

MAT2001

Statistics for Engineers

Module 2

Random Variables

Syllabus

Random variables:

Introduction -random variables-Probability mass Function, distribution and density functions - joint Probability distribution and joint density functions- Marginal, conditional distribution and density functions- Mathematical expectation, and its properties Covariance , moment generating function – characteristic function.

Probability Theory

Probability theory had its origin in the analysis of certain games of chance that were popular in the seventeenth century. It has since found applications in many branches of Science and Engineering and this extensive application makes it an important branch of study. Probability theory, as a matter of fact, is a study of random or unpredictable experiments and is helpful in investigating the important features of these random experiments.

- Random Experiment
- Sample Space
- Events

Random Experiment, Sample Space and Event

Problem (Non-deterministic)

Random Experiment

$S \rightarrow$ Sample Space (All possible outcomes)

$\Omega = P(S) =$ Power Set of S

$$= \{A : A \subseteq S\}$$

Event: \mapsto a subset of a sample space S

A is called an event, if $A \subseteq S$

$$(i.e.) A \in \Omega = P(S)$$

Mathematical Definition of Probability

Let E - Experiment with Sample Space S.

and let A - an event.

Here S is finite

$n(S)$ is finite

$$\overline{P(A) = \frac{n(A)}{n(S)} = \frac{|A|}{|S|}}$$

$n(A)$ = no. of elements in A
 $|A|$

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to } A}{\text{Exhaustive number of cases in } S}$$

Example:

$$\textcircled{1}. \quad S = \{H, T\} \Rightarrow |S| = 2$$

$$A = \{H\} \Rightarrow |A| = 1$$

$$P(A) = \frac{1}{2}$$

$$\textcircled{2}. \quad S = \{1, 2, \dots, 5\} \Rightarrow |S| = 5$$

$$A = \{1, 3, 5\} \Rightarrow |A| = 3$$

$$P(A) = \frac{1}{2}$$

Statistical Definition of Probability

n - Trials

A = Expected Event

n_A = No. of occurrences of A

$$P(A) = \lim_{n \rightarrow \infty} \frac{n_A}{n}$$

$$P(A) = \lim_{n \rightarrow \infty} \frac{n_A}{n}$$

Axiomatic Definition of Probability

Probability | Probability function |

Probability Measure

A function

$$P : \mathcal{P}(S) \longrightarrow [0, 1]$$

Let E -Exp's
with S.S.S

Let $A \subseteq S$

is called probability, if P satisfies

(i). $P(S) = 1$ [or $P(\emptyset) = 0$]

(ii). Addition theorem of Probability.

Mutually Exclusive Events

Let $A \times B$ are 2 events

A and B are said to be mutually exclusive events, if $A \cap B = \emptyset$

$$\text{(Or)} \quad P(A \cap B) = P(\emptyset) = 0$$

Addition Theorem of Probability

Let $A \times B$ be an events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Suppose $A \times B$ are mutually exclusive

then

$$P(A \cup B) = P(A) + P(B)$$

In general

Suppose $A_1, A_2, \dots, A_n, \dots$ are pair-wise
mutually exclusive events,

then

$$P(A_1 \cup A_2 \cup \dots \cup A_n \cup \dots) = P(A_1) + P(A_2) + \dots + \dots + P(A_n) + \dots$$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Sure Event

Sample Event

$$P(S) = 1$$

$\frac{1}{\emptyset}$

Impossible Event

Theorem 1

The probability of the impossible event is zero, i.e., if ϕ is the subset (event) containing no sample point, $P(\phi) = 0$.

Theorem 2

If \bar{A} is the complementary event of A , $P(\bar{A}) = 1 - P(A) \leq 1$.

Theorem 3

If A and B are any 2 events, $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$

Theorem 4

If $B \subset A$, $P(B) \leq P(A)$.

Conditional Probability

The conditional probability of an event B , assuming that the event A has happened, is denoted by $P(B/A)$ and defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P(A) \times P(B/A)$$

Conditional Probability

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ if } P(A) \neq 0$$

Example

$$S = \{1, 2, \dots, 6\}$$

$$E = \{2, 4, 6\} \Rightarrow P(E) = \frac{1}{2}$$

$$O = \{1, 3, 5\} \Rightarrow P(O) = \frac{1}{2}$$

$$A = \{1\} \quad A \cap O = \{1\} \Rightarrow P(A \cap O) = \frac{1}{6}$$

$$P(A) = \frac{1}{6}$$

$$P(A/O) = \frac{P(A \cap O)}{P(O)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

$$P(A/E) = \frac{P(A \cap E)}{P(E)} = \frac{P(\emptyset)}{\frac{1}{2}} = 0$$

Product theorem of probability

$$P(A \cap B) = P(A) \times P(B/A)$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

The product theorem can be extended to 3 events A , B and C as follows:

$$P(A \cap B \cap C) = P(A) \times P(B/A) \times P(C/A \text{ and } B)$$

Independent Events

$$P(A \cap B) = P(A) \times P(B/A)$$

If A and B are independent to each other,

$$P(B/A) = P(B)$$

$$P(A \cap B) = P(A) \times P(B)$$

Independent Events

If $P(A \cap B) = P(A) \times P(B)$,

the events A and B are said to be independent (pairwise independent).

In General,

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \times P(A_2) \times \dots \times P(A_n)$$

$$P\left(\bigcap_{i=1}^n A_i\right) = \prod_{i=1}^n P(A_i)$$

Theorem 1

If the events A and B are independent, the events \bar{A} and B (and similarly A and \bar{B}) are also independent.

$$\bar{A} = S - A = S/A$$

Theorem 2

If the events A and B are independent, then so are \bar{A} and \bar{B} .

Random Variables (X)

$$X: S \rightarrow \mathbb{R}$$

$$\begin{aligned} X(s) &= 1 \\ R_X &= \{1\} \end{aligned}$$

$$\{X \leq x\} = \{\underline{s \in S : X(s) \leq x}\}$$

$$P(\{X \geq x\}) = P(\{s \in S : X(s) = x\})$$

Example:

$$S = \{1, 2, 3, \dots, 6\}$$

$$A = \{1, 2, 3\}, B = \{4, 5, 6\}$$

$$P(A) \text{ vs } P(B) = ?$$

$$X: S \rightarrow \mathbb{R} \implies R_X = \{-3, 3\}$$

$$X(s) = \begin{cases} -3; & s \in \{1, 2, 3\} \\ +3; & s \in \{4, 5, 6\} \end{cases}$$

$$\begin{aligned} \{X = -3\} &= \{s \in S : X(s) = -3\} \\ &= \{1, 2, 3\} \end{aligned}$$

$$P(A) = P(X = -3)$$

$$P(B) = P(X = 3)$$

Random Variables

Definition: A random variable (abbreviatively RV) is a function that assigns a real number $X(s)$ to every element $s \in S$, where S is the sample space corresponding to a random experiment E .)

Hereafter, R_x will be referred to as **Range space**.

Similarly $\{X \leq x\}$ represents the subset $\{s: X(s) \leq x\}$ and hence an event associated with the experiment.

Discrete Random Variable

Continuous Random Variable

$X = \Sigma x_i$, where $x_i \in \{x_1, x_2, \dots, x_n\}$

$X = x$, where $x \in [a, b] \quad \forall a, b \in \mathbb{R}$

$\Rightarrow R_x$

Discrete Random Variable

$X - DRV$

$$X = x_1, x_2, \dots, x_n$$

$$x_i \in \mathbb{R}$$

Probability Mass function

x_i	x_1	...	x_n
p_i	p_1	...	p_n

(i) p_i 's ≥ 0

(ii). $\sum_{i=1}^n p_i = 1$

p_i 's are p.m.f

$\{x_i, p_i\}_{i=1}^n$ is called

prob. distribution of.

a DRV X

Continuous Random Variable

$X - CRV$

$$X = x \in [a, b] = R_x$$

$$[a, b] \subseteq \mathbb{R}$$

Prob. Density function

(p.d.f)

$f(x)$ is said p.d.f

if (i). $f(x) \geq 0$,

(ii). $\int_{R_x} f(x) dx = 1$

R_x

$\{x, f(x)\}$

$x \in R_x$ is

called $P.D.f.$ of a
CRV X

Discrete Random Variable

If X is a random variable (RV) which can take a finite number or countably infinite number of values, X is called a discrete RV.

