SIMILAR MATRICES

- Two matrices A and B are said to be similiar, if $\exists$ a non-singular ~~matrice~~ matrix P $\ni$

$$B = P^{-1}AP$$

Note:

1. If $B = P^{-1}AP$, then we say that B is obtained from a similarity transformation.

2. If $B = P^{-1}AP$, we write it as; $B \sim A$.

3. If $B \sim A$, then $A \sim B$.

Proof:

$$B = P^{-1}AP \qquad (B \sim A)$$

$$PBP^{-1} = P(P^{-1}AP)P^{-1}$$

$$= PP^{-1}.A.PP^{-1}$$

$$= I.A.I$$

$$PBP^{-1} = A$$

ie, $A = PBP^{-1}$

$$= (P^{-1})^{-1}BP^{-1}$$

Let $P^{-1} = M$

$\therefore$ $A = M^{-1}BM$

$\Rightarrow$ $\underline{\underline{A \sim B}}$.

4. Two similar matrices have the same eigen values.

Theorem: DIAGONALISATION

- Let $A_{n \times n}$ be a square matrix.

  $\lambda_1, \lambda_2, \dots, \lambda_n$: Eigen values.

  $x_1, x_2, \dots x_n$: Corresponding eigen vectors.

$M = [X_1 \quad X_2 \cdots \quad X_n]_{n \times n}$

Collection of eigen values = spectrum.

Collection of ind. eigen vectors = eigen space

Then; $\quad D = M^{-1} A M \quad ; \quad D \sim A$

where; $D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ is the diagonal matrix.

- We may also write D as ; $D(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Note :

- M in this case is called 'Model matrix' of A.

- ? Diagonalise $A = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$ by means of similarity transformations.

A : $\quad A = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$

char. eqn. is given by ;

$$|A - \lambda I| = 0 \implies \lambda^3 - D_1 \lambda^2 + D_2 \lambda - D_3 = 0$$

$D_1 = Tr(A) = 2 + 1 - 3 = 0$

$D_2 = \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -7 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}$

$= -5 - 6 - 2$

$= -13$

$D_3 = \det(A)$

$= 2(-3-2) - 2(-6) - 7(2)$

$= -10 + 12 - 14$

$= \underline{\underline{-12}}$

$\therefore$ Char. eqn is ;

$$\lambda^3 - 0\lambda^2 + (-13)\lambda - -12 = 0$$

$$\lambda^3 - 13\lambda + 12 = 0.$$

$\lambda = 1 \Rightarrow \lambda^3 - 13\lambda + 12 = 0$

$\Rightarrow \lambda - 1$ is a factor.

$$\begin{array}{r|l} & \lambda^2 + \lambda - 12 \\ \hline \lambda - 1 & \lambda^3 - 0\lambda^2 - 13\lambda + 12 \\ & \lambda^3 - \lambda^2 \\ \hline & \lambda^2 - 13\lambda \\ & \lambda^2 - \lambda \\ \hline & -12\lambda + 12 \\ & -12\lambda + 12 \\ \hline & 0 \end{array}$$

$\Rightarrow (\lambda - 1)(\lambda^2 + \lambda - 12) = 0$

$\Rightarrow (\lambda - 1)(\lambda + 4)(\lambda - 3) = 0$

$\Rightarrow \underline{\underline{\lambda_1 = 1, \ \lambda_2 = 3, \ \lambda_3 = -4}}$

Eigen vectors corresponding to λ:

$$\begin{bmatrix} 8-\lambda & 2 & -7 \\ 2 & 1-\lambda & 2 \\ 0 & 1 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>For $\lambda_1 = 1$</u>

$$\begin{bmatrix} 1 & 2 & -7 \\ 2 & 0 & 2 \\ 0 & 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 2 & -7 \\ 0 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & -7 \\ 2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{4} = \frac{-x_2}{16} = \frac{x_3}{-4}$$

$$\Rightarrow \cancel{x} \cdot \frac{x_1}{1} = \frac{x_2}{-4} = \frac{x_3}{-1}$$

$$\therefore \underline{\underline{X_1 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}}}$$

For $\lambda_2 = 3$

$$\begin{bmatrix} -1 & 2 & -7 \\ 2 & -2 & 2 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 2 & -7 \\ -2 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & -7 \\ 2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-10} = \frac{-x_2}{12} = \frac{x_3}{-2}$$

$$\Rightarrow \frac{x_1}{-5} = \frac{x_2}{-6} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$$

For $\lambda = -4$

$$\begin{bmatrix} 6 & 2 & -7 \\ 2 & 5 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 5 \\ 0 & 1 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{3} = -\frac{x_2}{2} = \frac{x_3}{2}$$

$$\therefore X_3 = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$$

Now, model matrix:

$$M = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

$$M = \begin{bmatrix} -1 & 5 & 3 \\ 4 & 4 & -2 \\ 1 & 1 & 4 \end{bmatrix}$$

$$\det(M) = -1(12 - -2) - 5(8 - -2) + 3(4-6)$$

$$= -14 - 50 - 6$$

$$= \underline{\underline{-70}}$$

$A_{11} = 14$	$A_{21} = -7$	$A_{31} = -28$
$A_{12} = -10$	$A_{22} = -5$	$A_{32} = 10$
$A_{13} = -2$	$A_{23} = +6$	$A_{33} = -26$

$$\therefore M^{-1} = \frac{-1}{70}\begin{bmatrix} 14 & -7 & -28 \\ -10 & -5 & 10 \\ -2 & 6 & -26 \end{bmatrix}$$

$$\therefore M^{-1}AM = \left\{ \frac{-1}{70}\begin{bmatrix} 14 & -7 & -28 \\ 10 & -5 & 10 \\ -2 & 6 & -26 \end{bmatrix}\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} -1 & 5 & 3 \\ 4 & 6 & -1 \\ 1 & 1 & 1 \end{bmatrix}\right.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\underline{\underline{M^{-1}AM = D}}$$

HW

• ? Find M that can diagonalise matrix;

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$ by similarity transformation.

Also verify $M^{-1}AM = D$.

ORTHOGONAL TRANSFORMATION

- For a real symmetric matrix A, the eigen vectors of A are mutually orthogonal.

Let $A_{n \times n}$ be a ~~s~~ real symmetric matrix;

$\lambda_1, \lambda_2, \dots \lambda_n$: Eigen values

$x_1, x_2, \dots x_n$: Eigen vectors.

Then; $x_1, x_2, \dots, x_n$ are mutually orthogonal.

ie, $x_1 \perp x_2$, $x_2 \perp x_3, \dots, x_n \perp x_1$.

- $\therefore$ $M = [x_1 \ x_2 \ \dots \ x_n]_{n \times n}$ is an orthogonal matrix.

$\Rightarrow$ $M^T = M^{-1}$

$$N = \left[\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \dots, \frac{x_n}{\|x_n\|} \right]_{n \times n}.$$

$\Rightarrow$ $MM^T = I = M^T M$. - Normalized model matrix

- $\therefore$ For $\boxed{\text{real symmetric matrix } A; \quad \cancel{M^T A M = D} \quad N^T A N = D}$

? Diagonalize the matrix A by means of an orthogonal transformation.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$$

: A is a real symmetric matrix.

Char. eqn. of A is;

$$|A - \lambda I| = 0$$

$$\lambda^3 - D_1 \lambda^2 + D_2 \lambda - D_3 = 0$$

$D_1 = Tr(A) = 4$

$$D_2 = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$= -3 + 1 + 1$

$= -1$

$D_3 = 2(1-4) - 1(1-2) + (-1)(-2-1)$

$= -6 + 1 + 1$

$= -4$

$\therefore$ char. eqn. is given by ;

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

$\lambda = -1 \Rightarrow -1 - 4 + 1 + 4 = 0$

$\Rightarrow (\lambda + 1)$ is a factor

$$\begin{array}{r|l} & \lambda^2 - 5\lambda + 4 \\ \lambda + 1 & \lambda^3 - 4\lambda^2 - \lambda + 4 \\ & \lambda^3 + \lambda^2 \\ \hline & -5\lambda^2 - \lambda \\ & -5\lambda^2 - 5\lambda \\ \hline & 4\lambda + 4 \\ & 4\lambda + 4 \\ \hline & 0 \end{array}$$

$\therefore (\lambda + 1)(\lambda^2 - 5\lambda + 4) = 0$.

$(\lambda + 1)(\lambda - 4)(\lambda - 1) = 0$

$\Rightarrow \lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 4$

Eigen vectors corresponding to eigen values are;

$$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>For $\lambda = 1$</u>

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-2} = \frac{-x_2}{-1} = \frac{x_3}{-1}$$

$$\Rightarrow X_1 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

For $\lambda = -1$

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{\cancel{1}\ 0} = \frac{-x_2}{-5} = \frac{x_3}{5}$$

$$\Rightarrow X_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 4$

$$\begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 1 & -1 \\ -3 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -1 \\ 1 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-5} = \frac{-x_2}{5} = \frac{x_3}{5}$$

$$\Rightarrow \quad X_3 = \begin{bmatrix} -5 \\ -5 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\therefore$ Model matrix ;

$$M = [X_1 \quad X_2 \quad X_3]$$

$$M = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$M^T = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\therefore M^T A M = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ 0 & -1 & -1 \\ 4 & 4 & -4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$\|X_1\| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\|X_2\| = \sqrt{(0)^2 + 1^2 + (-1)^2} = \sqrt{2}$$

$$\|X_3\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$$

$$\therefore N = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$ is the normalized matrix.

$$N^T = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$$

$$\therefore N^T A N = \begin{bmatrix} 2/\sqrt{6} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

HW

ANS: $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Char. eqn. is given by ;

$$|A - \lambda I| = 0$$

$$\Rightarrow \lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$$

$$D_1 = Tr(A) = 2 + 3 + 2 = 7$$

$$D_2 = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 4 + 3 + 4$$

$$= \underline{\underline{11}}$$

$$D_3 = \det(A)$$

$$= 2(6-2) - 2(2-1) + 1(2-3)$$

$$= 8 - 2 - 1$$

$$= \underline{\underline{5}}$$

$\therefore$ Char. eqn is;

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0.$$

For $\lambda = 1 \Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$\Rightarrow \lambda - 1$ is a factor of char. eqn

$$\begin{array}{r|l} & \lambda^2 - 6\lambda + 5 \\ \lambda - 1 & \lambda^3 - 7\lambda^2 + 11\lambda - 5 \\ & \lambda^3 - \lambda^2 \\ \hline & -6\lambda^2 + 11\lambda \\ & -6\lambda^2 + 6\lambda \\ \hline & 5\lambda - 5 \\ & 5\lambda - 5 \\ \hline & 0 \end{array}$$

$\therefore (\lambda - 1)(\lambda^2 - 6\lambda + 5) = 0$

$\Rightarrow (\lambda - 1)(\lambda - 1)(\lambda - 5) = 0$

$\Rightarrow \underline{\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 5}$

are the eigen values of A.

