{5\%}$ $(v_1 = 2, v_2 = 2) = 19.00$
Since $F_0 (= 19.93) > F_{5\%} (= 19.00)$ for the burners, there is significant difference between the burners.

Incidentally, since $F_0 > F_{5\%}$ for the rows, the difference between the days is significant and since $F_0 < F_{5\%}$ for the columns, the difference between the engine is not significant.

Example 10

Analyse the variance in the following Latin square of yields (in kgs) of paddy where A, B, C, D denote the different methods of cultivation

D122	A121	C123	B122
B124	C123	A122	D125
A120	B119	D120	C121
C122	D123	B121	A122

Examine whether the different methods of cultiyation have given significantly different yields.

We subtract 120 from the given values and work out with the new values of x_{ij}.

i \ j	1	2	3	4	T_i	T_i^2/n	$\sum_j x_{ij}^2$
1	D2	A1	C3	B2	8	16	18
2	B4	C3	A2	D5	14	49	54
3	A0	B-1	D0	C1	0	0	2
4	C2	D3	B1	A2	8	16	18
T_j	8	6	6	10	$T = 30$	$\sum T_i^2/n = 81$	92
T_j^2/n	16	9	9	25	$\sum T_j^2/n = 59$		
$\sum_i x_{ij}^2$	24	20	14	34	92		

Rearranging the data according to the letters, we have

Letter	x_k				T_k	T_k^2/n
A	1	2	0	2	5	6.25
B	2	4	-1	1	6	9.00
C	3	3	1	2	9	20.25
D	2	5	0	3	10	25.00
		Total			30	60.50

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 92 - \frac{30^2}{16} = 35.75$$

$$Q_1 = \frac{1}{n}\sum T_i^2 - \frac{T^2}{N} = 81 - 56.25 = 24.75$$

$$Q_2 = \frac{1}{n}\sum T_j^2 - \frac{T^2}{N} = 59 - 56.25 = 2.75$$

$$Q_3 = \frac{1}{n}\sum T_k^2 - \frac{T^2}{N} = 60.50 - 56.25 = 4.25$$

$$Q_4 = Q - Q_1 - Q_2 - Q_3 = 35.75 - (24.75 + 2.75 + 4.25)$$

$$= 4.0$$

ANOVA table

S.V.	*S.S.*	*d.f.*	*M.S.*	F_0
Between rows	$Q_1 = 24.75$	$n-1=3$	8.25	
Between columns	$Q_2 = 2.75$	$n-1=3$	0.92	
Between letters	$Q_3 = 4.25$	$n-1=3$	1.42	$\frac{1.42}{0.67} = 2.12$
Residual	$Q_4 = 4.0$	$(n-1)(n-2)=6$	0.67	–
Total	$Q = 35.75$	$n^2-1=15$	–	–

From the F-tables, $F_{5\%}$ $(\nu_1 = 3, \nu_2 = 6) = 4.76$.
Since $F_0 (= 2.12) < F_{5\%}$ $(= 4.76)$ with respect to the letters, the difference between the methods of cultivation is not significant.

Exercise 10

Part A (Short answer questions)

1. What do you mean by the term 'experiment' in Design of experiments?
2. What motivated the adoption of design of experiments technique in scientific problems?
3. What is the aim of the design of experiments?
4. Distinguish between experimental and extraneous variables.
5. Name the basic principles of experimental design.
6. What do you mean by experimental group and control group?
7. What are the techniques frequently used in the local control of extraneous variables?
8. Name three basic designs of experiment.
9. What do you mean by analysis of variance.
10. Explain completely randomised design briefly.
11. Write down the format of the ANOVA table for one factor of classification.
12. Explain randomised block design briefly.
13. Write down the format of the ANOVA table for two factors of classification.
14. Explain Latin square design briefly.
15. Is a 2 × 2 Latin square design possible? Why?
    [**Hint** : No, as the degree of freedom for the residual variation is zero]
16. Write down the format of ANOVA table for three factors of classification.
17. Compare RBD and LSD.
18. What is the main advantage of LSD over RBD?

