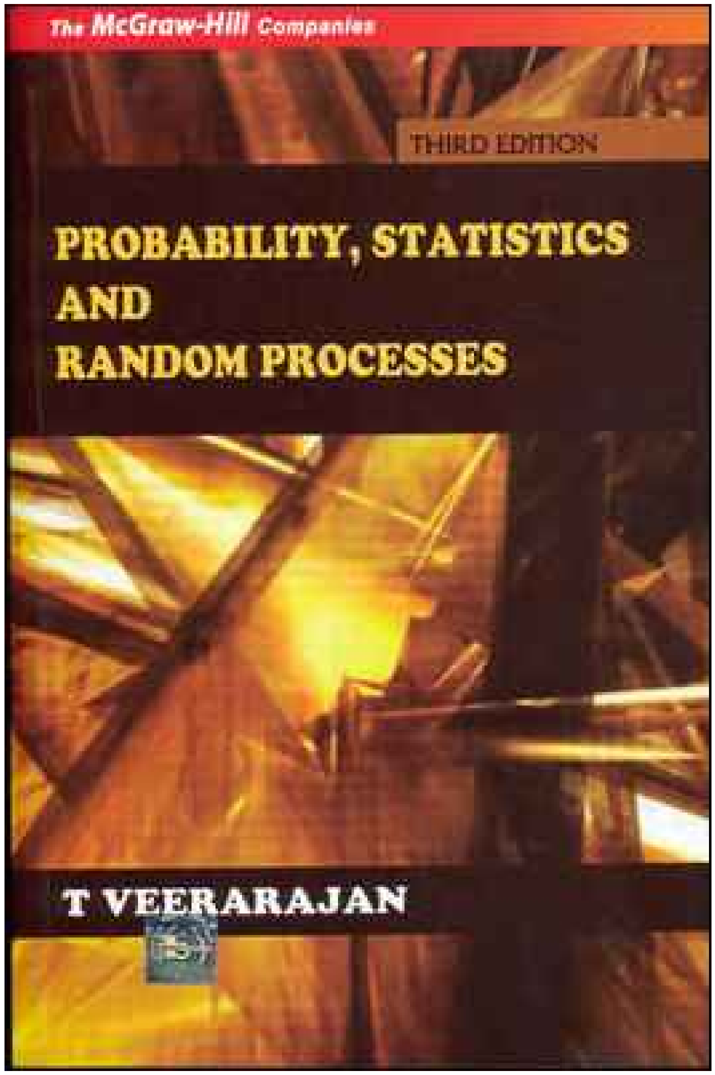
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THIRD EDITION
PROBABILITY, STATISTICS
AND
RANDOM PROCESSES
T VEERARAJAN

Probability, Statistics and Random Processes

Third Edition

About the Author

T Veerarajan is currently heading the Department of Science and Humanities, Sree Sowdambika College of Engineering Aruppukotai, Tamil Nadu. A Gold Medalist from Madras University, he has had a brilliant academic career all through. He has 47-years of teaching experience at undergraduate and postgraduate levels in various established engineering colleges in Tamil Nadu including Anna University, Chennai.

He is the author of several textbooks. Some of them are titled as:

Engineering Mathematics for first year

Engineering Mathematics for III Semester

Discrete Mathematics

Numerical Methods (Sigma Series)

Numerical Methods: With programs in C

All these books which are primarily meant for BE/B. Tech courses, are published by Tata McGraw-Hill Publishing Company, New Delhi. His prime area of interest includes Random Processes which has inspired him to write this book on Probability, Statistics and Random Processes.

Probability, Statistics and Random Processes

Third Edition

T Veerarajan
Professor of Mathematics and Head
Science and Humanities Department
Sree Sowdambika College of Engineering
Aruppukottai, Tamil Nadu

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Preface to the Third Edition

I am extremely gratified by the overwhelming response shown for the first two editions of this book.

Though the previous editions catered to the requirements of engineering students as per the Anna University syllabus, a recent survey of the syllabi followed by many universities across the country made me revise the book. The two topics 'Reliability Engineering' and 'Statistical Quality Control' have been removed, as the mechanical engineering students study them along with other subjects. The topic 'Statistical Averages' has been thoroughly revised—more concepts about basics of statistical measures have been included and more number of worked examples and exercise problems have been added. Various chapters have been rearranged.

The Online Learning Centre of the book can be accessed at http://www.mhhe.com./veerarajan/psrp3e and contains the following material for students:

- useful web links for further reading
- Chapters on
  —Reliability Engineering
  —Statistical Quality Control

I am extremely grateful to the Management Chockalingapuram Devangar Varthaga Sangam, Aruppukottai, which has sponsored Sree Sowdambika College of Engineering, Aruppukottai, in which I am presently working, for the support extended to me in this project.

I wish to express my thanks to Prof. M Jegan Mohan, Principal, Sree Sowdambika College of Engineering for the keen interest shown and constant encouragement given to me while preparing this revised edition of the book.

While preparing this edition of the book, care has been taken to completely eliminate a few errors which were present in the previous edition.

Critical evaluation and suggestions for further improvement of the book are most welcome.

T VEERARAJAN

Preface to the First Edition

This book is intended for a one-semester course in Probability, Statistics and Random Processes for undergraduate level Electrical/Electronics/Computer Engineering and Information Technology students. The various chapters also cover the syllabus content of 'Probability and Statistics' introduced recently for the BE (CSE and IT) course by the Madurai Kamaraj University of Tamil Nadu.

Most engineering students, who are used to a deterministic outlook of Physics and Engineering problems, find the theory of probability unreliable, vague and difficult. This is due to inadequate understanding of the basic concepts of probability theory and the wrong impression that the subject is an advanced branch of Mathematics.

The book is written in such a manner that beginners may develop an interest in the subject and may find it useful to pursue their studies. Basic concepts and proofs of theorems are explained in as lucid a manner as possible. Although the theory of probability is developed rigorously based on measure theory, it is developed in this book by a simple set-theory approach.

As engineering students find it easier to generalize specific results and examples than to specialize general results, considerable attention is devoted to working of problems. Nearly 300 problems including those with applications to communication theory are worked out in various chapters. Unless the students become personally involved in solving exercises, they cannot really develop an understanding and appreciation of the ideas and a familiarity with the pertinent techniques. Hence, in addition to a large number of short-answer questions under Part-A, over 350 problems have been given under Part-B of the Exercises in various chapters. Answers are provided at the end of every chapter.

Though chapters 7 and 8 are meant for Electrical/Electronics Engineering students, the other chapters that deal with probability theory, random variables, probability distributions and statistics may be useful to students of any discipline, such as those doing MCA and M.Sc courses who have Mathematical Statistics as a subject of their study.

I am sure that the students and the faculty will find this book very useful.

Critical evaluation and suggestions for improvement of the book will be highly appreciated and gratefully acknowledged.

I am extremely grateful to Dr K V Kuppusamy, Chairman, and Mr K Senthil Ganesh, Managing Trustee, RVS Educational Trust, Dindigul, for the support extended to me in this project.

I wish to express my thanks to Dr K M Karuppannan, Principal, RVS College of Engineering and Technology, Dindigul, for the appreciative interest shown and constant encouragement given to me while writing this book.

I am thankful to my publishers, Tata McGraw-Hill Publishing Company Limited, New Delhi, for their painstaking efforts and cooperation in bringing out this book in a short span of time.

I would also like to thank Dr A Rangan, Professor, Department of Mathematics, IIT Madras; Dr S Leela Devi, Professor and Head, Department of Mathematics J J College of Engineering and Technology, Tiruchirapalli; and Mr Sitharselvan and Mr Muthuraman of Bannari Amman Institute of Technology for reviewing and providing valuable suggestions during the developmental stages of the book.

I have great pleasure in dedicating this book to my beloved students, past and present.

T VEERARAJAN

1 Probability Theory

Probability theory had its origin in the analysis of certain games of chance that were popular in the seventeenth century. It has since found applications in many branches of Science and Engineering and this extensive application makes it an important branch of study. Probability theory, as a matter of fact, is a study of random or unpredictable experiments and is helpful in investigating the important features of these random experiments.

Random Experiment

An experiment whose outcome or result can be predicted with certainty is called a deterministic experiment. For example, if the potential difference E between the two ends of a conductor and the resistance R are known, the current I flowing in the conductor is uniquely determined by Ohm's law, $I = \frac{E}{R}$.

Although all possible outcomes of an experiment may be known in advance, the outcome of a particular performance of the experiment cannot be predicted owing to a number of unknown causes. Such an experiment is called a random experiment.

Whenever a fair 6-faced cubic dice is rolled, it is known that any of the 6 possible outcomes will occur, but it cannot be predicted what exactly the outcome will be, when the dice is rolled at a point of time.

Although the number of telephone calls received in a board in a 5-min. interval is a non-negative integer, we cannot predict exactly the number of calls received in the next 5-min. In such situations we talk of the chance or the probability of occurrence of a particular outcome, which is taken as a quantitative measure of the likelihood of the occurrence of the outcome.

Mathematical or Apriori Definition of Probability

Let S be the sample space (the set of all possible outcomes which are assumed equally likely) and A be an event (a sub-set of S consisting of possible outcomes) associated with a random experiment. Let $n(S)$ and $n(A)$ be the number of elements of S and A. Then the probability of event A occurring, denoted as $P(A)$, is defined by

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to } A}{\text{Exhaustive number of cases in } S}$$

For example, the probability of getting an even number in the die tossing experiment is 0.5, as $S = \{1, 2, 3, 4, 5, 6\}$, $E = \{2, 4, 6\}$, $n(S) = 6$ and $n(E) = 3$.

Statistical or Aposteriori Definition of Probability

Let a random experiment be repeated n times and let an event A occur n_A times out of the n trials. The ratio $\frac{n_A}{n}$ is called the relative frequency of the event A. As n increases, $\frac{n_A}{n}$ shows a tendency to stabilise and to approach a constant value. This value, denoted by $P(A)$, is called the probability of the event A, i.e., $P(A) = \lim_{n \to \infty} \frac{n_A}{n}$.