Eigen vectors corresponding to λ;

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\underline{\lambda_1, \lambda_2 = 1}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_1 \\ R_2 \to R_2 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 + x_3 = 0$

$x_1 = 1, x_3 = 1 \Rightarrow x_2 = -1$

$x_1 = 2 \quad x_3 = 0 \Rightarrow x_2 = -1$

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ -1 \\ +1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

For $\lambda = 5$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$\Rightarrow \quad X_3 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Model matrix $M = [X_1 \quad X_2 \quad X_3]$

$$\Rightarrow \quad M = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$ can diagonalize A.

$$\det(M) = 1(-1-0) - 2(-1-1) + 1(0-1)$$

$$= -1 + 4 + 1$$

$$= \underline{\underline{4}}$$

$A_{11} = -1$ | $A_{21} = -2$ | $A_{31} = 3$

$A_{12} = -(-1-1) = 2$ | $A_{22} = 0$ | $A_{32} = -2$

$A_{13} = 1$ | $A_{23} = +2$ | $A_{33} = 1$

$$\therefore \; M^{-1} = \frac{1}{|M|} (Adj(M))$$

$$= \frac{1}{4} \begin{bmatrix} -1 & -2 & 3 \\ 2 & 0 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$M^{-1} A M = \frac{1}{4} \begin{bmatrix} -1 & -2 & 3 \\ 2 & 0 & -2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -1 & -2 & 3 \\ 2 & 0 & -2 \\ 5 & 10 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

$$M^{-1} A M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D$$ is verified.

HW

Q) Diagonalize the matrix; $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ by means of orthogonal transformation.

QUADRATIC FORMS

- A homogeneous polynomial of 2^{nd} degree in any no. of variables is called a quadratic form.

Eg:

1. $x^2 + y^2$: QF in 2 variables
2. $x^2 + y^2 + z^2$: QF, 3 variables
3. $xy + yz + xz$: QF, 3 variables
4. $x + y + z$: Not QF.
5. $x^2 + xy + y^2$: QF, 2 variables
6. $x^2 + xy + tz$: QF, 4 variables.
7. $x^2 + y^2 + z$: Not QF.

Std. form of QF

$$
\begin{aligned}
Q = {} & c_{11} x_1^2 + c_{12} x_1 x_2 + \cdots + c_{1n} x_1 x_n \\
& c_{21} x_2 x_1 + c_{22} x_2^2 + \cdots + c_{2n} x_2 x_n \\
& \vdots \\
& c_{n1} x_n x_1 + c_{n2} x_n x_2 + \cdots + c_{nn} x_n^2
\end{aligned}
$$

	x_1	$x_2 \cdots$	x_n
x_1	$c_{11} x_1^2$	$c_{12} x_1 x_2 \cdots$	$x_1 x_n$
x_2	$x_2 x_1$	$x_2^2 \cdots$	$x_2 x_n$
$\vdots$	$\vdots$		
x_n	$x_n x_1$	$x_n x_2 \cdots$	x_n^2

$$\Rightarrow \quad Q = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_i x_j$$

Define $a_{ij} = \frac{1}{2} [c_{ij} + c_{ji}]$ To make the matrix symmetric.

$$\therefore \quad \boxed{Q = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j, \quad \text{where } a_{ij} = a_{ji}}$$

Then; $A = a_{ij}$ is a symmetric matrix.

Matrix Notation of QF

$$\boxed{Q = x^T A x, \quad A = (a_{ij}) \text{ where } a_{ij} = a_{ji} \text{ is a symmetric matrix.}}$$

X : Variable vector.

ie, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $X^T = [x_1 \quad x_2 \quad \dots \quad x_n]$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$ where; $a_{ij} = a_{ji}$

•? Find the matrix form of QF;

$$Q = 2x_1^2 - 3x_1x_2 + 4x_2^2$$

A:

	x_1	x_2
x_1	2	$-3/2$
x_2	$-3/2$	4

∴ Matrix form;

$$A = \begin{bmatrix} 2 & -3/2 \\ -3/2 & 4 \end{bmatrix}$$ is a symmetric matrix

•? Find the matrix form of; ~~$Q = x_1^2 + 3x_2$~~

$$Q = x_1^2 + 3x_2^2 + 6x_3^2 - 2x_1x_2 + 6x_1x_3 + 5x_2x_3$$

A:

	x_1	x_2	x_3
x_1	1	-1	3
x_2	-1	3	$5/2$
x_3	3	$5/2$	6

∴ Matrix form; $A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 3 & 5/2 \\ 3 & 5/2 & 6 \end{bmatrix}$

Q. Find the quadratic form of $x^T A x$ of the matrix:

$$A = \begin{bmatrix} 3 & 1/2 & 0 \\ 1/2 & 0 & 6 \\ 0 & 6 & -7 \end{bmatrix}$$

A. $QF = 3x^2 + 0y^2 - 7z^2 + \left(\frac{1}{2}+\frac{1}{2}\right)xy + (0+0)xz + (6+6)yz$

$= 3x^2 - 7z^2 + xy + 12yz$

A. Given,
$$A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$ is a real symmetric matrix.

Char. eqn. of given matrix;

$|A - \lambda I| = 0$

$\Rightarrow \lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

$D_1 = Tr(A) = 2+6+2 = 10$

$D_2 = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix}$

$= 12 + (\overset{-12}{\cancel{-8}}) + 12$

$= \cancel{20}\ \cancel{16}\ \cancel{16}\ 12$

$D_3 = \det(A) = 2(12-0) - 0 + 4(0-24)$

$= 24 - \cancel{48}\ 96$

$= \cancel{-24}\ \ -72$

$\therefore$ Char. eqn. is;

$\lambda^3 - 10\lambda^2 + \overset{12}{\cancel{20}}\lambda \overset{+72}{\cancel{+24}} = 0.$

~~When $\lambda = 2 \Rightarrow 8 - 40$~~

~~When $\lambda = 6 \Rightarrow 216 - 360 + 120 + 24 = 0$~~

~~$\Rightarrow \lambda - 6$ is a factor.~~

$\cancel{(\lambda - 6)(\lambda^2 - 4\lambda - 4) = 0}$

$$\cancel{\lambda - 6 \,\big)\, \lambda^3 - 10\lambda^2 + 20\lambda + 24 \,\big(\, \lambda^2 - 4\lambda - 4}$$

When $\lambda = -2$;

$8 - 40 - 24 + 72 = 0$.

$\Rightarrow \lambda = -2$ is a soln.

$\Rightarrow \lambda + 2 = 0$ is a factor.

ie, $(\lambda + 2)(\lambda^2 - 12\lambda + 36) = 0$

$$\lambda + 2 \,\big)\, \lambda^3 - 10\lambda^2 + 12\lambda + 72 \,\big(\, \lambda^2 - 12\lambda + 36$$

$$\lambda^3 + 2\lambda^2$$

$$-12\lambda^2 + 12\lambda$$

$$-12\lambda^2 - 24\lambda$$

$$36\lambda + 72$$

$$36\lambda + 72$$

$$0$$

$\Rightarrow (\lambda + 2)(\lambda - 6)^2 = 0$

$\Rightarrow \lambda_1 = -2, \; \lambda_2 = \lambda_3 = 6$

are the eigen values of A.

Eigen vectors corresponding to A;

$(A - \lambda I)\, x = 0$, x is the eigen vector

$$\Rightarrow \begin{bmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

When $\lambda_1 = -2$

$$\begin{bmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x_1 + 4x_3 = 0$, $8x_2 = 0 \Rightarrow x_2 = 0$

let $x_1 = 1$, $x_3 = -1$

$\therefore$ Eigen vector; $x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

When, $\lambda_2 = \lambda_3 = 6$

$$\begin{bmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$x_2 = 0$.

$x_1 - x_3 = 0 \Rightarrow x_1 = x_3$

$\Rightarrow$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

LINEAR TRANSFORMATION OF QUADRATIC FORM

- Let $Q = X^T A X$ be a QF — (1) | A is symmetric

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

and; $X = PY$ — (2)

$\therefore Q = (PY)^T A (PY)$

$= Y^T P^T A P Y$

$= Y^T (P^T A P) Y$

Let $B = P^T A P$.

$\therefore \boxed{Q = Y^T B Y}$ — (3)

IMP

B is symmetric ;

$B^T = (P^T A P)^T$

$= P^T A^T (P^T)^T$

$= P^T A P$ $\quad$ ($\because A$ is symmetric)

$B^T = B$.

$\Rightarrow$ B is symmetric verified.

$\therefore$ B is a QF.

ie, linear transformation of a QF is also a QF

OR

QF is invariant under linear transformation.

- Let $Q = X^T A X$ be a QF.

Let P be an orthogonal matrix $\Rightarrow P^T = P^{-1}$

$X = PY$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Then; $Q = X^T A X$

$= (PY)^T A (PY)$

$= Y^T \underbrace{P^T A P}_{D \ (\because P \text{ - orthogonal})} Y$

$= Y^T D Y$

$$= \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\boxed{q = y_1^2 \lambda_1 + y_2^2 \lambda_2 + \cdots + y_n^2 \lambda_n}$$

IMP is the canonical form of QF

- Let A be a square matrix.

