

Bell's theorem stands and falls with absolute space

Bart Jongejan^{1*}

¹*Copenhagen, Denmark, .

Corresponding author(s). E-mail(s): bart.jongejan@gmail.com;

Abstract

I propose a reformulation of the Bohm-Bell thought experiment that is in the spirit of Leibniz and rejects Newton's concept of absolute space as a container for the experimental setup. I hypothesise that space is composed of relations, including those encountered in Bohm-Bell experiments, and that hidden variables determine the results of spin component measurements. This proposal leads to a conclusion that contradicts J.S. Bell's theorem, which states that no hidden variable theory can both reproduce the predictions of Quantum Mechanics and obey Einstein's separability requirement. According to this proposal, the maximum degree to which Bell's inequality is violated is an indicator of the number of spatial dimensions. A statistical analysis of simulated results of two Bohm-Bell experiments clarifies the proposal, with hidden variables made visible in the second experiment.

Keywords: Bell's theorem, hidden variables, Tsirelson's bound

1 Introduction

From the beginning of the twentieth century, two contrasting, groundbreaking theories arose within physics: the general theory of relativity and quantum mechanics (QM). Both theories passed milestones in Albert Einstein's 'miraculous year' 1905 with the presentation of the special theory of relativity and the explanation of the photoelectric effect.

About twenty years later, QM had developed in an unusual direction: it had become a probabilistic theory, unable to predict or explain the outcome of a single trial in an experiment. There is another probabilistic physical theory, statistical mechanics. That

theory describes heat, temperature, entropy and gas pressure in terms of the mechanical movements of myriads of particles. QM, on the other hand, has no underlying detailed description to offer.

In 1935, Einstein, Podolsky and Rosen [1] called into question whether QM can be considered a complete theory, to which Niels Bohr answered that it can [2]. With other people in the roles formerly taken by Einstein and Bohr, this discussion has never completely subsided. The mainstream position in physics is to hesitate between accepting Bohr's point of view and adopting no particular perspective.

Some researchers have published proposals to add extra, hidden variables to create a version of QM that explains the movements of individual particles. The most notable of such proposals is the Broglie–Bohm theory [3], which, however, conflicts with the conceptual foundation of the special and general theories of relativity, specifically the principle of locality.

In 1964, J.S. Bell [4] presented a proof that no local hidden variable theory can reproduce the predictions of QM. Since that publication, experiments have corroborated the quantum mechanical predictions. These experiments are generally seen as evidence that Bell's conclusion is correct and that Einstein was mistaken.

In my opinion, Bell's conclusion is overreaching. What Bell proved is that it is not just a Stern-Gerlach (S-G) device that is unable to measure two or more spin components simultaneously. On the contrary, *no apparatus* will ever be able to do that, because Bell's theorem shows that such an apparatus would imply statistics of measurement outcomes that do not agree with experimentally corroborated predictions of QM. Bell has *not* proved that no local hidden variable can explain why a measurement develops the way it does. This is an important amendment of Bell's theorem, because the formalism of QM gives no such explanation. The *collapse of the wave function* is not well understood and is the subject of much debate.

In the following sections, I describe the Bohm-Bell thought experiment in a way that is more detached from physical practicalities than Bell's description. The thought experiment can be reified as a Bell-type experiment with spin- $\frac{\hbar}{2}$ particles, but also as a similar experiment in fewer or more spatial dimensions than the known three.

Sec. 2 summarises J.S. Bell's reasoning that led to his conclusion that local hidden variable theories cannot reproduce the predictions made by QM. Sec. 3 is about terminology and concepts used in this text. Sec. 4 describes a Bell-type experiment that has outcomes that could have been produced by a real, though unrealistically noiseless and errorless experiment. Sec. 5 repeats the experiment, but this time with the cause of the outcomes in full sight. Sec. 6 presents the theoretical foundation, or hidden variable theory, of the experiment. Sec. 7 maintains there is a contradiction between Bell's conclusion and the findings in this paper. Sec. 8 discusses the theoretical status of the current work and suggests a direction for future research. Sec. 9 brings us to the conclusion. Appendix A is the exposition of the algorithm that was used to produce the data that was the subject of statistical analysis in the previous sections. Appendix B explains how the maximum degree of violation of the inequality formulated by Clauser, Horne, Shimony and Holt (CHSH) can be computed for any number of spatial dimensions.

2 Bell's thought experiment

Bell's reasoning departs from a thought experiment with a system consisting of a pair of spin- $\frac{\hbar}{2}$ particles in the singlet state. In that state, the spins of the two particles are opposite to each other so that the system as a whole in all respects looks the same from every direction. This isotropy is reflected in the quantum mechanical description of the system. The two particles separate from each other while remaining entangled in this state, where they are each other's opposites. As long as the particles stay undisturbed, their spin axes fall outside the quantum mechanical description of the system. The particles travel some distance in opposite directions towards two assistants in the experiment, Alice and Bob. It is essential that Alice and Bob have free will and cannot communicate with each other. Alice and Bob operate apparatuses with which they measure a spin component of the particle arriving in their respective laboratories. The apparatuses have several settings. Alice and Bob determine the actual settings of their apparatuses, which, in turn, determine which spin components are measured.

A spin component is a scalar, a single number. The measurement of a spin component does not reveal the orientation of a spin axis, because for that, one would need two numbers: one for its longitude and another for its latitude. A spin component is measured relative to another object that has an orientation. The measured spin component is maximal if the orientation of the instrument aligns with the spin axis. The measured value is positive if the orientation of the instrument and the spin axis make a sharp angle, zero at 90° , and negative when the angle is obtuse. If the spinning particle has a large spin, the measured spin component can take many values. For a spin- $\frac{\hbar}{2}$ particle, the situation is a bit different. It has a spin so tiny that there are only two possible outcomes for the result of a spin component measurement. In the context of Bell's theorem, these outcomes are usually designated -1 and $+1$.

A spin component can be measured with an apparatus consisting of an S-G device and two or more particle detectors. An S-G device consists of a magnet with a strong, concentrated magnetic field near one of its poles and a weaker, dispersed field near the opposite pole. Due to the resulting inhomogeneity of the field in the gap between the poles, the magnet bends the path of a passing spinning particle either towards the stronger or towards the weaker field, depending on how the magnetic moment of the spinning particle interacts with the magnetic field. When the particle emerges on the other side of the gap, it is registered by one of the detectors.

Since multiple magnetic fields cannot exist at the same spot simultaneously, Alice and Bob can measure no more than one spin component each. Even so, since the particles are entangled, Alice's measurement tells something about Bob's particle that Bob could not have measured if he had chosen to measure some other spin component of his particle. Similarly, Bob's measurement discloses something about Alice's particle that she could not have measured if she had chosen to measure another spin component of her particle. Einstein, contemplating a similar thought experiment, stated that QM is not a complete theory, since Alice's measurement uncovered an element of reality about Bob's particle that QM did not describe.

In a famous paper [4], J.S. Bell investigated whether Einstein could be right. He asked whether the introduction of extra, perhaps hidden, variables can lead to a successor of QM that provides more detail, while still in agreement with the predictions

of QM and obeying the requirement of locality. Bell proposed that any viable local hidden variable theory must state that the expectation value P of the product of Alice's and Bob's measurements is computed with an equation of the form Eq. (1). The vectors \hat{a} and \hat{b} denote the orientations of Alice's and Bob's S-G devices. A shared variable λ travels with each of the two particles. The outcome of Alice's measurement is $A(\hat{a}, \lambda)$, and the outcome of Bob's measurement is $B(\hat{b}, \lambda)$. Because the singlet state is isotropic, the spin of Bob's particle is the opposite of the spin of Alice's particle. This is taken into account by setting $B(\hat{x}, \lambda) = -A(\hat{x}, \lambda)$. The expressions $A(\hat{a}, \lambda)$ and $B(\hat{b}, \lambda)$ are either +1 or -1. Bell allowed a probability measure ϱ in the integrand, which, however, should not depend on Alice and Bob's settings, but solely on the value of the hidden variable.

$$\begin{aligned} P^{LHV}(\hat{a}, \hat{b}) &= \int \varrho(\lambda) A(\hat{a}, \lambda) B(\hat{b}, \lambda) d\lambda \\ &= - \int \varrho(\lambda) A(\hat{a}, \lambda) A(\hat{b}, \lambda) d\lambda \end{aligned} \quad (1)$$

Bell derived an inequality that should apply to any hidden variable expression of the form Eq. (1), here in the formulation by CHSH [5]:

$$|S| \leq 2 \quad (2)$$

where

$$S = P(\hat{a}_1, \hat{b}_1) + P(\hat{a}_2, \hat{b}_1) + P(\hat{a}_2, \hat{b}_2) - P(\hat{a}_1, \hat{b}_2) \quad (3)$$

The equations Eqs. (2) and (3) can be experimentally tested. The symbols \hat{a}_1 and \hat{a}_2 in Eq. (3) denote two possible settings of Alice's S-G device. The symbols \hat{b}_1 and \hat{b}_2 denote two possible settings of Bob's S-G device. Alice and Bob choose one of the two available settings before the particles arrive at their S-G instruments. Alice and Bob do not know each other's choices. Alice measures either -1 or +1, and so does Bob. This procedure is repeated many times. Each time, Alice and Bob inform Carol about the settings and results, and each time Carol multiplies Alice and Bob's results. She creates sums of the products for each of the four combinations of settings. Finally, Carol divides each sum by its number of terms. The resulting averages $P(\hat{a}_i, \hat{b}_j)$ are fractional numbers in the interval [-1, 1].

The possible outcomes of spin component measurements have equal probability in Bell's thought experiment. The four possible *combinations* of outcomes, (-1, -1), (-1, +1), (+1, -1), and (+1, +1), are not all equally probable, except if the S-G devices are at right angles to each other. Thus, $P(\hat{a}_1, \hat{b}_1)$, $P(\hat{a}_2, \hat{b}_1)$, $P(\hat{a}_2, \hat{b}_2)$, and $P(\hat{a}_1, \hat{b}_2)$ have generally different values, depending on the relative orientations of \hat{a}_1 to \hat{b}_1 , \hat{a}_2 to \hat{b}_1 , \hat{a}_2 to \hat{b}_2 , and \hat{a}_1 to \hat{b}_2 . These experimentally obtained averages can be compared with the expectation values that are predicted by QM or by a hidden variable theory. According to QM, the expectation value is proportional to the cosine of the angle between the S-G devices. See Eq. (4).

$$P^{QM}(\hat{a}_i, \hat{b}_j) = \langle \hat{\sigma} \cdot \hat{a}_i \hat{\sigma} \cdot \hat{b}_j \rangle = -\cos(\angle_{a_i b_j}) \quad (4)$$

If we insert the quantum mechanical expectation value Eq. (4) into the CHSH expression Eq. (3), we find that Eq. (2) is violated for some choices of the settings $\hat{a}_1, \hat{b}_1, \hat{a}_2$ and \hat{b}_2 . The maximum violation of Eq. (2) is found when $\angle_{a_1 b_1} = \angle_{a_2 b_1} = \angle_{a_2 b_2} = \frac{\pi}{4}$ and $\angle_{a_1 b_2} = \frac{3\pi}{4}$:

$$\begin{aligned} S &= \langle \hat{\sigma} \cdot \hat{a}_1 \hat{\sigma} \cdot \hat{b}_1 \rangle + \langle \hat{\sigma} \cdot \hat{a}_2 \hat{\sigma} \cdot \hat{b}_1 \rangle + \langle \hat{\sigma} \cdot \hat{a}_2 \hat{\sigma} \cdot \hat{b}_2 \rangle - \langle \hat{\sigma} \cdot \hat{a}_1 \hat{\sigma} \cdot \hat{b}_2 \rangle \\ &= -\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) = -2\sqrt{2} \end{aligned} \quad (5)$$

The value $2\sqrt{2}$ is known as Tsirelson's bound. See [6].