For example, if we want to find the probability that a spare part produced by a machine is defective, we study the record of defective items produced by the machine over a considerable period of time. If, out of 10,000 items produced, 500 are defective, it is assumed that the probability of a defective item is 0.05.

Note *From both the definitions, it is obvious that $0 \le P(A) \le 1$. If A is an impossible event, $P(A) = 0$. Conversely, if $P(A) = 0$, then A can occur in a very small percentage of times in the long run. On the other hand, if A is a certain event, $P(A) = 1$. Conversely, if $P(A) = 1$, then A may fail to occur in a very small percentage of times in the long run.*

Axiomatic Definition of Probability

Let S be the sample space and A be an event associated with a random experiment. Then the probability of the event A, denoted by $P(A)$, is defined as a real number satisfying the following axioms.

(i) $0 \le P(A) \le 1$

(ii) $P(S) = 1$

(iii) If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$ and

(iv) If $A_1, A_2, \ldots, A_n, \ldots$ are a set of mutually exclusive events, $P(A_1 \cup A_2 \cup \ldots \cup A_n \ldots) = P(A_1) + P(A_2) + \ldots + P(A_n) + \ldots$

The term *mutually exclusive* used in the above definition can be explained as follows. A set of events is said to be mutually exclusive if the occurrence of any one of them excludes the occurrence of the others. Two events A and B are mutually exclusive if A occurs and B does not occur and vice versa. In other words, A and B cannot occur simultaneously, i.e., $P(A \cap B) = 0$.

In the development of the probability theory, all results are derived directly or indirectly using only the axioms of probability, as can be seen from the following theorems.

Theorem 1 The probability of the impossible event is zero, i.e., if ϕ is the subset (event) containing no sample point, $P(\phi) = 0$.

Proof The certain event S and the impossible event ϕ are mutually exclusive.

Hence $\quad P(S \cup \phi) = P(S) + P(\phi)$ [Axiom (iii)].

But $\quad S \cup \phi = S$.

$\therefore \quad P(S) = P(S) + P(\phi)$

$\therefore \quad P(\phi) = 0$

Theorem 2 If $\overline{A}$ is the complementary event of A, $P(\overline{A}) = 1 - P(A) \le 1$.

Proof A and $\overline{A}$ are mutually exclusive events, such that $A \cup \overline{A} = S$.

$\therefore \quad P(A \cup \overline{A}) = P(S)$

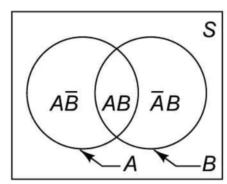
$\quad = 1$ [Axiom (ii)]

i.e., $\quad P(A) + P(\overline{A}) = 1$ [Axiom (iii)]

$\therefore \quad P(\overline{A}) = 1 - P(A)$

Since $P(A) \ge 0$, it follows that $P(\overline{A}) \le 1$.

Theorem 3 If A and B are any 2 events, $P(A \cup B) = P(A) + P(B) - P(A \cap B) \le P(A) + P(B)$



Proof A is the union of the mutually exclusive events $A\overline{B}$ and AB and B is the union of the mutually exclusive events $\overline{A}B$ and AB.

$\therefore \quad P(A) = P(A\overline{B}) + P(AB)$ [Axiom (iii)]

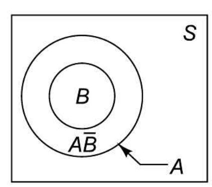
and $\quad P(B) = P(\overline{A}B) + P(AB)$ [Axiom (iii)]

$\therefore \quad P(A) + P(B) = [P(A\overline{B}) + P(AB) + P(\overline{A}B) + P(AB)]$

$\quad = P(A \cup B) + P(A \cap B)$

The result follows. Clearly, $P(A) + P(B) - P(A \cap B) \le P(A) + P(B)$]

Theorem 4 If $B \subset A$, $P(B) \le P(A)$.



Proof B and $A\overline{B}$ are mutually exclusive events such that $B \cup A\overline{B} = A$.

$\therefore \quad P(B \cup A\overline{B}) = P(A)$

i.e. $\quad P(B) + P(A\overline{B}) = P(A)$ [Axiom (iii)]

$\therefore \quad P(B) \le P(A)$

Note *In probability theory developed using the classical definition of probability, theorem 3 above is termed as Addition theorem of probability as applied to any 2 events. The theorem can be extended to any 3 events A, B and C as follows:*

$$P(A \cup B \cup C) = P(\text{at least one of } A, B \text{ and } C \text{ occurs})$$
$$= P(A) + P(B) + P(C) - P(A \cap B)$$
$$- P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

In the classical approach, probability axiom (iii) is termed as addition theorem of probability as applied to 2 mutually exclusive events, which is proved in the following way.

Let the total number of cases (outcomes) be n, of which n_A are favourable to the event A and n_B are favourable to the event B.

Therefore the number of cases favourable to A or B, i.e., $A \cup B$ is $(n_A + n_B)$, since the events A and B are disjoint.

$$\therefore \quad P(A \cup B) = \frac{n_A + n_B}{n} = \frac{n_A}{n} + \frac{n_B}{n} = P(A) + P(B)$$

Conditional Probability

The conditional probability of an event B, assuming that the event A has happened, is denoted by $P(B/A)$ and defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

For example, when a fair dice is tossed, the conditional probability of getting '1', given that an odd number has been obtained, is equal to 1/3 as explained below:

$$S = \{1, 2, 3, 4, 5, 6\};\ A = \{1, 3, 5\};\ B = \{1\}$$

$$\therefore \quad P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{1}{3}$$

As per the definition given above, $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/6}{1/2} = \dfrac{1}{3}$.

Rewriting the definition of conditional probability, we get $P(A \cap B) = P(A) \times P(B/A)$. This is sometimes referred to as *Product theorem of probability*, which is proved as follows.

Let n_A, n_{AB} be the number of cases favourable to the events A and $A \cap B$, out of the total number n of cases.

$$\therefore \quad P(A \cap B) = \frac{n_{AB}}{n} = \frac{n_A}{n} \times \frac{n_{AB}}{n_A} = P(A) \times P(B/A)$$

The product theorem can be extended to 3 events A, B and C as follows:

$$P(A \cap B \cap C) = P(A) \times P(B/A) \times P(C/A \text{ and } B)$$

The following properties are easily deduced from the definition of conditional probability:

1. If $A \subset B$, $P(B/A) = 1$, since $A \cap B = A$
2. If $B \subset A$, $P(B/A) \geq P(B)$, since $A \cap B = B$, and $\dfrac{P(B)}{P(A)} \geq P(B)$, as $P(A) \leq P(S) = 1$

3. If A and B are mutually exclusive events, $P(B/A) = 0$, since $P(A \cap B) = 0$
4. If $P(A) > P(B)$, $P(A/B) > P(B/A)$ (MKU — Apr. 96)
5. If $A_1 \subset A_2$, $P(A_1/B) \leq P(A_2/B)$ (BU — Apr. 96))

Independent Events

A set of events is said to be independent if the occurrence of any one of them does not depend on the occurrence or non-occurrence of the others.

When 2 events A and B are independent, it is obvious from the definition that $P(B/A) = P(B)$. If the events A and B are independent, the product theorem takes the form $P(A \cap B) = P(A) \times P(B)$. Conversely, if $P(A \cap B) = P(A) \times P(B)$, the events A and B are said to be independent (pairwise independent). The product theorem can be extended to any number of independent events: If $A_1, A_2, \ldots, A_n$ are n independent events.

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) \times P(A_2) \times \ldots \times P(A_n)$$

When this condition is satisfied, the events $A_1, A_2, \ldots, A_n$ are also said to be *totally independent*. A set of events $A_1, A_2, \ldots, A_n$ is said to be *mutually independent* if the events are totally independent when considered in sets of 2, 3, $\ldots$, n events.

In the case of more than 2 events, the term 'independence' is taken as 'total independence' unless specified otherwise.

Theorem 1 If the events A and B are independent, the events $\overline{A}$ and B (and similarly A and $\overline{B}$) are also independent.

Proof The events $A \cap B$ and $\overline{A} \cap B$ are mutually exclusive such that $(A \cap B) \cup (\overline{A} \cap B) = B$.

$\therefore \quad P(A \cap B) + P(\overline{A} \cap B) = P(B)$ (by addition theorem)

$$\begin{aligned} \therefore \quad P(\overline{A} \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)\,P(B) \text{ (by product theorem)} \\ &= P(B)\,[1 - P(A)] \\ &= P(\overline{A})\,P(B) \end{aligned}$$

Theorem 2 If the events A and B are independent, then so are $\overline{A}$ and $\overline{B}$.

(MU — Apr. 96)

Proof
$$\begin{aligned} P(\overline{A} \cap \overline{B}) &= P(\overline{A \cup B}) = 1 - P(A \cup B) & (1) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \text{ (by addition theorem)} \\ &= 1 - P(A) - P(B) + P(A) \times P(B) \\ &\qquad \text{(since } A \text{ and } B \text{ are independent)} \\ &= [1 - P(A)] - P(B)\,[1 - P(A)] \\ &= P(\overline{A}) \times P(\overline{B}) & (2) \end{aligned}$$

Note *From (1) and (2), it follows that when the events A and B are independent,* $P(A \cup B) = 1 - P(\overline{A}) \times P(\overline{B})$.

Worked Examples 1(A)

Example 1 *A fair coin is tossed 4 times. Define the sample space corresponding to this random experiment. Also give the subsets corresponding to the following events and find the respective probabilities:*

(a) *More heads than tails are obtained.*

(b) *Tails occur on the even numbered tosses.*

Solution S = {HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}

(a) Let A be the event in which more heads occur than tails.

Then A = {HHHH, HHHT, HHTH, HTHH, THHH}

(b) Let B be the event in which tails occur in the second and fourth tosses.