  $\lambda_1, \lambda_2 \cdots \lambda_n$: Eigen values of A

  $x_1, x_2 \cdots x_n$: Eigen vectors corresponding to $\lambda_1, \lambda_2, \cdots, \lambda_n$.

  $M = [x_1, x_2, \cdots, x_n]$ be the model matrix.

  $N = \left[\dfrac{x_1}{\|x_1\|}, \dfrac{x_2}{\|x_2\|}, \cdots, \dfrac{x_n}{\|x_n\|} \right]$ is the normalized model matrix.

  Then, N is orthogonal.

  $\therefore$ P can be replaced by N.

  $\therefore$ Linear transformation $X = PY$ becomes;

$$\boxed{X = NY}$$

Nature of QF

- When the QF, $Q = X^T A X$ is reduced to canonical form (CF), it will contain 'r' no. of terms.

  where; $\boxed{r = \text{Rank}(A)}$

- The terms in CF may be +ve or -ve depending on λ is +ve or -ve.

- The no. of +ve terms is called, index of Q denoted by 'p'

- The diff. b/w no. of +ve and -ve terms is called signature of QF denoted by s.

$$s = p - (r - p)$$

$$\boxed{s = 2p - r}$$

- The nature of QF, $Q = X^T A X$ in n-variables.

  i) +ve definite : All eigen values are +ve

  ii) -ve definite : All eigen values are -ve

  iii) +ve semi-definite : Some eigen values are +ve and some are zero.

  iv) -ve semi-definite : At least one eigen value is zero all others are -ve.

  v) Indefinite : In all other cases.

VVI

- ? Reduce the QF ; $Q = x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to CF through an orthogonal transformation and hence show that it is +ve-semi-definite.

A: $Q = x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$

Matrix form of given QF;

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Then, given, QF is;

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Char. eqn. of A is given by;

$$|A - \lambda I| = 0$$

$\Rightarrow \quad \lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$ where;

$D_1 = Tr(A) = 1 + 2 + 1 = 4$

$$D_2 = \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$= 1 + 1 + 1$

$= \underline{\underline{3}}$

$D_3 = Det(A)$

$= 1(2-1) + 1(-1) + 0$

$= 1 - 1$

$= \underline{\underline{0}}$

$\therefore$ Char. eqn. is;

$$\lambda^3 - 4\lambda^2 + 3\lambda = 0$$

$$\lambda(\lambda^2 - 4\lambda + 3) = 0$$

$$\lambda(\lambda - 3)(\lambda - 1) = 0$$

$\Rightarrow$ $\underline{\underline{\lambda_1 = 0, \ \lambda_2 = 3, \ \lambda_3 = 1}}$ are the eigen values of A.

Eigen vectors corresponding to eigen values;

$$\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 0$:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{-x_2}{1} = \frac{x_3}{1}.$$

$$\Rightarrow X_1 = \begin{bmatrix} +1 \\ +1 \\ -1 \end{bmatrix}$$

For $\lambda_2 = 1$;

$$\begin{bmatrix} 0 & -1 & 0 \\ -01 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = 0$; $-x_1 + x_2 + x_3 = 0$

$$x_1 = x_3$$

Let $x_1 = 1$; $x_3 = 1$

$$\therefore \underline{\underline{X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}}$$

For $\lambda_3 = 3$

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 0 \\ -1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{-x_2}{-\cancel{+}2} = \frac{x_3}{1}$$

$$\Rightarrow x_3 = \begin{bmatrix} -1 \\ +2 \\ +1 \end{bmatrix}$$

$\therefore$ Model matrix;

$$M = [x_1 \quad x_2 \quad x_3]$$

$$M = \begin{bmatrix} -1 & 1 & -1 \\ -1 & 0 & +2 \\ 1 & 1 & +1 \end{bmatrix}$$

$$\|x_1\| = \sqrt{(-1)^2 + (-1)^2 + (1^2)} = \sqrt{3}$$

$$\|x_2\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\|x_3\| = \sqrt{1^2 + (-2)^2 + (-1)^2} = \sqrt{6}$$

$\therefore$ Normalized model matrix;

$$N = \begin{bmatrix} +1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ +1/\sqrt{3} & 0 & +2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & +1/\sqrt{6} \end{bmatrix}$$ is orthogonal.

$$N^T = \begin{bmatrix} +1/\sqrt{3} & +1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & +2/\sqrt{6} & +1/\sqrt{6} \end{bmatrix}$$

$$N^T A N = \begin{bmatrix} +1/\sqrt{3} & +1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & +2/\sqrt{6} & +1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} N$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 4/\sqrt{6} & 3/\sqrt{6} \end{bmatrix} \begin{bmatrix} +1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ +1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \underline{\underline{D}}$$

$\therefore$ Canonical form is;

$$Y^T D Y \quad \text{where; } X = NY, \; Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [y_1\, y_2\, y_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cancel{2/2} & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \frac{2}{\sqrt{2}} y_2^2 + \frac{6}{\sqrt{20}} y_3^2$$

$$= \underline{\underline{0\, y_1^2 + y_2^2 + 3 y_3^2}}$$

$\therefore$ One of the eigen value is 0 and all ot are +ve;

the given QF is +ve semi-definite

- Identity transformation: Transforming $X \to X$. Matrix to transform it is I.

$\rightarrow \quad X = NY$

$q = X^T A X$

$= (NY)^T A (NY)$

$= Y^T \underbrace{N^T A N}_{D} Y$

$q = Y^T D Y$

~~Determine the~~ Reduce to canonical form and hence find index, signature and nature of QF :

i) $x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_1x_3$

ii) $2x_1^2 + x_2^2 - 3x_3^2 + 12x_1x_2 - 8x_2x_3 - 4x_3x_1$

~~SINGULAR VALUE DECOMPOSITION~~

i) Given; QF is; $x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_1x_3$

This can be written in matrix form;

AX, where $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Char. eqn. of A is given by;

$|A - \lambda I| = 0$

$\Rightarrow \lambda^3 - D_1 \lambda^2 + D_2 \lambda - D_3 = 0$, where;

$D_1 = Tr(A) = 1 + 3 + 6 = 10$

$D_2 = \begin{vmatrix} 3 & 1 \\ 1 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$

$= 17 + 2 + 2$

$= \underline{\underline{21}}$

$D_3 = Det(A) =$

$= 1(18-1) - 1(6-2) + 2(1-6)$

$= 17 - 4 - 10$

$= \underline{\underline{3}}$

$\therefore$ Char. eqn. is given by;

$$\lambda^3 - 10\lambda^2 + 21\lambda - 3 = 0.$$

$\lambda = 2 \Rightarrow |A - \lambda I| = 7 \neq 0$

$\lambda = 3 \Rightarrow |A - \lambda I| = -3 \neq 0.$

$\because f(2)$ and $f(3)$ are of opp. signs, the soln. :

b/w 2 and 3.

(iii) $2x_1^2 + x_2^2 - 3x_3^2 + 12x_1x_2 - 8x_2x_3 - 4x_1x_3$ is the QF.

The given QF can be written as;

Ax, where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$

$\therefore A = A^T \Rightarrow A$ is a real symmetric matrix

Char. eqn of A is given by;

$|A - \lambda I| = 0 \Rightarrow \lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$ where;

$D_1 = Tr(A) = 2 + 1 - 3 = 0$

$$D_2 = \begin{vmatrix} 1 & -4 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ -2 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 6 \\ 6 & 1 \end{vmatrix}$$

$= -3 - 16 + (-6 - 4) + (2 - 36)$

$= -19 - 10 - 34$

$= \underline{\underline{-63}}$

$D_3 = Det(A)$

$= 2(-3-16) - 6(-18-8) - 2(-24 - -2)$

$= -38 + 156 + 44$

$= \underline{\underline{162}}$

$\therefore$ Char. eqn is;

$\lambda^3 - 63\lambda - 162 = 0$

$\lambda = -3 \Rightarrow -27 + 189 - 162 = 0$

$\Rightarrow \lambda + 3$ is a factor.

$\therefore (\lambda + 3)(\lambda^2 - 3\lambda - 54) = 0$

$$\begin{array}{r|l} & \lambda^2 - 3\lambda - 54 \\ \hline \lambda + 3 & \lambda^3 - 63\lambda - 162 \\ & \lambda^3 + 3\lambda^2 \\ \hline & -3\lambda^2 - 63\lambda \\ & -3\lambda^2 - 9\lambda \\ \hline & -54\lambda - 162 \\ & -54\lambda - 162 \\ \hline & 0 \end{array}$$

$\Rightarrow (\lambda+3)(\lambda-9)(\lambda+6) = 0$

$\Rightarrow \lambda_1 = -3 \; ; \; \lambda_2 = -6 \; ; \; \lambda_3 = 9$

Eigen vectors corresponding to eigen values;

$$(A - \lambda I)\, x = 0$$

$$\begin{bmatrix} 2-\lambda & 6 & -2 \\ 6 & 1-\lambda & -4 \\ -2 & -4 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = -3$

$$\begin{bmatrix} 5 & 6 & -2 \\ 6 & 4 & -4 \\ -2 & -4 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\cancel{\begin{vmatrix} 5 & 6 \\ 6 & 4 \end{vmatrix}}\ \begin{vmatrix} 6 & -2 \\ 4 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & -2 \\ 6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & 6 \\ 6 & 4 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-16} = \frac{-x_2}{+8} = \frac{x_3}{-16}$$

$$\Rightarrow X_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

For $\lambda_2 = -6$

$$\begin{bmatrix} 8 & 6 & -2 \\ 6 & 7 & -4 \\ -2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 6 & -2 \\ 7 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 8 & -2 \\ 6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & 6 \\ 6 & 7 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-10} = \frac{-x_2}{-20} = \frac{x_3}{20}$$

$$\Rightarrow x_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

For $\lambda_3 = 9$:

$$\begin{bmatrix} -7 & 6 & -2 \\ 6 & -8 & -4 \\ -2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{\begin{vmatrix} 6 & -2 \\ -8 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -7 & -2 \\ 6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 6 \\ 6 & -8 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-40} = \frac{-x_2}{40} = \frac{x_3}{20}$$

$$\Rightarrow x_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$\therefore$ Modal matrix :

$$M = [x_1 \quad x_2 \quad x_3]$$

$$M = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$\|x_1\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$

$\|x_2\| = 3$

$\|x_3\| = 3$

$\therefore$ Normalized model matrix;

$$N = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \text{ is orthogonal}$$

Apply transformation; $X = NY$

$\cancel{N^T D}$, $N^T A N = D$

$Q = Y^T D Y$ is the canonical form

$$N^T = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$$

$$N^T A N = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} -6 & 3 & -6 \\ 6 & -12 & -12 \\ 18 & 18 & -9 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} -27 & 0 & 0 \\ 0 & -54 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$= \underline{\begin{bmatrix} -3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{bmatrix}} = D$$

$\therefore Q = Y^T D Y$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$= -3y_1^2 \cancel{+} - 6y_2^2 + 9y_3^2$ is the canonical form.

