DIGITAL ASSIGNMENT-I

E2-TE2 Slot

Probability & Statistics

PMAT5011

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1. A box contains 5 black, 7 red and 6 green balls. Three balls are drawn at this box one after another. What is the probability that the three balls are

(i) all black

(ii) of different colors

(iii) 2 black and 1 green color.

Solution

Total no of balls $= 5+7+6 = 18$

Total ways to draw 3 out of 18

$$^{18}C_3 = \frac{18\times17\times16}{3\times2\times1} = 816$$

(i) P(3 black) =

no. of ways to choose 3 out of 5 black

$$= {}^5C_3 = \frac{5\times4\times3}{3\times2\times1} = 10$$

$$P(3\text{ black}) = \frac{^5C_3}{^{18}C_3} = \frac{10}{816} = \frac{5}{408} = 0.0123$$

(ii) P(all different)

$$= \frac{^5C_1\times{}^7C_1\times{}^6C_1}{^{18}C_3} = \frac{5\times7\times6}{816} = \frac{35}{136} = 0.2574$$

(iii) P(2 black, 1 green)

$$= \frac{^5C_2\ {}^6C_1}{^{18}C_3} = \frac{10\times6}{816} = \frac{5}{68} = 0.0735$$

Q2. A card is drawn from a standard deck of 52 cards, and without replacing it, a second card is drawn. If the first card is a heart. What is the probability of second card being a heart?

Solution Total cards = 52
Hearts in deck = 13

$P\left(\frac{\text{second card is heart}}{\text{First card is heart}}\right) = \frac{12}{51}$

$= \frac{4}{17} = \underline{0.2353}$

Q3. A urn contains 10 white and 3 black balls. Another urn contains 3 white & 5 black balls. Two balls are drawn at random from first urn and placed in the second urn, and then 1 ball is taken at random from the later. What is the prob. that it is a white ball?

First urn : 10 white + 3 black = $\underline{13 \text{ balls}}$
Second urn = 3 white + 5 black = $\underline{8 \text{ balls}}$

Both balls drawn at white

Choose 2 white balls from $10 = {}^{10}C_2 = \frac{10 \times 9}{2 \times 1} = 45$

Choose any 2 balls from $13 = {}^{13}C_2 = \frac{13 \times 12}{2 \times 1} = 78$

Probability of drawing 2 white balls $= \frac{{}^{10}C_2}{{}^{13}C_2}$

$= \frac{45}{78} = \frac{15}{26} = \underline{0.5769}$

One white and one black ball

Probability of choosing 1 white and 1 black ball =

$P(1 \text{ white}, 1 \text{ black}) = \frac{{}^{10}C_1\, {}^{3}C_1}{{}^{13}C_2} = \frac{10 \times 3}{78} = \frac{5}{13}$
$= 0.3846$

Both balls are black

Probability of drawing 2 black balls =

$P(2 \text{ black}) = \frac{{}^{3}C_2}{{}^{13}C_2} = \frac{3}{78} = \frac{1}{26}$

If both transfer balls white,

$P(\text{white from urn } 2 / 2 \text{ white}) = \frac{5}{10} = \frac{1}{2}$

If 1 white & 1 black transferred,

$P(\text{white from Urn } 2 / 1 \text{ white } 1 \text{ black}) = \frac{4}{10} = \frac{2}{5}$

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If both transferred balls black

P(white from urn 2 / 2 black transfer) $= \frac{3}{10}$

Total Prob. of Drawing white balls,

$$P(\text{white ball}) = \frac{15}{26} \times \frac{1}{2} + \frac{5}{13} \times \frac{2}{5} + \frac{1}{26} \times \frac{3}{10}$$

$$= \frac{15}{52} + \frac{10}{65} + \frac{3}{260} = \frac{59}{130} = \underline{0.4538}$$

Q)4. Find value of a $P(X<3)$, $P(1.3 \leq X \leq 6.7)$, $P(\frac{1}{2} < X < \frac{15}{2} \mid X>3)$, cdf, SD, $E(2X \pm 3)$ and $Var(2X \pm 3)$ of discrete RV (X) with the foll. Prob distribution.