Then B = {HTHT, HTTT, TTHT, TTTT}

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{16}; \; P(B) = \frac{n(B)}{n(S)} = \frac{1}{4}$$

Example 2 *There are 4 letters and 4 addressed envelopes. If the letters are placed in the envelopes at random, find the probability that (i) none of the letters is in the correct envelope and (ii) at least 1 letter is in the correct envelope, by explicitly writing the sample space and the event spaces.*

Solution Let the envelopes be denoted by A, B, C and D and the corresponding letters by a, b, c and d.

S = {(Aa, Bb, Cc, Dd), (Aa, Bb, Cd, Dc), (Aa, Bc, Cb, Dd),
(Aa, Bc, Cd, Db), (Aa, Bd, Cb, Dc), (Aa, Bd, Cc, Db),
(Ab, Ba, Cc, Dd), (Ab, Ba, Cd, Dc), (Ab, Bc, Ca, Dd),
(Ab, Bc, Cd, Da), (Ab, Bd, Ca, Dc), (Ab, Bd, Cc, Da),
(Ac, Ba, Cb Dd), (Ac, Ba, Cd, Db), (Ac, Bb, Ca, Dd),
(Ac, Bb, Cd, Da), (Ac, Bd, Ca, Db), (Ac, Bd, Cb, Da),
(Ad, Ba, Cb, Dc), (Ad, Ba, Cc, Db), (Ad, Bb, Ca, Dc),
(Ad, Bb, Cc, Da), (Ad, Bc, Ca, Db), (Ad, Bc, Cb, Da)}

where 'Aa' means that the letter 'a' is placed in the envelope A.

Let E_1 denote the event in which none of the letters is in the correct envelope.

Then E_1 = {(Ab, Ba, Cd, Dc), (Ab, Bc, Cd, Da), (Ab, Bd, Ca, Dc),
(Ac, Ba, Cd, Db), (Ac, Bd, Ca, Db), (Ac, Bd, Cb, Da),
(Ad, Ba, Cb, Dc), (Ad, Bc, Ca, Db), (Ad, Bc, Cb, Da)}

Let E_2 denote the event in which at least one of the letters is in the correct envelope.

We note that E_2 is the complement of E_1. Therefore E_2 consists of all the elements of S except those in E_1.

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{9}{24} = \frac{3}{8} \text{ and } P(E_2) = 1 - P(E_1) = \frac{5}{8}$$

Example 3 *A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. Two articles are chosen from the lot at random (without replacement). Find the probability that (i) both are good, (ii) both have major defects, (iii) at least 1 is good, (iv) at most 1 is good, (v) exactly 1 is good, (vi) neither has major defects and (vii) neither is good.*

Solution Although the articles may be drawn one after the other, we can consider that both articles are drawn simultaneously, as they are drawn without replacement.

(i) $P(\text{both are good}) = \dfrac{\text{No. of ways drawing 2 good articles}}{\text{Total no. of ways of drawing 2 articles}}$

$$= \frac{10C_2}{16C_2} = \frac{3}{8}$$

(ii) $P(\text{both have major defects})$

$$= \frac{\text{No. of ways of drawing 2 articles with major defects}}{\text{Total no. of ways}}$$

$$= \frac{2C_2}{16C_2} = \frac{1}{120}$$

(iii) $P(\text{at least 1 is good}) = P(\text{exactly 1 is good or both are good})$

$$= P(\text{exactly 1 is good and 1 is bad or both are good})$$

$$= \frac{10C_1 \times 6C_1 + 10C_2}{16C_2} = \frac{7}{8}$$

(iv) $P(\text{atmost 1 is good}) = P(\text{none is good or 1 is good and 1 is bad})$

$$= \frac{10C_0 \times 6C_2 + 10C_1 \times 6C_1}{16C_2} = \frac{5}{8}$$

(v) $P(\text{exactly 1 is good}) = P(\text{1 is good and 1 is bad})$

$$= \frac{10C_1 \times 6C_1}{16C_2} = \frac{1}{2}$$

(vi) $P(\text{neither has major defects})$

$$= P(\text{both are non-major defective articles})$$

$$= \frac{14C_2}{16C_2} = \frac{91}{120}$$

(vii) $P(\text{neither is good}) = P(\text{both are defective})$

$$= \frac{6C_2}{16C_2} = \frac{1}{8}$$

Example 4 *From 6 positive and 8 negative numbers, 4 numbers are chosen at random (without replacement) and multiplied. What is the probability that the product is positive?*

Solution If the product is to be positive, all the 4 numbers must be positive or all the 4 must be negative or 2 of them must be positive and the other 2 must be negative.

No. of ways of choosing 4 positive numbers = $6C_4 = 15$.

No. of ways of choosing 4 negative numbers = $8C_4 = 70$.

No. of ways of choosing 2 positive and 2 negative numbers

$$= 6C_2 \times 8C_2 = 420.$$

Total no. of ways of choosing 4 numbers from all the 14 numbers

$$= 14C_4 = 1001.$$

P(the product is positive)

$$= \frac{\text{No. of ways by which the product is positve}}{\text{Total no. of ways}}$$

$$= \frac{15 + 70 + 420}{1001} = \frac{505}{1001}$$

Example 5 *A box contains tags marked 1, 2, ..., n. Two tags are chosen at random without replacement. Find the probability that the numbers on the tags will be consecutive integers.*

Solution If the numbers on the tags are to be consecutive integers, they must be chosen as a pair from the following pairs.

$$(1, 2); (2, 3); (3, 4); \ldots; (n - 1, n)$$

No. of ways of choosing any one pair from the above $(n - 1)$ pairs = $(n - 1)\, C_1 = n - 1$.

Total No. of ways of choosing 2 tags from the n tags = nC_2.

$\therefore$ Required probability = $\dfrac{n-1}{\dfrac{n(n-1)}{2}} = \dfrac{2}{n}$

Example 6 *If n biscuits are distributed at random among m children, what is the probability that a particular child receives r biscuits, where r < n?*

(MKU — Nov. 96)

Solution The first biscuit can be given to any one of the m children, i.e., in m ways.

Similarly the second biscuit can be given in m ways.

Therefore 2 biscuits can be given in m^2 ways.

Extending, n biscuits can be distributed in m^n ways. The r biscuits received by the particular child can be chosen from the n biscuits in nC_r ways. If this child has got r biscuits, the remaining $(n - r)$ biscuits can be distributed among the remaining $(m - 1)$ children in $(m - 1)^{n-r}$ ways.

$\therefore$ No. of ways of distributing in the required manner

$$= nC_r(m - 1)^{n-r}$$

$\therefore$ Required probability = $\dfrac{nC_r\,(m-1)^{n-r}}{m^n}$

Example 7 *If $P(A) = P(B) = P(AB)$, show that $P\left(A\overline{B} + \overline{A}B\right) = 0$ [$AB \equiv A \cap B$].*

Solution By addition theorem,

$$P(A \cup B) = P(A) + P(B) - P(AB) \tag{1}$$

From the Venn diagram on page (3), it is clear that

$$A \cup B = A\overline{B} + \overline{A}B + AB$$

$$\therefore \quad P(A \cup B) = P(A\overline{B}) + P(\overline{A}B) + P(AB) \text{ (by probability axiom)} \tag{2}$$

Using the given condition in (1),

$$P(A \cup B) = P(AB) \tag{3}$$

From (2) and (3), $P(A\overline{B}) + P(\overline{A}B) = 0$

Example 8 *If A, B and C are any 3 events such that $P(A) = P(B) = P(C) = 1/4$, $P(A \cap B) = P(B \cap C) = 0$; $P(C \cap A) = 1/8$. Find the probability that at least 1 of the events A, B and C occurs.*

Solution P(at least one of A, B and C occurs) $= P(A \cup B \cup C)$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) \tag{1}$$

Since $P(A \cap B) = P(B \cap C) = 0$, $P(A \cap B \cap C) = 0$. Equation (1) becomes

$$P(A \cup B \cup C) = \frac{3}{4} - 0 - 0 - \frac{1}{8} = \frac{5}{8}$$

Example 9 *Solve Example 5, if the tags are chosen at random with replacement.*

Solution If the tag with '1' is chosen in the first draw, the tag with '2' must be chosen in the second draw. Probability for each $= 1/n$.

$\therefore$ P('1' in the first draw and '2' in the second draw) $= 1/n^2$ (Product theorem)

Similarly, P('n' in the first draw and '$n - 1$' in the second draw $= 1/n^2$.

If the number drawn first is '2', the number drawn second may be '1' or '3'.

Probability of drawing consecutive numbers in this case

$$= \frac{1}{n} \times \frac{2}{n} = \frac{2}{n^2}$$

Similarly, when the first number drawn is '3', '4', ..., '$(n - 1)$' probability of drawing consecutive numbers will be $2/n^2$.

All the above possibilities are mutually exclusive.

$$\therefore \quad \text{Required probability} = \frac{1}{n^2} + \frac{1}{n^2} + (n-2) \times \frac{2}{n^2} = \frac{2(n-1)}{n^2}$$

Example 10 *A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?*

Solution Let A = one of the tubes drawn is good and B = the other tube is good.

$$P(A \cap B) = P(\text{both tubes drawn are good})$$

$$= \frac{6C_2}{10C_2} = \frac{1}{3}$$

Knowing that one tube is good, the conditional probability that the other tube is also good is required, i.e., $P(B/A)$ is required.

By definition,

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{6/10} = \frac{5}{9}$$

Example 11 *Two defective tubes get mixed up with 2 good ones. The tubes are tested, one by one, until both defectives are found. What is the probability that the last defective tube is obtained on (i) the second test, (ii) the third test and (iii) the fourth test?*

Solution Let D represent defective and N represent non-defective tube.