Rank $(r) = 3$

Index ; $p = 1$

Signature ; $s = 2p - r$

$= 2 - 3$

$= -1$

Nature: Indefinite ($\because$ there are +ve and -ve eigen values).

DECOMPOSITION OF MATRICES

- Let A be an $n \times n$ ~~by~~ matrix.

  Eigen values : $\lambda_1, \lambda_2, \dots \lambda_n$

  Eigen vectors : $x_1, x_2, \dots, x_n$

  $M = [x_1 \ x_2 \ \dots \ x_n]$ is the model matrix.

Then; $M^{-1} A M = D$ (similarity transformation)

$\Rightarrow A = M D M^{-1}$

Construct an orthogonal matrix N $\because$ $M^{-1} \to N^T$ (Gram-Smith Process)

$\therefore A = N D N^T$

$$= \begin{bmatrix} Y_1 & Y_2 & \dots & Y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} Y_1^T \\ Y_2^T \\ \vdots \\ Y_n^T \end{bmatrix}$$

$$= \lambda_1 Y_1 Y_1^T + \lambda_2 Y_2 Y_2^T \cdots + \lambda_n Y_n Y_n^T$$

which is called the ~~specto~~ spectral decomposition.

Note:

1. YY^T is a rank - 1 - matrix.

Thus, by spectral decomposition, we are writing A terms of rank (1) matrices.

• 2 Find spectral decomposition of $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$

A: Given; $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$

Char. eqn; $|A - \lambda I| = 0$

$$\Rightarrow \lambda^2 - Tr(A)\lambda + Det(A) = 0$$

$Tr(A) = 11$

$Det(A) = 7 \cdot 4 - \cancel{28} \; 2(2)$

$= 28 - 4$

$= \underline{\underline{24}}$

$\therefore \lambda^2 - 11\lambda + 24 = 0$

$(\lambda - 8)(\lambda - 3) = 0$

$$\underline{\underline{\lambda_1 = 8, \quad \lambda_2 = 3}}$$

when; $\lambda_1 = 8$;

$$\cancel{A}x = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad R_2 \rightarrow R_2 + 2R_1$$

$$\sim \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 + 2x_2 = 0$

When; $x_2 = 1$; $x_1 = 2$

$$\therefore \quad X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

When $\lambda_2 = 3$

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad R_2 \to R_2 - 2R_1$$

$$\cancel{4x_1 + 2x_2 = 0} \qquad \sim \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\cancel{2x_1 - x_2 = 0} \quad \Rightarrow$$

$$\Rightarrow \quad 4x_1 + 2x_2 = 0 \Rightarrow 2x_1 + x_2 = 0$$

$$\cancel{4x + 2x_2 = 0} \qquad \therefore \quad X_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\cancel{4x_1 - 6x_2 = 0}$$

$$\cancel{8x_2 = 0} \Rightarrow \cancel{x_2 = 0}$$

$$\therefore \quad M = [X_1 \quad X_2]$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\| X_1 \| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$\| X_2 \| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\therefore \quad N = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \qquad N^T = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$\therefore \quad A = N D N^T$$

$$= \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 16 & -8 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 35 & 10 \\ 10 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$$

$$= 8\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} + 3\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

is the spectal decomposition of A.

•? Find the spectal decomposition of;

$$A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

A: $A = A^T$ is a real symmetric matrix.

Char. eqn. of A;

$$|A - \lambda I| = 0$$

$$\Rightarrow \lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0 \quad \text{where;}$$

$D_1 = Tr(A) = 17$

$$D_2 = \begin{vmatrix} 6 & -1 \\ -1 & 5 \end{vmatrix} + \begin{vmatrix} 6 & -1 \\ -1 & 5 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 6 \end{vmatrix}$$

$$= 29 + 29 + 32$$

$$= 90$$

$$D_3 = 6(30-1) + 2(-10-1) - 1(8)$$

$$= 6\cdot 29 - 22 - 8 = \underline{144}$$

$\Rightarrow \lambda^3 - 17\lambda^2 + 90\lambda - 144 = 0$

when $\lambda = 3 \Rightarrow 27 - 17.9 + 90.3 - 144 = 0$

$\Rightarrow \lambda = 3$ is a factor

$$\begin{array}{r|l} & \lambda^2 - 14\lambda + 48 \\ \lambda - 3 & \lambda^3 - 17\lambda^2 + 90\lambda - 144 \\ & \lambda^3 - 3\lambda^2 \\ \hline & -14\lambda^2 + 90\lambda \\ & -14\lambda^2 + 42\lambda \\ \hline & 48\lambda - 144 \\ & 48\lambda - 144 \\ \hline & 0 \end{array}$$

$\therefore (\lambda - 3)(\lambda^2 - 14\lambda + 48) = 0$

$(\lambda - 3)(\lambda - 8)(\lambda - 6) = 0$

$\therefore \underline{\lambda_1 = 8, \quad \lambda_2 = 6, \quad \lambda_3 = 3}$

are the eigen values of A.

Eigen vectors corresponding to eigen values,

$$\begin{bmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = 8$;

$$\begin{bmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{\begin{vmatrix} -2 & -1 \\ -1 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -1 \\ -1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -1 & -1 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{5} = \frac{-x_2}{5} = \frac{x_3}{0}$$

$$\Rightarrow X_1 = \begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 6$;

$$\begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{\begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & -1 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{2} = \frac{-x_2}{-2} = \frac{x_3}{-4}$$

$$\Rightarrow X_2 = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

For $\lambda_3 = 3$;

$$\begin{bmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{\begin{vmatrix} -2 & -1 \\ 3 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 3 & -1 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{5} = \frac{-x_2}{-5} = \frac{x_3}{5}$$

$$\Rightarrow X_3 = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ Model matrix;

$$M = [X_1 \quad X_2 \quad X_3]$$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

Normalized model matrix ;

$$N = \left[\frac{x_1}{\|x_1\|} \quad \frac{x_2}{\|x_2\|} \quad \frac{x_3}{\|x_3\|} \right]$$

$$\|x_1\| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$$

$$\|x_2\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

$$\|x_3\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\therefore \quad N = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$ is orthogonal.

$$N^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$\therefore \quad A = N D N^T$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{\sqrt{2}} & \frac{6}{\sqrt{6}} & \frac{3}{\sqrt{3}} \\ \frac{-8}{\sqrt{2}} & \frac{6}{\sqrt{6}} & \frac{3}{\sqrt{3}} \\ 0 & \frac{-12}{\sqrt{6}} & \frac{3}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

$$= 3\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} + 6\begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}\begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & \frac{-2}{\sqrt{6}} \end{bmatrix}$$

$$+ 3\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

= the spectral decomposition of A.

$$= 8 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} + 6 \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & \ldots \end{bmatrix}$$

$$+ 3 \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

is the spectral decomposition of A.

Note:

- If eigen vectors of a matrix is not linearly independent, diagonalization is not possible.

ie, if $A_{n \times n}$ with eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$.

If eigen ~~values~~ ~~corres~~ vectors corresponding to eigen $x_1, x_2, \ldots, x_n$ are not all independent,

then; ~~M~~ = Model matrix doesnot exist.

$\therefore$ Diagonalization is not possible.

$\therefore$ Spectral decomposition is also not possible.

SINGULAR VALUE DECOMPOSITION (SVD)

- Let $A_{n \times n}$ and the model matrix exists, then spectral decomposition is possible.

- When we decompose a rectangular matrix $A_{m \times n}$; we use Singular Value Decomposition (SVD)

Note:

$$\boxed{A = U \Sigma V^T}$$

where;

U : Normalized Model matrix of $A \cdot A^T$

V : Normalized Model matrix of $A^T \cdot A$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ 0 & \cdots & \sigma_n & \cdots 0 \\ 0 & \cdots & 0 & \cdots 0 \end{bmatrix}_{m \times n}$$ where; $\boxed{\sigma_i^2 = \lambda_i}$

- $\sigma_i = \sqrt{\lambda_i}$ (Eigen values of $A^T A$) is called the singular value.

$\therefore$ The decomposition is called SVD.

- $AA^T \to$ Symmetric.
- $A^T A \to$ Symmetric.

} Eigen vectors of AA^T and A^TA are orthogonal.

- Find the SVD of $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$

Given; $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

is symmetric square matrix

$$A^T A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

is real symmetric matrix.

To find U : Normalized Model matrix of AA^T.

$$A \cdot A^T = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

Char. eqn : $\lambda^2 - 22\lambda + 120 = 0$

$$(\lambda - 12)(\lambda - 10) = 0$$

$\Rightarrow$ $\underline{\underline{\lambda_2 = 10}}$, $\underline{\underline{\lambda_1 = 12}}$. - Descending order of eigen values.

$\lambda_1 > \lambda_2 > \dots$

For $\lambda_2 = 10$;

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x_1 + x_2 = 0$

$\Rightarrow$ $x_1 = -x_2$

Let $x_1 = +1$; $x_2 = -1$

$$\therefore \underline{\underline{X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}}$$

For $\underline{\lambda_1 = 12}$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$$\sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $-x_1 + x_2 = 0$

$\Rightarrow$ $x_1 = x_2$

Let $x_1 = 1$; $\Rightarrow$ $x_2 = 1$.