Since, according to Bell and CHSH, the inequality Eq. (2) is *not* violated by *any* hidden variable theory that offers a local causal explanation of the actual outcome of a spin component measurement, such theories *must* fail to reproduce the expectation value $P^{QM}(\hat{a}_i, \hat{b}_j)$. The orthodox interpretation of Bell's theorem is that no hidden variable theory can reproduce the predictions of QM, unless such a theory breaks a cornerstone of Einstein's theory of Relativity and one of the assumptions needed to arrive at the inequality Eq. (2), locality, or, as Einstein preferred, separability [7].

3 Vectors, settings, labels, and key choices

Bell calls $\hat{a}_1, \hat{a}_2, \hat{b}_1$ and \hat{b}_2 both ‘setting’ and ‘unit vector’. These two concepts are regarded as distinct in the current text. Vectors never exclude other vectors. *Setting* is an ambiguous concept. On the one hand, we can say that an apparatus has so and so many settings. In another context, the word is used to denote the actual position or orientation of an apparatus. To make the distinction clearer, I have adopted ‘label’ for the first sense of *setting* and ‘key choice’ (or ‘choice of key’) for the second sense. Labels are never incompatible; key choices are. Vectors in a space with two or more dimensions are fully determined by two or more abstract components with respect to a background reference system. A label requires a concrete, distinctive identity. In the first sections of this text, I use ‘label’ and ‘key choice’ and do not mention vectors. Near the end of this paper, in Appendix A, I explain how a computer program generates the data. According to the algorithm, each setting corresponds to a unit vector that has coordinates in a d -dimensional space and a label, or identity.

4 Carol's experiment, told in pictures and text

4.1 Carol, Alice, Bob, two apparatuses with many keys, and a notepad

Carol is the chief experimenter. Carol has enrolled two assistants in her experiment, Alice and Bob. The assistants are in separate rooms and cannot communicate with each other. There is also an apparatus in each room. See Figs. 1a and 1b. Alice's apparatus has $K = 30$ keys on the front, labelled a_1, a_2, \dots, a_K . Bob's almost identical apparatus also has $M = 30$ keys. They are labelled b_1, b_2, \dots, b_M . To make the assistants believe they are participating in a psychological test, Carol tells them that their

task is to press arbitrary keys, one at a time, and that from time to time the yellow lamp on top of the apparatus will light up. Each time the yellow lamps light up, Alice and Bob must quickly check the narrow slit on the left-hand side of the front panels of their apparatuses, before the light turns off again.

4.2 Alice and Bob do *green* and *not-green* observations.

In the slit to the left of the keys, Alice and Bob will sometimes observe a colour. We call that a *green* observation. At other times, the colour is absent; let us call that a *not-green* observation. After each observation, the assistant must send a telegraph message to Carol. The message must report which identifier was written on the pressed key and whether *green* or *not-green* was observed. This procedure is repeated thirty million times. The assistants are encouraged to spread their key choices over all keys, but otherwise they are free to choose any key each time. Carol, who has a notepad

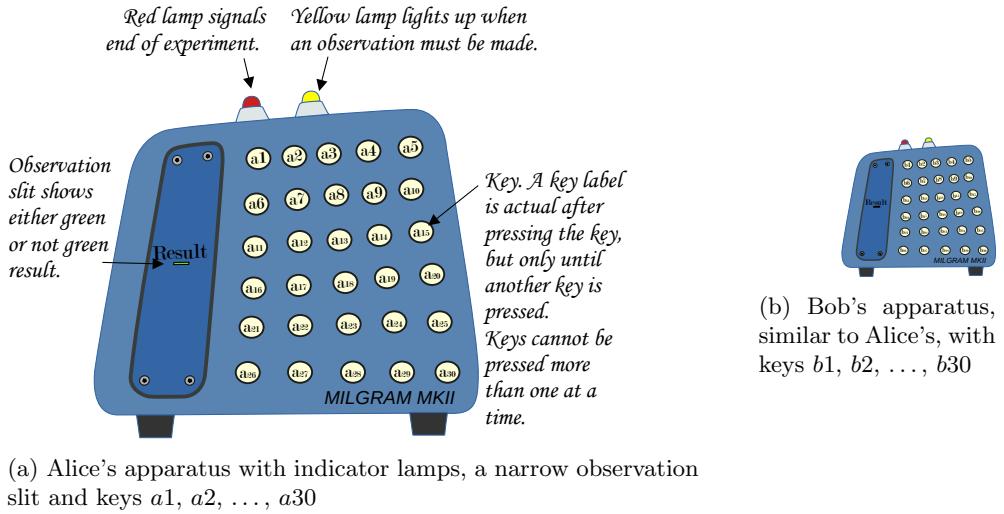


Fig. 1: Apparatuses of Alice and Bob

with columns for Alice and Bob's trials (see Fig. 2), writes down the results in the order in which she obtains them. After thirty million trials, the red lamp on Alice's apparatus turns on, signalling to Alice to put down her work. The same happens on Bob's side. Carol's notepad is filled to the last line with results from thirty million trials, each trial consisting of two key choices and two colour observations.

Carol wants to know whether the observations by Alice and Bob are statistically correlated. If they are, then for what reason? Is there a cause-and-effect relation?

4.3 Statistical dependencies - preparation

Carol's notepad has thirty million lines. Each line represents a pair of measurements, one telegraphed in from Alice and the other from Bob. A line has fields in four columns:

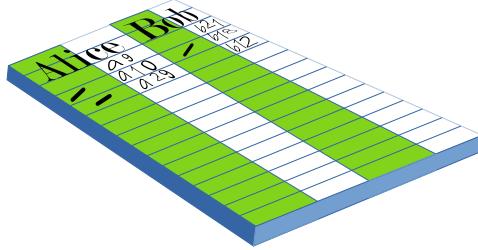


Fig. 2: Carol's notepad with two columns for Alice's observations and key choices, and two columns for Bob's

there are always two key identifiers, one for Alice's apparatus and the other for Bob's, and then there are two fields that can contain a mark, one for Alice's colour observation and one for Bob's. Each of the four columns in the notepad can be regarded as a (discrete) variable. Alice's and Bob's key choices are two variables, and their colour observations are the other two variables.

For the analysis that follows, we need some notation to address the colour observation variables and key choice variables. If X stands for A (Alice) or B (Bob), n stands for the line number in Carol's notepad, K and M for the number of keys on Alice's and Bob's apparatuses, then we refer to a key variable with $s_X(n)$ and to a colour observation variable with $g_X(n)$. See Eq. (6).

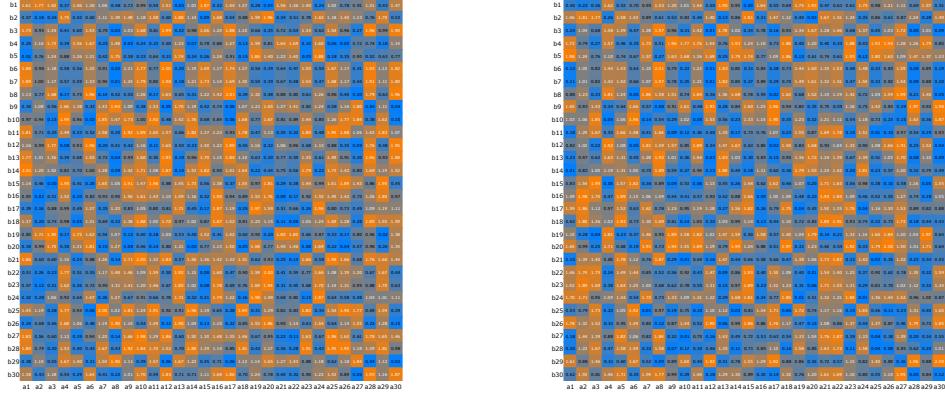
$$\begin{aligned}
 A &= \text{Alice} \\
 B &= \text{Bob} \\
 n &\in \{1, 2, \dots, N\} \\
 X &\in \{A, B\} \\
 s_A(n) &\in \{a_1, a_2, \dots, a_K\} \\
 s_B(n) &\in \{b_1, b_2, \dots, b_M\} \\
 g_X(n) &\in \{\text{green}, \text{not-green}\}
 \end{aligned} \tag{6}$$

We will often use a fraction F . In this context, a fraction is always a number in the interval $[0, 1]$. With the Kronecker function Eq. (7), we define F as in Eq. (8).

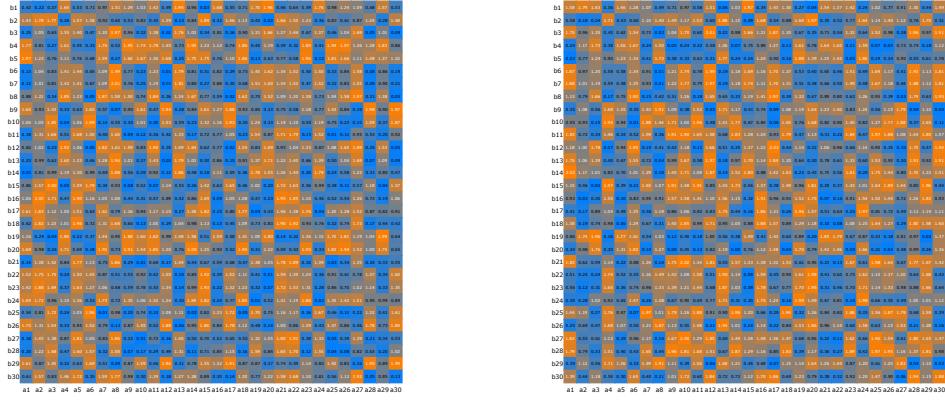
$$\delta_{v(n), v} = \begin{cases} 1 & \text{if } v(n) = v \\ 0 & \text{if } v(n) \neq v \end{cases} \tag{7}$$

$$\begin{aligned}
 F(v_1, v_2, \dots) &= \frac{\text{marginal distribution of discrete variables } v_1, v_2, \dots}{\text{number of trials } N} \\
 &= \frac{\sum_{n=1}^N \delta_{v_1(n), v_1} \delta_{v_2(n), v_2} \dots}{N}
 \end{aligned} \tag{8}$$

For example, if there are 987,345 trials where Alice has chosen key $a24$ and Bob observes *green*, and disregarding which key Bob has chosen and which colour Alice observes, then $F(s_A = a24, g_B = \text{green}) = \frac{987,345}{30,000,000}$. If Alice, Bob and Carol were to

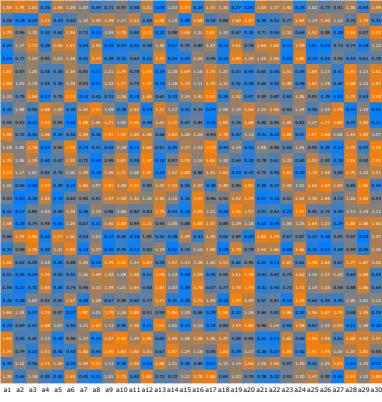


(a) $s_A = \text{not-green}$, $s_B = \text{not-green}$



(b) $s_A = \text{not-green}$, $s_B = \text{green}$

(c) $s_A = \text{green}$, $s_B = \text{not-green}$



(d) $s_A = \text{green}$, $s_B = \text{green}$

Fig. 3: Distribution of the co-occurrence of Alice and Bob observing *not-green* or *green* for each of Alice and Bob's 30×30 combinations of key choices

continue the experiment beyond 30,000,000 trials, they could use fraction values as the best estimates for the probabilities of future trials.