Probability Function

If X is a discrete RV which can take the values x_1, x_2, x_3, \dots such that $P(X = x_i) = p_i$, then p_i is called the *probability function* or *probability mass function* or *point probability function*, provided p_i ($i = 1, 2, 3, \dots$) satisfy the following conditions:

(i) $p_i \geq 0$, for all i , and

(ii) $\sum_i p_i = 1$

The collection of pairs $\{x_i, p_i\}$, $i = 1, 2, 3, \dots$, is called the *probability distribution of the RV X*, which is sometimes displayed in the form of a table as given below:

$X = x_i$	$P(X = x_i)$
x_1	p_1
x_2	p_2
\vdots	\vdots
x_r	p_r
\vdots	\vdots

Continuous Random Variable

If X is an RV which can take all values (i.e., *infinite number* of values) in an interval, then X is called a *continuous* RV.

Probability Density Function

If X is a continuous RV such that

$$P\left\{x - \frac{1}{2} dx \leq X \leq x + \frac{1}{2} dx\right\} = f(x)dx$$

then $f(x)$ is called the *probability density function* (shortly denoted as pdf) of X , provided $f(x)$ satisfies the following conditions:

(i) $f(x) \geq 0$, for all $x \in R_x$, and

$$(ii) \int_{R_X} f(x)dx = 1$$

Moreover, $P(a \leq X \leq b)$ or $P(a < X < b)$ is defined as

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

When X is a continuous RV

$$P(X = a) = P(a \leq X \leq a) = \int_a^a f(x)dx = 0$$

Cumulative Distribution Function (cdf)

If X is an R V, discrete or continuous, then $P(X \leq x)$ is called the *cumulative distribution function* of X or *distribution function* of X and denoted as $F(x)$.

If X is discrete,

$$F(x) = \sum_j p_j$$
$$x_j \leq x$$

If X is continuous,

$$F(x) = P(-\infty < X \leq x) = \int_{-\infty}^x f(x) dx$$

Properties of the cdf $F(x)$

1. $F(x)$ is a non-decreasing function of x , i.e., if $x_1 < x_2$, then $F(x_1) \leq F(x_2)$.
2. $F(-\infty) = 0$ and $F(\infty) = 1$.
3. If X is a discrete R V taking values x_1, x_2, \dots , where $x_1 < x_2 < x_3 < \dots < x_{i-1} < x_i < \dots$, then $P(X = x_i) = F(x_i) - F(x_{i-1})$.
4. If X is a continuous R V, then $\frac{d}{dx} F(x) = f(x)$, at all points where $F(x)$ is differentiable.

Special Distributions

Discrete Distributions

- 1. Binomial Distribution*
- 2. Poisson Distribution*

Continuous Distributions

- 1. Normal Distribution*
- 2. Exponential Distribution*
- 3. Gamma Distribution*
- 4. Weibull Distribution*

$X - RV$

$$\underset{\text{Mean}}{\frac{E(X)}{\sum p_i}} = \frac{\sum_{i=1}^n x_i p_i}{\sum p_i} = \sum_{i=1}^n x_i p_i \quad ; \text{ if } X \text{ is DRV}$$

$$\int x f(x) dx \quad ; \text{ if } X \text{ is CRV}$$

$$E(X^2) = \sum_{i=1}^n x_i^2 p_i ; \text{ if } X \text{ is DRV}$$

$$\int_a^{\infty} x^2 f(x) dx ; \text{ if } X \text{ is CRV}$$

Let $g(x)$ be a function on a RV X .

$$E(g(x)) = \begin{cases} \sum_{i=1}^n g(x_i) p_i & ; \text{ if } X \text{ is DRV} \\ \int_a^{\infty} g(x) f(x) dx & ; \text{ if } X \text{ is CRV} \end{cases}$$

Suppose $g(x) = x$ (Identity function)
 $E(g(x)) = E(X)$

Suppose $g(x) = x^2$
then $E(g(x)) = E(X^2)$

Properties:

$$E(aX) = a E(X)$$

$$E(ax+by) = a E(X) + b E(Y)$$

$$E(X+Y) = E(X) + E(Y)$$

$$E(1) = 1$$

$$E(a) = a$$

$$\text{Variance} = \frac{1}{N} \sum_{i=1}^N p_i \cdot \left(\sum_{j=1}^N x_j p_j \right)^2 - \left(\sum_{i=1}^N x_i p_i \right)^2 = E(X^2) - (E(X))^2$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = E((X - E(X))^2)$$

$$\sigma_X^2 = E[(X - E(X))^2]$$

$$SD(X) = \sigma_X = \sqrt{\text{Var}(X)}$$

Mathematical Expectation of One Dimensional Random Variable

$$E(g(x)) = \begin{cases} \sum_{i=1}^n g(x_i) \cdot p_i & ; \text{if } X \text{- DRV} \\ \int_{-\infty}^{\infty} g(x) \cdot f(x) dx & ; \text{if } X \text{- CRV} \end{cases}$$

choose $g(x) = x$

$$\rightarrow E(X) = \begin{cases} \sum_{i=1}^n x_i \cdot p_i & ; \text{if } X \text{- DRV} \\ \int_{-\infty}^{\infty} x \cdot f(x) dx & ; \text{if } X \text{- CRV} \end{cases}$$

Mean of X

choose: $g(x) = x \cdot x = x^2$

$$E(X^2) = \begin{cases} \sum_{i=1}^n x_i^2 \cdot p_i & ; \text{if } X \text{- DRV} \\ \int_{-\infty}^{\infty} x^2 \cdot f(x) dx & ; \text{if } X \text{- CRV} \end{cases}$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - (E(X))^2 \\ &= E[(X - E(X))^2] \end{aligned}$$

$$SD = +\sqrt{\text{Var}(X)}$$

Mathematical Expectation of One Dimensional Random Variable

Definitions: If X is a discrete RV, then *the expected value* or the mean value of $g(X)$ is defined as

$$E\{g(X)\} = \sum_i g(x_i)p_i,$$

where $p_i = P(X = x_i)$ is the probability mass function of X .

If X is a continuous RV with pdf $f(x)$, then

$$E\{g(X)\} = \int_{R_X} g(x)f(x)dx$$

Mean, Variance and Standard Deviation of a Random Variable

Two expected values which are most commonly used for characterising a RV X are its **mean** μ_X and **variance** σ_X^2 , which are defined as follows:

$$\mu_X = E(X)$$

$$= \sum_i x_i p_i, \text{ if } X \text{ is discrete}$$

$$= \int_{R_X} x f(x) dx, \text{ if } X \text{ is continuous}$$

$$\text{Var}(X) = \sigma_X^2 = E\{(X - \mu_X)^2\}$$

$$= \sum_i (x_i - \mu_X)^2 p_i, \text{ if } X \text{ is discrete}$$

$$\int_{R_X} (x - \mu_X)^2 f(x) dx, \text{ if } X \text{ is continuous}$$

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2$$

The square root of variance is called **the standard deviation**.

Properties of Expected Value :

$$1). E(aX) = a E(X)$$

$$2). \text{Var}(aX) = a^2 \text{Var}(X)$$

$$3). E(X+Y) = E(X) + E(Y)$$

4). If X and Y are independent RVs, then $E(XY) = E(X) \cdot E(Y)$

Example:

If the random variable X takes the values 1, 2, 3 and 4 such that $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)$, find the probability distribution and cumulative distribution function of X .

Solution:

Let $P(X = 3) = 30K$. Since $2P(X = 1) = 30K$, $P(X = 1) = 15K$.

Similarly $P(X = 2) = 10K$ and $P(X = 4) = 6K$.