$X = x$:	0	1	2	3	4	5	6	7	8
$P = P(X=x)$:	a	3a	5a	7a	9a	11a	13a	15a	17a

Total Prob. = 1

$$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$81a = 1 \text{ So, } \boxed{a = \frac{1}{81}}$$

$\underline{P(X<3)} = P(X=0) + P(X=1) + P(X=2)$

$$= a + 3a + 5a + 9a = \frac{9}{81} = \frac{1}{9} = \underline{0.1111}$$

$\underline{P(1.3 \leq X \leq 6.7)} = P(2 \leq X \leq 6)$

$$= P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$$

$$= 5a + 7a + 9a + 11a + 13a = 45a$$

$$= \frac{45}{81} = \frac{5}{9} = \underline{0.5555}$$

$\underline{P(\frac{1}{2} \leq X \leq \frac{15}{2} / X>3)} = P(1 < X < 15/2 \,/\, X>3)$

$$= \frac{P((1<X<7) \cap (X>3))}{P(X>3)}$$

$x = 4, 5, 6, 7$

$x = 45678$

$$= \frac{33a}{65a} = \frac{33}{65} = \underline{0.5077}$$

$\underline{\text{C.D.F.}}$,

$$F(x) \begin{cases} \frac{1}{81} & x = 0 \\ 4/81 & x \leq 1 \\ 1/9 & x \leq 2 \\ 16/81 & x \leq 3 \\ 25/81 & x \leq 4 \\ 4/9 & x \leq 5 \\ 49/81 & x \leq 6 \\ 64/81 & x \leq 7 \\ 1 & x \leq 8 \end{cases}$$

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Standard Deviation

$E(X) = 0\times a + 1\times 3a + 2\times 5a + 3\times 7a + 4\times 9a + 5\times 11a + 6\times 13a + 7\times 15a + 8\times 17a$

$= 444a = \frac{444}{81} = \frac{148}{27}$

$E(X^2) = 0 + 3a + 20a + 63a + 144a + 275a + 468a + 735a + 1088a$

$= 2796a$

$= \frac{2796}{81} = \frac{932}{27}$

$Var(X) = E(X^2) - [E(X)]^2 = \frac{932}{27} - \left(\frac{148}{27}\right)^2 = \frac{932}{27} - \frac{21904}{729}$

$\sigma^2 = 4.4719 \Rightarrow \boxed{\sigma = 2.1147}$

$E(2X \pm 3)$ and $Var(2X \pm 3)$

$E(2X \pm 3) = 2E(X) \pm 3 = \frac{2\times 148}{27} \pm 3$

$E(2X+3) = \frac{296+81}{27} = \frac{377}{27}$

$E(2X-3) = \frac{296-81}{27} = \frac{215}{27} = 7.9629$

$Var(2X \pm 3) = 4\,Var(X)$

$= 4\times\frac{3260}{729} = \frac{13040}{729} = 17.8875$

Q5) Suppose X is continuous RV, with the pdf given by

$$f(x) = \begin{cases} kxe^{-\lambda x}; & x \geq 0, \lambda > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

Find k, $P(X<5)$, $P(1\leq X\leq 100 / X \nleq 5)$, cumulative distribution function & SD of X.

$\int_0^\infty f(x)\,dx = 1$; $\int kxe^{-\lambda x}\,dx = 1 = k\cdot\frac{1}{\lambda^2}$

$\boxed{k = \lambda^2}$

$P(X<5) = \int_0^5 \lambda^2 x e^{-\lambda x}\,dx$

$= 1-(1+5\lambda)e^{-5\lambda}$

$P(X\leq 1) = \int_0^1 \lambda^2 x e^{-\lambda x}\,dx) = 1-(1+\lambda)e^{-\lambda}$

$P(1\leq X\leq 100/X\leq 5) = (1-(1+5\lambda)e^{-5\lambda}) - (1-(1+\lambda)e^{-\lambda})$

Cumulative Distribution Function (CDF)

$F(x) = \int_0^x \lambda^2 t e^{-\lambda t}\,dt$

$= 1-(1+\lambda x)e^{-\lambda x}$

SD of X,

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$E(X) = \int_0^\infty x f(x) dx = \int_0^\infty x \lambda^2 x e^{-\lambda x} dx = \frac{2}{\lambda}$

$E(X^2) = \int_0^\infty x^2 f(x) dx = \int_0^\infty x^2 \lambda^2 x e^{-\lambda x} dx$

$= \frac{6}{\lambda^2}$

$var(X) = E(X^2) - \cancel{[E(X)]^2}$

$= \frac{6}{\lambda^2} - \left(\frac{2}{\lambda}\right)^2 = \frac{2}{\lambda^2}$

$SD(X) = \sigma = \sqrt{var(X)} = \frac{\sqrt{2}}{\lambda}$

6.) Two dimensional RV (X, Y) has the joint prob. function given by $P(X=x, Y=y) = k(3x+5y)$, for $x = 0, 1, 2, 3$ and $y = 0, 1$