The first test Carol performs is to check that the apparatuses have been calibrated such that observing *green* is as probable as observing *not-green*. Carol counts the number of times Alice and Bob observe *green*, and she concludes that the fraction F of observations where Alice and Bob observe *green* is one-half of the total number of observations to within a small error. Using the notation introduced above, this is written as in Eq. (9).

$$F(g_A) = F(g_B) = \frac{1}{2} \pm 0.00025 \quad (9)$$

Carol sees no need for further calibration of either apparatus.

Next, Carol investigates statistical dependencies between two, three and four variables. There are $K \times M = 30 \times 30$ key combinations and 2×2 ways in which

Alice and Bob's pairs of colour observations could occur, so, in all, Carol needs $30 \times 30 \times 2 \times 2 = 3600$ separate tallies to count how often each combination of values of all four variables occurs. Since there are $N = 30,000,000$ lines in her notepad, Carol knows that each tally has a value of $\frac{3600}{30,000,000} = 8,333.\bar{3}$ on average, which makes quite accurate statistics possible. For combinations of two or three (marginal) variables, she needs fewer tallies. Those can be obtained by summing over the variable(s) that are not in the set of marginal variables. For example, to know how often Alice observes *green* while her key is a_{11} and Bob's key is b_{28} , Carol must sum two counts: one for that combination where Bob saw *green* and the other where Bob saw *not-green*. In this example, Bob's colour observation is marginalised out.

Carol organises all 3600 counts in a 4-dimensional $30 \times 30 \times 2 \times 2$ table. Fig. 3 shows one possible representation of this distribution table. The values in Fig. 3 are not the counts themselves, but normalised values that are computed by dividing each count by the average count. Cells in Fig. 3 with values close to 1.0 correspond to a count that is close to the average of $8333.\bar{3}$. Those cells have a greyish colour. Cell values that are lower than average are in shades of blue. Values that are higher than average are shown in a shade of orange. The upper limit to a cell's value is obtained if all cells are empty except one, in which case that single cell has a count of 30,000,000, corresponding to a value of $\frac{30,000,000}{8,333.\bar{3}} = 3,600$ in Fig. 3, but it turns out that the maximum count in Carol's table is barely twice the average value, or a little bit more than 2.00 in Fig. 3.

4.4 Are the key choices and colour observations independent variables?

Carol plans to count how often each combination of two, three or four variable values occurs in her notepad. Then she wants to investigate whether those counts, if deviating from the statistically expected average, can be reconciled with reasonable assumptions, namely that Alice and Bob exerted their free wills and the circumstance that they were unable to communicate with each other, nor their apparatuses.

Carol will discover that:

- ($ab__$): Alice's and Bob's key choices are statistically independent of each other,
- ($a_A__$), ($_b_B__$): Alice's (Bob's) colour observation is independent of her (his) key choice,
- ($a_B__$), ($_b_A__$): Alice's (Bob's) colour observation is independent of his (her) key choice,
- ($abA__$), ($ab_B__$): Alice's or Bob's colour observation is independent of the combination of their key choices,
- ($__AB__$): Alice's colour observation is independent of Bob's colour observation,
- ($a_AB__$), ($_bAB__$): the combination of Alice's and Bob's colour observations is independent of the choice of key by Alice or by Bob, and
- $\neg(abAB)$: Alice's key choice, Bob's key choice, Alice's colour observation and Bob's colour observation are strongly correlated.

The expressions before the colons ‘:’ are compact notations for the propositions following them. Lower case letters a and b denote the variables s_A and s_B for Alice's and

Bob's key choices. Capital letters A and B denote Alice's and Bob's colour observations g_A and g_B . An underscore fills the place of a variable that has been marginalised out.

4.4.1 (ab__): Alice's and Bob's key choices are statistically independent of each other

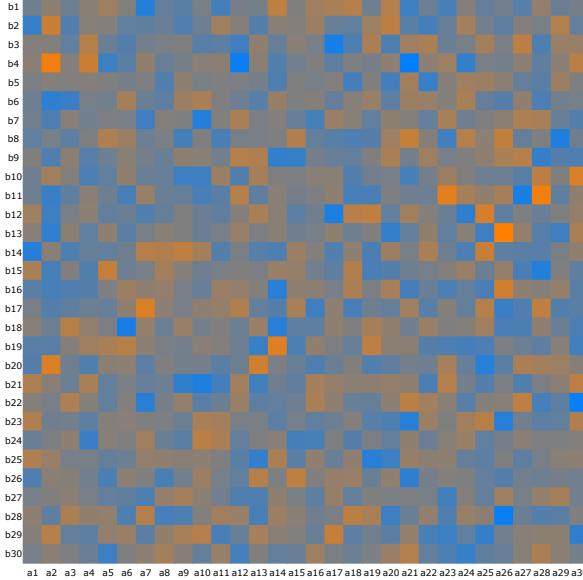


Fig. 4: ab__: Density map showing that Alice's and Bob's key choices are not statistically correlated

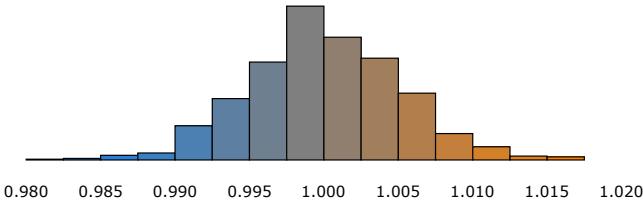


Fig. 5: ab__: Distribution of the relative frequencies of combinations of key choices

Carol first wants to know whether there are indications that Alice's and Bob's key choices were coordinated, either by Alice and Bob themselves or by some common cause. Carol must mitigate any such coordination.

Carol's method is as follows: For each of the 30×30 combinations of s_A and s_B , she computes the actual fraction $F(s_A, s_B)$ and compares that number with the product of the fractions $F(s_A)$ and $F(s_B)$. If the key choices were made independently of each

other, then $F(s_A, s_B)$ and $F(s_A) \times F(s_B)$ must be the same.

$$s_A \text{ and } s_B \text{ are independent} \implies F(s_A, s_B) = F(s_A) F(s_B) \quad (10)$$

Carol evaluates both sides of Eq. (10) and confirms that they are the same to within a small statistical deviation.

$$\frac{F(s_A, s_B)}{F(s_A) F(s_B)} = 1.0000 \pm 0.0051 \quad (11)$$

The fractions are indirectly displayed in Fig. 4. The colour of each cell in that figure indicates the cell's value. The distribution of all 30×30 cell values is shown in Fig. 5. The height of each column is proportional to the number of cells in Fig. 4 that have the same colour as the column. Frequencies are normalised such that the value 1.0 represents the expected frequency if all involved variables are independent. If the deviations from the expected value follow a normal distribution, as is the case here, they can be ascribed to the finite size of the sample space. By increasing the number of observations, the width of the distribution will shrink.

Since the relative frequencies have a normal distribution around the value 1.0, Carol concludes that there is no reason to suspect that Alice and Bob coordinated their key choices.

4.4.2 (a_A_), (_b_B): Alice's (Bob's) colour observation is independent of her (his) key choice

Carol considers the distribution of *green* observations by Alice and Bob. Carol already knows that Alice and Bob observe equally often *green* and *not-green* in the narrow display, but is that true for any choice of their keys? In other words, was the probability for Alice to observe *green* independent of the key she had hit? And is, *mutatis mutandis*, the same true for Bob? The low standard deviation in Eq. (12) tells Carol that it is safe to assume that the outcomes produced by an apparatus are not biased by the key choices made on that apparatus.

$$\begin{aligned} \frac{F(s_A, g_A)}{F(s_A) F(g_A)} &= 1.0000 \pm 0.0012 \\ \frac{F(s_B, g_B)}{F(s_B) F(g_B)} &= 1.0000 \pm 0.0011 \end{aligned} \quad (12)$$

4.4.3 (a_B), (_bA_): Alice's (Bob's) colour observation is independent of his (her) key choice

Now Carol is quite sure that there was no obvious causal relationship between Alice's key choice and the colour she would see (*green* or *not-green*) a while later, and that the same is true for Bob's key choice and Bob's colour observation. But what about Alice's key choice and Bob's colour observation, or Bob's key choice and Alice's colour observation? Were Alice and Bob's apparatuses connected, so that Alice's keys determined which colour Bob would see and so that Bob's keys determined which colour

Alice would see?

$$\begin{aligned}\frac{F(s_B, g_A)}{F(s_B) F(g_A)} &= 1.0000 \pm 0.0012 \\ \frac{F(s_A, g_B)}{F(s_A) F(g_B)} &= 1.0000 \pm 0.0011\end{aligned}\tag{13}$$

Since the standard deviation in Eq. (13) is small, there is no reason to assume that key choices of the other assistant influenced whether an assistant would observe *green* or *not-green*.

4.4.4 (abA_), (ab_B): Alice's or Bob's colour observation is independent of the combination of their key choices

Even though Alice's and Bob's choices, in isolation, do not determine whether Alice or Bob observes the green colour, the combination of Alice's and Bob's key choices might determine whether Alice (or Bob) observes *green*.

As before, the small standard deviation indicates that there is no correlation between Alice's and Bob's choices of key and whether Alice (or Bob) observes *green*. See Eq. (14).

$$\begin{aligned}\frac{F(s_A, s_B, g_A)}{F(s_A) F(s_B) F(g_A)} &= 1.0000 \pm 0.0076 \\ \frac{F(s_A, s_B, g_B)}{F(s_A) F(s_B) F(g_B)} &= 1.0000 \pm 0.0076\end{aligned}\tag{14}$$

4.4.5 (_AB): Alice's colour observation is independent of Bob's colour observation

Carol has already checked that a single colour observation is not statistically correlated with the key choices of Alice and Bob. Her next task is to check whether the colour observations themselves are correlated, and whether Alice's and Bob's key choices have an influence. First, Carol confirms that Alice's and Bob's observations, irrespective of key choices, are not correlated.

$$\frac{F(g_A, g_B)}{F(g_A) F(g_B)} = 1.000 \pm 0.019\tag{15}$$

4.4.6 (a_AB), (_bAB): the combination of Alice's and Bob's colour observations is independent of the choice of key by Alice or by Bob

Even though Alice's or Bob's key choice does not determine whether Alice or Bob observes *not-green* or *green*, it could be that Alice's choice is correlated with Alice's and Bob's observations in combination. It could, for example, be that some key choice by Alice or by Bob would dictate that either both slits display *green* or that none of them does, whereas another choice could dictate that either Alice's or Bob's slit displays *green*, but not both. Fig. 6 and Eqs. (16a) and (16b) reveal that the distribution



Fig. 6: a_{AB}: Co-occurrence frequencies of Alice's and Bob's colour observations and Alice's key choices

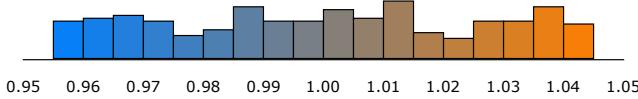


Fig. 7: a_{AB}: As Fig. 6, but here Alice and Bob have twice as many keys to choose from and do four times as many observations

of the co-occurrence of a key choice by either Alice or Bob and the colour observations by both Alice and Bob is not normal, and that the standard deviation is quite large.

$$\frac{F(s_A, g_A, g_B)}{F(s_A) F(g_A) F(g_B)} = 1.00 \pm 0.14 \quad (16a)$$

$$\frac{F(s_B, g_A, g_B)}{F(s_B) F(g_A) F(g_B)} = 1.00 \pm 0.11 \quad (16b)$$

In an upgraded setup, using apparatuses with 60 keys each and with four times as many trials, the distribution becomes narrower and the standard deviation smaller. See Fig. 7 and Eq. (17). Given enough keys and trials, the standard deviation can be made arbitrarily small. Carol concludes that Alice's and Bob's colour observations, together with either Alice's or Bob's key choice, are statistically independent.

$$\begin{aligned} \frac{F(s_A, g_A, g_B)}{F(s_A) F(g_A) F(g_B)} &= 1.000 \pm 0.026 \\ \frac{F(s_B, g_A, g_B)}{F(s_B) F(g_A) F(g_B)} &= 1.000 \pm 0.077 \end{aligned} \quad (17)$$

4.4.7 $\neg(abAB)$: Alice's key choice, Bob's key choice, Alice's colour observation and Bob's colour observation are strongly correlated.