(i) P(Second D in the II test) = P(D in the I test and D in the II test)

$$= P(D_1 \cap D_2), \text{ say}$$
$$= P(D_1) \times P(D_2) \text{ (by independence)}$$
$$= \frac{2}{4} \times \frac{1}{3} = \frac{1}{6}$$

(ii) P(second D in the III test) = $P(D_1 \cap N_2 \cap D_3$ or $N_1 \cap D_2 \cap D_3)$

$$= P(D_1 \cap N_2 \cap D_3) + P(N_1 \cap D_2 \cap D_3)$$
$$= \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2}$$
$$= \frac{1}{3}$$

(iii) P(second D in the IV test) = $P(D_1 \cap N_2 \cap N_3 \cap D_4) + P(N_1 \cap D_2 \cap N_3 \cap D_4) + P(N_1 \cap N_2 \cap D_3 \cap D_4)$

$$= \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times 1 + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times 1 + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times 1$$
$$= \frac{1}{2}$$

Example 12 *In a shooting test, the probability of hitting the target is 1/2 for A, 2/3 for B and 3/4 for C. If all of them fire at the target, find the probability that (i) none of them hits the target and (ii) at least one of them hits the target.*

Solution Let $A \equiv$ Event of A hitting the target, and so on.

$$P(\bar{A}) = \frac{1}{2}, P(\bar{B}) = \frac{1}{3}, P(\bar{C}) = \frac{1}{4}$$

$$P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\bar{A}) \times P(\bar{B}) \times P(\bar{C}) \text{ (by independence)}$$

ie., P(none hits the target) $= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} = \frac{1}{24}$

P(at least one hits the target)

$$= 1 - P(\text{none hits the target})$$

$$= 1 - \frac{1}{24} = \frac{23}{24}$$

Example 13 *A and B alternately throw a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, show that his chance of winning is 30/61.* (BU — Apr. 96)

Solution Throwing 6 with 2 dice ≡ Getting 6 as the sum of the numbers shown on the upper faces of the 2 dice.

$$P(\text{throwing 6 with 2 dice}) = \frac{5}{36}$$

$$P(\text{throwing 7 with 2 dice}) = \frac{1}{6}$$

Let $A \equiv$ Event of A throwing 6.

Let $B \equiv$ Event of B throwing 7.

A plays in the first, third, fifth, ..., trials.

Therefore A will win, if he throws 6 in the first trial or third trial or in subsequent (odd) trials.

$$\therefore \quad P(A \text{ wins}) = P(A \text{ or } \bar{A}\,\bar{B}\,A \text{ or } \bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A \text{ or } \ldots)$$

$$= P(A) + P(\bar{A}\,\bar{B}\,A) + P\left(\bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A\right) + \ldots \text{ (Addition theorem)}$$

$$= \frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)\frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)^2 \times \frac{5}{36} + \ldots \text{ upto } \infty$$

$$= \frac{5/36}{1 - (155/216)} \text{ (since the series is an infinite geometric series)}$$

$$= \frac{30}{61}$$

Example 14 *Show that $2^n - (n + 1)$ equations are needed to establish the mutual independence of n events.*

Solution n events are mutually independent, if they are totally independent when considered in sets of 2, 3, ..., n events.

Sets of r events can be chosen from the n events in nC_r ways.

To establish total independence of r events, say, $A_1, A_2, \ldots, A_r$ chosen in any one of the nC_r ways, we need one equation, namely, $P(A_1, A_2, \ldots, A_r) = P(A_1) \times P(A_2) \ldots \times P(A_r)$.

Therefore to establish total independence of all the nC_r sets, each of r events, we need nC_r equations.

Therefore the number of equations required to establish mutual independence

$$= \sum_{r=2}^{n} nC_r$$

$$= (nC_0 + nC_1 + nC_2 + \dots + nC_n) - (1 + n)$$
$$= (1 + 1)^n - (n + 1)$$
$$= 2^n - (n + 1)$$

Example 15 *Two fair dice are thrown independently. Three events A, B and C are defined as follows.*

(a) Odd face with the first die
(b) Odd face with the second die
(c) Sum of the numbers in the 2 dice is odd. Are the events A, B and C mutually independent? (MKU — Apr. 97)

Solution $P(A) = \frac{3}{6} = \frac{1}{2}$; $P(B) = \frac{3}{6} = \frac{1}{2}$

The outcomes favourable to the event C are (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5) and so on.

$$\therefore \quad P(C) = \frac{1}{2}$$

$$P(A \cap B) = P(B \cap C) = P(A \cap C) = \frac{1}{4}$$

$\therefore$ $P(A \cap B) = P(A)\, P(B)$, and so on

But $P(A \cap B \cap C) = 0$, since C cannot happen when A and B occur. Therefore $P(A \cap B \cap C) \neq P(A) \times P(B) \times P(C)$.

Therefore the events are pairwise independent, but not mutually independent.

Example 16 *If A, B and C are random subsets (events) in a sample space and if they are pairwise independent and A is independent of $(B \cup C)$, prove that A, B and C are mutually independent.* (MU — Nov. 96)

Solution Given:
$$P(AB) = P(A) \times P(B) \quad (1)$$
$$P(BC) = P(B) \times P(C) \quad (2)$$
$$P(CA) = P(C) \times P(A) \quad (3)$$
$$P[A\,(B \cup C)] = P(A) \times P(B \cup C) \quad (4)$$

Consider
$$P[A(B \cup C)] = P(AB \cup AC)$$
$$= P(AB) + P(AC) - P(AB \cap AC) \text{ (by addition theorem)}$$
$$= P(A) \times P(B) + P(A) \times P(C) - P(ABC) \text{ [by (1) and (3)]} \quad (5)$$

Therefore from (4) and (5), we get
$$P(ABC) = P(A) \times P(B) + P(A) \times P(C) - P(A) \times P(B \cup C)$$
$$= P(A) \times [P(B) + P(C) - P(B \cup C)]$$
$$= P(A) \times P(B \cap C) \text{ (by addition theorem)}$$
$$= P(A) \times P(B) \times P(C) \text{ [by (2)]} \quad (6)$$

From (1), (2), (3) and (6), the required result follows.

Exercise 1(A)

Part A (Short answer questions)

1. What is a random experiment? Give an example.
2. Give the apriori definition of probability with an example.
3. Give the aposteriori definition of probability with an example.
4. Give the relative frequency definition of probability with an example.
5. Define the sample space and an event associated with a random experiment with an example.
6. Give the axiomatic definition of probability.
7. State the axioms of probability.
8. What do you infer from the statements $P(A) = 0$ and $P(A) = 1$?
9. Define mutually exclusive events with an example.
   (**Example:** Getting an odd number and getting an even number when a 6-faced die is tossed are 2 mutually exclusive events.)
10. From a bag containing 3 red and 2 black balls, 2 balls are drawn at random. Find the probability that they are of the same colour.
11. When 2 cards are drawn from a well-shuffled pack of playing cards, what is the probability that they are of the same suit?
12. When A and B are 2 mutually exclusive events such that $P(A) = 1/2$ and $P(B) = 1/3$, find $P(A \cup B)$ and $P(A \cap B)$.
13. If $P(A) = 0.29$, $P(B) = 0.43$, find $P(A \cap \overline{B})$, if A and B are mutually exclusive.
14. When A and B are 2 mutually exclusive events, are the values $P(A) = 0.6$ and $P(A \cap \overline{B}) = 0.5$ consistent? Why?
15. Prove that the probability of an impossible event is zero (or prove that $P(\phi) = 0$.
16. Prove that $P(\overline{A}) = 1 - P(A)$, where $\overline{A}$ is the complement of A.
17. State addition theorem as applied to any 2 events. Extend it to any 3 events.
18. If $P(A) = 3/4$, $P(B) = 5/8$, prove that $P(A \cap B) \geq 3/8$.
19. A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace?
20. The probability that a contractor will get a plumbing contract is 2/3 and the probability that he will get an electric contract is 4/9. If the probability of getting at least one contract is 4/5, what is the probability that he will get both?
21. If $P(A) = 0.4$, $P(B) = 0.7$ and $P(A \cap B) = 0.3$, find $P\left(\overline{A} \cap \overline{B}\right)$.
22. If $P(A) = 0.35$, $P(B) = 0.75$ and $P(A \cup B) = 0.95$, find $P\left(\overline{A} \cup \overline{B}\right)$.
23. Prove that $P(A \cup B) \leq P(A) + P(B)$. When does the equality hold good?
24. If $B \subset A$, prove that $P(B) \leq P(A)$.
25. Give the definitions of joint and conditional probabilities with examples.
26. Give the definition of conditional probability and deduce the product theorem of probability.

27. If $A \subset B$, prove that $P(B/A) = 1$.
28. If $B \subset A$, prove that $P(B/A) \geq P(B)$.
29. If A and B are mutually exclusive events, prove that $P(B/A) = 0$
30. If $P(A) > P(B)$, prove that $P(A/B) > P(B/A)$.
31. If $A \subset B$, prove that $P(A/C) \leq P(B/C)$.
32. If $P(A) = 1/3$, $P(B) = 3/4$ and $P(A \cup B) = 11/12$, find $P(A/B)$ and $P(B/A)$.
33. When are 2 events said to be independent? Give an example for 2 independent events.
34. What is the probability of getting atleast 1 head when 2 coins are tossed?
35. When 2 dice are tossed, what is the probability of getting 4 as the sum of the face numbers?
36. If the probability that A solves a problem is 1/2 and that for B is 3/4 and if they aim at solving a problem independently, what is the probability that the problem is solved?
37. If $P(A) = 0.65$, $P(B) = 0.4$ and $P(A \cap B) = 0.24$, can A and B be independent events?
38. 15% of a firm's employees are BE degree holders, 25% are MBA degree holders and 5% have both the degrees. Find the probability of selecting a BE degree holder, if the selection is confined to MBAs.
39. In a random experiment, $P(A) = 1/12$, $P(B) = 5/12$ and $P(B/A) = 1/15$, find $P(A \cup B)$.
40. What is the difference between total independence and mutual independence?
41. Can 2 events be simultaneously independent and mutually exclusive? Explain.
42. If A and B are independent events, prove that $\overline{A}$ and B are also independent.
43. If A and B are independent events, prove that A and $\overline{B}$ are also independent.
44. If $P(A) = 0.5$, $P(B) = 0.3$ and $P(A \cap B) = 0.15$, find $P(A/\overline{B})$.
45. If A and B are independent events, prove that $\overline{A}$ and $\overline{B}$ are also independent.
46. If A and B are independent events, prove that
$$P(A \cup B) = 1 - P(\overline{A}) \times P(\overline{B}).$$
47. A and B toss a fair coin alternately with the understanding that the one who obtains the head first wins. If A starts, what is his chance of winning?