19. What is the total number of all possible Latin squares of order 3?
20. What is the total number of all possible Latin squares of order 4?

Part B

21. The following tables gives the yields of wheat from 16 plots, all of approximately equal fertility, when 4 varieties of wheat were cultivated in a completely randomised fashion. Test the hypothesis that the varieties are not significantly different.

Plot No.	:	1	2	3	4	5	6	7	8	9	10
Variety	:	A	B	D	C	B	C	A	D	B	D
Yield	:	32	34	29	31	33	34	34	26	36	30
Plot No.	:	11	12	13	14	15	16				
Variety	:	A	C	B	A	B	C				
Yield	:	33	35	37	35	35	32				

22. A random sample is selected from each of 3 makes of ropes and their breaking strength (in certain units) are measured with the following results:
I : 70, 72, 75, 80, 83
II : 60, 65, 57, 84, 87, 73
III : 100, 110, 108, 112, 113, 120, 107
Test whether the breaking strength of the ropes differ significantly.

23. The weights in gm of a number of copper wires, each of length 1 metre, were obtained. These are shown classified according to the dye from which they come:
D_1 : 1.30, 1.32, 1.36, 1.35, 1.32, 1.37
D_2 : 1.28, 1.35, 1.33, 1.34
D_3 : 1.32, 1.29, 1.31, 1.28, 1.33, 1.30
D_4 : 1.31, 1.29, 1.33, 1.31, 1.32
D_5 : 1.30, 1.32, 1.30, 1.33
Test the hypothesis that there is no difference between the mean weights of wires coming from different dyes.

24. It is suspected that four machines used in a canning operation fills cans to different levels on the average. Random samples of cans produced by each machine were taken and the fill (in ounces) was measured. The results are tabulated below:

Machine

A	*B*	*C*	*D*
10.20	10.22	10.17	10.15
10.18	10.27	10.22	10.27
10.36	10.26	10.34	10.28
10.21	10.25	10.27	10.40
10.25	–	–	10.30

Do the machines appear to be filling the cans at different average levels?

25. Different numbers of leaves were taken from each of 6 trees and the lengths measured. The following are the lengths in millimetres :

Tree	Lengths							
1	82	87	86	90	81	84		
2	85	84	91	92	88			
3	92	90	84	86	88	93	89	90
4	80	86	87	81	82	82		
5	87	86	88	90	85	86	87	
6	90	86	84	85	85	86	87	84

Can all these leaves be regarded as having come from the same species trees?

26. There are 3 typists working in an office. The times (in minutes) they spe[nd] for tea-break in addition to the allowed lunch tea break are observed a[nd] noted below:

A :	25	18	30	32	35	37	19			
B :	24	22	26	28	30	32	28	26		
C :	28	20	27	19	29	35	30	23	27	3

Can the difference in average times that the 3 typists spend for tea break i[s] attributed to chance variation?

27. Four machines *A*, *B*, *C*, *D* are used to produce a certain kind of cott[on] fabric. 4 Samples with each unit of size 100 square metres are select[ed] from the outputs of the machines at random and the number of flaws [in] each 100 square metres are counted, with the following results:

A	*B*	*C*	*D*
8	6	14	20
9	8	12	22
11	10	18	25
12	4	9	23

Do you think that there is a significant difference in the performance of t[he] four machines?

28. The following table shows the yield (in certain units) of lima beans on [2]0 plots of land subject to 4 different treatments, 5 plots per treatment. Set u[p] an analysis of variance table to test the significance of the differences b[e]tween the yields due to different treatments.

T_1 :	26.3	30.0	54.2	25.7	52.4
T_2 :	18.5	21.1	29.3	17.2	12.4
T_3 :	36.9	21.8	24.0	18.5	10.2
T_4 :	39.8	28.7	21.2	39.4	29.0

29. To test the significance of the variation of the retail prices of a certa[in] commodity in the 4 principal cities Mumbai, Kolkata, Delhi and Chenna[i], 7 shops were chosen at random in each city and the prices (in Rs.) o[b]served were as follows:

Mumbai :	100,	97,	91,	87,	87,	81,	79
Kolkata :	102,	100,	98,	97,	94,	86,	80
Delhi :	106,	102,	98,	86,	86,	84,	84
Chennai :	97,	95,	94,	92,	90,	86,	82

Do the data indicate that the prices in the 4 cities are significantly different?