To finalise the analysis of the experiment, Carol checks whether all four variables in combination, Alice's and Bob's key choices and Alice's and Bob's observations, are statistically dependent. See Eq. (18).

$$F(s_A, s_B, g_A, g_B) \stackrel{?}{=} F(s_A) F(s_B) F(g_A) F(g_B) \quad (18)$$

Carol counts how often (expressed as fractions) each of the $2 \times 2 \times 30 \times 30 = 3600$ combinations of all four variables occurs. See Fig. 3. Carol discovers that those fractions vary from zero to twice (in a few cases, even a little bit more) the value that she would

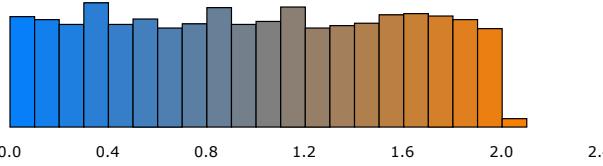


Fig. 8: abAB: Histogram over cooccurrence numbers in Fig. 3

expect if all four variables were statistically independent. See the histogram in Fig. 8. Unsurprisingly, the standard deviation is large. See Eq. (19).

$$\frac{F(s_A, s_B, g_A, g_B)}{F(s_A) F(s_B) F(g_A) F(g_B)} = 1.00 \pm 0.58 \quad (19)$$

The first twenty columns of the histogram in Fig. 8 have more or less the same height, but there is a handful of cells with values even higher than 2.0, though only very slightly, in each table shown in Fig. 3. That small number explains why the twenty-first column in Fig. 8 is so low. It is not an error that some cells display a value above 2.0. There are combinations of keys where Alice and Bob almost always make the same colour observations (or almost always opposite colour observations), which is twice as often as expected if all variables were independent. If, in addition, such key combinations occur slightly more often than expected statistically, $F(s_A, s_B) > F(s_A) F(s_B)$, then a value higher than 2.0 appears in Fig. 3. See e.g. cells (a10, b21) in Figs. 3b and 3d.

4.5 The expectation value $P^{obs}(s_A, s_B)$

In line with the conventions adopted by Bell and others, spin component measurements result either in the value +1 or the value -1. In this text, a *not-green* observation yields a value of +1, and a *green* observation yields a value of -1. We can then ask for the expected value $P^{obs}(s_A, s_B)$ of the product of Alice's and Bob's observations, given a pair (s_A, s_B) of key choices. See Eq. (20).

$$\begin{aligned} P^{obs}(s_A, s_B) = \frac{1}{F(s_A, s_B)} & \{ (-1)(+1)F(s_A, s_B, g_A = \text{green}, g_B = \text{not-green}) \\ & + (+1)(-1)F(s_A, s_B, g_A = \text{not-green}, g_B = \text{green}) \\ & + (-1)(-1)F(s_A, s_B, g_A = \text{green}, g_B = \text{green}) \\ & + (+1)(+1)F(s_A, s_B, g_A = \text{not-green}, g_B = \text{not-green}) \} \end{aligned} \quad (20)$$

$P^{obs}(s_A, s_B)$ is in the interval $[-1, +1]$.

Eq. (21) defines a correlation function $H^{obs}(s_A, s_B)$ that yields a value in the interval $[0, 2]$, where 0.0 signifies complete agreement between Alice's and Bob's colour observations, 2.0 signifies complete disagreement, and 1.0 signifies *no correlation*.

$$H^{obs}(s_A, s_B) = 1 - P^{obs}(s_A, s_B) \quad (21)$$

	a1	a2	a3	a4	a5	a6	a7	a8	a9	a10	a11	a12	a13	a14	a15	a16	a17	a18	a19	a20	a21	a22	a23	a24	a25	a26	a27	a28	a29	a30	
b1	1.80	1.79	1.64	0.37	1.47	1.30	1.05	0.49	0.72	0.99	0.59	1.51	0.06	1.04	1.96	1.34	1.45	1.30	0.28	0.04	1.54	1.37	1.41	0.25	1.05	0.78	0.91	1.32	0.44	1.48	
b2	0.58	0.19	0.24	1.74	0.43	0.60	1.10	1.40	1.48	1.18	1.57	0.61	1.88	1.15	0.10	1.69	0.55	0.88	1.60	1.98	0.35	0.51	0.78	0.24	1.16	1.40	1.13	0.77	1.72	0.52	
b3	1.76	0.96	1.34	0.43	0.61	1.53	0.72	0.03	1.04	1.69	0.61	1.98	0.23	0.98	1.67	1.21	1.85	1.10	0.67	0.35	0.72	0.55	1.34	0.63	1.55	0.97	0.29	1.95	0.99	1.92	
b4	0.25	1.15	1.74	0.39	1.55	1.67	0.25	1.49	0.05	0.74	0.23	0.58	1.26	0.08	0.77	0.89	1.27	0.13	1.61	0.81	1.62	1.68	0.11	1.59	0.07	0.05	0.73	0.74	0.19	1.14	
b5	0.03	0.77	1.24	0.90	1.26	1.33	0.42	1.74	0.38	0.33	0.64	0.32	1.76	0.24	0.26	1.24	0.91	0.16	1.88	1.39	1.25	1.43	0.06	1.88	0.19	0.35	0.90	0.54	0.63	0.78	
b6	1.88	0.97	1.19	0.58	0.57	1.36	0.91	0.01	1.22	1.77	0.78	2.00	0.20	1.19	1.60	1.17	1.72	1.27	0.55	0.40	0.65	0.47	1.51	0.51	1.66	1.17	0.42	1.92	1.13	1.82	
b7	1.89	1.00	1.19	0.58	0.59	1.34	0.93	0.01	1.23	1.76	0.80	1.99	0.19	1.20	1.73	1.13	1.71	1.32	0.51	0.39	0.67	0.49	1.53	0.48	1.68	1.18	0.45	1.91	1.12	1.80	
b8	1.13	0.75	1.67	0.17	0.75	1.95	0.15	0.42	0.51	1.27	0.18	1.65	0.66	0.33	1.22	1.41	0.99	1.39	1.32	0.48	0.90	0.80	0.62	1.27	0.96	0.40	0.03	1.79	0.63	1.96	
b9	0.35	1.08	0.57	1.67	1.37	0.35	1.43	1.93	1.10	0.39	1.53	0.05	1.73	1.18	0.41	0.75	0.09	1.08	1.21	1.66	1.26	1.42	0.83	1.25	0.58	1.17	1.81	0.05	1.10	0.04	
b10	0.96	0.95	0.15	1.96	0.97	0.02	1.86	1.46	1.71	1.00	1.96	0.48	1.42	1.77	0.68	0.85	0.06	1.66	0.77	1.68	0.81	0.90	1.45	0.86	1.27	1.77	1.86	0.36	1.64	0.11	
b11	1.82	0.71	0.35	1.48	0.34	0.52	1.60	0.36	1.92	1.88	1.05	1.59	0.67	1.84	1.28	1.24	0.94	1.78	0.47	1.13	0.31	0.22	1.85	0.48	1.99	1.90	1.06	1.46	1.81	1.08	
b12	1.18	0.99	1.78	0.08	0.98	1.96	0.20	0.41	0.42	1.17	0.11	1.66	0.51	0.34	1.38	1.22	1.99	0.95	0.45	1.17	0.32	1.07	0.97	0.67	1.13	0.91	0.36	0.10	1.76	0.48	1.96
b13	1.77	1.03	1.39	0.40	0.64	1.55	0.72	0.05	1.00	1.64	0.57	1.98	0.19	0.97	1.70	1.15	1.85	1.10	0.64	0.30	0.78	0.61	1.34	0.61	1.50	0.94	0.31	1.93	0.92	1.92	
b14	2.00	1.19	1.01	0.82	0.70	1.01	1.30	1.10	1.43	1.72	1.08	1.88	0.14	1.52	1.82	0.89	1.41	1.64	0.23	0.44	0.76	0.57	1.80	0.22	1.77	1.43	0.80	1.69	1.21	1.53	
b15	1.16	0.46	0.01	1.95	0.41	0.21	1.65	1.09	1.92	1.48	1.94	0.87	1.45	1.74	0.57	1.38	0.38	1.54	0.96	1.81	0.29	0.38	1.44	1.01	1.63	1.90	1.43	0.85	1.96	0.45	
b16	0.93	0.03	0.31	1.53	0.11	0.84	0.95	0.92	1.57	1.60	1.43	1.11	1.59	1.14	0.33	1.92	0.96	0.91	1.53	1.77	0.08	0.17	0.92	1.54	1.49	1.45	0.74	1.27	1.82	0.96	
b17	0.41	0.17	0.89	1.02	0.49	1.37	0.36	1.22	0.85	1.06	0.82	0.83	1.73	0.45	0.17	1.85	1.21	0.26	1.97	1.58	0.52	0.65	0.25	1.96	0.81	0.73	0.49	1.11	1.19	1.11	
b18	1.38	0.20	0.75	0.99	0.66	1.29	0.69	0.32	1.40	1.87	1.00	1.71	0.97	1.03	0.87	1.87	1.56	0.98	1.27	1.17	0.11	0.08	1.08	1.25	1.68	1.27	0.28	1.81	1.57	1.59	
b19	0.85	1.72	1.96	0.17	1.78	1.63	0.55	1.05	0.12	0.41	1.84	0.08	0.54	0.40	1.50	1.41	1.43	0.60	0.91	0.21	1.86	1.79	0.67	0.87	0.33	0.18	0.81	0.97	0.02	1.37	
b20	0.34	1.05	0.20	0.30	1.33	1.82	0.10	1.28	0.98	0.46	1.44	0.84	1.22	0.01	0.76	1.12	1.49	0.85	1.69	0.79	1.41	1.47	0.08	0.23	0.24	0.04	0.48	0.99	0.27	1.36	
b21	1.86	0.61	0.60	1.16	0.23	0.88	1.25	0.14	1.72	1.99	1.33	1.84	0.56	1.57	1.35	1.41	1.33	1.51	0.62	0.95	0.24	1.14	1.68	0.60	1.97	1.66	0.68	1.77	1.67	1.45	
b22	0.51	0.26	0.24	1.76	0.52	0.55	1.16	1.49	1.45	1.09	1.58	0.51	1.91	1.15	0.08	1.61	0.47	0.93	1.60	2.00	0.42	0.60	0.77	1.64	1.10	1.39	1.22	0.66	1.68	0.43	
b23	0.57	0.12	0.32	1.63	0.37	0.74	0.96	1.32	1.40	1.22	1.46	0.68	1.86	1.03	0.09	1.78	0.68	0.78	1.69	1.94	0.31	0.46	0.69	1.72	1.15	1.31	0.99	0.86	1.68	0.65	
b24	0.31	0.28	1.05	0.92	0.65	1.47	0.27	1.28	0.67	0.92	0.68	0.78	1.71	0.32	0.20	1.76	1.24	0.16	1.99	1.49	0.68	0.81	0.15	1.99	0.65	0.57	0.49	1.05	1.01	1.12	
b25	1.46	1.20	0.28	1.77	0.96	0.07	2.00	1.03	1.81	1.26	1.90	0.92	0.90	1.98	1.20	0.66	0.29	1.95	0.31	1.27	0.84	0.84	1.85	0.35	1.55	1.89	1.78	0.69	1.60	0.40	
b26	0.25	0.69	0.47	1.69	1.05	0.50	1.21	1.88	1.14	0.55	1.49	0.12	1.94	1.03	1.14	1.15	0.23	0.85	1.53	1.87	0.95	1.15	0.62	1.57	0.64	1.14	1.55	0.22	1.28	0.16	
b27	1.82	0.56	0.61	1.13	0.19	0.95	1.18	0.15	1.68	2.00	1.79	1.85	1.60	1.50	1.20	1.48	1.37	1.46	1.68	0.96	0.22	0.11	1.62	0.67	1.96	1.61	0.61	1.80	1.67	1.47	
b28	1.80	0.79	0.32	1.53	0.41	0.43	1.68	0.44	1.93	1.83	1.71	1.53	0.68	1.86	1.30	1.15	0.86	1.84	0.41	1.16	0.37	0.28	1.89	0.43	0.97	1.93	1.17	1.38	1.81	0.99	
b29	0.39	1.12	0.55	1.69	1.32	0.32	1.49	1.92	1.12	0.40	1.55	0.07	1.69	1.22	0.45	0.70	0.07	1.13	1.14	1.64	1.27	1.43	0.87	1.19	0.61	1.19	1.84	0.04	1.12	0.03	
b30	1.39	0.44	1.18	0.55	0.30	0.66	0.41	0.22	1.02	1.71	0.60	1.83	0.72	0.72	1.12	1.71	1.87	0.70	1.24	0.79	0.40	0.33	0.91	1.20	1.45	0.90	0.07	1.96	1.16	1.68	