Part B

48. Write the sample space associated with the experiment of tossing 3 coins at a time and the event of getting heads from the first 2 coins. Also find the corresponding probability.

49. Items coming off a production line are marked defective (D) or non-defective (N). Items are observed and their condition listed. This is continued until 2 consecutive defectives are produced or 4 items have been checked, whichever occurs first. Describe a sample space for this experiment.
50. An urn contains 2 white and 4 black balls. Two balls are drawn one by one without replacement. Write the sample space corresponding to this experiment and the subsets corresponding to the following events.
    (a) The first ball drawn is white.
    (b) Both the balls drawn are black.
    Also find the probabilities of the above events.
51. A box contains three 10-Ω resistors labelled R_1, R_2 and R_3 and two 50-Ω resistors labelled R_4 and R_5. Two resistors are drawn from this box without replacement. List all the outcomes of this random experiment as pairs of resistors. Also list the outcomes associated with the following events and hence find the corresponding probabilities.
    (a) Both the resistors drawn are 10-Ω resistors.
    (b) One 10-Ω resistor and one 50-Ω resistor are drawn.
    (c) One 10-Ω resistor is drawn in the first draw and one 50-Ω resistor is drawn in the second draw.
52. A box contains 3 white balls and 2 black balls. We remove at random 2 balls in succession. What is the probability that the first removed ball is white and the second is black? (BDU — Apr. 96)
53. An urn contains 3 white balls, 4 red balls and 5 black balls. Two balls are drawn from the urn at random. Find the probability that (i) both of them are of the same colour and (ii) they are of different colours.
54. One integer is chosen at random from the numbers 1, 2, 3, ..., 100. What is the probability that the chosen number is divisible by (i) 6 or 8 and (ii) 6 or 8 or both?
55. If there are 4 persons A, B, C and D and if A tossed with B, then C tossed with D and then the winners tossed. This process continues till the prize is won. What are the probabilities of each of the 4 to win? (MKU — Nov. 96)
56. Ten chips numbered 1 through 10 are mixed in a bowl. Two chips are drawn from the bowl successively and without replacement. What is the probability that their sum is 10?
57. A bag contains 10 tickets numbered 1, 2, ..., 10. Three tickets are drawn at random and arranged in ascending order of magnitude. What is the probability that the middle number is 5?
58. Two fair dice are thrown independently. Four events A, B, C and D are defined as follows:
    A: Even face with the first dice.
    B: Even face with the second dice.
    C: Sum of the points on the 2 dice is odd.

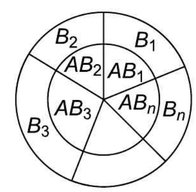
D: Product of the points on the 2 dice exceeds 20.
Find the probabilities of the 4 events.

59. A box contains 4 white, 5 red and 6 black balls. Four balls are drawn at random from the box. Find the probability that among the balls drawn, there is at least 1 ball of each colour.
60. Four persons are chosen at random from a group consisting of 4 men, 3 women and 2 children. Find the chance that the selected group contains at least 1 child.
61. A committee of 6 is to be formed from 5 lecturers and 3 professors. If the members of the committee are chosen at random, what is the probability that there will be a majority of lecturers in the committee?
62. Twelve balls are placed at random in 3 boxes. What is the probability that the first box will contain 3 balls?
63. If A and B are any 2 events, show that $P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$. (MU — Apr. 96)
64. A and B are 2 events associated with an experiment. If $P(A) = 0.4$ and $P(A \cup B) = 0.7$, find $P(B)$ if (i) A and B are mutually exclusive (ii) A and B are independent.
65. If $P(A + B) = 5/6$, $P(AB) = 1/3$ and $P(\overline{B}) = 1/2$, prove that the events A and B are independent.
66. If $A \subset B$, $P(A) = 1/4$ and $P(B) = 1/3$, find $P(A/B)$ and $P(B/A)$.
67. m objects are selected from n objects ($m < n$). What is the probability that the selection contains a particular object that was present in the n given objects?
68. What is the probability that there will be 53 sundays in (i) a leap year and (ii) a non-leap year?
69. If the probability that a communication system has high selectivity is 0.54 and the probability that it will have high fidelity is 0.81 and the probability that it will have both is 0.18, find the probability that (i) a system with high fidelity will also have high selectivity and (ii) a system with high selectivity will also have high fidelity.
70. An electronic assembly consists of two subsystems A and B. From previous testing procedures, the following probabilities are assumed to be known: $P(A \text{ fails}) = 0.20$, $P(A \text{ and } B \text{ fail}) = 0.15$ and $P(B \text{ fails alone}) = 0.15$. Evaluate (i) $P(A \text{ fails alone})$ and (ii) $P(A \text{ fails}/B \text{ has failed})$.
71. A consignment of 15 tubes contains 4 defectives. The tubes are selected at random, one by one, and examined. Assuming that the tubes tested are not put back, what is the probability that the ninth one examined is the last defective? (BDU — Apr. 97)
72. A card is drawn from a 52-card deck, and without replacing it, a second card is drawn. The first and second cards are not replaced and a third card is drawn.
    (a) If the first card is a heart, what is the probability of second card being a heart?

(b) If the first and second cards are hearts, what is the probability that the third card is the king of clubs? (BU — Nov. 96)

73. A pair of dice are rolled once. Let A be the event that the first die has a 1 on it, B the event that the second die has a 6 on it and C the event that the sum is 7. Are A, B and C independent? (BDU — Nov. 96)
74. A problem is given to 3 students whose chances of solving it are 1/2, 1/3 and 1/4. What is the probability that (i) only one of them solves the problem and (ii) the problem is solved. (MU — Nov. 96)
75. A and B alternately cut a pack of cards and the pack is shuffled after each cut. If A starts and the game is continued until one cuts a diamond, what is the chance that A wins at his second cut?
76. Players X and Y roll a pair of dice alternately. The player who rolls 11 first wins. If X starts, find his chance of winning.
77. Three persons A, B and C draw in succession from a bag containing 8 red and 4 white balls until a white ball is drawn. What is the probability that C draws the white ball?

Theorem of Total Probability



If $B_1, B_2, \ldots, B_n$ be a set of exhaustive and mutually exclusive events, and A is another event associated with (or caused by) B_i, then

$$P(A) = \sum_{i=1}^{n} P(B_i)\, P(A/B_i)$$

Proof The inner circle represents the event A. A can occur along with (or due to) $B_1, B_2, \ldots, B_n$ that are exhaustive and mutually exclusive.

$\therefore$ $AB_1, AB_2, \ldots, AB_n$ are also mutually exclusive.

$\therefore$ $A = AB_1 + AB_2 + \ldots + AB_n$ (by addition theorem)

$\therefore$ $P(A) = P(\Sigma\, AB_i)$

$= \Sigma\, P(AB_i)$ (since $AB_1, AB_2, \ldots, AB_n$ are mutually exclusive)

$$= \sum_{i=1}^{n} P(B_i) \times P(A/B_i)$$

Bayes' Theorem or Theorem of Probability of Causes

If $B_1, B_2, \ldots, B_n$ be a set of exhaustive and mutually exclusive events associated with a random experiment and A is another event associated with (or caused by) B_i, then

$$P(B_i/A) = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}, \quad i = 1, 2, \ldots, n$$

Proof $P(B_i \cap A) = P(B_i) \times P(A/B_i) = P(A) \times P(B_i/A)$

$$\therefore \quad P(B_i/A) = \frac{P(B_i) \times P(A/B_i)}{P(A)}$$

$$= \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}, \; i = 1, 2, 3, \ldots, n$$

Worked Examples 1(B)

Example 1 *A bolt is manufactured by 3 machines A, B and C. A turns out twice as many items as B, and machines B and C produce equal number of items. 2% of bolts produced by A and B are defective and 4% of bolts produced by C are defective. All bolts are put into 1 stock pile and 1 is chosen from this pile. What is the probability that it is defective?*

Solution Let A = the event in which the item has been produced by machine A, and so on.

Let D = the event of the item being defective.

$$P(A) = \frac{1}{2}, \; P(B) = P(C) = \frac{1}{4}$$

$$P(D/A) = P(\text{an item is defective, given that } A \text{ has produced it})$$

$$= \frac{2}{100} = P(D/B)$$

$$P(D/C) = \frac{4}{100}$$

By theorem of total probability,

$$P(D) = P(A) \times P(D/A) + P(B) \times P(D/B) + P(C) \times P(D/C)$$

$$= \frac{1}{2} \times \frac{2}{100} + \frac{1}{4} \times \frac{2}{100} + \frac{1}{4} \times \frac{4}{100}$$

$$= \frac{1}{40}$$

Example 2 *An urn contains 10 white and 3 black balls. Another urn contains 3 white and 5 black balls. Two balls are drawn at random from the first urn and placed in the second urn and then 1 ball is taken at random from the latter. What is the probability that it is a white ball?*

Solution The two balls transferred may be both white or both black or 1 white and 1 black.

Let B_1 = event of drawing 2 white balls from the first urn, B_2 = event of drawing 2 black balls from it and B_3 = event of drawing 1 white and 1 black ball from it.

Clearly B_1, B_2 and B_3 are exhaustive and mutually exclusive events.

Let A = event of drawing a white ball from the second urn after transfer.

$$P(B_1) = \frac{10C_2}{13C_2} = \frac{15}{26}; \; P(B_2) = \frac{3C_2}{13C_2} = \frac{1}{26}; \; P(B_3) = \frac{10 \times 3}{13C_2} = \frac{10}{26},$$

$$\begin{aligned} P(A/B_1) &= P(\text{drawing a white ball/2 white balls have been transferred}) \\ &= P(\text{drawing a white ball/urn II contains 5 white and 5 black balls}) \\ &= \frac{5}{10} \end{aligned}$$

Similarly, $P(A/B_2) = \dfrac{3}{10}$ and $P(A/B_3) = \dfrac{4}{10}$.