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen vectors are; $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore$ $U' =$ Model matrix of AA^T

$$U' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\|x_1\| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\|x_2\| = \sqrt{1^2+(-1)^2} = \sqrt{2}$$

$$\therefore \quad U = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} \end{bmatrix}$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Now, to find V: Normalized model matrix of A^TA

$$A^TA = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

char. eqn. is given by;

$$\lambda^3 - D_1 \lambda^2 + D_2 \lambda - D_3 = 0$$

$$D_1 = 10 + 10 + 2 = 22$$

$$D_2 = \begin{vmatrix} 10 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 10 & 2 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 10 & 0 \\ 0 & 10 \end{vmatrix}$$

$$= 4 + 16 + 100$$

$$= 120$$

$$D_3 = 10(20-16) - 0 + 2(0-20)$$

$$= 40 - 40$$

$$= \underline{0}$$

$$\therefore \quad \lambda^3 - 22\lambda^2 + 120\lambda = 0$$

$$\lambda(\lambda^2 - 22\lambda + 120) = 0$$

$$\lambda(\lambda - 12)(\lambda - 10) = 0$$

$$\Rightarrow \quad \lambda_1 = 12 \ ; \quad \lambda_2 = 10 \ ; \quad \lambda_3 = 0$$

Eigen vectors corresponding to λ :

For $\lambda = 12$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 0 & 2 \\ -2 & 4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 2 \\ 0 & 4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix}}$$

$$\Rightarrow \quad \frac{x_1}{+4} = \frac{-x_2}{-8} = \frac{x_3}{4}$$

$$\Rightarrow \quad X_1 = \begin{bmatrix} +4 \\ +8 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ +1 \end{bmatrix}$$

For $\lambda_2 = 10$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} 0 & 4 \\ 4 & -8 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 4 \\ 2 & -8 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 0 \\ 2 & 4 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-16} = \frac{-x_2}{-8} = \frac{x_3}{0}$$

$$\Rightarrow X_2 = \begin{bmatrix} -16 \\ +8 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

For $\lambda_3 = 0$

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_3}{\begin{vmatrix} 10 & 0 \\ 0 & 10 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 10 & 2 \\ 0 & 4 \end{vmatrix}} = \frac{x_1}{\begin{vmatrix} 0 & 2 \\ 10 & 4 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-20} = \frac{-x_2}{40} = \frac{x_3}{100}$$

$$\Rightarrow X_3 = \begin{bmatrix} -20 \\ -40 \\ 100 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$$

$\therefore$ Model matrix ; $M = [X_1 \quad X_2 \quad X_3]$

$$M = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & -5 \end{bmatrix}$$

$$\|X_1\| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

$$\|X_2\| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}$$

$$\|X_3\| = \sqrt{1^2 + 2^2 + (-5)^2} = \sqrt{30}$$

$\therefore$ V = Normalized model matrix of $A^T A$

$$V = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{5} & 1/\sqrt{30} \\ 2/\sqrt{6} & -1/\sqrt{5} & 2/\sqrt{30} \\ -1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 1/\sqrt{30} & 2/\sqrt{30} & -5/\sqrt{30} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ SVD of A ;

$$A = U \Sigma V^T$$

$$= \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2\times 3} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 1/\sqrt{30} & 2/\sqrt{30} & -5/\sqrt{30} \end{bmatrix}$$

$$= \sqrt{12}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} +$$

$$\sqrt{10}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \end{bmatrix} +$$

$$= \frac{6}{\sqrt{12}}\begin{bmatrix} 1/\sqrt{12} & \frac{2}{\sqrt{12}} & -1/\sqrt{12} \\ 1/\sqrt{12} & 2/\sqrt{12} & -1/\sqrt{12} \end{bmatrix} + \sqrt{10}\begin{bmatrix} 2/\sqrt{10} & -1/\sqrt{10} & 0 \\ -2/\sqrt{10} & 1/\sqrt{10} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 0 \\ -2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ -1 & 3 & -1 \end{bmatrix}$$

$A = U \Sigma V^T$

$\because V$ is orthogonal;

$V^T = V^{-1}$

$A = U \Sigma V^{-1}$

$AV = U \Sigma$

$A v_i = u_i \sigma_i, \forall i$

$\therefore$ $\boxed{u_i = \frac{A v_i}{\sigma_i}, \sigma_i \neq 0}$

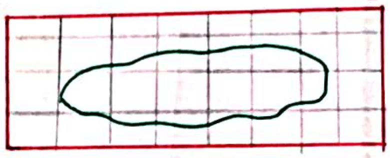
HW

? Find SVD of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

APPLICATION OF DECOMPOSITION OF MATRICES

1.

- Consider an image that can be ~~repre~~ separated by $m \times n$ grid. The image ~~canvas~~ carries $0 - 255$ grey scale. ($2^8 = 256$)



$m \times n$

- Each can be represented by a numerical value and hence image becomes a matrix of real numbers of order $m \times n$.

Now; $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \dots + \sigma_n u_n v_n^T$

where; σ_i : Singular value of A.

- We write σ_i in the descending order and neglect those σ_i that are not significant.

Eg:

$$A = \begin{bmatrix} aa & bb & cc \\ aa & bb & cc \\ aa & bb & cc \end{bmatrix}_{3\times 6} \rightarrow \begin{bmatrix} a & a & b & b & c & c \\ a & a & b & b & c & c \\ a & a & b & b & c & c \end{bmatrix}$$

$$= \overset{X}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{1\times 3} \overset{Y}{\begin{bmatrix} aa & bb & cc \end{bmatrix}}_{3\times 6}$$

To transmit A, we need to transmit only x

Instead of transmitting $6 \times 3 = 18$ data, we need

transfer only $6+3 = 9$ data.

$\boxed{\text{Eg}}$:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix}_{4 \times 5}$$

$$= \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}}_{Y}$$

Instead of transmitting $4 \times 5 = 20$ data elements, we

to transmit only $\underset{X}{4} + \underset{Y}{5} = 9$ elements.

HW

ANS: Given, $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$,

$$A^T = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & 4 \\ -4 & 4 & -8 \\ 4 & -8 & 8 \end{bmatrix}$$ is real symmetric matrix

$$A^T A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

~~MOD-4~~ :

~~VECTOR SPACES~~

$$= \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$ is real symmetric.

ie, AA^T and A^TA ~~is~~ are orthogonal.

For AA^T :

$$AA^T = \begin{bmatrix} 2 & -4 & 4 \\ -4 & 4 & -8 \\ 4 & -8 & 8 \end{bmatrix}$$

char. eqn is given by ;

$$\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$$

where ; $D_1 = Tr(AA^T)$

$$= 2 + 4 + 8 = \underline{\underline{14}}$$

$$D_2 = \begin{vmatrix} 4 & -8 \\ -8 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} + \begin{vmatrix} 2 & -4 \\ -4 & 4 \end{vmatrix}$$

$$= -32 + 0 - 8$$

$$= \underline{\underline{-40}}$$

$$D_3 = Det(AA^T)$$

$$= 2(32 - 64) + 4(-32 - -32) + 4(32 - 16)$$

$$= -64 + 0 + 64$$

$$= \underline{\underline{0}}$$

$\therefore$ char. eqn. is ;

$$\lambda^3 - 14\lambda^2 - 40\lambda = 0$$

$$\lambda(\lambda^2 - 14\lambda \cancel{+} -40) = 0$$

$$\lambda(\lambda - 4)(\lambda - 10) = 0$$

$\therefore$ Eigen values are ; $\underline{\underline{\lambda_1 = 10 \; ; \; \lambda_2 = 4 \; ; \; \lambda_3 = 0}}$

Eigen vectors corresponding to eigen values ;

$\underline{\lambda_1 = 10}$

$$\begin{bmatrix} -8 & -4 & 4 \\ -4 & -6 & -8 \\ 4 & -8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2 \begin{bmatrix} -4 & -2 & 2 \\ -2 & -3 & -4 \\ 2 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & -2 & 2 \\ -2 & -3 & -4 \\ 2 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -4 & -2 \\ -2 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -4 & 2 \\ -2 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -4 & -2 \\ -2 & -3 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{\cancel{1}8} = \frac{-x_2}{20} = \frac{x_3}{8}$$

$$\Rightarrow \underline{\underline{X_1 = \begin{bmatrix} 8 \\ -20 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 2 \end{bmatrix}}}$$

$\underline{\lambda = 4}$

$$\begin{bmatrix} -2 & -4 & 4 \\ -4 & 0 & -8 \\ 4 & -8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -4 & 4 \\ 0 & -8 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 4 \\ -8 & -8 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & -4 \\ -4 & 0 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{32} = \frac{-x_2}{48} = \frac{x_3}{-16}$$

$$\Rightarrow X_2 = \begin{bmatrix} 32 \\ -48 \\ -16 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$

$\lambda_3 = 0$

$$\begin{bmatrix} 2 & -4 & 4 \\ -4 & 4 & -8 \\ 4 & -8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{\begin{vmatrix} -4 & 4 \\ 4 & -8 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 4 \\ -4 & -8 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 4 \\ -4 & 4 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{16} = \frac{-x_2}{0} = \frac{x_3}{24}$$

$$\Rightarrow X_3 = \begin{bmatrix} 16 \\ 0 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$\therefore$ Model matrix of AA^T;

$$M = [X_1 \quad X_2 \quad X_3]$$

$$= \begin{bmatrix} 2 & -2 & 2 \\ -5 & 4 & 0 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\|X_1\| = \sqrt{2^2 + (-5)^2 + 2^2} = \sqrt{33}$$

$$\|X_2\| = \sqrt{(-2^2) + 4^2 + 1^2} = \sqrt{21}$$

$\|x_3\| = \sqrt{2^2+0^2+3^2} = \sqrt{13}$

$\therefore$ U = Normalized model matrix of AA^T

$$= \begin{bmatrix} \frac{X_1}{\|X_1\|} & \frac{X_2}{\|X_2\|} & \frac{X_3}{\|X_3\|} \end{bmatrix}$$

$$U = \begin{bmatrix} 2/\sqrt{33} & -2/\sqrt{21} & 2/\sqrt{13} \\ -5/\sqrt{33} & 4/\sqrt{21} & 0 \\ 2/\sqrt{33} & 1/\sqrt{21} & 3/\sqrt{13} \end{bmatrix} \quad \text{—— (1)}$$

For A^TA :

$$A^TA = \begin{bmatrix} +9 & -9 \\ -9 & 9 \end{bmatrix}$$

Char. eqn. is given by;

$$\lambda^2 - (Tr(A))\lambda + Det(A) = 0$$

$$\lambda^2 - 18\lambda + (81-81) = 0$$

$$\Rightarrow \lambda(\lambda - 18) = 0$$

$\Rightarrow$ $\lambda_1 = 18$; $\lambda_2 = 0$ are the eigen values.