30. Steel wire was made by 4 manufacturers A, B, C and D. In order to compare their products, 10 samples were randomly drawn from a batch of wires made by each manufacturer and the strength of each piece of wire was measured. The (coded) values are given below:

A : 55, 50, 80, 60, 70, 75, 40, 45, 80, 70
B : 70, 80, 85, 105, 65, 100, 90, 95, 100, 70
C : 70, 60, 65, 75, 90, 40, 95, 70, 65, 75
D : 90, 115, 80, 70, 95, 100, 105, 90, 100, 60

Carry out an analysis of variance and give your conclusions.

31. A randomised block experiment was laid out (with 4 blocks, each block containing 4 plots) to test 4 varieties of manure A, B, C, D and the yields per acre are given as below. Test for the significance of the difference among the 4 varieties of manure.

Block I	A155	B152	C157	D156
Block II	B152	C150	D156	A154
Block III	C156	D153	A161	B162
Block IV	D153	A154	B156	C155

32. The following table gives the gains in weights of 4 different types of pigs fed on 3 different rations over a period. Test whether
(i) the difference in the rations significant
(ii) the 4 types of pigs differ significantly in gaining weight.

	Types of pig			
Ration	I	II	III	IV
A	13.8	15.7	16.0	20.2
B	8.7	11.8	9.0	12.9
C	12.0	16.5	13.3	12.5

33. Four experiments determine the moisture content of samples of a powder, each observer taking a sample from each of six consignments. The assessments are given below:

	Consignment					
Observer	1	2	3	4	5	6
1	9	10	9	10	11	11
2	12	11	9	11	10	10
3	11	10	10	12	11	10
4	12	13	11	14	12	10

Perform an analysis of variance on these data and discuss whether there is any significant difference between consignments or between observers.

34. In order to compare three burners B_1, B_2 and B_3, one observation is made on each burner on each of four successive days. The data are tabulated below:

	B_1	B_2	B_3
Day 1	21	23	24
Day 2	18	17	23
Day 3	18	21	20
Day 4	17	20	22

Perform an analysis of variance on these data and find whether the difference between (i) the days and (ii) the burners significant at 5% LOS.

35. A company appoints 4 salesmen A, B, C and D and observes their sales in 3 seasons summer, winter and monsoon. The figures (in lakhs of Rs) are given in the following table :

	Salesmen			
Season	A	B	C	D
Summer	36	36	21	35
Winter	28	29	31	32
Monsoon	26	28	29	29

Carry out an analysis of variance.

36. The following data represent the numbers of units of production per day turned out by 4 different workers using 5 different types of machines:

	Machine type				
Worker	A	B	C	D	E
1	4	5	3	7	6
2	6	8	6	5	4
3	7	6	7	8	8
4	3	5	4	8	2

On the basis of this information, can it be concluded that (i) the mean productivity is the same for different machines (ii) the workers do not differ with regard to productivity?

37. The number of automobiles arriving at 4 toll gates were recorded for a 2 hours time period (10 A.M. to 12 noon) for each of six working days. The data are as follows :

Day	Gate 1	Gate 2	Gate 3	Gate 4
Mon	200	228	212	301
Tues	208	230	215	305
Wed	225	240	228	288
Thur	223	242	224	212
Fri	228	210	235	215
Sat	220	208	245	200

Determine whether the rate of arrival (i) is the same at each toll gate (ii) differs significantly during the six days or not.

38. The following table gives the number of refrigerators sold by 4 salesmen in 3 months:

	Salesman			
Months	*I*	*II*	*III*	*IV*
May	50	40	48	39
June	46	48	50	45
July	39	44	40	39

Determine whether (i) there is any difference in average sales made by the four salesmen (ii) the sales differ with respect to different months.