By theorem of total probability,

$$\begin{aligned} P(A) &= P(B_1) \times P(A/B_1) + P(B_2) \times P(A/B_2) + P(B_3) \times P(A/B_3) \\ &= \frac{15}{26} \times \frac{5}{10} + \frac{1}{26} \times \frac{3}{10} + \frac{10}{26} \times \frac{4}{10} \\ &= \frac{59}{130} \end{aligned}$$

Example 3 *In a coin tossing experiment, if the coin shows head, 1 dice is thrown and the result is recorded. But if the coin shows tail, 2 dice are thrown and their sum is recorded. What is the probability that the recorded number will be 2?*

(BDU — Apr. 96)

Solution When a single dice is thrown, $P(2) = \dfrac{1}{6}$.

When 2 dice are thrown, the sum will be 2, only if each die shows 1.

$\therefore$ $P(\text{getting 2 as sum with 2 dice}) = \dfrac{1}{6} \times \dfrac{1}{6} = \dfrac{1}{36}$ (since independence)

By theorem of total probability,

$$\begin{aligned} P(2) &= P(H) \times P(2/H) + P(T) \times P(2/T) \\ &= \frac{1}{2} \times \frac{1}{6} + \frac{1}{2} \times \frac{1}{36} \\ &= \frac{7}{72} \end{aligned}$$

Example 4 *A bag contains 5 balls and it is not known how many of them are white. Two balls are drawn at random from the bag and they are noted to be white. What is the chance that all the balls in the bag are white?*

Solution Since 2 white balls have been drawn out, the bag must have contained 2, 3, 4 or 5 white balls.

Let B_1 = Event of the bag containing 2 white balls, B_2 = Event of the bag containing 3 white balls, B_3 = Event of the bag containing 4 white balls and B_4 = Event of the bag containing 5 white balls.

Let A = Event of drawing 2 white balls.

$$P(A/B_1) = \frac{2C_2}{5C_2} = \frac{1}{10},\ P(A/B_2) = \frac{3C_2}{5C_2} = \frac{3}{10}$$

$$P(A/B_3) = \frac{4C_2}{5C_2} = \frac{3}{5},\ P(A/B_4) = \frac{5C_2}{5C_2} = 1$$

Since the number of white balls in the bag is not known, B_i's are equally likely.

$\therefore$ $$P(B_1) = P(B_2) = P(B_3) = P(B_4) = \frac{1}{4}$$

By Bayes' theorem,

$$P(B_4/A) = \frac{P(B_4) \times P(A/B_4)}{\sum_{i=1}^{4} P(B_i) \times P(A/B_i)} = \frac{\frac{1}{4} \times 1}{\frac{1}{4} \times \left(\frac{1}{10} + \frac{3}{10} + \frac{3}{5} + 1\right)} = \frac{1}{2}$$

Example 5 *There are 3 true coins and 1 false coin with 'head' on both sides. A coin is chosen at random and tossed 4 times. If 'head' occurs all the 4 times, what is the probability that the false coin has been chosen and used?*

Solution $P(T) = P(\text{the coin is a true coin}) = \frac{3}{4}$

$P(F) = P(\text{the coin is a false coin}) = \frac{1}{4}$

Let A = Event of getting all heads in 4 tosses

Then $P(A/T) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$ and $P(A/F) = 1$.

By Bayes' theorem,

$$P(F/A) = \frac{P(F) \times P(A/F)}{P(F) \times P(A/F) + P(T) \times P(A/T)}$$

$$= \frac{\frac{1}{4} \times 1}{\frac{1}{4} \times 1 + \frac{3}{4} \times \frac{1}{16}} = \frac{16}{19}$$

Example 6 *For a certain binary, communication channel, the probability that a transmitted '0' is received as a '0' is 0.95 and the probability that a transmitted '1' is received as '1' is 0.90. If the probability that a '0' is transmitted is 0.4, find the probability that (i) a '1' is received and (ii) a '1' was transmitted given that a '1' was received.*

Solution Let A = the event of transmitting '1', $\overline{A}$ = the event of transmitting '0', B = the event of receiving '1' and, $\overline{B}$ = the event of receiving '0'.

Given: $P(\overline{A}) = 0.4$, $P(B/A) = 0.9$ and $P(\overline{B}/\overline{A}) = 0.95$

$\therefore$ $P(A) = 0.6$ and $P(B/\overline{A}) = 0.05$

By the theorem of total probability

$$P(B) = P(A) \times P(B/A) + P(\overline{A}) \times P(B/\overline{A})$$
$$= 0.6 \times 0.9 + 0.4 \times 0.05$$
$$= 0.56$$

By Bayes' theorem,

$$P(A/B) = \frac{P(A) \times P(B/A)}{P(B)} = \frac{0.6 \times 0.9}{0.56} = \frac{27}{28}$$

Exercise 1(B)

Part A (Short answer questions)

1. State the theorem of total probability.
2. Bag I contains 2 red and 1 black balls and bags II contains 3 red and 2 black balls. What is the probability that a ball drawn from one of the bags is red?
3. State Bayes' theorem on inverse probability.
4. Bag I contains 2 white and 3 black balls and bag II contains 4 white and 1 black balls. A ball chosen at random from one of the bags is white. What is the probability that it has come from bag I?
5. Five men out of 100 and 25 women out of 1000 are colour-blind. A colour-blind person is chosen at random. What is the probability that the person is a male? (Assume males and females are in equal numbers).

Part B

6. There are 2 bags one of which contains 5 red and 8 black balls and the other 7 red and 10 black balls. A ball is drawn from one or the other of the 2 bags. Find the chance of drawing a red ball.
7. In a bolt factory, machines A, B and C produce 25, 35 and 40% of the total output, respectively. Of their outputs, 5, 4 and 2%, respectively, are defective bolts. If a bolt is chosen at random from the combined output, what is the probability that it is defective? If a bolt chosen at random is found to be defective, what is the probability that it was produced by B or C? (MU — Apr. 96)
8. A box contains 2000 components of which 5% are defective. A second box contains 500 components of which 40% are defective. Two other boxes contain 1000 components, each with 10% defective components. We select at random one of the above boxes and remove from it at random a single component.
   (a) What is the probability that the component is defective?
   (b) Finding that the selected component is defective, what is the probability that it was drawn from box 2? (MSU — Apr. 96)

9. There are 4 candidates for the office of the highway commissioner; the respective probabilities that they will be selected are 0.3, 0.2, 0.4 and 0.1, and the probabilities for a project's approval are 0.35, 0.85, 0.45 and 0.15, depending on which of the 4 candidates is selected. What is the probability of the project getting approved? (MKU — Apr. 97)
10. In a binary communication system a '0' or '1' is transmitted. Because of noise in the system, a '0', can be received as a '1' with probability p and a '1' can be received as a '0' also with probability p. Assuming that the probability that a '0' is transmitted is p_0, and that a '1' is transmitted is q_0 $(= 1 - p_0)$ find the probability that a '1' was transmitted when a '1' is received.
11. A bag contains 7 red and 3 black marbles, and another bag contains 4 red and 5 black marbles. One marble is transferred from the first bag into the second bag and then a marble is taken out of the second bag at random. If this marble happens to be red, find the probability that a black marble was transferred.
12. The probability that a student passes a certain exam is 0.9, given that he studied. The probability that he passes the exam without studying is 0.2. Assume that the probability that the student studies for an exam is 0.75. Given that the student passed the exam, what is the probability that he studied?
13. Urn I has 2 white and 3 black balls, urn II has 4 white and 1 black balls and urn III has 3 white and 4 black balls. An urn is selected at random and a ball drawn at random is found to be white. Find the probability that urn I was selected. (MKU — Apr. 96)
14. Suppose that coloured balls are distributed in 3 boxes as follows:

	Box 1	*Box 2*	*Box 3*
Red	2	4	3
White	3	1	4
Blue	5	3	5

A box is selected at random from which a ball is selected at random and it is observed to be red. What is the probability that box 3 was selected? (MU — Nov. 96)
15. Three urns contain 3 white, 1 red and 1 black balls; 2 white, 3 red and 4 black balls; 1 white, 3 red and 2 black balls respectively. One urn is chosen at random and from it 2 balls are drawn at random. If they are found to be 1 red and 1 black ball, what is the probability that the first urn was chosen?
16. An urn contains 10 red and 3 black balls. Another urn contains 3 red and 5 black balls. Two balls are transferred from the first urn to the second urn, without noticing their colour. One ball is now drawn from the second urn and it is found to be red. What is the probability that 1 red and 1 black ball were transferred?

17. Box 1 contains 1000 bulbs of which 10% are defective. Box 2 contains 2000 bulbs of which 5% are defective. Two bulbs are drawn (without replacement) from a randomly selected box. (i) Find the probability that both bulbs are defective and (ii) assuming that both are defective, find the probability that they came from box 1.
18. The chance that a doctor A will diagnose a disease x correctly is 60%. The chance that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of doctor A, who had disease x, died. What is the chance that his disease was diagnosed correctly?
19. The chances of A, B and C becoming the general manager of a certain company are in the ratio 4:2:3. The probabilities that the bonus scheme will be introduced in the company if A, B and C become general manager are 0.3, 0.7 and 0.8 respectively. If the bonus scheme has been introduced, what is the probability that A has been appointed as general manager?

Bernoulli's Trials

Let us consider n independent repetitions (trials) of a random experiment E. If A is an event associated with E such that $P(A)$ remains the same for the repetitions, the trials are called *Bernoulli's trials.*

Theorem If the probability of occurrence of an event (probability of success) in a single trial of a Bernoulli's experiment is p, then the probability that the event occurs exactly r times out of n independent trials is equal to $nC_r\, q^{n-r} p^r$, where $q = 1 - p$, the probability of failure of the event.

Proof Getting exactly r successes means getting r sucesses and $(n - r)$ failures simultaneously.