Since $\sum p_i = 1$, $15K + 10K + 30K + 6K = 1$.

$$\therefore K = \frac{1}{61}$$

The probability distribution of X is given in the following table:

$X = i$	1	2	3	4
p_i	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

The cdf $F(x)$ is defined as $F(x) = P(X \leq x)$. Accordingly the cdf for the above distribution is found out as follows:

When $x < 1$, $F(x) = 0$

When $1 \leq x < 2$, $F(x) = P(X = 1) = \frac{15}{61}$

When $2 \leq x < 3$, $F(x) = P(X = 1) + P(X = 2) = \frac{25}{61}$

When $3 \leq x < 4$, $F(x) = P(X = 1) + P(X = 2) + P(X = 3) = \frac{55}{61}$

When $x \geq 4$, $F(x) = P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4) = 1$.

Example:

A random variable X has the following probability distribution.

$x:$	-2	-1	0	1	2	3
$p(x):$	0.1	K	0.2	$2K$	0.3	$3K$

- (a) Find K , (b) Evaluate $P(X < 2)$ and $P(-2 < X < 2)$, (c) find the cdf of X and
(d) evaluate the mean of X .

Solution:

(a) Since $\sum P(x) = 1$, $6K + 0.6 = 1$

$$\therefore K = \frac{1}{15}$$

\therefore the probability distribution becomes

x	: -2	-1	0	1	2	3
$p(x)$: $1/10$	$1/15$	$1/5$	$2/15$	$3/10$	$1/5$

$$(b) P(X < 2) = P(X = -2, -1, 0 \text{ or } 1)$$

$$= P(X = -2) + P(X = -1) + P(X = 0) + P(X = 1)$$

[since the events $(X = -2)$, $(X = -1)$ etc. are mutually exclusive]

$$= \frac{1}{10} + \frac{1}{15} + \frac{1}{5} + \frac{2}{15} = \frac{1}{2}$$

$$P(-2 < X < 2) = P(X = -1, 0 \text{ or } 1)$$

$$= P(X = -1) + P(X = 0) + P(X = 1)$$

$$= \frac{1}{15} + \frac{1}{5} + \frac{2}{15} = \frac{2}{5}$$

Solution (Continued):

(c) $F(x) = 0$, when $x < -2$

$$= \frac{1}{10}, \text{ when } -2 \leq x < -1$$

$$= \frac{1}{6}, \text{ when } -1 \leq x < 0$$

$$= \frac{11}{30}, \text{ when } 0 \leq x < 1$$

$$= \frac{1}{2}, \text{ when } 1 \leq x < 2$$

$$= \frac{4}{5}, \text{ when } 2 \leq x < 3$$

$$= 1, \text{ when } 3 \leq x$$

(d) The mean of X is defined as $E(X) = \Sigma xp(x)$

$$\therefore \text{Mean of } X = \left(-2 \times \frac{1}{10}\right) + \left(-1 \times \frac{1}{15}\right) + \left(0 \times \frac{1}{5}\right)$$

$$+ \left(1 \times \frac{2}{15}\right) + \left(2 \times \frac{3}{10}\right) + \left(3 \times \frac{1}{5}\right)$$

$$= -\frac{1}{5} - \frac{1}{15} + \frac{2}{15} + \frac{3}{5} + \frac{3}{5} = \frac{16}{15}$$

Exercise:

Find the value of a , $P(X < 3)$, cumulative distribution function, mean, variance and standard deviation of the discrete random variable (X) with the following probability distribution.

$X = x:$	0	1	2	3	4	5	6	7	8
$p = P(X = x):$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

Exercise:

A random variable X has the following probability distribution.

$x :$	0	1	2	3	4	5	6	7
$p(x) :$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2 + K$

Find (i) the value of K , (ii) $P(1.5 < X < 4.5 | X > 2)$

Example:

$$\text{If } p(x) = \begin{cases} x e^{-x^2/2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

- (a) show that $p(x)$ is a pdf (of a continuous RV X .)
- (b) find its distribution function $P(x)$.

Example:

$$\text{If } p(x) = \begin{cases} x e^{-x^2/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

- (a) show that $p(x)$ is a pdf (of a continuous RV X).
 (b) find its distribution function $F(x)$.

a)

$$(i) I = \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx$$

$$I = \int_0^{\infty} x \cdot e^{-x^2/2} dx$$

$$\begin{aligned} \text{Let } t &= x^2 \Rightarrow x = \sqrt{t} & x=0 \Rightarrow t=0 \\ dt &= 2x dx & x=\infty \Rightarrow t=\infty \end{aligned}$$

$$I = \frac{1}{2} \int_0^{\infty} e^{-t/2} dt$$

$$\begin{aligned} I &= \frac{1}{2} \left[-\frac{e^{-t/2}}{1/2} \right]_0^{\infty} \\ &= - \left[0 - 1 \right] \\ &= 1 \end{aligned}$$

$$F(x) = \int_{-\infty}^x f(x) dx$$

$$-\infty < x < 0, F(x) = 0$$

$$0 \leq x < \infty, F(x) = \int_0^x f(x) dx$$

$$= \int_0^x x \cdot e^{-x^2/2} dx$$

$$F(x) = 1 - e^{-x^2/2} \quad [t=x^2]$$

$$F(x) = \begin{cases} 0; & -\infty < x < 0 \\ 1 - e^{-x^2/2}; & 0 \leq x < \infty \end{cases}$$

Solution:

(a) If $p(x)$ is to be a pdf, $p(x) \geq 0$ and

$$\int_{R_X} p(x) dx = 1$$

Obviously, $p(x) = xe^{-x^2/2} \geq 0$, when $x \geq 0$

$$\begin{aligned} \text{Now } \int_0^\infty p(x) dx &= \int_0^\infty xe^{-x^2/2} dx = \int_0^\infty e^{-t} dt \text{ (putting } t = x^2/2\text{)} \\ &= 1 \end{aligned}$$

$\therefore p(x)$ is a legitimate pdf of a RV X .

$$F(x) = P(X \leq x) = \int_0^x f(x) dx$$

$\therefore F(x) = 0$, when $x < 0$

and $F(x) = \int_0^x xe^{-t^2/2} dt = 1 - e^{-x^2/2}$, when $x \geq 0$.

Example:

If the density function of a continuous RV X is given by

$$\begin{aligned}f(x) &= ax, & 0 \leq x \leq 1 \\&= a, & 1 \leq x \leq 2 \\&= 3a - ax, & 2 \leq x \leq 3 \\&= 0, & \text{elsewhere}\end{aligned}$$

- (i) find the value of a
- (ii) find the cdf of X

Solution:

(i) Since $f(x)$ is a pdf, $\int_{R_x} f(x)dx = 1$.

i.e., $\int_0^3 f(x)dx = 1$

i.e., $\int_0^1 axdx + \int_1^2 adx + \int_2^3 (3a - ax)dx = 1$

i.e., $2a = 1$

$\therefore a = \frac{1}{2}$

(ii) $F(x) = P(X \leq x) = 0$, when $x < 0$

$$F(x) = \int_0^x \frac{x}{2} dx = \frac{x^4}{4}, \text{ when } 0 \leq x \leq 1$$

$$= \int_0^1 \frac{x}{2} dx + \int_1^x \frac{1}{2} dx = \frac{x}{2} - \frac{1}{4} \text{ when } 1 \leq x \leq 2$$

$$\begin{aligned} &= \int_0^1 \frac{x}{2} dx + \int_1^2 \frac{1}{2} dx + \int_2^x \left(\frac{3}{2} - \frac{x}{2}\right) dx = \frac{3}{2}x - \frac{x^2}{4} - \frac{5}{4}, \text{ when } 2 \leq x \leq 3 \\ &= 1, \text{ when } x > 3 \end{aligned}$$

Example:

A line of length a units is divided into two parts. If the first part is of length X , find $E(X)$, $\text{var}(X)$ and $E\{X(a - X)\}$.