(i) Find value of k

(ii) Find all marginal & conditional distribution of X & Y

(iii) Find the Prob. distribution of Z, mean & variance of Z, where $Z = X+Y$

Solution

(i) $\sum_{x=0}^{3} \sum_{y=0}^{1} P(X=x, Y=y) = 1$

$\sum_{x=0}^{3} \sum_{y=0}^{1} k(3x+5y) = 1$

for $x=0$: $k(3.0+5\times0) + k(3\times0+5\times1)$

$= k(0) + k(5) = 5k$

For $x=1$: $k(3\times1+5\times1) + k(3\times1+5\times1)$

$= k(3) + k(8) = 11k$

For $x=2$: $k(3\times2+5\times0) + k(3\times2+5\times1)$

$= k(6) + k(11) = 17k$

For $x=3$: $k(3\times3+5\times0) + k(3\times3+5\times1)$

$= k(9) + k(14) = 23k$

$5k + 11k + 17k + 23k = 56k$

$k = \frac{1}{56}$

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(ii) Marginal Distribution

$$P(X=x) = \sum_{y=0}^{1} P(X=x, Y=y)$$

For $X=0$, $P(X=0) = k(0)+k(5) = 5k = 5\times\frac{1}{56} = \frac{5}{56}$.

For $X=1$, $P(X=1) = k(3)+k(8) = 11k = \frac{11}{56} = \underline{0.1964}$.

For $X=2$, $P(X=2) = 17k = \frac{17}{56} = \underline{0.3036}$

For, $X=3$, $P(X=3) = 23k = \frac{23}{56} = \underline{0.4107}$

Marginal Distribution of X,

$P(X=0) = 5/56$, $P(X=1) = 11/56$, $P(X=2) = \frac{17}{56}$

$P(X=3) = 23/56$

Marginal Distribution of Y

$$P(Y=y) = \sum_{x=0}^{3} P(X=x, Y=y)$$

$P(X=0) = k(0)+k(3)+k(6)+k(9)$

$= 18k = 18\times\frac{1}{56} = \frac{9}{28} = \underline{0.3214}$

$P(X=1) = k(5)+k(8)+k(11)+k(14)$

$= 38k = \frac{19}{28} = \underline{0.6786}$

(iii) Conditional Dist of X given Y

$$P(X=x/Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

For $(Y=0)$

$P(X=0/Y=0) = \frac{k(0)}{P(Y=0)} = 0$

$P(X=1/Y=0) = \frac{k(3)}{P(Y=0)} = \frac{3k}{9/28} = \frac{28}{168} = \frac{1}{6}$

$= \underline{0.1667}$

$P(X=2/Y=0) = \frac{k(6)}{P(Y=0)} = \frac{6k}{9/28} = \frac{56}{168} = \frac{1}{3} = \underline{0.3333}$

$P(X=3/Y=0) = \frac{k(9)}{P(Y=0)} = \frac{9k}{9/28} = \frac{28}{168} = \frac{1}{2} = \underline{0.5}$

For $(Y=1)$

$P(X=0/Y=1) = \frac{k(5)}{P(Y=1)} = \frac{5k}{19/28} = \frac{5\times 28}{19\times 56} = \frac{5}{38}$

$= \underline{0.1316}$

$P(X=1/Y=1) = \frac{k(8)}{P(Y=1)} = \frac{8k}{19/28} = \frac{8}{38} = \frac{4}{19}$

$= \underline{0.2105}$

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$P(X=2/Y=1) = \frac{k(11)}{P(Y=1)} = \frac{11k}{19/28}$

$= \frac{11}{38} = \underline{0.2895}$

$P(X=3/Y=1) = \frac{k(14)}{P(Y=1)} = \frac{14k}{P(Y=1)} = \frac{14}{38} = \frac{7}{19}$