Fig. 9: Correlation table for pairs of key choices by Alice and Bob, indicating to what degree their observations are the same, white signifying perfect correlation and black signifying perfect anti-correlation, and quadruples of cells marked red or green indicating CHSH-inequality-violating correlations



Fig. 10: Histogram over correlation values, 0 (white) signifying perfect correlation and 2 (black) signifying perfect anti-correlation

The correlation values for all key combinations are shown in Fig. 9. The histogram Fig. 10 shows how the values (and shades of grey) are distributed in Fig. 9. Note the similarity between the correlation table Fig. 9 and the co-occurrence frequency table Fig. 3, and the corresponding similarity between the correlation histogram Fig. 10 and the co-occurrence frequency histogram Fig. 8. The only difference is that correlations are restricted to the interval [0, 2], while co-occurrence frequencies, in theory, have a much higher upper bound.

Remarkably, all columns in Fig. 10 have approximately the same height. Fig. 10 demonstrates that all amounts of correlation occur equally often. Moreover, if we look more closely at Fig. 9, we see that each column and each row is populated with random

a3	a6	a21	a22	a17	a5	a30	a2	a15	a28	a12	a19	a24	a8	a1	a16	a27	a13	a23	a10	a18	a25	a7	a14	a20	a26	a9	a11	a4	a29		
b15	0.01	0.21	0.29	0.38	0.38	0.41	0.45	0.46	0.57	0.57	0.87	0.96	1.01	1.09	1.16	1.38	1.43	1.45	1.44	1.48	1.54	1.63	1.65	1.74	1.81	1.90	1.92	1.94	1.95	1.96	
b10	0.15	0.02	0.81	0.90	0.06	0.95	0.11	0.95	0.68	0.38	0.48	0.77	0.81	1.46	0.96	0.85	1.86	1.42	1.45	1.00	1.66	1.27	1.86	1.77	1.68	1.77	1.71	1.96	1.64		
b2	0.24	0.60	0.35	0.51	0.55	0.43	0.56	0.19	0.10	0.77	0.61	1.60	1.60	1.40	0.58	1.69	1.13	1.88	0.78	1.18	0.88	1.16	1.10	1.15	1.98	1.40	1.48	1.57	1.74	1.72	
b22	0.24	0.55	0.42	0.60	0.47	0.52	0.43	0.26	0.08	0.66	0.51	1.60	1.64	1.49	0.51	1.61	1.22	1.91	0.77	1.09	0.91	1.10	1.16	1.15	2.00	1.39	1.45	1.58	1.76	1.68	
b25	0.28	0.07	0.84	0.84	0.29	0.96	0.40	1.20	1.20	0.69	0.92	0.31	0.35	1.03	1.46	0.66	1.78	1.90	1.85	1.26	1.95	1.55	2.00	1.98	1.27	1.89	1.81	1.90	1.77	1.60	
b16	0.31	0.84	0.08	0.17	0.96	0.11	0.96	0.03	0.33	1.27	1.11	1.53	1.54	0.92	0.93	1.92	0.74	1.59	0.92	1.60	0.91	1.49	0.95	1.14	1.77	1.45	1.57	1.43	1.53	1.82	
b23	0.32	0.74	0.31	0.46	0.68	0.37	0.65	0.12	0.09	0.88	0.68	1.69	1.72	1.32	0.57	1.78	0.99	1.86	0.69	1.22	0.76	1.15	0.96	1.03	1.94	1.31	1.40	1.46	1.63	1.68	
b28	0.32	0.43	0.37	0.28	0.86	0.41	0.99	0.75	1.30	1.38	1.53	0.41	0.43	0.44	1.80	1.15	1.17	0.68	1.89	1.83	1.84	1.97	1.68	1.89	1.16	1.93	1.71	1.53	1.81		
b11	0.35	0.52	0.31	0.22	0.94	0.34	1.08	0.71	1.28	1.46	1.59	0.47	0.48	0.36	1.82	1.24	1.05	0.67	1.85	1.88	1.78	1.99	1.60	1.84	1.13	1.90	1.92	1.65	1.48	1.81	
b26	0.47	0.50	0.95	1.15	0.23	1.06	0.16	0.69	0.14	0.22	0.12	1.53	1.57	1.88	0.25	1.15	1.55	1.94	0.62	0.55	0.85	0.64	1.21	1.05	1.87	1.14	1.14	1.49	1.69	1.28	
b29	0.55	0.32	1.27	1.43	0.07	1.39	0.03	1.12	0.45	0.04	0.07	1.14	1.19	1.92	0.39	0.70	1.84	1.69	0.87	0.40	1.13	0.61	1.49	1.22	1.64	1.19	1.12	1.55	1.69	1.12	
b9	0.57	0.35	1.26	1.42	0.09	1.37	0.04	1.08	0.41	0.05	0.05	1.21	1.25	1.93	0.35	0.75	1.81	1.73	0.83	0.39	1.08	0.58	1.43	1.18	1.66	1.17	1.10	1.53	1.67	1.10	
b21	0.60	0.88	0.24	0.14	1.33	0.23	1.45	0.61	1.35	1.77	1.84	0.62	0.60	0.14	1.86	1.41	0.68	0.56	1.68	1.99	1.53	1.97	1.25	1.57	0.95	1.66	1.72	1.33	1.16	1.67	
b27	0.61	0.95	0.22	0.11	1.37	0.19	1.47	0.56	1.29	1.80	1.85	0.68	0.67	0.15	1.82	1.48	0.61	0.60	2.00	1.46	1.96	1.18	1.50	0.96	1.61	1.68	1.29	1.13	1.67		
b18	0.75	1.29	0.11	0.08	1.56	0.06	1.59	0.20	0.87	1.83	1.71	1.27	1.25	0.32	1.38	1.87	0.28	0.97	1.08	1.87	0.91	1.68	0.69	1.03	1.17	1.27	1.40	1.00	0.99	1.57	
b17	0.80	1.37	0.52	0.65	1.21	0.49	1.11	0.17	0.17	1.11	0.83	1.97	1.96	1.22	0.41	1.85	0.49	1.73	0.25	1.06	0.26	0.81	0.36	0.45	1.58	0.73	0.85	0.82	1.02	1.19	
b14	1.01	1.01	0.76	0.57	1.41	0.70	1.53	1.19	1.82	1.69	1.88	0.23	0.22	0.10	2.00	0.89	0.80	0.14	1.80	1.72	1.64	1.77	1.30	1.52	0.44	1.43	1.43	1.08	0.82	1.21	
b24	1.05	1.47	0.68	0.81	1.24	0.65	1.12	0.29	0.20	1.05	0.78	1.99	1.99	1.28	0.31	1.76	0.49	1.71	0.15	0.92	0.16	0.65	0.27	0.32	1.49	0.57	0.67	0.68	0.92	1.01	
b30	1.18	1.66	0.40	0.33	1.87	0.30	1.88	0.44	1.12	1.96	1.83	1.84	1.24	1.20	0.22	1.39	1.71	0.07	0.72	0.91	1.74	0.70	1.45	0.41	0.72	0.79	0.9	1.03	0.60	0.55	1.16
b6	1.19	1.36	0.65	0.47	1.72	0.57	1.82	0.97	1.69	1.92	2.00	0.55	0.51	0.01	1.88	1.17	0.42	0.20	1.51	1.77	1.27	1.68	0.91	1.19	0.40	1.17	1.23	0.78	0.58	1.13	
b7	1.19	1.34	0.67	0.49	1.71	0.59	1.80	1.00	1.73	1.91	1.99	0.51	0.48	0.01	1.89	1.13	0.45	0.19	1.53	1.76	1.32	1.68	0.93	1.20	0.39	1.18	1.21	0.80	0.58	1.12	
b5	1.24	1.33	1.25	1.41	1.91	1.26	1.76	0.77	0.26	0.54	0.32	1.88	1.88	1.74	0.03	1.24	0.90	1.76	0.06	0.33	0.16	0.19	0.42	0.24	1.39	1.35	0.38	0.64	0.90	0.63	
b3	1.34	1.53	0.72	0.55	1.85	0.61	1.92	0.96	1.67	1.95	1.99	0.67	0.63	0.03	1.76	1.21	0.29	0.23	1.34	1.69	1.10	1.55	0.72	0.98	0.35	0.97	1.04	0.61	0.43	0.99	
b13	1.39	1.55	0.78	0.61	1.85	0.68	1.92	1.03	1.70	1.93	1.98	0.64	0.61	0.03	1.77	1.15	0.31	0.19	1.34	1.64	1.10	1.50	0.72	0.97	0.30	0.92	1.00	0.57	0.40	0.92	
b1	1.64	1.30	1.54	1.37	1.45	1.47	1.48	1.79	1.98	1.32	1.51	0.28	0.25	0.49	1.60	0.34	0.91	0.06	1.41	0.99	1.30	1.03	1.05	1.04	0.04	0.78	0.72	0.59	0.37	0.44	
b8	1.67	1.95	0.90	0.80	1.99	0.76	1.96	0.78	1.22	1.79	1.65	1.32	1.27	0.42	1.13	1.41	0.03	0.66	0.62	1.27	0.39	0.96	0.15	0.33	0.48	0.40	0.51	0.18	0.17	0.63	
b4	1.74	1.67	1.62	1.68	1.27	1.55	1.19	1.19	0.77	0.74	0.58	1.61	1.59	1.47	0.25	0.89	0.73	1.26	0.11	0.24	0.13	0.07	0.25	0.08	0.81	0.05	0.05	0.23	0.39	0.19	
b20	1.79	1.82	1.41	1.47	1.49	1.32	1.36	1.01	0.76	0.99	0.81	1.69	1.69	1.26	0.34	1.12	0.48	1.22	0.08	0.46	0.05	0.22	0.10	0.01	0.79	0.04	0.08	0.14	0.30	0.27	
b12	1.78	1.96	1.07	0.97	1.99	0.94	1.96	0.95	1.38	1.76	1.66	1.17	1.13	0.41	1.18	1.22	0.10	0.51	0.67	1.17	0.45	0.91	0.20	0.34	0.32	0.36	0.42	0.11	0.08	0.48	
b19	1.96	1.63	1.86	1.79	1.43	1.78	1.37	1.72	1.50	0.97	1.00	0.91	0.87	1.05	0.85	0.41	0.81	0.54	0.67	0.41	0.60	0.33	0.55	0.40	0.21	0.18	0.12	0.18	0.17	0.02	

Fig. 11: Correlation table representing the same data as Fig. 9, but columns and rows permuted in such a way that the upper row and the leftmost column have monotonically increasing values, starting from the top-left corner

values from the full spectrum of correlations. In a sense, all keys are equal; there are no obvious preferred keys.