Solution:

Since the positions of the point of division are equally likely, X is uniformly distributed in $(0, a)$.

$$\therefore f(x) = \frac{1}{a}$$

$$E(X) = \int_0^a xf(x) dx = \frac{1}{a} \int_0^a x dx = \frac{a}{2}$$

$$E(X^2) = \int_0^a x^2 f(x) dx = \frac{a^2}{3}$$

$$\therefore \text{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{a^2}{3} - \frac{a^2}{4} = \frac{a^2}{12}$$

$$E\{X(a - X)\} = a E(X) - E(X^2) = \frac{a^2}{3} - \frac{a^2}{4} = \frac{a^2}{12}$$

Exercise:

A continuous RV X that can assume any value between $x = 2$ and $x = 5$ has a density function given by $f(x) = k(1 + x)$. Find $P(X < 4)$.

Exercise:

A continuous RV X has a pdf $f(x) = kx^2e^{-x}$; $x >$. Find k , mean and variance.

Exercise:

A continuous RV has a pdf $f(x) = 3x^2$; $0 \leq x \leq 1$. Find a and b such that

- (i) $P(X \leq a) = P(X > a)$ and
- (ii) $P(X > b) = 0.05$

Exercise:

Suppose X is a continuous random variable with the probability density function given

$$\text{by } f(x) = \begin{cases} kx, & 0 \leq x \leq 2; \\ 2k, & 2 \leq x \leq 4, \\ 6k - kx, & 4 \leq x \leq 6. \end{cases}$$

Find the value of k , cumulative distribution function, mean, variance and standard deviation of X .

Two-Dimensional Random Variables

Definitions: Let S be the sample space associated with a random experiment E . Let $X = X(s)$ and $Y = Y(s)$ be two functions each assigning a real number to each outcomes $s \in S$. Then (X, Y) is called a *two-dimensional random variable*.

If the possible values of (X, Y) are finite or countably infinite, (X, Y) is called a *two-dimensional discrete RV*. When (X, Y) is a two-dimensional discrete RV the possible values of (X, Y) may be represented as (x_i, y_j) , $i = 1, 2, \dots, m, \dots; j = 1, 2, \dots, n, \dots$.

If (X, Y) can assume all values in a specified region R in the xy -plane, (X, Y) is called a *two-dimensional continuous RV*.

Probability Function of (X, Y)

If (X, Y) is a two-dimensional discrete RV such that $P(x = x_i, y = y_j) = p_{ij}$, then p_{ij} is called the *probability mass function* or simply the *probability function* of (X, Y) provided the following conditions are satisfied.

(i) $p_{ij} \geq 0$, for all i and j

(ii) $\sum_j \sum_i p_{ij} = 1$

The set of triples $\{x_i, y_j, p_{ij}\}$, $i = 1, 2, \dots, m, \dots, j = 1, 2, \dots, n, \dots$, is called *the joint probability distribution of (X, Y)* .

Joint Probability Density Function

If (X, Y) is a two-dimensional continuous RV such that,

$$P\left\{x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2} \text{ and } y - \frac{dy}{2} \leq Y \leq y + \frac{dy}{2}\right\} = f(x, y) dx dy, \text{ then } f(x, y) \text{ is}$$

called *the joint pdf* of (X, Y) , provided $f(x, y)$ satisfies the following conditions.

(i) $f(x, y) \geq 0$, for all $(x, y) \in R$, where R is the range space.

(ii) $\iint_R f(x, y) dx dy \equiv 1$.

Moreover if D is a subspace of the range space R , $P\{(X, Y) \in D\}$ is defined as

$$P\{(X, Y) \in D\} = \iint_D f(x, y) dx dy. \text{ In particular}$$

$$P\{a \leq X \leq b, c \leq Y \leq d\} = \int_c^d \int_a^b f(x, y) dx dy$$

Cumulative Distribution Function

If (X, Y) is a two-dimensional RV (discrete or continuous), then $F(x, y) = P\{X \leq x \text{ and } Y \leq y\}$ is called *the cdf of (X, Y)* .

In the discrete case,

$$F(x, y) = \sum_j \sum_i p_{ij} \quad y_j \leq y, x_i \leq x$$

In the continuous case,

$$F(x, y) = \int_{-\infty}^y \int_{-\infty}^x f(x, y) dx dy$$

Properties of $F(x, y)$

- (i) $F(-\infty, y) = 0 = F(x, -\infty)$ and $F(\infty, \infty) = 1$
- (ii) $P\{a < X < b, Y \leq y\} = F(b, y) - F(a, y)$
- (iii) $P\{X \leq x, c < Y < d\} = F(x, d) - F(x, c)$
- (iv) $P\{a < X < b, c < Y < d\} = F(b, d) - F(a, d) - F(b, c) + F(a, c)$
- (v) At points of continuity of $f(x, y)$

$$\frac{\partial^2 F}{\partial x \partial y} = f(x, y)$$

Marginal Probability Distribution

$$P(X = x_i) = P\{(X = x_i \text{ and } Y = y_1) \text{ or } (X = x_i \text{ and } Y = y_2) \text{ or etc.}\}$$

$$= p_{i1} + p_{i2} + \dots = \sum_j p_{ij}$$

$P(X = x_i) = \sum_j p_{ij}$ is called the *marginal probability function of X*. It is defined

for $X = x_1, x_2, \dots$ and denoted as P_{i*} . The collection of pairs $\{x_i, p_{i*}\}, i = 1, 2, 3, \dots$ is called the *marginal probability distribution of X*.

Similarly the collection of pairs $\{y_j, p_{*j}\}, j = 1, 2, 3, \dots$ is called the *marginal probability distribution of Y*, where $p_{*j} = \sum_i P_{ij} = P(Y = y_j)$.

In the continuous case,

$$P\left(x - \frac{1}{2}dx \leq X \leq x + \frac{1}{2}dx, -\infty < Y < \infty\right)$$

$$= \int_{-\infty}^{\infty} \int_{x - \frac{1}{2}dx}^{x + \frac{1}{2}dx} f(x, y) dx dy$$

$$\begin{aligned} &= \left[\int_{-\infty}^{\infty} f(x, y) dy \right] dx \quad [\text{since } f(x, y) \text{ may be treated a constant in } \\ &\quad (x - 1/2dx, x + 1/2dx)] \\ &= f_X(x)dx, \text{ say} \end{aligned}$$

$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$ is called the *marginal density of X*.

Similarly, $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$ is called the *marginal density of Y*.

Note

$$P(a \leq X \leq b) = P(a \leq X \leq b, -\infty < Y < \infty)$$

$$= \int_{-\infty}^{\infty} \int_a^b f(x, y) dx dy$$

$$= \int_a^b \left[\int_{-\infty}^{\infty} f(x, y) dy \right] dx = \int_a^b f_X(x) dx$$

Similarly, $P(c \leq Y \leq d) = \int_c^d f_Y(y) dy$

Conditional Probability Distribution

$P\{X = x_i / Y = y_j\} = \frac{P\{X = x_i, Y = y_j\}}{P\{Y = y_j\}} = \frac{p_{ij}}{p_{*j}}$ is called *the conditional probability function of X, given that Y = y_j*.

The collection of pairs, $\left\{x_i, \frac{p_{ij}}{p_{*j}}\right\}, i = 1, 2, 3, \dots,$

is called *the conditional probability distribution of X, given Y = y_j*.

Similarly, the collection of pairs, $\left\{Y_j, \frac{p_{ij}}{p_{*i}}\right\}, j = 1, 2, 3, \dots$, is called the *conditional probability distribution of Y given X = x_i*. In the continuous case,

$$\begin{aligned} & P\left\{x - \frac{1}{2} dx \leq X \leq x + \frac{1}{2} dx / Y = y\right\} \\ &= P\left\{x - \frac{1}{2} dx \leq X \leq x + \frac{1}{2} dx / y - \frac{1}{2} dy \leq Y \leq y + \frac{1}{2} dy\right\} \\ &= \frac{f(x, y) dx dy}{f_Y(y) dy} = \left\{\frac{f(x, y)}{f_Y(y)}\right\} dx. \end{aligned}$$

$\frac{f(x, y)}{f_Y(y)}$ is called *the conditional density of X, given Y*, and is denoted by $f(x/y)$.