$\therefore$ P(getting r successes and $n - r$ failures) $= p^r q^{n-r}$ (since the n trials are independent) (by product theorem).

The trials, from which the successes are obtained, are not specified. There are nC_r ways of choosing r trials for successes. Once the r trials are chosen for successes, the remaining $(n - r)$ trials should result in failures.

These nC_r ways are mutually exclusive. In each of these nC_r ways, P(getting exactly r successes) $= p^r q^{n-r}$.

Therefore, by the addition theorem, the required probability $= nC_r \times q^{n-r} \times p^r$.

De Moivre–Laplace Approximation

A result which is useful when a large number of terms of the form $nC_r\, q^{n-r} p^r$ is required to be summed up, is given below without proof.

If the probability of getting exactly r successes out of n Bernoulli's trials is denoted by $P_n(r)$, then

$$\sum_{r=r_1}^{r_2} P_n(r) = \sum_{r=r_1}^{r_2} nC_r\, q^{n-r} p^r \text{ is approximately equal to } \int_{r_1-\frac{1}{2}}^{r_2+\frac{1}{2}} y\, dx,$$

where $y = \frac{1}{\sqrt{2\pi npq}} e^{-(x-np)^2/2npq,}$ which is the density of a normal distribution with mean np and variance npq.

As the reader is familiar with normal distribution, it can be easily seen that

$$\sum_{r=r_1}^{r_2} P_n(r) = \int_{t_1}^{t_2} \phi(t)dt$$

where $\phi(t)$ is the standard normal density and $t_1 = \frac{r_1 - np - 1/2}{\sqrt{npq}}$ and $t_2 = \frac{r_2 - np + 1/2}{\sqrt{npq}}$. Now $\int_{t_1}^{t_2} \phi(t)dt$ can be computed using the Table of areas under normal curve.

Generalisation of Bernoulli's Theorem Multinomial Distribution

If $A_1, A_2, \dots, A_k$ are exhaustive and mutually exclusive events associated with a random experiment such that $P(A_i$ occurs$) = p_i$, where $p_1 + p_2 + \dots + p_k = 1$, and if the experiment is repeated n times, then the probability that A_1 occurs r_1 times, A_2 occurs r_2 times, …, A_k occurs r_k times is given by

$$P_n(r_1, r_2, \dots, r_k) = \frac{n!}{r_1!\, r_2! \dots r_k!}\, p_1^{r_1} \times p_2^{r_2} \dots p_k^{r_k}$$

where $r_1 + r_2 + \dots + r_k = n$.

Proof The r_1 trials in which the event A_1 occurs can be chosen from the n trials in nC_{r_1} ways. The remaining $(n - r_1)$ trials are left over for the other events.

The r_2 trials in which the event A_2 occurs can be chosen from the $(n - r_1)$ trials in $(n - r_1)C_{r_2}$ ways.

The r_3 trials in which the event A_3 occurs can be chosen from the $(n - r_1 - r_2)$ trials in $(n - r_1 - r_2)C_{r_3}$ ways, and so on.

Therefore the number of ways in which the events $A_1, A_2, \dots, A_k$ can happen $= nC_{r_1} \times (n - r_1)C_{r_2} \times (n - r_1 - r_2)C_{r_3} \times \dots \times (n - r_1 - r_2 \dots - r_{k-1})C_{r_k}$ $= \frac{n!}{r_1!\, r_2! \dots r_k!}$.

Consider any one of the above ways in which the events $A_1, A_2, \dots, A_k$ occur.

Since the n trials are independent, $r_1, r_2, \dots, r_k$ trials are also independent.

$\therefore$ $P(A_1$ occurs r_1 times$) = p_1^{r1}$

$P(A_2$ occurs r_2 times$) = p_2^{r_2}$, and so on.

$\therefore$ $P(A_1$ occurs r_1 times, A_2 occurs r_2 times, …, A_k occurs r_k times$) = p_1^{r1} \times p_2^{r2} \times \dots \times p_k^{rk}$

Since the ways in which the events happen are mutually exclusive, the required probability is given by

$$P_n(r_1, r_2, \dots, r_k) = \frac{n!}{r_1!\, r_2! \dots r_k!}\, p_1^{r1} \times p_2^{r2} \times \dots \times p_k^{rk}$$

Worked Examples 1(C)

Example 1 *A coin is tossed an infinite number of times. If the probability of a head in a single toss is p, show that the probability that kth head is obtained at the nth tossing, but not earlier is* $(n-1)C_{k-1}\,p^k q^{n-k}$, *where* $q = 1 - p$.

Solution k heads should be obtained at the nth tossing, but not earlier.

Therefore $(k-1)$ heads must be obtained in the first $(n-1)$ tosses and 1 head must be obtained at the nth toss.

$$\therefore \quad \text{Required probability} = P[k-1 \text{ heads in } (n-1) \text{ tosses}] \times P(1 \text{ head in } 1 \text{ toss})$$

$$= (n-1)C_{k-1}p^{k-1}q^{n-k} \times p$$

$$= (n-1)C_{k-1}p^k q^{n-k}$$

Example 2 *Each of two persons A and B tosses 3 fair coins. What is the probability that they obtain the same number of heads?*

Solution $P(A$ and B get the same no. of heads)

$= P$(they get no head each or 1 head each or 2 heads each or 3 heads each)

$= P$(each gets 0 head) + P(each gets 1 head) + P(each gets 2 heads) + P(each gets 3 heads) (since the events are mutually exclusive)

$= P(A$ gets 0 head) $\times$ $P(B$ gets 0 head) + … (since A and B toss independently)

$$= \left[3C_0\left(\frac{1}{2}\right)^3\right]^2 + \left[3C_1\left(\frac{1}{2}\right)^3\right]^2 + \left[3C_2\left(\frac{1}{2}\right)^3\right]^2 + \left[3C_3\left(\frac{1}{2}\right)^3\right]^2$$

$$= \frac{1}{64}(1+9+9+1) = \frac{5}{16}$$

Example 3 *A coin with P(head) = p = 1 – q is tossed n times. Show that the probability that the number of heads obtained is even is* $0.5[1 + (q-p)^n]$.

Solution P(even no. of heads are obtained)

$$= P(0 \text{ head or } 2 \text{ heads or } 4 \text{ heads or } \ldots)$$

$$= nC_0q^np^0 + nC_2q^{n-2}p^2 + nC_4q^{n-4}p^4 + \ldots \tag{1}$$

Consider

$$1 = (q+p)^n = nC_0q^n p^0 + nC_1q^{n-1}p^1 + nC_2q^{n-2}p^2 + \ldots \tag{2}$$

$$(q-p)^n = nC_0q^np^0 - nC_1q^{n-1}p^1 + nC_2q^{n-2}p^2 - \ldots \tag{3}$$

Adding (2) and (3), we get

$$1 + (q-p)^n = 2[nC_0q^np^0 + nC_2q^{n-2}p^2 + nC_4q^{n-4}p^4 + \ldots] \tag{4}$$

Using (4) in (1), the required probability $= 0.5[1 + (q-p)^n]$.

Example 4 *If at least 1 child in a family with 2 children is a boy, what is the probability that both children are boys?* (MKU — Apr. 96)

Solution p = Probability that a child is a boy $= \dfrac{1}{2}$

$\therefore \qquad q = \frac{1}{2}$ and $n = 2$

P(at least one boy) = p(exactly 1 boy) + p(exactly 2 boys)

$$= 2C_1\left(\frac{1}{2}\right)^2 + 2C_2\left(\frac{1}{2}\right)^2 \qquad \text{(by Bernoulli's theorem)}$$

$$= \frac{3}{4}$$

P(both are boys/at least one is a boy)

$$= \frac{P(\text{both are boys} \cap \text{at least one is a boy})}{P(\text{at least one is a boy})}$$

$$= \frac{P(\text{both are boys})}{P(\text{at least one is a boy})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Example 5 *Find the probability of getting at least 60 heads, when 100 fair coins are tossed.*

Solution Since the coins are fair, $p = q = \frac{1}{2}$.

$n = 100$

$\therefore \qquad np = 50$ and $\sqrt{npq} = 5$

Required probability $P = \sum_{r=60}^{100} 100C_r\left(\frac{1}{2}\right)^{100}$

$$= \int_{1.9}^{10.1} \phi(t)\mathrm{d}t \simeq \int_{1.9}^{\infty} \phi(t)dt \text{ (by Demoivre–Laplace approximation)}$$

$$= 0.5 - \int_{0}^{1.9} \phi(t)dt = 0.5 - 0.4719 \text{ (from the Normal Table)}$$

$$= 0.0281$$

Example 6 *A fair dice is rolled 5 times. Find the probability that 1 shows twice, 3 shows twice and 6 shows once.*

Solution This problem is a direct application of Bernoulli's generalisation.

Here $n = 5$, $A_1 \equiv$ (getting 1), $A_3 \equiv$ (getting 3)

$A_6 =$ (getting 6), $k_1 = 2$, $k_3 = 2$ and $k_6 = 1$; $p_1 = p_3 = p_6 = 1/6$

$\therefore \qquad$ Required probability $= \frac{5!}{2!2!1!}\left(\frac{1}{6}\right)^2 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{1}{6}\right)^1$

$$= 0.0039$$

(It is to be noted that $k_2 = k_4 = k_5 = 0$, though $p_2 = p_4 = p_5 = \frac{1}{6}$.)

Exercise 1(C)

Part A (Short answer questions)

1. State Bernouilli's theorem on independent trials.
2. A fair coin is tossed 4 times. What is the probability of getting more heads than tails?
3. When 12 coins are tossed 256 times, how many times may one expect 8 heads and 4 tails?
4. If war breaks out on the average once in 25 years, find the probability that in 50 years at a stretch, there will be no war.
5. State the generalised form of Bernoulli's theorem on independent trials.
6. State De Moivre–Laplace theorem.