Fig. 11 is derived from Fig. 9. Rows and columns have been permuted such that the cell with the lowest value ends up in the upper left corner, and the leftmost column and top row are sorted. What Fig. 11 tells more clearly than Fig. 9 is that the cells do not just have completely random values in the interval [0, 2], but that there is some organisation: every row and column that is near the border is more sorted than rows and columns that include cells in the centre of the figure.

4.6 Violation of the CHSH inequality

We see cells surrounded by red or green squares in Figs. 9 and 11. The green cells are:

$$\begin{aligned}
 H^{obs}(a3, b2) &: 0.236972 \\
 H^{obs}(a24, b2) &: 1.642167 \\
 H^{obs}(a24, b20) &: 1.752507 \\
 H^{obs}(a3, b20) &: 1.668873
 \end{aligned} \tag{22}$$

By subtracting these values from 1, we obtain the expectation values $P^{obs}(a_i, b_j)$ (see Eq. (21)) that have values in the interval $[-1, 1]$, where 1 signifies perfect agreement and -1 signifies perfect non-agreement:

$$\begin{aligned} P^{obs}(a3, b2) &: 0.763028 \\ P^{obs}(a24, b2) &: -0.642167 \\ P^{obs}(a24, b20) &: -0.752507 \\ P^{obs}(a3, b20) &: -0.668873 \end{aligned} \quad (23)$$

With the CHSH expression Eq. (3) in mind, we subtract the first number from the sum of the other three, and we get:

$$\begin{aligned} -P^{obs}(a3, b2) + P^{obs}(a24, b2) + P^{obs}(a24, b20) + P^{obs}(a3, b20) = \\ -0.763028 - 0.642167 - 0.752507 - 0.668873 = -2.826575 \end{aligned} \quad (24)$$

This is the largest violation of the CHSH inequality and is close to Tsirelson's bound, which is approximately 2.828427.

The red cells are:

$$\begin{aligned} H^{obs}(a7, b4) &: 0.247834 \\ H^{obs}(a10, b4) &: 0.239565 \\ H^{obs}(a10, b30) &: 0.405787 \\ H^{obs}(a7, b30) &: 1.708426 \end{aligned} \quad (25)$$

Normalised to the interval $[-1, 1]$:

$$\begin{aligned} P^{obs}(a7, b4) &: 0.752166 \\ P^{obs}(a10, b4) &: 0.760435 \\ P^{obs}(a10, b30) &: 0.594213 \\ P^{obs}(a7, b30) &: -0.708426 \end{aligned} \quad (26)$$

Subtracting $P^{obs}(a7, b30)$ from the sum of the other three numbers, we again get a number very close to Tsirelson's bound:

$$\begin{aligned} P^{obs}(a7, b4) + P^{obs}(a10, b4) + P^{obs}(a10, b30) - P^{obs}(a7, b30) = \\ 0.752166 + 0.760435 + 0.708426 + 0.594213 = 2.815240 \end{aligned} \quad (27)$$

There are many more quadruplets of cells containing values that, once substituted in Eq. (3), violate the CHSH inequality Eq. (2). This shows that the data indeed contain correlations that violate the CHSH inequality to the same degree as QM does.

4.7 Mismatched observations are not correlated.

To see whether the statistical dependence only exists because Alice's and Bob's observations come in pairs, Carol intentionally combines Alice's and Bob's observations

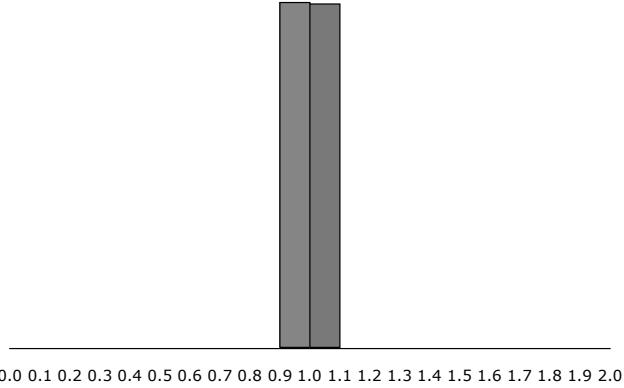


Fig. 12: Histogram showing the fractions of combinations of key choices and the outcomes of colour observations relative to the expected fraction if all variables were independent

in the wrong way by pairing Alice's first observation with Bob's second observation, Alice's second with Bob's third, etc., finally combining Alice's last observation with Bob's first. All statistics are approximately the same as before, except that all four variables are uncorrelated. Compare Fig. 10, which relates to matched observations, and Fig. 12, which relates to mismatched observations.

The statistical dependence between paired observations (Figs. 9 and 10) is not a manifestation of magic, provided that there is a physical explanation. Without an explanation in terms of something Alice and Bob, or their apparatuses, have in common, it is not possible to understand why Figs. 10 and 12 are so different.

5 Unhiding the hidden green fluid

5.1 The fluid-filled glass tube behind the slit

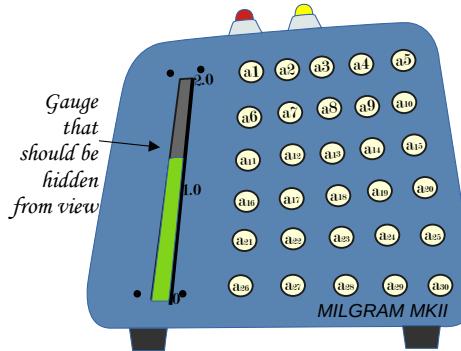


Fig. 13: Alice's apparatus, cover over the fluid level gauge removed

Let us return to Alice's apparatus. Fig. 13 shows her apparatus without the cover with the slit that sometimes displayed a green colour and sometimes didn't. Behind the cover is a glass pipe filled with a green fluid. There is a scale along the full length of the glass pipe, running from 0.0 at the lowest level via 1.0 in the middle to 2.0 at the top of the glass pipe. If the fluid level G_A or G_B was above the 1.0 mark, the colour visible through the slit was green. Otherwise, one just saw the indistinct background behind the glass. The slits of Alice's and Bob's apparatuses implement step functions A and B of the green fluid levels G_A and G_B as in Eq. (28).

$$g_A = A(G_A) = \begin{cases} \text{not-green} & \text{if } 0 \leq G_A \leq 1 \\ \text{green} & \text{if } 1 < G_A \leq 2 \end{cases} \quad (28)$$

$$g_B = B(G_B) = \begin{cases} \text{not-green} & \text{if } 0 \leq G_B \leq 1 \\ \text{green} & \text{if } 1 < G_B \leq 2 \end{cases}$$

Let us also remove the lid covering the gauge of Bob's apparatus and ask Alice and Bob to do another thirty million observations. This time, they must observe and report the height of the column of green fluid. Their readings are numbers between 0.0 and 2.0, which they must pass to Carol using their telegraph equipment, together with the labels on the last pressed keys. Instead of placing check marks in some fields in the green column, Carol now writes numbers between 0.0 and 2.0 in all green column fields as indications of the level of the green fluid in Alice's and Bob's apparatuses.

5.2 Carol's statistical investigation of Alice's and Bob's combined fluid level observations

Carol repeats all statistical tests. Previously, the colour observations were discrete variables. This time, they are continuous. Carol applies coarse-graining to transform the continuous variables into discrete variables that can be used in a statistical analysis in the same way as the former binary colour observations.

5.2.1 Alice's and Bob's green fluid levels are uniformly distributed.

Is any fluid level as probable as any other? To measure this, Carol divides the interval $[0, 2]$ into twenty equally sized subintervals, $[0.0, 0.1]$, $[0.1, 0.2]$, $[0.2, 0.3]$, etc. There is nothing special about the number 20 other than that it conveniently divides the interval into bins with a subinterval size of $\frac{1}{10}$. Carol could have chosen any other number, provided that it is much smaller than the number of trials. Each trial delivers two fluid levels. Each of these fluid levels is put in one of the twenty bins. Iterating over all trials, all fluid levels are binned. For example, if a million fluid levels are in the subinterval $[1.2, 1.3]$, then Carol puts the value 1,000,000 in the bin labelled 13. So, whereas Carol earlier counted how many times Alice and Bob observed *not-green* and *green*, she now counts the number of fluid levels in the first bin, the second bin, and so on, until the last bin. Carol has essentially increased the number of bins from just two (*not-green* and *green*) to a higher number, in recognition of the fact that Alice and Bob see the whole column of green fluid.

Carol notices that each bin is equally filled, with only small variations.

$$\frac{F(G_X)}{\frac{1}{\#\text{bins}}} = 1.00000 \pm 0.00036 \quad (29)$$

Since Alice and Bob each have thirty keys and Alice's and Bob's fluid levels are coarse-grained into twenty bins, there are $30 \times 30 \times 20 \times 20 = 360,000$ different combinations.

Carol should not choose a much higher number of bins, because with 30,000,000 trials, each of the 360,000 combinations occurs, on average, only $\frac{30,000,000}{360,000} = 83.\bar{3}$ times. This number, which is not high for a statistical analysis, would decrease even more if the number of bins were increased.

5.2.2 (a_A_), (_b_B): Alice's (Bob's) green fluid level is not correlated with her (his) key choice

Carol finds that the level of green fluid in assistant X 's apparatus is not dependent on which key X chooses to press, as the small standard deviations in Eqs. (30a) and (30b) prove.

$$\frac{F(s_A, G_A)}{F(s_A) F(G_A)} = 1.0000 \pm 0.0044 \quad (30\text{a})$$

$$\frac{F(s_B, G_B)}{F(s_B) F(G_B)} = 1.0000 \pm 0.0042 \quad (30\text{b})$$

Carol concludes that Alice's or Bob's key choice does not single-handedly determine the fluid level of their apparatus.

5.2.3 (a_B_), (_bA_): Alice's (Bob's) green fluid level is not correlated with his (her) key choice

It is also not the case that the level of green fluid in the apparatus of assistant X depends on which key the other assistant Y chooses. See Eqs. (31a) and (31b) and Figs. 14 and 15.

$$\frac{F(s_A, G_B)}{F(s_A) F(G_B)} = 1.0000 \pm 0.0044 \quad (31\text{a})$$

$$\frac{F(s_B, G_A)}{F(s_B) F(G_A)} = 1.0000 \pm 0.0043 \quad (31\text{b})$$

Alice's key choice does not determine Bob's green fluid level and vice versa. Nor does Alice's green fluid level restrict Bob's key choice.