~~Ei~~ Eigen vectors corresponding to eigen values;

$\lambda_1 = 18$:

$$\begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_1 \to \frac{-1}{9}R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -x_2$$

let $x_1 = 1 \Rightarrow x_2 = -1$.

$\therefore \quad X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 0$

$$\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_1 \to \frac{R_1}{9} \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

let $x_1 = 1 \Rightarrow x_2 = 1$.

$\therefore \quad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ Model matrix ; $M = [X_1 \quad X_2]$

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$\|X_1\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\|X_2\| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Then ;

$V =$ Normalized model matrix of $A^T A$

$$= \begin{bmatrix} \frac{X_1}{\|X_1\|} & \frac{X_2}{\|X_2\|} & \frac{X_3}{\|X_3\|} \end{bmatrix}$$

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{—— (2)}$$

$\therefore$ SVD of A is given by;

$$A = U \Sigma V^T$$

$$= \underbrace{\begin{bmatrix} 2/\sqrt{33} & -2/\sqrt{21} & 2/\sqrt{13} \\ -5/\sqrt{33} & 4/\sqrt{21} & 0/\sqrt{13} \\ 2/\sqrt{33} & 1/\sqrt{21} & 3/\sqrt{13} \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} \sqrt{18} & \cancel{0} & 0 \\ 0 & \cancel{0} & 0 \\ 0 & & 0 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{2\times 2}$$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = A.$$

MOD4 :

VECTOR SPACES

Set : Set is a primitive concept.

Binary operation : $* : S \times S \to S$ is a binary operation iff :

$$*(s_1, s_2) = s_1 * s_2 \in S \quad ; \quad s_1, s_2 \in S$$

Group : A non-empty set G ~~is said to be a group~~ with binary operation $*$ on G, then $(G, *)$ ~~then~~; is a said to be a group if the following properties are satisfied :

i) For $\forall a, b \in G$, then $a * b \in G$ - closure

$\boxed{Eg}$: $(\mathbb{R}, +)$ is closed.

$(\mathbb{Z}, \div)$ is not closed.

ii) Associative ppty : Let $a, b, c \in G$, then ;

$$(a * b) * c = a * (b * c), \quad \forall a, b, c \in G$$

iii) Existence of Identity : For $\forall a \in G$, $\exists e \in G$, s.t.

$$a * e = a = e * a$$

iv) Existence of inverse : For $\forall a \in G$, $\exists a^{-1} \in G$ s.t.

$$a * a^{-1} = a^{-1} * a = e$$

$\boxed{Eg}$:

1) $(\mathbb{R}, +)$ forms a group. $\mathbb{R}$: Real numbers
+ : Usual addition.

2) $(\mathbb{Z}^+, +)$ does not form a group.
$\because$ Identity element, inverse doesnot exist.

3) $(\mathbb{Z}, +)$ forms a group.

4) $(\mathbb{Z}^*, -)$ doesnot form a group.

$\because$ It is not associative.

$$1 - (2-4) = 3$$

$$(1-2) - 4 = -5$$

$$\Rightarrow 1 - (2-4) \neq (1-2) - 4.$$

Eg: The permutation of number $(1,2,3)$ forms a group.

(Symmetry of equilateral Δ^{le})

verify.

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad \sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad \sigma_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

i) Closure : $\sigma_2 \circ \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$= \underline{\underline{\sigma_5}}$$

ii)

Abelian Group

A ~~fro~~ commutative group is called Abelian Group.

if $(G, *)$ is a group and ~~a∈G~~ if, $a, b \in G$.

$(G, *)$ is said to be abelian if ;

$$a * b = b * a, \quad \forall a, b \in G.$$

Eg: $\{\mathbb{Z}, +\}$ is an abelian group.

Field

$(F, +, \cdot)$ is said to be a field if :

i) $(F, +)$ is an abelian group.

ii) $(F / \{0\}, \cdot)$ is an abelian group $\rightarrow F - \{0\}$

iii) $+, \cdot$ must be related (distributive law)

Eg: i) $(\mathbb{R}, +, \cdot) \Rightarrow \mathbb{R}_{(+,\cdot)}$ - is a field.

2) $\mathbb{Q}_{(+,\cdot)}$ forms a field.

3) $\mathbb{C}_{(+,\cdot)}$ forms a field.

Note:

1. In a field, distributive law follows.

ie, $a \cdot (b+c) = a \cdot b + a \cdot c$, $\forall a, b, c \in \mathbb{F}$.

and; $(a+b) \cdot c = a \cdot c + b \cdot c$, $\forall a, b, c \in \mathbb{F}$.

2. In general, an element of a field is called a scalar. ~~are~~

3. $\mathbb{R}$ ~~is~~ the set of real numbers is a field and is the background feild for most of the real problem

VECTOR SPACES

- Let V be a non-empty set, $\mathbb{F}$ be field.

Define two operations denoted as $+$ ~~and~~ within V and $\cdot$ b/w V and F (external / scalar multiplication) such that;

$v_1 + v_2 \in V$, $\forall v_1, v_2 \in V$ (closure)

$\alpha \cdot v \in V$, $\forall \alpha \in \mathbb{F}$, $v \in V$

Then, V is said to be a vector space over $\mathbb{F}$ w.r.t addition and scalar multiplication if:

1) $(V, +)$ is an abelian group. (5 ppties)

2) $\forall v \in V$, $\alpha \in F$; $\alpha \cdot v \in V$.

$\Rightarrow$ closed w.r.t scalar multiplication.

3) $\alpha \cdot (v_1 + v_2) = \alpha \cdot v_1 + \alpha \cdot v_2$, $\forall \alpha \in \mathbb{F}$ and v_1, v_2

4) $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta v$, $\forall \alpha, \beta \in F$ and $v \in V$

5. $\alpha \cdot (\beta \cdot v) = \beta \cdot (\alpha \cdot v) = (\alpha\beta) \cdot v$, $\forall \alpha, \beta \in \mathbb{F}$ and $v \in V$

6. $1 \in \mathbb{F}$, $v \in V \Rightarrow 1 \cdot v = v$, $\forall v \in V$

identity in fied

$\boxed{\text{Eg}}$:

Euclidean space.

1) $\mathbb{R}^2 = \{ (x,y) : x, y \in \mathbb{R} \}$ is a vector space ~~for~~ over $\mathbb{R}$ for usual addition, and scalar multiplicat?. defined as;

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$$

$$\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2.$$

$$\alpha \cdot (x_1, y_1) = (\alpha x_1, \alpha y_1) \in \mathbb{R}, \quad \forall (x_1, y_1) \in \mathbb{R}^2, \alpha \in \mathbb{R}.$$

i) $\mathbb{R}^2$ is closed w.r.t addition.

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$$

ii) Let $(x_1, y_1) \in \mathbb{R}^2$, $(x_2, y_2) \in \mathbb{R}^2$.

$(x_3, y_3) \in \mathbb{R}^2$.

Then; $((x_1, y_1) + (x_2, y_2)) + (x_3, y_3)$

$= \cancel{(x_1, y_1)} \; (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3))$. (Point-wise addition)

$= (x_1, y_1) + ((x_2, y_2) + (x_3, y_3))$ ($\because \mathbb{R}$ is associative wrt $+$)

$\therefore V$ is associative w.r.t addition.

$\therefore \exists \ (0,0) \in V$ s.t. $(0,0) + \cancel{(x,y)} = \cancel{\ } \ v + (0,0) = v \in V$

$\therefore$ Existence of identity.

iv) Let $(x,y) \in V$, $(-x,-y) \in V$.

$(x,y) + (-x,-y) = (x + (-x), \ y + (-y))$

$= (0,0)$

For $\forall (x,y) \in V$, $\exists \ (-x,-y) \in V$ s.t.

$v_1 + v_2 = (0,0)$ - identity.

$\therefore v_2 = v_1^{-1}$

$\therefore$ Existence of inverse.

v) Let, $v_1 = (x_1, y_1)$

$v_2 = (x_2, y_2)$

$v_1 + v_2 = (x_1 + x_2, \ y_1 + y_2)$

$v_2 + v_1 = (x_2 + x_1, \ y_2 + y_1)$

$= (x_1 + x_2, \ y_1 + y_1)$ ($\because$ point-wise addition (addition - commutative))

$v_2 + v_1 = v_1 + v_2$.

$\therefore$ For $\forall \ v_1, v_2 \in V$,

$v_1 + v_2 = v_2 + v_1$

$\therefore V = \mathbb{R}^2$ is commutative w.r.t $+$.

$\because$ All these properties are satisfied,

$(\mathbb{R}^2, +)$ is an abelian group.

Now, ~~α~~ let $\alpha \in \mathbb{R}$, $v = (x, y) \in \mathbb{R}^2$, $\beta \in \mathbb{R}$

$$\alpha \cdot v = \alpha \cdot (x_1, y_1)$$

$$= (\alpha x, \alpha y) \in \mathbb{R}^2$$

$\Rightarrow \mathbb{R}^2$ is closed w.r.t scalar multiplication.