39. Four different drugs have been developed for a certain disease. These drugs are used in 3 different hospitals and the results, given below, show the number of cases of recovery from the disease per 100 people who have taken the drugs :

	D_1	D_2	D_3	D_4
H_1	19	8	23	8
H_2	10	9	12	6
H_3	11	13	13	10

What conclusions can you draw based on an analysis of variance ?

40. The following table gives the additional hours of sleep due to 3 soporofic drugs A, B, C tried on one patient each from 4 different age groups. Examine whether age has got any significant effect on the gain in sleep. Also examine whether the 3 drugs are similar in their effects or not

	Age group			
Drug	30–40	40–50	50–60	60–70
A	2.0	1.2	1.0	0.3
B	1.1	0.8	0.0	− 0.1
C	1.5	1.3	0.9	0.1

41. The following table gives the results of experiments on 4 varieties of a crop in 5 blocks of plots. Prepare the ANOVA table to test the significance of the difference between the yields of the 4 varieties :

Variety	B_1	B_2	B_3	B_4	B_5
A	32	34	33	35	37
B	34	33	36	37	35
C	31	34	35	32	36
D	29	26	30	28	29

42. In the table given below are the yields of 6 varieties of a crop in a 4 replicate RBD experiment. Analyse the data:

	Varieties					
Replicates	1	2	3	4	5	6
1	18.5	15.7	16.2	14.1	13.0	13.6
2	11.7	14.25	12.9	14.4	14.9	12.5
3	15.4	14.6	15.5	20.3	18.4	21.5
4	16.5	18.6	12.7	15.7	16.5	18.0

43. Analyse the variance in the following Latin square:

A8	C18	B9
C9	B18	A16
B11	A10	C20

44. Analyse the variance in the following Latin square:

20 B	17 C	25 D	34 A
23 A	21 D	15 C	24 B
24 D	26 A	21 B	19 C
26 C	23 B	27 A	22 D

45. A varietal trial was conducted on wheat with 4 verities A, B, C, D in a Latin square design. The plan of the experiment and the per plot yield are given below.

C25	B23	A20	D20
A19	D19	C21	B18
B19	A14	D17	C20
D17	C20	B21	A15

Analyse the data and interpret the result.

46. The following is the Latin square layout of a design when 4 varieties of seeds are tested. Set up the analysis of variance table and state your conclusions.

A105	B95	C125	D115
C115	D125	A105	B105
D115	C95	B105	A115
B95	A135	D95	C115

47. The table given below shows the yield of a certain crop in kgs per plot. The letters A, B, C, D refer to 4 different manurial treatments. Carry out an analysis of variance.

A260	B300	C335	D371
B280	A300	D300	C410
D320	C345	B340	A254
C372	D395	A290	B328

48. The following results were obtained in a textile experiment to compare the effects of sizing treatments A, B, C, D on the number of warps breaking per hour. Is the difference between the treatments significant?

Loom

period	1	2	3	4
1	A 54	B 31	C 70	D 45
2	B 59	A 23	D 100	C 22
3	C 40	D 41	B 74	A 33
4	D 83	C 29	A 100	B 28

49. An agricultural experiment on the Latin square plan gave the following results for the yield of wheat per acre, letters corresponding to varieties.

A16	B10	C11	D9	E9
E10	C9	A14	B12	D11
B15	D8	E8	C10	A18
D12	E6	B13	A13	C12
C13	A11	D10	E7	B14

Discuss the variation of yield with each of the factors corresponding to the rows and columns.

50. The following is a Latine square design of five treatments:

A13	B9	C21	D7	E6
D9	E8	A15	B7	C16
B11	C17	D8	E10	A17
E8	A15	B7	C10	D7
C11	D9	E8	A11	B15

Analyse the data and interpret the results.

ANSWERS

Exercise 10

19. 12 20. 576

21. $Q_1 = 46.08$, $Q_2 = 73.67$, $F_0 = 2.50$, $F_{5\%} = 3.49$; Difference between varieties not significant.