5.2.4 (abA_), (ab_B): Neither Alice's nor Bob's green fluid level is correlated with the combination of their key choices

Even though Alice's or Bob's choices, in isolation, do not determine the level of green fluid in Alice's or Bob's apparatus, it could be that the combination of Alice's and Bob's key choices determines the level of fluid when Alice or Bob makes an observation.

$$F(s_X, s_Y, G_X) \stackrel{?}{=} F(s_X) F(s_Y) F(G_X) \quad (32)$$

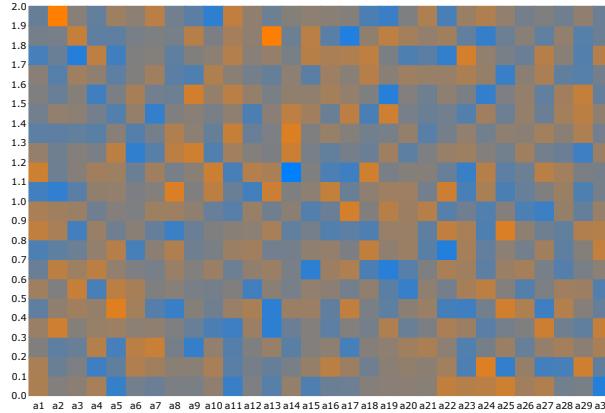


Fig. 14: a...B: Distribution showing (lack of) statistical dependence between the level of green fluid in Bob's apparatus and the key Alice chooses

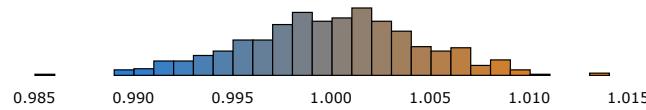


Fig. 15: a...B: Histogram showing distribution of occupation numbers in Fig. 14

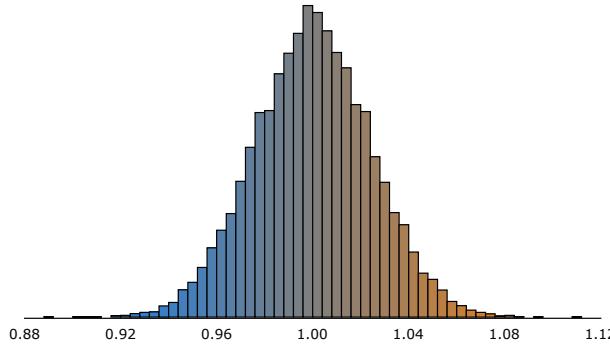


Fig. 16: abA...: Statistical dependence between Alice's and Bob's key choices and Alice's green fluid level

The histogram Fig. 16 shows, and the small standard deviation proves that there is no statistical dependence:

$$\begin{aligned} \frac{F(s_A, s_B, G_A)}{F(s_A) F(s_B) F(G_A)} &= 1.000 \pm 0.024 \\ \frac{F(s_A, s_B, G_B)}{F(s_A) F(s_B) F(G_B)} &= 1.000 \pm 0.024 \end{aligned} \quad (33)$$

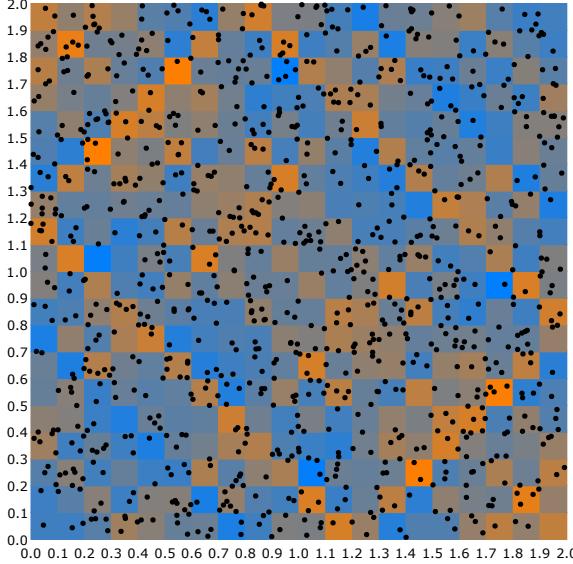


Fig. 17: --AB : Density map of combinations of fluid levels

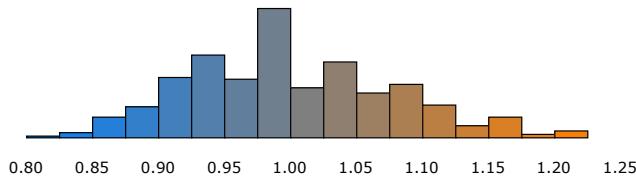


Fig. 18: --AB : Distribution of the occupation numbers in Fig. 17

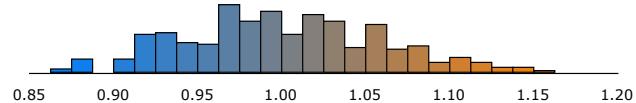


Fig. 19: --AB : Distribution if Alice and Bob have twice as many keys to choose from

5.2.5 (--AB): Alice's and Bob's green fluid levels are not correlated

Carol plots how often each of the 20×20 combinations of coarse-grained fluid levels occurs. See the density map in Fig. 17. On top of the coloured cells, Carol plots the first 1,000 (out of a total of thirty million) observations as black dots. The position of each dot has two components, both in the interval $[0, 2]$, see the scales below and to the left of the density map. The horizontal component is the level of the green fluid in Alice's apparatus. The vertical component is Bob's green fluid level. There are diagonal bands visible in the coloured cells. There is some statistical dependence, but not enough to make a clear pattern appear in the distribution of the 1,000 dots. The standard deviation is not small. See Eq. (34). The height of a bar in the histogram Fig.

[18](#) indicates how often densities occur in Fig. [17](#) that are printed against backgrounds with colour shades similar to the colour of the bar.

In an experiment with apparatuses that have twice as many keys, it is found that the statistical dependence diminishes as more keys are added. The standard deviation is smaller, 0.058, if Alice and Bob have 60 keys to choose from. See the histogram Fig. [19](#), which is slightly more concentrated around the central value 1.0 than Fig. [18](#).

$$\frac{F(G_A, G_B)}{F(G_A) F(G_B)} = 1.000 \pm 0.077 \quad (34)$$

5.2.6 (a_AB), (_bAB): Alice's green fluid level, Bob's green fluid level, and either Alice's or Bob's key choice are not correlated

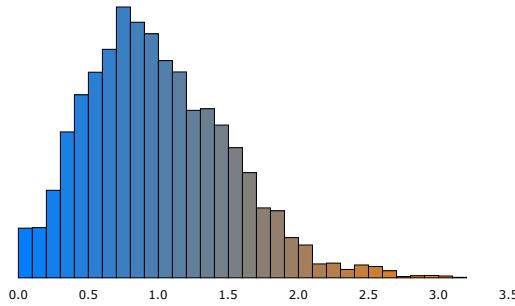


Fig. 20: a_AB: Distribution of co-occurrence of Alice's key choice and Alice's and Bob's green fluid levels

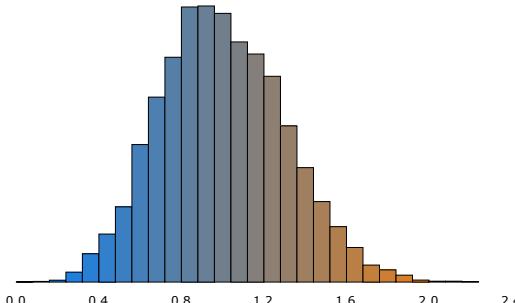


Fig. 21: a_AB: Distribution of co-occurrence of Alice's key choice and Alice's and Bob's green fluid levels if Alice and Bob have twice as many keys to choose from

Is there a statistical dependence between Alice's key choice and Alice's and Bob's green fluid heights in combination? Could, for example, some key choice dictate that the heights are the same? The standard deviation in Eq. (35) is quite large, so a

statistical dependence cannot be excluded yet.

$$\begin{aligned} \frac{F(s_A, G_A, G_B)}{F(s_A) F(G_A) F(G_B)} &= 1.00 \pm 0.51 \\ \frac{F(s_B, G_A, G_B)}{F(s_B) F(G_A) F(G_B)} &= 1.00 \pm 0.48 \end{aligned} \quad (35)$$

Fig. 20 shows a broad distribution, with a peak near 1.0. Carol's later experiment with twice as many keys on each apparatus indicates that the statistical dependence eventually will disappear if more and more keys are added to the apparatuses. The standard deviation in Fig. 21, which refers to the experiment with apparatuses that have 60 keys, is already somewhat lower, 0.30.

5.2.7 $\neg(abAB)$: Alice's green fluid level, Bob's green fluid level, Alice's key choice, and Bob's key choice are strongly correlated

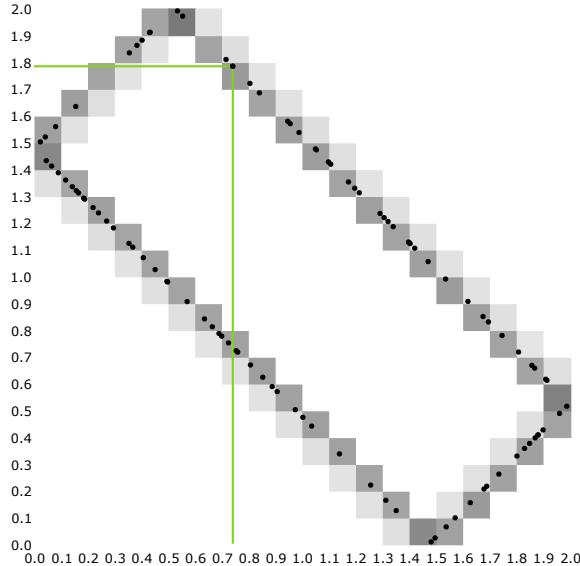


Fig. 22: abAB: Statistical dependence between the levels of green fluid in Alice's and Bob's gauges for key choices (a_{10}, b_{15}) , black dots indicating the first 100 trials and horizontal and vertical lines representing Alice's and Bob's green fluid columns observed during a single trial

Carol notices that all four variables in combination, the two fluid levels and the two key choices, are strongly correlated. The density map Fig. 22 visualises the statistical dependence between the green fluid levels for a particular, but arbitrary, pair of keys (a_{10}, b_{15}) .

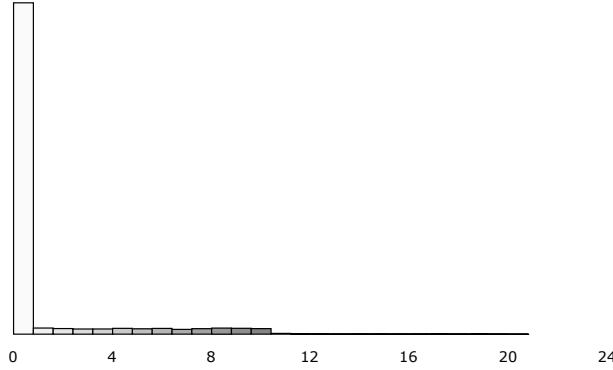


Fig. 23: abAB: Distribution of the occupation numbers in Fig. 22

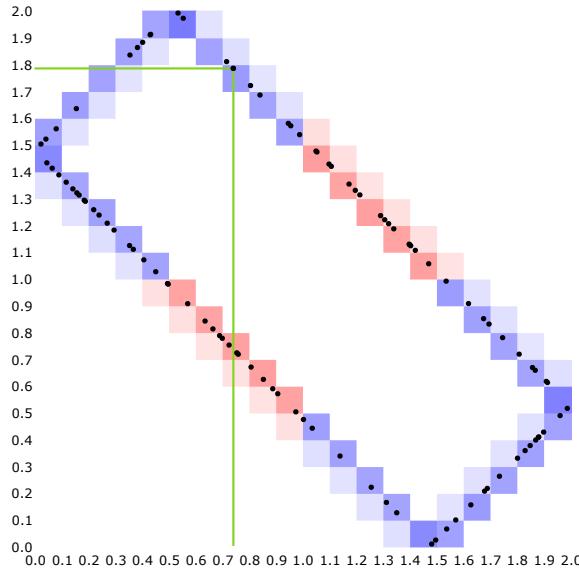


Fig. 24: The same density map as Fig. 22, shades of red for trials where Alice and Bob make the same colour observations, shades of blue for trials where Alice and Bob make different observations

The high bar near the value 0 in the histogram Fig. 23 shows that almost all combinations of the four variables never occur. The remaining combinations appear many times more often (> 20 times in Fig. 23) than statistically expected if the variables were independent.