$= \underline{0.3684}$

(iv) **Probability Distribution, Mean & Variance of $Z = X+Y$**

$P(Z=0) = P(X=0, Y=0) = 0$

$P(Z=1) = P(X=0, Y=1) = 5k = \underline{5/56}$

$P(Z=2) = P(X=1, Y=0) + P(X=1, Y=1)$

$= 3k + 8k = 11/56 = \underline{0.1964}$

$P(Z=3) = P(X=2, Y=0) + P(X=2, Y=1)$

$= 6k + 11k = 17/56 = \underline{0.3036}$

$P(Z=4) = P(X=3, Y=0) + P(X=3, Y=1)$

$= 9k + 14k = \frac{23}{56} = \underline{0.4107}$

Mean of Z,

$E(Z) = \sum_z z P(Z=z) = 1 \times \frac{5}{56} + 2 \times \frac{11}{56} + 3 \times \frac{17}{56} + 4 \times \frac{23}{56}$

$E(Z) = \frac{5}{56} + \frac{22}{56} + \frac{51}{56} + \frac{92}{56}$

$= \frac{170}{56} = \frac{85}{28} \approx \underline{3.04}.$

Variance of Z,

$Var(Z) = \sigma_Z^2 = [E(Z^2) - E(Z)]^2$

$E(Z^2) = \sum_z z^2 P(Z=z) = 1^2 \times \frac{5}{56} + 2^2 \times \frac{11}{56} + 3^2 \times \frac{17}{56} + 4^2 \times \frac{23}{56}$

$E(Z^2) = \frac{5}{56} + \frac{44}{56} + \frac{153}{56} + \frac{368}{56} = \frac{570}{56}$

$= \frac{285}{28}$

$Var(Z) = \frac{285}{28} - \left(\frac{85}{28}\right)^2 = \frac{285}{28} - \frac{7225}{784}$

$= \frac{285}{28} - \frac{7225}{784}$

$= \frac{7980 - 7225}{784} = \frac{755}{784} = \underline{0.9630}$

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Q7) Two Random variable X and Y have the following joint probability density function:-

$$f_{xy}(x,y) = \begin{cases} 2-x-y, & 0 \le x \le 1, 0 \le y \le 1; \\ 0, & \text{otherwise} \end{cases}$$

Find (i) $P(X > \frac{1}{2})$, $P(Y < \frac{1}{2})$ and $P(X > \frac{1}{2} / Y < \frac{1}{2})$

(ii) Marginal Probability density funⁿ of X & Y

(iii) Conditional density function of X and Y.

(iv) Variance Var (X) and Var(Y)

(v) Co-Variances between X & Y.

Soln

$$f_{xy}(x,y) = \begin{cases} 2-x-y, & 0 \le x \le 1; 0 \le y \le 1; \\ 0, & \text{otherwise} \end{cases}$$

(i) $P(X > \frac{1}{2}) = \int_{1/2}^{1} \int_{0}^{1} (2-x-y)\,dy\,dx = \left[(2-x)y - \frac{y^2}{2}\right]_0^1$

$= \left[1.5x - \frac{x^2}{2}\right]_{1/2}^{1} = 0.375$

- $P(Y < \frac{1}{2}) = \int_0^1 \int_0^{1/2} (2-x-y)\,dy\,dx = \left[(2-x)y - \frac{y^2}{2}\right]_0^{1/2}$

$= \int_0^1 \left(\left(1 - \frac{x}{2}\right) - \frac{1}{8}\right) = 0.625$

$P(X > \frac{1}{2} / Y < \frac{1}{2}) = \int_{1/2}^{1} \int_0^{1/2} (2-x-y)\,dy\,dx$

$= \int_{1/2}^{1} \left(\left(1 - \frac{x}{2}\right) - \frac{1}{8}\right) dx = 0.25$

$P(X > \frac{1}{2} / Y < \frac{1}{2}) = \cancel{\int_{1/2}^{1} \int_0^{1/2} (2-x-y)\,dy\,dx}$

$= \dfrac{P(X > \frac{1}{2}, Y < \frac{1}{2})}{P(Y < \frac{1}{2})}$

$= \dfrac{0.25}{0.625} = \underline{\underline{0.4}}$

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(ii) **Marginal Probability Density Functions of X and Y.**

$f_X(x) = \int_0^1 (2-x-y)\,dy = \frac{3}{2} - x$ for $0 \leq x \leq 1$

$f_Y(y) = \int_0^1 (2-x-y)\,dx = \frac{3}{2} - y$ for $0 \leq y \leq 1$

(iii) **Conditional Density Functions**, for $0 \leq x; y \leq 1$

$$f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2-x-y}{f_Y(y)} = \frac{2-x-y}{\frac{3}{2}-y}$$

$$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{2-x-y}{f_X(x)} = \frac{2-x-y}{\frac{3}{2}-x}$$