Similarly, $\frac{f(x, y)}{f_X(x)}$ is called *the conditional density of Y, given X*, and is denoted by $f(y/x)$.

Independent RVs

If (X, Y) is a two-dimensional discrete RV such that $P\{X = x_i | Y = y_j\} = P(X = x_i)$ i.e., $\frac{P_{ij}}{P_{*j}} = p_{i*}$, i.e., $P_{ij} = p_{i*} \times p_{*j}$ for all i, j then X and Y are said to be independent RVs.

Similarly if (X, Y) is a two-dimensional continuous RV such that $f(x, y) = f_X(x) \times f_Y(y)$, then X and Y are said to be independent RVs.

Random Vectors

Definitions: A vector $X: [X_1, X_2, \dots, X_n]$ whose components X_i are RVs is called a *random vector*. (X_1, X_2, \dots, X_n) can assume all values in some region R_n of the n -dimensional space. R_n is called the *range space*.

Mathematical Expectation of Two Dimensional Random Variable

Expected Values of a Two-Dimensional RV

If (X, Y) is a two-dimensional discrete RV with joint probability mass function p_{ij} , then $E\{g(X, Y)\} = \sum_j \sum_i g(x_i, y_i) p_{ij}$.

If (X, Y) is a two-dimensional continuous RV with joint pdf $f(x, y)$, then

$$E\{g(X, Y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy$$

Example:

Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denotes the number of red balls drawn, find the joint probability distribution of (X, Y) .

Solution:

As there are only 2 white balls in the box, X can take the values 0, 1 and 2 and Y can take the values 0, 1, 2 and 3.

$$\begin{aligned} P(X=0, Y=0) &= P(\text{drawing 3 balls none of which is white or red}) \\ &= P(\text{all the 3 balls drawn are black}) \\ &= 4C_3/9C_3 = \frac{1}{21} \end{aligned}$$

$$\begin{aligned} P(X=0, Y=1) &= P(\text{drawing 1 red and 2 black balls}) \\ &= \frac{3C_1 \times 4C_2}{9C_3} = \frac{3}{14} \end{aligned}$$

$$\text{Similarly, } P(X=0, Y=2) = \frac{3C_2 \times 4C_1}{9C_3} = \frac{1}{7}; P(X=0, Y=3) = \frac{1}{84}$$

$$P(X=1, Y=0) = \frac{1}{7}; P(X=1, Y=1) = \frac{2}{7}; P(X=1, Y=2) = \frac{1}{14};$$

$$P(X=1, Y=3) = 0 \text{ (since only 3 balls are drawn)}$$

$$P(X=2, Y=0) = \frac{1}{21}; P(X=2, Y=1) = \frac{1}{28}; P(X=2, Y=2) = 0;$$

$$P(X=2, Y=3) = 0$$

The joint probability distribution of (X, Y) may be represented in the form of a table as given below:

X	Y			
	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

Example:

For the bivariate probability distribution of (X, Y) given below, find $P(X \leq 1)$, $P(Y \leq 3)$, $P(X \leq 1, Y \leq 3)$, $P(X \leq 1/Y \leq 3)$, $P(Y \leq 3/X \leq 1)$ and $P(X + Y \leq 4)$.

$X \backslash Y$	1	2	3	4	5	6
0	0	0	$1/32$	$2/32$	$2/32$	$3/32$
1	$1/16$	$1/16$	$1/8$	$1/8$	$1/8$	$1/8$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/64$

Solution:

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$= \sum_{j=1}^6 P(X = 0, Y=j) + \sum_{j=1}^6 P(X = 1, Y=j)$$

$$= \left(0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{2}{32} + \frac{3}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= \frac{1}{4} + \frac{5}{8} = \frac{7}{8}$$

$$P(Y \leq 3) = P(Y = 1) + P(Y = 2) + P(Y = 3)$$

$$= \sum_{i=0}^2 P(X = i, Y = 1) + \sum_{i=0}^2 P(X = i, Y = 2)$$

$$+ \sum_{i=0}^2 P(X = i, Y = 3)$$

$$= \left(0 + \frac{1}{16} + \frac{1}{32}\right) + \left(0 + \frac{1}{16} + \frac{1}{32}\right) + \left(\frac{1}{32} + \frac{1}{8} + \frac{1}{64}\right)$$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$$

$$P = \frac{3}{32} = \left(0 + 0 + \frac{1}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8}\right) = \frac{9}{32}$$

$$P(X \leq 1 | Y \leq 3) = \frac{P(X \leq 1, Y \leq 3)}{P(Y \leq 3)} = \frac{9/32}{23/64} = \frac{18}{23}$$

$$P(Y \leq 3 | X \leq 1) = \frac{P(X \leq 1, Y \leq 3)}{P(X \leq 1)} = \frac{9/32}{7/8} = \frac{9}{28}$$

$$P(X + Y \leq 4) = \sum_{j=1}^4 P(X = 0, Y=j) + \sum_{j=1}^3 P(X = 1, Y=j) + \sum_{j=1}^2 P(X = 2, Y=j)$$

$$= \frac{3}{32} + \frac{1}{4} + \frac{1}{16} = \frac{13}{32}$$

Example:

The joint probability mass function of (X, Y) is given by $p(x, y) = k(2x + 3y)$, $x = 0, 1, 2$; $y = 1, 2, 3$. Find all the marginal and conditional probability distributions. Also find the probability distribution of $(X + Y)$.

Solution:

The joint probability distribution of (X, Y) is given below. The relevant probabilities have been computed by using the given law.

X	Y		
	1	2	3
0	$3k$	$6k$	$9k$
1	$5k$	$8k$	$11k$
2	$7k$	$10k$	$13k$

$$\sum_{j=1}^3 \sum_{i=0}^2 p(x_i, y_j) = 1$$

i.e., the sum of all the probabilities in the table is equal to 1.

$$\text{i.e., } 72k = 1.$$

$$\therefore k = \frac{1}{72}$$

Solution (Continued):

Marginal Probability Distribution of X : $\{i, p_{i*}\}$

$X = i$	$p_{i*} = \sum_{j=1}^3 p_{ij}$
0	$p_{01} + p_{02} + p_{03} = \frac{18}{72}$
1	$p_{11} + p_{12} + p_{13} = \frac{24}{72}$
2	$p_{21} + p_{22} + p_{23} = \frac{30}{72}$
Total = 1	

Marginal Probability Distribution of Y : $\{j, p_{*j}\}$

$Y = j$	$p_{*j} = \sum_{i=0}^2 p_{ij}$
1	15/27
2	24/72
3	33/72
Total = 1	

Solution (Continued):

Conditional distribution of X , given $Y = 1$, is given by $\{i, P(X = i|Y = 1)\} = \{i, P(X = i, Y = 1)/P(Y = 1)\}$
 $= \{i, p_{i1}/p_{*1}\}, i = 0, 1, 2.$

The tabular representation is given below:

$X = i$	p_{i1}/p_{*1}
0	$3k/15k = \frac{1}{5}$
1	$5k/15k = \frac{1}{3}$
2	$7k/15k = \frac{7}{15}$
	Total = 1

The other conditional distributions are given below:

C.P.D. of X , given $Y = 2$	
$X = i$	p_{i2}/p_{*2}
0	$\frac{6k}{24k} = \frac{1}{4}$
1	$\frac{8k}{24k} = \frac{1}{3}$
2	$\frac{10k}{24k} = \frac{5}{12}$
	Total = 1

C.P.D. of X , given $Y = 3$	
$X = i$	p_{i3}/p_{*3}
0	$\frac{9k}{33k} = \frac{3}{11}$
1	$\frac{11k}{33k} = \frac{1}{3}$
2	$\frac{13k}{33k} = \frac{13}{33}$
	Total = 1

Solution (Continued):