22. $Q_1 = 5838.4$, $Q_2 = 1126$, $F_0 = 38.89$, $F_{5\%} = 3.68$; Breaking strengths of ropes differ significantly.

23. $Q_1 = 35.98$, $Q_2 = 99.38$, $F_0 = 1.81$, $F_{5\%} = 2.87$; Mean weights of wires do not differ significantly.

24. $Q_1 = 44.44$, $Q_2 = 696$, $F_0 = 2.98$, $F_{5\%} = 3.35$; No, the machines appear to fill at same level.

25. $Q_1 = 151.95$, $Q_2 = 255$, $F_0 = 4.17$, $F_{5\%} = 2.50$; Leaves have not come from the same species.

26. $Q_1 = 2.52$, $Q_2 = 29.27$, $F_0 = 11.62$, $F_{5\%} = 19.45$; Difference may be attributed to chance variation.

27. $Q_1 = 540.65$, $Q_2 = 85.75$, $F_0 = 25.21$, $F_{5\%} = 3.49$; Performances of the machines differ significantly.

28. $Q_1 = 34845.93$, $Q_2 = 10032.78$, $F_0 = 3.47$, $F_{5\%} = 3.24$; Treatments give significantly different yields.

29. $Q_1 = 94.97$, $Q_2 = 1446.03$, $F_0 = 1.9$, $F_{5\%} = 8.64$; Prices do not differ significantly.

30. $Q_1 = 5151$, $Q_2 = 8348$, $F_0 = 7.41$, $F_{5\%} = 8.60$; Strengths of wire do not differ significantly.

31. $Q_1 = 42.75$, $Q_2 = 6.75$, $Q_3 = 96.25$, $F_0 = 4.75$, $F_{5\%} = 8.82$; Difference between manures is not significant.

32. $Q_1 = 3393.59$, $Q_2 = 878.44$, $Q_3 = 344.36$,
F_0 (rows) = 9.85 and $F_{5\%} = 5.14$,
F_0 (columns) = 2.55 and $F_{5\%} = 4.76$;
Difference between rations significant. Difference between pigs is not significant.

33. $Q_1 = 13.13$, $Q_2 = 9.71$, $Q_3 = 13.12$,
F_0 (rows) = 5.03 and $F_{5\%} = 3.29$,
F_0 (columns) = 2.23 and $F_{5\%} = 5.05$;
Difference between observers is significant, Difference between consignments is not significant.

34. $Q_1 = 22.00$, $Q_2 = 28.17$, $Q_3 = 14.50$,
F_0 (rows) = 3.03 and $F_{5\%} = 4.76$,
F_0 (columns) = 5.83 and $F_{5\%} = 5.14\%$;
Difference between days is not significant; Difference between burners is significant.

35. $Q_1 = 32$, $Q_2 = 42$, $Q_3 = 136$,
F_0 (rows) = 1.42 and $F_{5\%} = 19.33$,
F_0 (columns) = 1.62, and $F_{5\%} = 8.94$;
Differences between seasons and between salesmen are not significant.

36. $Q_1 = 22.0$, $Q_2 = 12.8$, $Q_3 = 30.0$;
F_0 (rows) = 2.93 and $F_{5\%} = 3.49$;
F_0 (columns) = 1.28 and $F_{5\%} = 3.26$;
Differences between the workers and between machine types are not significant.

37. $Q_1 = 2279.83$, $Q_2 = 1470.05$, $Q_3 = 820.12$,

F_0 (rows) = 1.80 and $F_{5\%} = 4.64$,
F_0 (columns) = 1.79 and $F_{5\%} = 3.29$;
Differences between the days and between the gates are not significant.

38. $Q_1 = 109.5$, $Q_2 = 42.0$, $Q_3 = 64.5$,
F_0 (rows) = 5.09 and $F_{5\%} = 5.14$,
F_0 (columns) = 1.30 and $F_{5\%} = 4.76$;
Differences between the months and between salesmen are not significant.

39. $Q_1 = 55.17$, $Q_2 = 113.0$, $Q_3 = 89.5$,
F_0 (rows) = 1.85 and $F_{5\%} = 5.14$;
F_0 (columns) = 2.52 and $F_{5\%} = 4.76$;
Differences between the hospitals and between the drugs are not significant.