• $\alpha(v_1 + v_2)$

let $\alpha \in \mathbb{R}$; $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in \mathbb{R}^2$

$$\alpha \cdot (v_1 + v_2) = \alpha \cdot ((x_1, y_1) + (x_2, y_2))$$

$$= \alpha \cdot (x_1 + x_2, y_1 + y_2)$$

$$= (\alpha(x_1 + x_2), \alpha(y_1 + y_2))$$ ($\mathbb{R}^2$ is closed under scalar mul)

$$= (\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2)$$ (Point-wise multiplication)

$$= (\alpha x_1, \alpha y_1) + (\alpha x_2, \alpha y_2)$$

$$= \alpha(x_1, y_1) + \alpha(x_2, y_2)$$

$$\alpha \cdot (v_1 + v_2) = \alpha v_1 + \alpha_2 v_2; \quad \forall \alpha \in \mathbb{R} \text{ and } v_1, v_2 \in \mathbb{R}^2$$

• $(\alpha + \beta) v = (\alpha + \beta) \cdot (x, y)$

$$= ((\alpha + \beta) x, (\alpha + \beta) y)$$

$$= (\alpha x + \beta x, \alpha y + \beta y)$$ (Real no. multiplicat^n)

$$= (\alpha x, \alpha y) + (\beta x + \beta y)$$ (Point-wise addition)

$$= \alpha(x, y) + \beta(x, y)$$ (scalar multiplicat^n)

$$= \alpha v + \beta v$$

$\bullet)$ $\alpha(\beta v) = \alpha(\beta(x, y))$

$= \alpha(\beta x, \beta y)$

$\alpha(\beta v) = (\alpha\beta x, \alpha\beta y)$ — (1)

$(\alpha v)\beta = (\alpha(x,y)) \cdot \beta$

$= (\alpha x, \alpha y) \cdot \beta$

$= (\alpha\beta x, \alpha\beta y)$ — (2) (multiplicat^n is commutative in $\mathbb{R}$)

$(\alpha\beta)v = \alpha\beta(x,y)$

$= (\alpha\beta x, \alpha\beta y)$ — (3)

From (1), (2) and (3);

$\alpha(\beta v) = \cancel{\beta}\,(\alpha v)\beta = (\alpha\beta)v.$

$\bullet)$ ~~Let~~ Consider $1 \in \mathbb{R}$

$1 \cdot (x, y) = (x \cdot 1, y \cdot 1)$

$= (x, y), \quad \forall (x,y) \in V.$

$\because$ All the properties are satisfied,

$\mathbb{R}^2$ is a vector ~~field~~ space over $\mathbb{R}$.

$\bullet$? Show that $\mathbb{R}^3$ is a vector space over $\mathbb{R}$ ~~wi~~ w.r.t nor addition and scalar multiplication.

$\bullet$? Show than $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ w.r.t nor addition and scalar multiplication.

Eg

1) $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a,b,c,d \in \mathbb{R} \right\}$ for vs over $\mathbb{R}$ w.r.t usual addition and point-wise multiplication.

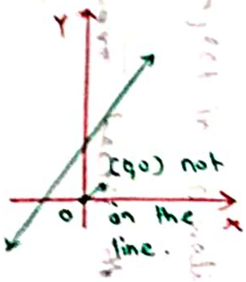
2) $M_{m\times n} = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} / a_{ij} \in \mathbb{R} \right\}$ is a vector space over $\mathbb{R}$ for matrix addition and scalar multiplication.

3) $\mathbb{F}^n$ is a vector space over $\mathbb{F}$, where $\mathbb{F}$: field.

4) Any line in x-y plane which doesnot pass through origin.

If we take set of all points on that line is not a vector space over $\mathbb{R}$.

($\because$ (0,0) - identity is not in the set).



• Similarly, a plane (S) in ~~Eu~~ Euclidean space ($\mathbb{R}^3$), which is not passing through the origin wont be a vector space ($\because$ $(0,0,0) \notin S$)

5) Q_1, Q_2, Q_3, Q_4 are quarter planes.

i) Q_1, Q_2, Q_3, Q_4 individually are not vector space.

$\because$ For $\forall x \in Q_i$, $-x \notin Q_i \Rightarrow x$ have no inverse in Q_i

$\boxed{\text{Eg}}$: $(1,1) \in Q_1$, but $(-1,-1) \in Q_3$
$\notin Q_1$

$\therefore$ (1,1) doesn't have inverse in Q_1.

ii) $Q_1 \cup Q_3$ is not a vector space.

Take: $(3,2) \in Q_1$

$(-2,-3) \in Q_3$

$(3,2) + (-2,-3) = (1,-2) \notin Q_1 \cup Q_3$

$\therefore$ Addition is not closed in $Q_1 \cup Q_3$.

Note:

1) $L = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} / a,b,c \in \mathbb{R} \right\}$ under usual addition and scalar multiplication forms a vector space over R.

2) $M = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} / a,b,c \in \mathbb{R} \right\}$ is a vector space. over R

3) $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} / a,b \in \mathbb{R} \right\}$ is a vector space over R

4) $M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a,b,c,d \in \mathbb{R} \ ; \ ad-bc \neq 0 \right\}$ doesnot form a vector space.

~~If~~ $\cancel{ad-bc \neq 0}$

For $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ - identity element

$ad-bc = 0$

If $ad-bc \neq 0$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin M$.

$\therefore$ M doesn't have additive identity.

$\therefore$ M is not a vector space.

5) $S = \{ A \in M_{2\times 2} : \det A = 0 \}$ is not a vector space

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in S \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in S.$$

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin S$$

$\because$ $\det(A+B) = 1 \neq 0.$

$\therefore$ S is not a vector space.

SUBSPACE

- Let V be a vectorspace over F, let $W \subseteq V$. Then;

W is said to be a ~~sup~~ subspace of V if W itself form a vector space over F.

Remark :

- To check whether W is a subspace of V, it is enough to check only the following 2 conditions:

1) $\forall\, w_1, w_2 \in W, \quad w_1 + w_2 \in W.$

2) $\forall\, w \in W, \ \alpha \in F;$

$$\alpha . (w) \in W$$

We check closure under addition, and scalar multiplication

OR

For ~~$\alpha \in$~~ $\forall\, \alpha, \beta \in F$, $\forall\, w_1, w_2 \in W$,

$$\alpha w_1 + \cancel{\alpha} \beta w_2 \in W$$

IMP

GRAM - SCHMIDT PROCESS

- To construct a set of orthogonal vectors from a set of linearly independent vectors.
- Ortho - normalization ~~we~~ process
- The normal model matrix;

$$N = \left[\frac{x_1}{\|x_1\|} \quad \frac{x_2}{\|x_2\|} \quad \cdots \quad \frac{x_n}{\|x_n\|} \right]$$ is called the ortho-normal matrix.

- Let $\{u_1, u_2, \dots, u_n\}$ be a set of linearly independent vectors.

First, we construct a set of orthogonal vectors say: $\{v_1, v_2, \dots, v_n\}$ and then normalize it get the orthonormal set of vectors. $\{w_1, w_2, \dots, w_n\}$

Note:

If u and v are orthogonal, $u.v = 0$ OR $\langle u,v \rangle = 0$.

or

Steps:

~~Step 1~~:

We have $\{u_1, u_2, \dots, u_n\}$ as LI vectors.

Fix ; $v_1 = u_1$

Define ; $w_1 = \dfrac{v_1}{\|v_1\|}$

dot product = scalar quantity.

Now ; $v_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$$w_2 = \frac{v_2}{\|v_2\|}$$

$\langle \alpha u_1 + \beta u_2 , v \rangle$

$= \alpha \langle u_1 , v \rangle +$

$\beta \langle u_2 , v \rangle$

Claim : $v_2 \perp v_1$

$$\langle v_2 , v_1 \rangle = \langle u_2 - \underbrace{\frac{\langle u_2 , v_1 \rangle}{\langle v_1 , v_1 \rangle}}_{\text{scalar}} v_1 , v_1 \rangle$$

scalar

$$= \langle u_2 , v_1 \rangle - \frac{\langle u_2 , v_1 \rangle}{\cancel{\langle v_1 , v_1 \rangle}} \cdot \cancel{\langle v_1 , v_1 \rangle}$$

$$= \langle u_2 , v_1 \rangle - \langle u_2 , v_1 \rangle$$

$\langle a , a \rangle = \|a\|^2$

$$= \underline{\underline{0}}$$

$\therefore$ v_1 , v_2 are orthogonal.

$\therefore$ w_1 , w_2 are orthonormal

Now; $v_3 = u_3 - \frac{\langle u_3 , v_1 \rangle}{\langle v_1 , v_1 \rangle} v_1 - \frac{\langle u_3 , v_2 \rangle}{\langle v_2 , v_2 \rangle} v_2$

$$w_3 = \frac{v_3}{\|v_3\|}$$

Claim : $v_3 \perp v_1$, $v_3 \perp v_2$.

$$\langle v_3 , v_1 \rangle = u_3 - \frac{\langle u_3 , v_1 \rangle}{\langle v_1 , v_1 \rangle} v_1 - \frac{\langle u_3 , v_2 \rangle}{\langle v_2 , v_2 \rangle} v_2 , v_1$$

$$= \langle u_3 , v_1 \rangle - \frac{\langle u_3 , v_1 \rangle}{\cancel{\langle v_1 , v_1 \rangle}} \cdot \cancel{\langle v_1 , v_1 \rangle} - \frac{\langle u_3 , v_2 \rangle}{\langle v_2 , v_2 \rangle} \cdot \langle v_2 , v_1 \rangle$$

$(\because v_2 \perp v_1)$

$\langle v_1 , v_2 \rangle = 0$

$$= \langle u_3 , v_1 \rangle - \langle u_3 , v_1 \rangle - \frac{\langle u_3 , v_2 \rangle}{\langle v_2 , v_2 \rangle} \cdot 0$$

$$= \underline{\underline{0}}$$

$\therefore \; v_3 \perp v_1 \; \Rightarrow v_1, v_3$ is orthogonal

$\Rightarrow w_3$ is orthonormal.