C.P.D. of Y , given $X = 0$

$Y = j$	p_{oj}/p_{o*}
1	$\frac{3k}{18k} = \frac{1}{6}$
2	$\frac{6k}{18k} = \frac{1}{3}$
3	$\frac{9k}{18k} = \frac{1}{2}$
	Total = 1

C.P.D. of Y , given $X = 1$

$Y = j$	p_{1j}/p_{1*}
1	$\frac{5k}{24k} = \frac{5}{24}$
2	$\frac{8k}{24k} = \frac{1}{3}$
3	$\frac{11k}{24k} = \frac{11}{24}$
	Total = 1

C.P.D. of Y , given $X = 2$

$Y = j$	p_{2j}/p_{2*}
1	$\frac{7k}{30k} = \frac{7}{30}$
2	$\frac{10k}{30k} = \frac{1}{3}$
3	$\frac{13k}{30k} = \frac{13}{30}$
	Total = 1

Solution (Continued):

Probability distribution of $(X + Y)$	
$(X + Y)$	P
1	$p_{01} = \frac{3}{72}$
2	$p_{02} + p_{11} = \frac{11}{72}$
3	$p_{03} + p_{12} + p_{21} = \frac{24}{72}$
4	$p_{13} + p_{22} = \frac{21}{72}$
5	$p_{23} = \frac{13}{72}$
	Total = 1

Exercise:

If X denotes the number of aces and Y denotes the number of queens obtained when 2 cards are randomly drawn without replacement from a standard deck of cards, then obtain the joint probability distribution of (X, Y) .

Exercise:

The following table represents the joint probability distribution of the two dimensional discrete random variable (X, Y) .

Y	X		
	1	2	3
1	1/12	1/6	0
2	0	1/9	1/5
3	1/18	1/4	2/15

Find all the marginal and conditional probability distributions.

Example:

The joint pdf of a two-dimensional RV (X, Y) is given by $f(x, y) = xy^2 + \frac{x^2}{8}$,
 $0 \leq x \leq 2$, $0 \leq y \leq 1$.

Compute $P(X > 1)$, $P(Y < \frac{1}{2})$, $P(X > 1 | Y < 1/2)$

$P(Y < \frac{1}{2} | X > 1)$, $P(X < Y)$ and $P(X + Y \leq 1)$.

Solution:

Here the rectangle defined by $0 \leq x \leq 2$, $0 \leq y \leq 1$ is the range space R . R_1, R_2, \dots are event spaces.

$$(i) P(X > 1) = \int \int f(x, y) dx dy$$

$$= \int_0^1 \int_0^2 \left(xy^2 + \frac{x^2}{8} \right) dx dy = \frac{19}{24}$$

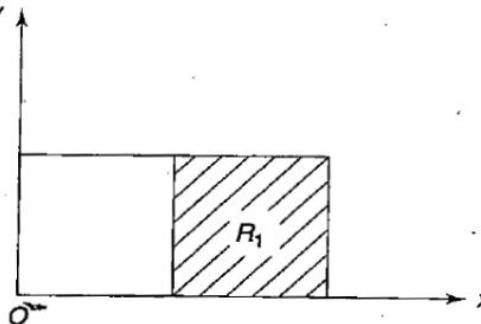


Fig.

$$(ii) P(Y < 1/2) = \int_{R_2} \left(xy^2 + \frac{x^2}{8} \right) dx dy$$

$$= \int_0^{1/2} \int_0^2 \left(xy^2 + \frac{x^2}{8} \right) dx dy$$

$$= \frac{1}{4}$$

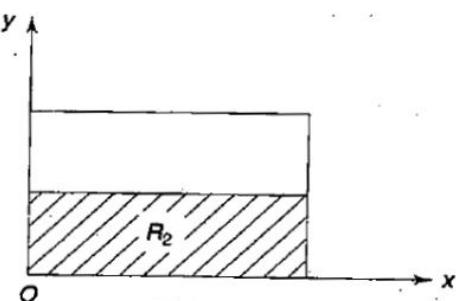


Fig.

Solution (Continued):

$$\begin{aligned}
 \text{(iii)} \quad P(X > 1, Y < 1/2) &= \int_{R_3} \left(xy^2 + \frac{x^2}{8} \right) dx dy \\
 &\quad \left(x > 1 \& y < \frac{1}{2} \right) \\
 &= \int_0^{1/2} \int_1^2 \left(xy^2 + \frac{x^2}{8} \right) dx dy \\
 &= \frac{5}{24}
 \end{aligned}$$

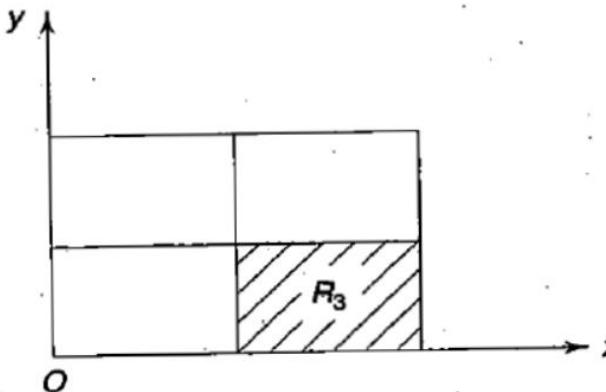


Fig.

$$\text{(iv)} \quad P\left(X > 1, Y < \frac{1}{2}\right) = \frac{P\left(X > 1, Y < \frac{1}{2}\right)}{P\left(Y < \frac{1}{2}\right)} = \frac{5/24}{1/4} = \frac{5}{6}$$

$$\text{(v)} \quad P\left(Y < \frac{1}{2} / X > 1\right) = \frac{P\left(X > 1, Y < \frac{1}{2}\right)}{P(X > 1)} = \frac{5/24}{19/24} = \frac{5}{19}$$

Solution (Continued):

$$\begin{aligned}
 \text{(vi)} \quad P(X < Y) &= \int_{R_4} \int \left(xy^2 + \frac{x^2}{8} \right) dx dy \\
 &\quad (x < y) \\
 &= \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8} \right) dx dy = \frac{53}{480}
 \end{aligned}$$

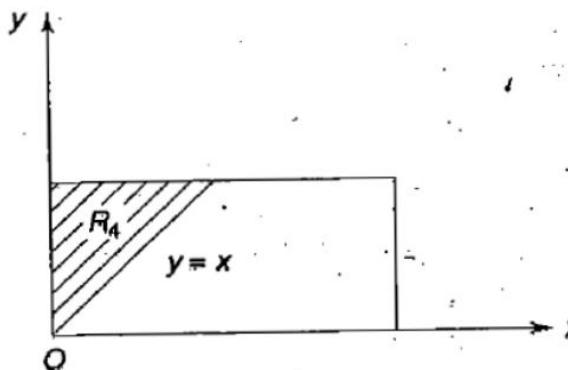


Fig.

$$\begin{aligned}
 \text{(vii)} \quad P(X + Y \leq 1) &= \int_{R_5} \int \left(xy^2 + \frac{x^2}{8} \right) dx dy \\
 &\quad (x + y \leq 1) \\
 &= \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8} \right) dx dy = \frac{13}{480}
 \end{aligned}$$

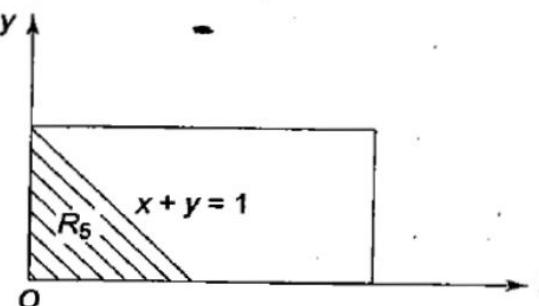


Fig.

Example:

The joint pdf of the RV (X, Y) is given by $f(x, y) = k x y e^{-(x^2 + y^2)}$ $x > 0, y > 0$.
Find the value of k and prove also that X and Y are independent.