40. $Q_1 = 98.17$, $Q_2 = 341.58$, $Q_3 = 25.17$,
F_0 (rows) = 11.69 and $F_{5\%} = 5.14$,
F_0 (columns) = 27.11 and $F_{5\%} = 4.76$;
Age has significant effect on the gain in sleep; Drugs differ significantly in their effect.

41. $Q_1 = 134.0$, $Q_2 = 21.7$, $Q_3 = 29.5$,
F_0 (row) = 18.16 and $F_{5\%} = 3.49$,
F_0 (columns) = 2.21 and $F_{5\%} = 3.26$;
Difference between the yields of 4 varieties is significant.

42. $Q_1 = 56.76$, $Q_2 = 12.58$, $Q_3 = 80.18$,
F_0 (rows) = 3.30 and $F_{5\%} = 3.34$,
F_0 (columns) = 2.27 and $F_{5\%} = 4.65$;
Difference between the varieties is significant.

43. $Q_1 = 11.56$, $Q_2 = 68.23$, $Q_3 = 29.56$, $Q_4 = 68.21$,
F_0 (rows) = 5.90, F_0 (cols.) = 1, F_0 (letters) = 2.31,
$F_{5\%}$ (for all) = 19.0;
The differences between rows, between columns and between letters are not significant.

44. $Q_1 = 34.19$, $Q_2 = 22.69$, $Q_3 = 141.19$, $Q_4 = 96.87$, F_0 (rows) = 1.42 and $F_{5\%} = 8.94$; F_0 (columns) = 2.14 and $F_{5\%} = 8.94$; F_0(letters) = 2.91 and $F_{5\%} = 4.76$; Differences between rows, between columns and between letters are not significant.

45. $Q_1 = 46.5$, $Q_2 = 7.5$, $Q_3 = 48.5$, $Q_4 = 10.5$,
F_0 (rows) = 8.86, F_0 (columns) = 1.43, F_0 (letters) = 9.24, $F_{5\%} = 4.76$.
Difference between varieties is significant.

46. $Q_1 = 2$, $Q_2 = 4$, $Q_3 = 22$, $Q_4 = 60$, F_0 (rows) = 15, F_0 (columns) = 7.5, F_0 (letters) = 1.36, $F_{5\%} = 8.94$; Difference between rows is significant, but differences between columns and between letters are not significant.

47. $Q_1 = 2540.5$, $Q_2 = 2853.75$, $Q_3 = 18690$, $Q_4 = 7515.75$, F_0 (rows) = 1.48, and $F_{5\%} = 8.94$; F_0 (columns) = 1.32 and $F_{5\%} = 8.94$; F_0 (letters) = 4.97 and $F_{5\%} = 4.76$; Differences between rows and between columns are not significant, but difference between treatments is significant.

48. $Q_1 = 376$, $Q_2 = 8184$, $Q_3 = 1547.5$, $Q_4 = 284.5$
F_0 (rows) = 2.64, F_0 (columns) = 57.53, F_0 (letters) = 10.88; $F_{5\%}$ (for all) = 4.76; Difference between periods is not significant; Differences between looms and between treatments are significant.

49. $Q_1 = 2.16$, $Q_2 = 66.56$, $Q_3 = 122.56$, $Q_4 = 5.28$, F_0 (rows) = 1.2, F_0 (columns) = 37.8, F_0 (letters) = 69.6, $F_{5\%}$ (for all) = 3.26; Difference between rows is not significant, but differences between columns and between varieties are significant.

50. $Q_1 = 26$, $Q_2 = 34$, $Q_3 = 224.4$, $Q_4 = 103.6$, F_0 (rows) = 1.33 and $F_{5\%} = 5.91$, F_0 (columns) = 1.02 and $F_{5\%} = 5.91$, F_0 (letters) = 6.50 and $F_{5\%} = 3.26$; Differences between rows and between columns are not significant, but difference between treatments is significant.