Now;

$$\langle v_3, v_2 \rangle = \left\langle u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 \; , \; v_2 \right\rangle$$

$$u_2 - \frac{\cancel{\langle u_2, v_1\rangle}}{\cancel{\langle v_1, v_1\rangle}} v_1 \Big\rangle$$

$$= \langle u_3, v_2 \rangle - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} \overset{0}{\cancel{\langle v_1, v_2\rangle}} - \quad (\because v_1 \perp v_2)$$

$$\frac{\langle u_3, v_2\rangle}{\cancel{\langle v_2, v_2\rangle}} \cancel{\langle v_2, v_2\rangle}$$

$$= \langle u_3, v_2 \rangle - \langle u_3, v_2 \rangle$$

$$= \underline{\underline{0}}$$

$\therefore \; v_3 \perp v_2 \; \Rightarrow v_2$ and v_3 are orthogonal.

$\Rightarrow w_3$ is orthonormal.

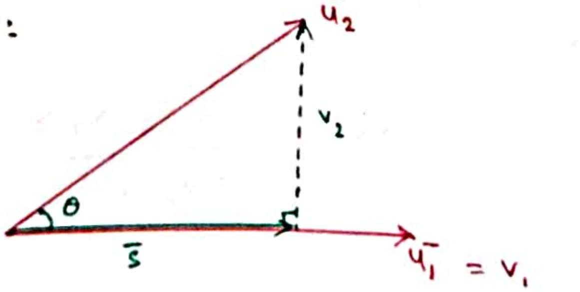
Similarly, by proving for each vector ;

$\{v_1, v_2, \ldots, v_n\}$ are orthogonal vectors.

$\therefore \{w_1, w_2, \ldots, w_n\}$ are orthonormal vectors.

This is called Gram-Schmidt orthonormalization process.

__Note:__



$$\bar{s} + \bar{v}_2 = \bar{u}_2$$

$$v_2 = \bar{u}_2 - s$$

$$= \text{[illegible]}$$

$\therefore \; s = \dfrac{\langle v_1 \cdot u_2 \rangle}{\langle v_1 \cdot v_1 \rangle} v_1$ —

= Orthogonal projection of u_2 on v_1.

$\therefore \; v_2 = u_2 -$ orth. proj. of u_2 on v_1

Q. $u_1 = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$, $u_2 = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix}$

Find orthonormal matrices using Gram - Schmidt process.

A: $u_1^T \cdot u_2 = \begin{bmatrix} 6 & 3 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \neq 0$

$u_2^T u_3 = \begin{bmatrix} 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} \neq 0$

$u_3^T u_1 = \begin{bmatrix} -1 & -5 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} \neq 0$.

$\therefore \; u_1, u_2, u_3$ are not orthogonal.

Now; $v_1 = u_1 = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$

$v_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$= \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} - \dfrac{27}{81} \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} - \dfrac{\cancel{81}}{\cancel{81}} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 2 \\ -2 \\ -81 \end{bmatrix}}}$

$\langle u_2, v_1 \rangle =$

$(4i - j + k) \cdot (6i + 3j + 6k)$

$= 24 - 3 + 6$

$= \underline{\underline{27}}$

$\langle v_1, v_1 \rangle = |v_1|^2$

$= \left(\sqrt{6^2 + 3^2 + 6^2}\right)^2$

$= \underline{\underline{81}}$

$$v_1^T \cdot v_2 = \begin{bmatrix} 6 & 3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

$$= 12 - 6 - 6$$

$$= \underline{\underline{0}}$$

$$\therefore \; v_1 \perp v_2$$

Now;

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} \cdot v_2$$

$$= \begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} - \frac{-15}{81} \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} - \frac{7}{9} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} + \frac{5}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \frac{7}{9} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} -4 \\ 19 \\ 17 \end{bmatrix}$$

$$= \cancel{\frac{1}{9} \begin{bmatrix} -13 \\ -26 \\ 26 \end{bmatrix}} \quad \frac{1}{9} \begin{bmatrix} -13 \\ -26 \\ 26 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} //$$

$\langle u_3 \cdot v_1 \rangle$

$= (-i - 5j + k) \cdot (6i + 3j + 6k)$

$= -6 - 15 + 6$

$= \underline{\underline{-15}}$

& $\langle v_1, v_1 \rangle$

$= |v_1|^2 = 81$

~~$\langle u_3, v_2 \rangle =$~~

~~$(-i - 5j + k) \cdot (4i - j + k)$~~

~~$= -4 + 5 + 1$~~

~~$= \underline{\underline{2}}$~~

~~$\langle v_2, v_2 \rangle =$~~

$\langle u_3, v_2 \rangle =$

$(-i - 5j + k) \cdot (2i - 2j + k)$

$= \underline{\underline{7}}$

$\langle v_2, v_2 \rangle = 2^2 + (-2)^2 +$

$= \underline{9}$

$$\therefore \quad w_1 = \frac{v_1}{\|v_1\|}$$

$$= \frac{1}{\sqrt{81}} \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

$$= \frac{3}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$w_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{1}{3} \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

$$\therefore \quad W = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}$$

$$W = \frac{1}{3} \begin{bmatrix} 2 & 2 & -1 \\ 1 & -2 & -2 \\ 2 & -1 & 2 \end{bmatrix}$$ is orthonormal vectors.

Eg :

1) We know that $\mathbb{R}^3$ is a vector space over $\mathbb{R}$.

$$W = \{ \underbrace{(0,a,b)}_{\to\ Y\text{-}Z \text{ plane}} : a,b \in \mathbb{R} \} \subseteq \mathbb{R}^3$$ is a subspace of

~~let~~ let $\alpha, \beta \in \mathbb{R}$, $w_1, w_2 \in W$.

$w_1 = (0, a_1, b_1)$

$w_2 = (0, a_2, b_2)$

Now; $\alpha w_1 + \beta w_2 = \alpha(0, a_1, b_1) + \beta(0, a_2, b_2)$

$= (0, \alpha a_1, \alpha b_1) + (0, \beta a_2, \beta b_2)$

$= (0, \alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2)$

$\alpha w_1 + \beta w_2 = (0, a_3, b_3) \in W$

$\because a_3 = \alpha a_1 + \beta a_2$ and $b_3 = \alpha b_1 + \beta b_2 \in \mathbb{R}$.

$\therefore \alpha w_1 + \beta w_2 \in W$

$\Rightarrow$ W is closed under addition and scalar multiplication.

$\therefore$ W is a ~~sup~~ subspace of $\mathbb{R}^3$

• ? let $W_1 = \{ \cancel{(a,b,c)}\ (1,b,c) , b,c \in \mathbb{R} \}$.

Is W_1 a subspace of $\mathbb{R}^3$.

A : W is not a ~~sup~~ subspace of $\mathbb{R}^3$

$\because$ Additive identity $(0,0,0) \notin W$.

Q. $W = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} / a, b, c \in \mathbb{R} \right\} \subset M_{2\times 2}$

Is W a subspace ~~or~~ of $M_{2\times 2}$

A: Let $\alpha, \beta \in \mathbb{R}$

$$w_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & 0 \end{bmatrix} \in W$$

$$w_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & 0 \end{bmatrix} \in W.$$

Then;

$$\alpha w_1 + \beta w_2 = \alpha \begin{bmatrix} a_1 & b_1 \\ c_1 & 0 \end{bmatrix} + \beta \begin{bmatrix} a_2 & b_2 \\ c_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & 0 \end{bmatrix} + \begin{bmatrix} \beta a_2 & \beta b_2 \\ \beta c_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 & 0 \end{bmatrix}$$

$\because \alpha, \beta, a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$

$\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2 \in \mathbb{R}.$

$$\therefore \alpha w_1 + \beta w_2 = \begin{bmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 & 0 \end{bmatrix} \in W.$$

ie, W is ~~cl~~ closed under addition and scalar multiplication.

$\therefore$ W is ~~sub~~ subspace of $M_{2\times 2}$

Q. Show: $P_n(x) = \{ a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n / a_i \in \mathbb{R} \}$ $i = 0, \dots, n$

is a vector space over $\mathbb{R}$.

$P_n[x] \cong \mathbb{R}^{n+1}$
isomorphic.

• ? Prove that $P_2(x)$ is a subspace of $P_3(x)$ over $\mathbb{R}$.

A $P_2(x) = \{ (a_0 + a_1 x + a_2 x^2) : a_0, a_1, a_2 \in \mathbb{R} \}$

$P_3(x) = \{(a_0 + a_1 x + a_2 x^2 + a_3 x^3) : a_i \in \mathbb{R} \quad i = 0, \ldots 3\}$

We know; $P_3(x)$ is a vector space over $\mathbb{R}$.

let $\alpha, \beta \in \mathbb{R}$

$$P_1 = a_0 + a_1 x + a_2 x^2 \in P_2(x)$$

$$P_2 = b_1 + b_2 x + b_2 x^2 \in P_2(x)$$

Then;

$$\alpha P_1 + \beta P_2 = \alpha (a_0 + a_1 x + a_2 x^2) + \beta (b_0 + b_1 x + b_2 x^2)$$

$$= (\alpha a_0 + \alpha a_1 x + \alpha a_2 x^2) + (\beta b_0 + \beta b_1 x + \beta b_2 x^2)$$

$$= ((\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1) x + (\alpha a_2 + \beta b_2) x^2)$$

$$\alpha P_1 + \beta P_2 = k_0 + k_1 x + k_2 x^2$$

where; $k_0, k_1, k_2 \in \mathbb{R}$ $(\because a_0, a_1, a_2, b_0, b_1, b_2, \alpha, \beta \in \mathbb{R})$

$\therefore$ $\alpha P_1 + \beta P_2 = k_0 + k_1 x + k_2 x^2 \in P_2(x)$

$\therefore$ $P_2(x)$ is closed under addition and scalar multiplication.

$\therefore$ $P_2(x)$ is a subspace over $\mathbb{R}$.