Solution:

Here the range space is the entire first quadrant of the xy -plane.
By the property of the joint pdf

$$\iint_{x>0, y>0} kxy e^{-(x^2+y^2)} dx dy = 1$$

i.e., $k \int_0^\infty ye^{-y^2} dy \int_0^\infty xe^{-x^2} dx = 1$

i.e., $\frac{k}{4} = 1$

$\therefore k = 4$

Now $f_X(x) = \int_0^\infty 4x e^{-x^2} \times ye^{-y^2} dy = 2x e^{-x^2}$, $x > 0$

Similarly, $f_Y(y) = 2ye^{-y^2}$, $y > 0$.

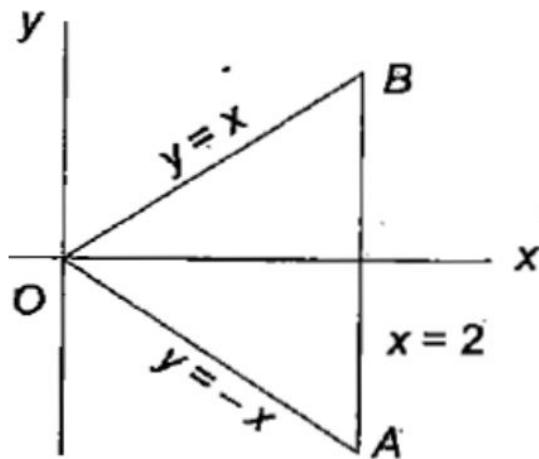
Now $f_X(x) \times f_Y(y) = 4xy e^{-(x^2+y^2)} = f(x, y)$

\therefore The RVs x and y are independent.

Example:

Given $f_{XY}(x, y) = cx(x - y)$, $0 < x < 2$, $-x < y < x$, and 0 elsewhere, (a) evaluate c , (b) find $f_X(x)$, (c) $f_{Y|X}(y/x)$ and (d) $f_Y(y)$.

Solution:



Here the range space is the area within the triangle OAB (shown in the figure), defined by $0 < x < 2$ and $-x < y < x$.

(a) By the property of jpdf

$$\int \int_{\Delta OAB} cx(x-y) dx dy = 1$$

ΔOAB

$$\int_0^2 \int_{-x}^x cx(x-y) dy dx = 1$$

$$\text{i.e.,} \quad 8c = 1$$

$$\therefore c = \frac{1}{8}$$

Solution (Continued):

$$(b) f_X(x) = \int_{-x}^x \frac{1}{8} x (x - y) dy \\ = \frac{x^3}{4}, \text{ in } 0 < x < 2$$

$$(c) f(y/x) = \frac{f(x, y)}{f_X(x)} = \frac{1}{2x^2} (x - y), -x < y < x$$

$$(d) f_Y(y) = \int_{-y}^2 \frac{1}{8} x (x - y) dx, \text{ in } -2 \leq y \leq 0 \\ = \int_y^2 \frac{1}{8} x (x - y) dx, \text{ in } 0 \leq y \leq 2$$

$$\text{i.e., } f_Y(y) = \begin{cases} \frac{1}{3} - \frac{y}{4} + \frac{5}{48} y^3, & \text{in } -2 \leq y \leq 0 \\ \frac{1}{3} - \frac{y}{4} + \frac{1}{48} y^3, & \text{in } 0 \leq y \leq 2 \end{cases}$$

Exercise:

If the joint probability density function of a two-dimensional random variable (X, Y) is given by

$$f(x, y) = \begin{cases} k(6 - x - y), & 0 < x < 2, 2 < y < 4; \\ 0, & \text{otherwise,} \end{cases}$$

then find (i). the value of k , (ii). $P(Y < 3)$, (iii). $P(X < 1, Y < 3)$, (iv). $P(X + Y < 3)$, and (v). $P(X < 1 | Y < 3)$.

Exercise:

If the joint density for the random variables (X, Y) , where X is the unit temperature change and Y is the proportion of spectrum shift that a certain atomic particle produces, is given by

$$f(x, y) = \begin{cases} cxy^2, & 0 < x < y < 1; \\ 0, & \text{otherwise,} \end{cases}$$

then find (i). the value of c , (ii). $E(XY)$, (iii). $f_X(x)$, (iv). $f_Y(y)$ and (v). $f_{Y/X}(y/x)$.

Covariance

As the variance $E\{X - E(X)\}^2$ measures the variations of the R.V. X from its mean value $E(X)$, the quantity $E\{[X - E(X)][Y - E(Y)]\}$ measures the simultaneous variation of two R.V.'s X and Y from their respective means and hence it is called *the covariance of X , Y* and denoted as $\text{Cov}(X, Y)$.

$\text{Cov}(X, Y) = E\{[X - E(X)][Y - E(Y)]\}$ is also called the *product moment* of X and Y and is also denoted as $p(X, Y)$.

$$\text{Cov}(X, Y) = E\{[X - E(X)][Y - E(Y)]\} = E(XY) - E(X) \cdot E(Y)$$

$$\text{Cov}(X, Y) = \frac{1}{n} \sum x_i y_i - \frac{1}{n} \sum x_i \cdot \frac{1}{n} \sum y_i$$

Example:

Consider the joint probability distribution

		Y		
		-1	0	1
		0	0	1/3
X	0	1/3	0	1/3
	1	1/3	0	1/3

Find the co-variance between X and Y .

Solution:

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0.$$

Example:

Let X and Y be continuous random variables with joint pdf

$$f_{X,Y}(x, y) = 3x, \quad 0 \leq y \leq x \leq 1,$$

and zero otherwise. Find the co-variance between X and Y .

Solution:

The marginal pdfs, expectations and variances of X and Y are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^x 3xdy = 3x^2, \quad 0 \leq x \leq 1,$$

$$\Rightarrow E_{f_X}[X] = \int_{-\infty}^{\infty} xf_X(x)dx = \int_0^1 x \times 3x^2 dx = \left[\frac{3}{4}x^4 \right]_0^1 = \frac{3}{4},$$

$$E_{f_X}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^1 x^2 \times 3x^2 dx = \left[\frac{3}{5}x^5 \right]_0^1 = \frac{3}{5},$$

$$\Rightarrow Var_{f_X}[X] = E_{f_X}[X^2] - \{E_{f_X}[X]\}^2 = \frac{3}{5} - \left\{ \frac{3}{4} \right\}^2 = \frac{3}{80}.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_y^1 3xdx = \left[\frac{3}{2}x^2 \right]_y^1 = \frac{3}{2}(1-y^2), \quad 0 \leq y \leq 1,$$

$$\Rightarrow E_{f_Y}[Y] = \int_{-\infty}^{\infty} yf_Y(y)dy = \int_0^1 y \times \frac{3}{2}(1-y^2)dy = \left[\frac{3}{2} \left(\frac{y^2}{2} - \frac{y^4}{4} \right) \right]_0^1 = \frac{3}{2} \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{3}{8},$$

$$E_{f_Y}[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)dy = \int_0^1 y^2 \times \frac{3}{2}(1-y^2)dy = \left[\frac{3}{2} \left(\frac{y^3}{3} - \frac{y^5}{5} \right) \right]_0^1 = \frac{3}{2} \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{1}{5},$$

$$\Rightarrow Var_{f_Y}[Y] = E_{f_Y}[Y^2] - \{E_{f_Y}[Y]\}^2 = \frac{1}{5} - \left\{ \frac{3}{8} \right\}^2 = \frac{19}{320},$$

and to compute the covariance we also need to compute $E_{f_{X,Y}}[XY]$

Solution (Continued):

$$\begin{aligned} E_{f_{X,Y}}[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x,y) dy dx = \int_0^1 \int_0^x xy \times 3xy dy dx \\ &= \int_0^1 \left\{ \int_0^x y dy \right\} 3x^2 dx = \int_0^1 \left[\frac{y^2}{2} \right]_0^x 3x^2 dx = \int_0^1 \frac{x^2}{2} \times 3x^2 dx \\ &= \frac{3}{2} \left[\frac{x^5}{5} \right]_0^1 = \frac{3}{10}, \end{aligned}$$

$$\Rightarrow Cov_{f_{X,Y}}[X,Y] = E_{f_{X,Y}}[XY] - E_{f_X}[X] E_{f_Y}[Y] = \frac{3}{10} - \frac{3}{4} \times \frac{3}{8} = \frac{3}{160}$$

Exercise:

Let X and Y be discrete random variables with joint mass function defined by

$$f_{X,Y}(x,y) = \frac{1}{4}, \quad (x,y) \in \{(0,0), (1,1), (1,-1), (2,0)\},$$

and zero otherwise. Find the co-variance between X and Y .