Part B

7. A binary number (composed only of the digits '0' and '1') is made up of n digits. If the probability of an incorrect digit appearing is p and that errors in different digits are independent of one another, find the probability of forming an incorrect number.
8. Suppose that twice as many items are produced (per day) by machine 1 as by machine 2. However 4% of the items from machine 1 are defective while machine 2 produces only about 2% defectives. Suppose that the daily output of the 2 machines is combined. A random sample of 10 items is taken from the combined output. What is the probability that this sample contains 2 defectives?
9. Binary digits are transmitted over a noisy communication channel in blocks of 16 binary digits. The probability that a received binary digit is in error because of channel noise is 0.1. If errors occur in various digit positions within a block independently, find the probability that the number of errors per block is greater than or equal to 5.
10. A company is trying to market a digital transmission system (modem) that has a bit error probability of 10^{-4}, and the bit errors are independent. The buyer will test the modem by sending a known message of 10^4 digits and checking the received message. If more than 2 errors occur, the modem will be rejected. Find the probability that the customer will buy the company's modem.
11. A fair coin is tossed 10,000 times. Find the probability that the number of heads obtained is between 4900 and 5100, using DeMoivre-Laplace approximation.
12. Over a period of 12h, 180 calls are made at random. What is the 'probability' that in a 4 h interval the number of calls is between 50 and 70? Use DeMoivre-Laplace approximation.

$$\left[\textbf{Hint: } P(\text{a particular call occurs in the 4-h interval}) = p = \frac{4}{12} = \frac{1}{3}\right].$$

13. A random experiment can terminate in one of 3 events A, B and C with probabilities 1/2, 1/4 and 1/4 respectively. The experiment is repeated 6 times. Find the probability that the events A, B and C occur once, twice and thrice respectively.
14. A throws 3 fair coins and B throws 4 fair coins. Find the chance that A will throw more number of heads than would B.
15. In a large consignment of electric bulbs 10% are defective. A random sample of 20 bulbs is taken for inspection. Find the probability that (i) exactly 3 of them are defective, (ii) at most 3 of them are defective and (iii) at least 3 of them are defective.
16. A lot contains 1% defective items. What should be the number of items in a random sample so that the probabihty of finding at least 1 defective in it is at least 0.95?
17. In a precision bombing attack there is a 50% chance that any one bomb will hit the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% or more chance of completely destroying the target?

Answers

Exercise 1(A)

10. P(both balls are of the same colour)
$= P$(both balls are red or both are black)
$= P$(both are red) + P(both are black)
$= \dfrac{3C_2}{5C_2} + \dfrac{2C_2}{5C_2} = \dfrac{2}{5}$
11. Required probability = P(both are spades) + P(both are clubs) + P(both are hearts) + P(both are diamonds)
$= 4 \times \dfrac{13C_2}{52C_2} = \dfrac{4}{17}$
12. $P(A \cup B) = P(A) + P(B) = 5/6$; $P(A \cap B) = 0$, by definition.
13. When A and B are mutually exclusive, $P(A \cap \overline{B}) = P(A) = 0.29$
14. $P(A) = P(A \cap B) + P(A \cap \overline{B}) = 0 + P(A \cap \overline{B})$ i.e., $0.6 = 0.5$, which is absurd. Hence the given values are inconsistent.
18. P(any event) ≤ 1; $P(A \cup B) \le 1$; $P(A) + P(B) - P(A \cap B) \le 1$
$\therefore P(A \cap B) \ge \dfrac{3}{4} + \dfrac{5}{8} - 1 \left(= \dfrac{3}{8}\right)$
19. $P(S \cup A) = P(S) + P(A) - P(S \cap A)$
$= \dfrac{13}{52} + \dfrac{4}{52} - \dfrac{1}{52} = \dfrac{4}{13}$
20. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore P(A \cap B) = \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{14}{45}$

21. $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$
$$= 1 - [0.4 + 0.7 - 0.3] = 0.2$$

22. $P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B)$

$= 1 - [P(A) + P(B) - P(A \cup B)] = 0.85$

23. Equality holds good, when A and B are mutually exclusive events.
25. The probability for the simultaneous occurrence of two events A and B is called the joint probability of A and B. Probability of getting 2 heads, when 2 coins are tossed, is an example of joint probability.
32. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$; $P(A \cap B) = \frac{1}{3} + \frac{3}{4} - \frac{11}{12} = \frac{1}{6}$

$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{2}{9}$; $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{2}$

34. Required probability= $1 - P$(both tails)

$$= 1 - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$$

35. Required probability = P (1 from I dice and 3 from II dice or 2 from 1 and 2 from II or 3 from I and 1 from II)

$= \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} = \frac{1}{12}$

36. Required probability = $1 - P$ (the problem is not solved)

$$= 1 - \frac{1}{2} \times \frac{1}{4} = \frac{7}{8}$$

37. No, since $P(A \cap B) \neq P(A) \times P(B)$
38. $P(BE/MBA) = \frac{P(BE \cap MBA)}{P(MBA)} = \frac{0.05}{0.25} = 0.2$
39. $P(A \cap B) = P(A) \times P(B/A) = \frac{1}{12} \times \frac{1}{15} = \frac{1}{180}$

$P(A \cup \mathrm{B}) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{12} + \frac{5}{12} - \frac{1}{180} = \frac{89}{180}$

41. A and B are independent, if $P(A \cap B) = P(A) \times P(B)$. They are mutually exclusive, if $P(A \cap B) = 0$.
They are both independent and mutually exclusive if $P(A) \times PB) = 0$, i.e., if $P(A) = 0$ or $P(B) = 0$ or $P(A) = 0$ and $P(B) = 0$. The third case is trivial. Hence A and B can be both independent and mutually exclusive, provided either of the events is an impossible event.

44. $P(A \cap B) = P(A) \times P(B)$

$\therefore$ A and B are independent. Hence A and $\overline{B}$ are also independent.

$\therefore$ $P(A/\overline{B}) = P(A) = 0.5$.

47. $P = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^4 + \ldots + \infty = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}$

48. $\frac{1}{4}$

49. $\{DD, NDD, DNDD, DNDN, DNND, DNNN, NDND, NDNN, NNDD, NNDN, NNND, NNNN\}$

50. (i) $\frac{1}{3}$ (ii) $\frac{2}{5}$

51. (i) $\frac{3}{10}$ (ii) $\frac{3}{5}$ (iii) $\frac{3}{10}$

52. $\frac{6}{25}$ (if the ball is replaced), $\frac{3}{10}$ (if the ball is not replaced).

53. (i) $\frac{19}{66}$ (ii) $\frac{47}{66}$

54. (i) $\frac{1}{5}$ (ii) $\frac{6}{25}$

55. $\frac{1}{4}$ 56. $\frac{4}{45}$ 57. $\frac{1}{6}$

58. $P(A) = \frac{1}{2} = P(B) = P(C)$; $P(D) = \frac{1}{6}$

59. $\frac{48}{91}$ 60. $\frac{13}{18}$ 61. $\frac{9}{14}$ 62. $12\, C_3 \times 2^9/3^{12}$

64. (i) 0.3 (ii) 0.5

66. $\frac{3}{4}$ 67. $\frac{m}{n}$

68. (i) $\frac{2}{7}$ (ii) $\frac{1}{7}$

69. (i) $\frac{2}{9}$ (ii) $\frac{1}{3}$

70. (i) 0.05 (ii) 0.50

71. $\frac{8}{195}$

72. (a) $\frac{12}{51}$ (b) $\frac{1}{50}$

73. Pairwise independent, but not totally independent

74. (i) $\frac{11}{24}$ (ii) $\frac{3}{4}$

75. $\frac{9}{64}$ 76. $\frac{18}{35}$ 77. $\frac{7}{33}$

Exercise 1(B)

2. P (red ball) = P (red ball from bag I) + P (red ball from bag II)

$$= \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{3}{5} = \frac{19}{30}$$

4. $$P(B_1/W) = \frac{P(B_1) \times P(W/B_1)}{P(B_1) \times P(W/B_1) + P(B_2) \times P(W/B_2)}$$

$$= \frac{\frac{1}{2} \times \frac{2}{5}}{\frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{4}{5}} = \frac{1}{3}$$

5. $P(M) = P(F) = \frac{1}{2}$; $P(B/M) = \frac{1}{20}$; $P(B/F) = \frac{1}{40}$

By Bayes' theorem, $$P(M/B) = \frac{\frac{1}{2} \times \frac{1}{20}}{\frac{1}{2} \times \frac{1}{20} + \frac{1}{2} \times \frac{1}{40}} = \frac{2}{3}$$

6. $\frac{88}{221}$

7. (i) $\frac{69}{2000}$ (ii) $\frac{44}{69}$

8. (i) $\frac{13}{80}$ (ii) $\frac{8}{13}$

9. 0.47

10. $\dfrac{(1-p)q_0}{(1-p)q_0 + pp_0}$

11. $\frac{12}{47}$

12. $\frac{27}{29}$

13. $\frac{14}{57}$

14. $\frac{5}{19}$

15. $\frac{3}{25}$

16. $\frac{20}{59}$

17. (i) 0.0062 (ii) 0.8005

18. $\frac{6}{13}$

19. $\frac{6}{25}$

Exercise 1(C)

2. Required probability = P (getting exactly 3 or 4 heads)

$$= 4C_3 \left(\frac{1}{2}\right)^4 + 4C_4 \left(\frac{1}{2}\right)^4 = \frac{5}{16}$$

3. Required number = $256 \times 12C_8 \left(\frac{1}{2}\right)^{12} \cong 31$

4. $p = \dfrac{1}{25}$, $q = \dfrac{24}{25}$, $n = 50$; $P = 50C_0 \left(\dfrac{1}{25}\right)^0 \left(\dfrac{24}{25}\right)^{50} = \left(\dfrac{24}{25}\right)^{50}$

7. $1-(1-p)^n$ 8. 0.0381 9. 0.017 10. 0.9197

11. 0.9545 12. 0.9876 13. 0.0293 14. $\dfrac{29}{128}$

15. (i) 0.1898 (ii) 0.8655 (iii) 0.3243

16. 299 17. 11