(iv) **Variance Var(X) and Var(Y)**

$$E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x\left(\frac{3}{2}-x\right)dx = \frac{5}{12}$$

$$E(X^2) = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 x^2\left(\frac{3}{2}-x\right)dx = \frac{1}{4}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{11}{144} = \underline{0.0763}$$

$$E(Y) = \int_0^1 y f_Y(y)\,dy, \quad E(Y^2) = \int_0^1 y^2 f_Y(y)\,dy$$

$$Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{11}{144} = \underline{0.0763}$$

(v) **Covariance between X and Y**

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$E(X,Y) = \int_0^1 \int_0^1 xy f_{XY}(x,y)\,dy\,dx$$

$$E(X,Y) \Rightarrow y\left[\int_0^1 (2x - x^2 - xy)\,dx\right] = y\left(1 - \frac{1}{3} - \frac{y}{2}\right) = y\left(\frac{2}{3} - \frac{y}{2}\right)$$

$$\int_0^1 y\left(\frac{2}{3} - \frac{y}{2}\right)dy = \frac{2}{3}\int_0^1 y\,dy - \frac{1}{2}\int_0^1 y^2\,dy :$$

$$\frac{2}{3}\left[\frac{y^2}{2}\right]_0^1 - \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 = \frac{1}{6}$$

$$Cov(X,Y) = \frac{1}{6} - \frac{5}{12} \times \frac{5}{12} = -\frac{1}{144} = \underline{-6.944 \times 10^{-3}}$$

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8. If X represents the outcome, when a fair die is tossed, then find the moment generating function of X and hence find E(X) and Var(X).

MGF, $M_x(t) = E(e^{tX})$

possible outcomes of X are 1, 2, 3, 4, 5 & 6, each with probability $\frac{1}{6}$.

So,

$$M_x(t) = \sum_{x=1}^{6} P(X=x) \cdot e^{tx} = \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$$

Taking first derivative of $M_x(t)$ at $t=0$,

$E(X) = M'_x(0)$

$$M'_x(t) = \frac{1}{6}(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t})$$

at $t=0$:

$$M'_x(0) = \frac{1}{6}(1+2+3+4+5+6) = \frac{1}{6} \times 21 = 3.5$$

$$E(X) = 3.5$$

Now, $Var(X) = E(X^2) - (E(X))^2$

$$M''_x(t) = \frac{1}{6}(e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t})$$

at $t=0$:

$$M''_x(0) = \frac{1}{6}(1+4+9+16+25+36) = \frac{1}{6} \times 91$$

$$= 15.1667$$

So, $E(X^2) = 15.1667$.

$$Var(X) = 15.1667 - (3.5)^2 = 15.1667 - 12.25$$

$$= 2.9167$$

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9. Compute the coefficient of correlation between the foll. two set of measures X and Y and hence obtain the equation of regression lines.

X	Y	X^2	Y^2	XY
65	67	4225	4489	4355
67	68	4489	4624	4556
66	68	4356	4624	4488
71	70	5041	4900	4970
67	64	4489	4096	4288
70	67	4900	4489	4690
68	72	4624	5184	4896
69	70	4761	4900	4830
$\Sigma X =$ 538	$\Sigma Y =$ 486	$\Sigma X^2 =$ 36214	$\Sigma Y^2 =$ 36842	ΣXY = 42686

$$\bar{X} = \frac{538}{8} = 67.25$$

$$\bar{Y} = \frac{486}{8} = 60.75$$

$$r = \frac{n\Sigma XY - \Sigma X \Sigma Y}{\sqrt{(n\Sigma X^2 - (\Sigma X)^2)(n\Sigma Y^2 - (\Sigma Y)^2)}}$$

$$= \frac{8 \times 42686 - 538 \times 486}{\sqrt{(8 \times 36214 - 538^2)(8 \times 36842 - 486^2)}}$$

$$r = 20.18$$

$$b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x} = \frac{79200}{268} \approx 297.0$$

Y on X

$$y - \bar{Y} = r \cdot \frac{\sigma_y}{\sigma_x}(X - \bar{X})$$

$$\boxed{Y = 297.0X - 13969.5}$$

$$b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y} = 1.36 \quad \left(\because \frac{79720}{58540} \approx 1.36\right)$$

X on Y

$$X - \bar{X} = r \cdot \frac{\sigma_x}{\sigma_y}(Y - \bar{Y})$$

$$\boxed{X = 1.36Y - 15.35}$$