Exercise:

Two random variables X and Y have the following joint probability density function :

$$f(x, y) = \begin{cases} 2 - x - y & ; 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & , \text{ otherwise} \end{cases}$$

Find

- (i) Marginal probability density functions of X and Y .
- (ii) Conditional density functions.
- (iii) $\text{Var}(X)$ and $\text{Var}(Y)$.
- (iv) Co-variance between X and Y .

Moment Generating Function (MGF)

Moment Generating Function (MGF) of a RV X (discrete or continuous) is defined as $E(e^{tX})$, where t is a real variable and denoted as $M(t)$.

If X is discrete, then $M(t) = \sum_r e^{tx_r} p_r$,

where X takes the values x_1, x_2, x_3, \dots , with probabilities p_1, p_2, p_3, \dots

If X is a continuous RV with density function $f(x)$, then

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Properties of MGF

(Proofs of the properties are omitted, as the proofs of the corresponding properties of characteristic function will be given later.)

$$1. M(t) = \sum_{n=0}^{\infty} t^n E(X^n)/n!$$

i.e., $E(X^n) = \mu'_n$ is the co-efficient of $\frac{t^n}{n!}$ in the expansion of $M(t)$ in series of powers of t .

$$2. \mu'_n = E(X^n) = \left[\frac{d^n}{dt^n} M(t) \right]_{t=0}$$

3. If the MGF of X is $M_X(t)$ and if $Y = aX + b$, then $M_Y(t) = e^{bt}M_X(at)$.

4. If X and Y are independent RVs and $Z = X + Y$, then $M_Z(t) = M_X(t)M_Y(t)$.

Example:

If X represents the outcome, when a fair die is tossed, find the MGF of X and hence find $E(X)$ and $\text{Var}(X)$.

Solution:

The probability distribution of X is given by

$$p_i = p(X = i) = \frac{1}{6}, i = 1, 2, \dots, 6$$

$$\begin{aligned}M(t) &= \sum_i e^{tx_i} p_i = \sum_{i=1}^6 e^{ti} p_i \\&= \frac{1}{6} (e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})\end{aligned}$$

$$E(X) = [M'(t)]_{t=0} = \frac{7}{2}$$

$$E(X^2) = [M''(t)]_{t=0}$$

$$= \frac{1}{6} [e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}]_{t=0} = \frac{91}{6}$$

$$\text{Var}(x) = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

Example:

If a RV X has the MGF $M(t) = \frac{3}{3-t}$, obtain the standard deviation of X .

Solution:

$$M(t) = \frac{3}{3\left(1 - \frac{t}{3}\right)} = 1 + t/3 + t^2/9 + \dots + \infty$$

$$E(X) = \text{coefficient of } \frac{t}{[1]} \text{ in (1)} = \frac{1}{3}$$

$$E(X^2) = \text{coefficient of } \frac{t^2}{[2]} \text{ in (1)} = \frac{2}{9}$$

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{1}{9}$$

$$\sigma_X = \frac{1}{3}$$

Example:

If $x = 1, 2, 3, \dots$ has the geometric distribution $f(x) = pq^{x-1}$, where $q = 1 - p$, show that the moment generating function is

$$M(t) = \frac{pe^t}{1 - qe^t}.$$

Find $E(x)$.

Solution:

The moment generating function of x is

$$\begin{aligned} M(t) &= \sum_{x=1}^{\infty} e^{xt} pq^{x-1} = \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x \\ &= pe^t \sum_{x=0}^{\infty} (qe^t)^x = \frac{pe^t}{1 - qe^t}. \end{aligned}$$

To find $E(x)$, we may use the quotient rule to differentiate the expression $M(t)$ with respect to t . This gives

$$\frac{dM(t)}{dt} = \frac{(1 - qe^t)pe^t - pe^t(-qe^t)}{(1 - qe^t)^2}.$$

Setting $t = 0$ gives $E(x) = 1/p$.

Example:

Find the MGF of the binomial distribution and hence find its mean and variance.

Binomial distribution is given by

$$p_r = p(X = r) = nC_r p^r q^{n-r}, \quad r = 0, 1, 2, \dots, n$$

$$p + q = 1$$

Solution:

$$M(t) = \sum_{r=0}^n e^{tr} p_r$$

$$= \sum_{r=0}^n e^{tr} n C_r p^r q^{n-r}$$

$$= \sum_{r=0}^n n C_r (p e^t)^r q^{n-r}$$

$$= (p e^t + q)^n$$

$$M'(t) = n(p e^t + q)^{n-1} \times p e^t$$

$$M''(t) = np[(p e^t + q)^{n-1} \times e^t + (n-1)(p e^t + q)^{n-2} p e^{2t}]$$

$$E(X) = M'(0) = np \quad (\text{since } p + q = 1)$$

$$E(X^2) = M''(0) = np [1 + (n-1)p]$$

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2$$

$$= np - np^2$$

$$= npq$$

Exercise:

Find the MGF of a RV which is uniformly distributed over $(-1, 2)$ and hence find its mean and variance.

Characteristic Function (CF)

Characteristic function of a RV X (discrete or continuous) is defined as $E(e^{i\omega X})$ and denoted as $\phi(\omega)$.

If X is a discrete RV that can take the values x_1, x_2, \dots , such that $P(X = x_r) = p_r$, then

$$\phi(\omega) = \sum_r e^{i\omega x_r} p_r$$

If X is a continuous RV with density function $f(x)$, then

$$\phi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x) dx$$

Properties of Characteristic Function

1. $\mu'_n = E(X^n)$ = the coefficient of $\frac{i^n \omega^n}{n!}$ in the expansion of $\phi(\omega)$ in series of ascending powers of $i\omega$.

2. $\mu'_n = \frac{1}{i^n} \left[\frac{d^n}{d\omega^n} \phi(\omega) \right]_{\omega=0}$

3. If the characteristic function of a RV X is $f_X(\omega)$ and if $Y = aX + b$, then

$$\phi_Y(\omega) = e^{ib\omega} \phi_X(a\omega)$$

4. If X and Y are independent RVs, then

$$\phi_{X+Y}(\omega) = \phi_X(\omega) \times \phi_Y(\omega)$$

Example:

Find the characteristic function of the geometric distribution given by $P(X = r) = q^r p$, $r = 0, 1, 2, \dots, \infty$, $p + q = 1$.

Hence find the mean and variance.

Solution:

$$\phi(\omega) = \sum_{r=0}^{\infty} e^{i\omega r} pq^r$$

$$= p \sum_{r=0}^{\infty} (qe^{i\omega})^r = p(1 - qe^{i\omega})^{-1}$$

$$\phi^{(1)}(\omega) = p(1 - qe^{i\omega})^{-2} i q e^{i\omega}$$

$$\phi^{(2)}(\omega) = i^2 pq [2(1 - q e^{i\omega})^{-3} q e^{i2\omega} + (1 - q e^{i\omega})^{-2} e^{i\omega}]$$

$$E(X) = \frac{1}{i} \phi^{(1)}(0) = \frac{q}{p} \text{ and } E(X^2) = \frac{1}{i^2} \phi^{(2)}(0) = \frac{q}{p^2} (1 + q)$$

$$\mu_X = \frac{q}{p} \text{ and } \sigma_X^2 = \frac{q}{p^2}$$

Exercise:

Find the CF of X whose pdf is given by $f(x) = \frac{1}{2} e^{-|x|}$, $-\infty < x < \infty$.