The positions of the first 100 trials (out of 33373) where Alice had chosen key a_{10} and Bob had chosen key b_{15} , are plotted in Fig. 22 as black dots. Each dot represents the levels of two columns of green fluid. For example, there is a dot at $(0.742349, 1.781968)$, indicating that Alice's green fluid level was $G_A = 0.742349$ and

Bob's $G_B = 1.781968$. The horizontal green line depicts Alice's green fluid G_A . The longer vertical green line depicts Bob's green fluid level G_B .

The dots are, without exception, restricted to the perimeter of a rectangle. It is almost as though Alice, if she knew her own green fluid level and both her own and Bob's key choices, would be able to predict or even determine Bob's green fluid level, but this is never the case, except if her green fluid level was reaching one of the corners of the rectangle. In those four cases, Bob's green fluid level is predictable with certainty. In all other cases, there are two possibilities, and the probability that Alice guesses the correct height of Bob's green fluid is only 50%.

For other key pairs, the diagram looks similar, but with different proportions between adjacent sides. The corners of each rectangle always touch the sides of a square spanned by the points $(0, 0)$, $(2, 0)$, $(2, 2)$ and $(0, 2)$, but the points where the rectangle touches the surrounding square are unique to the actual pair of key choices. For each pair of key choices, we can identify these points by a constant $H^{obs}(s_A, s_B)$ as in Eq. (36).

$$\begin{aligned} \text{Bottom: } & (H^{obs}(s_A, s_B), 0) \\ \text{Left: } & (0, H^{obs}(s_A, s_B)) \\ \text{Top: } & (2 - H^{obs}(s_A, s_B), 2) \\ \text{Right: } & (2, 2 - H^{obs}(s_A, s_B)) \end{aligned} \quad (36)$$

5.3 The expectation value $P^{obs}(s_A, s_B)$

Fig. 24 is the same density map as Fig. 22, but displays one extra piece of information. Dots plotted in the red cells denote trials where Alice and Bob see the same colour in the slits of their apparatuses. So, Alice and Bob either both see no colour (lower left quadrant) or they both see a green colour through the slits of their apparatuses (upper right quadrant). The observations in the blue fields are different. In the upper left quadrant are those observations where Bob observes *green* and Alice observes *not-green*. The observations in the lower right quadrant are those where Alice observes *green* and Bob observes *not-green*.

To compute the expectation value $P^{obs}(a_{10}, b_{15})$, we must count the number of trials in the red fields, subtract the number of trials in the blue fields, and divide the result by the number of trials. See Eq. (37).

$$\begin{aligned} P^{obs}(s_A, s_B) &= \frac{\sum_{g_A=0}^1 \sum_{g_B=0}^1 (-1)^{g_A+g_B} F(s_A, s_B, g_A, g_B)}{F(s_A, s_B)} \\ &= \frac{\sum_{n=1}^N A(G_A(n)) B(G_B(n)) \delta_{s_A(n), s_A} \delta_{s_B(n), s_B}}{F(s_A, s_B)} \end{aligned} \quad (37)$$

The trials are randomly distributed along the perimeter of the rectangle, with a uniform density. As the number of trials increases, the result is approached by integration of $A(G_A) B(G_B)$ along the perimeter of the rectangle over all allowed combinations of levels of green fluid (G_A, G_B) while keeping the key choices (a_{10}, b_{15}) fixed. By doing

this integration for all combinations of keys, we find all values of the expectation value $P^{obs}(a, b)$. The next section is devoted to this integration.

6 Local Hidden Variables that explain the outcome of each trial

6.1 The Loop of Four

In this section, we make an abstract detour and connect the thought experiment as described in the previous sections with the theorem by J.S. Bell. Fig. 25 depicts three

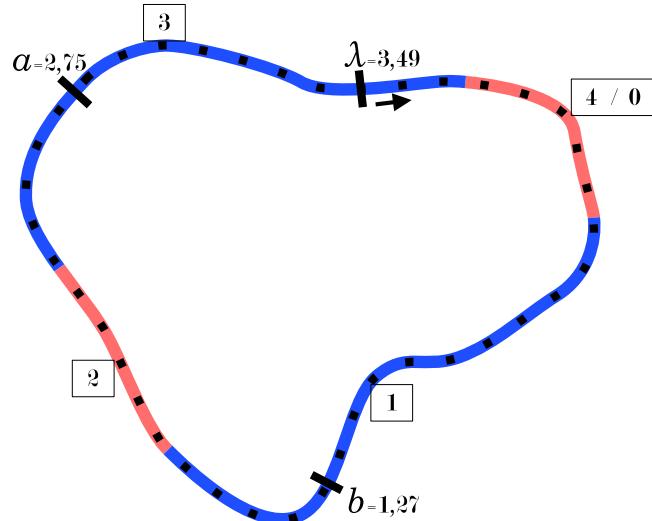


Fig. 25: Points λ , a and b , on a loop with a circumference of 4, red parts being either farther from or closer to both a and b than 1

points λ , a , and b on a loop that has a circumference of 4. We define a scale along the loop that goes from 0 to 4, so each of the three points has a coordinate value in the interval $[0, 4]$. Every pair $(x, y) \in \{(a, \lambda), (a, b), (b, \lambda)\}$ divides the loop into two sections. The lengths of the two sections always add up to 4.

Points against a red background in Fig. 25 are either farther from both a and b than 1 or closer to both a and b than 1. For the points with a blue background, the opposite is the case.

Figs. 26a and 26b show the same loop as Fig. 25. In each figure, the length of the green loop segment is equal to the height of the column of green fluid in the corresponding apparatus. The position of the observation slit is also represented.

Figs. 25 and 26 resemble Fig. 24 in several ways.

- They do all show closed paths: a rectangle in Fig. 24 and a loop in Figs. 25 and 26.
- They have three special points: Fig. 24 has the black dot where the two green line segments meet, the left corner of the rectangle, where Alice's green fluid level

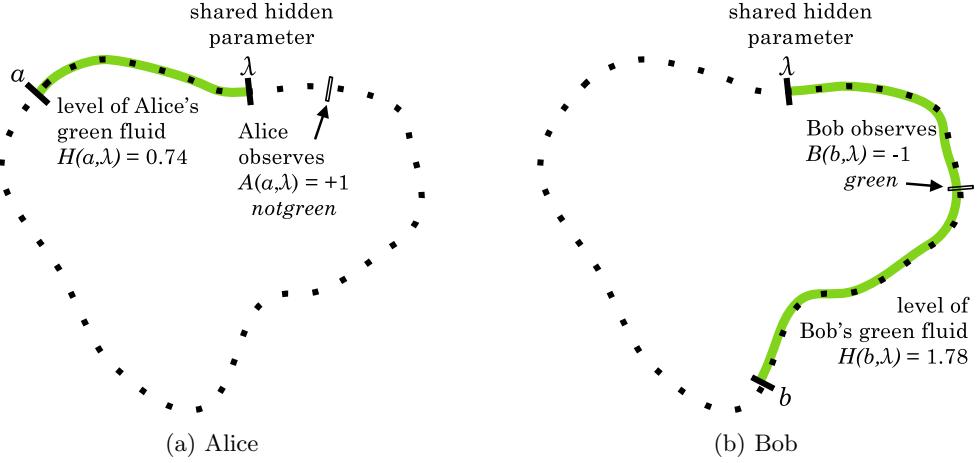


Fig. 26: Observations by Alice and Bob as explained by a shared hidden parameter

is zero, and the bottom corner of the rectangle, where Bob's green fluid level is zero. Figs. 25 and 26 have λ , a , and b .

- Points against a red background in Fig. 24 correspond to points against a red background in Fig. 25, and points against a blue background in Fig. 24 correspond to points against a blue background in Fig. 25.
- Horizontal and vertical green lines no longer than 2 can be drawn from the vertical and horizontal axes to every point of the rectangle in Fig. 24. Any special point in Fig. 25 can be reached in one way by two curves no longer than 2 that follow the outline of the loop, starting from the two other special points. For example, in Figs. 26a and 26b, λ is reached from a and b by curves that are no longer than 2.

But there are also differences.

- The rectangle in Fig. 24 has a perimeter of $4\sqrt{2}$. The loop in Fig. 25 has a circumference of just 4.
- The green lines in Fig. 24 cross the 2-dimensional plane, while the green curves in Figs. 26a and 26b stay within the confines of the loop, which is 1-dimensional.
- In Fig. 24, each dot is specified by two numbers, the green fluid levels of Alice and Bob. In Fig. 25, every point has only a single coordinate, since the loop is 1-dimensional.

The differences notwithstanding, the loop picture and the rectangle picture are isomorphic. Anything of physical importance that can be expressed in one picture can also be conveyed in the other. Therefore, we hypothesise that we can compute the expectation value P^{obs} by computing a corresponding expectation value P^{loop} . Until the hypothesis is proven to be false, we assume that the sum of all products of colour observation pairs in Fig. 24 is approached by an integral along the full loop in Fig. 25, using a single scalar parameter λ that runs from 0 to 4 from an arbitrarily chosen origin point in an arbitrary direction, clockwise or anticlockwise. We will compute the expectation value $P^{loop}(a, b)$ and compare that with the expectation value $P^{obs}(s_A, s_B)$

that Carol had found by analysing the results from a large number of trials in her experiment. The comparison must give answers to four questions:

1. What is the connection between the key choices s_A and s_B on the one hand and the numbers a and b on the other hand?
2. Are the minimum and maximum values of P^{loop} and P^{obs} the same?
3. Are the distributions of P^{loop} and P^{obs} the same?
4. Is the CHSH inequality violated to the same degree as in Carol's experiment?

To arrive at the aforementioned integral, we first need to define the separation H and the step function A that determines the result of a colour observation, given the value of H .

Let x and y be two points on the loop. Let $|x - y|$ be the absolute value of the difference between x and y . Define the separation $H^{\text{loop}}(x, y)$ between x and y as the length of the shortest loop section between x and y :

$$H^{\text{loop}}(x, y) = \min(|y - x|, 4 - |y - x|) \quad (38)$$

For example, in Fig. 25

$$\begin{aligned} H^{\text{loop}}(a, b) &= |1.27 - 2.75| = 1.48 \\ H^{\text{loop}}(a, \lambda) &= |3.49 - 2.75| = 0.74 \\ H^{\text{loop}}(b, \lambda) &= 4 - |3.49 - 1.27| = 1.78 \end{aligned} \quad (39)$$

The value 1.48 of $H^{\text{loop}}(a, b)$ in Eq. (39) is not only the separation between a and b . It is also the level of green fluid in Bob's or Alice's apparatus when the level of green fluid in the other apparatus is zero. This happens when λ coincides with a or b , respectively. These two exceptional cases correspond to the left and bottom corners of the rectangle in Fig. 24. With Eq. (36), it follows therefore that for any pair of key choices (s_A, s_B) , there is a (non-unique) pair of points (a, b) such that Eq. (40) is valid:

$$H^{\text{obs}}(s_A, s_B) = H^{\text{loop}}(a, b) \quad (40)$$

The separations $H^{\text{loop}}(a, \lambda)$ and $H^{\text{loop}}(b, \lambda)$, which are the lengths of the green curves in Figs. 26a and 26b, vary over the interval $[0, 2]$. According to our hypothesis, they correspond to the levels of the green fluid in Alice's and Bob's apparatuses. See Eq. 41.

$$\begin{aligned} G_A &\cong H^{\text{loop}}(a, \lambda) \\ G_B &\cong H^{\text{loop}}(b, \lambda) \end{aligned} \quad (41)$$

We have now defined how the separations H in the loop correspond to levels of green fluid in a trial during Carol's experiment.

Next, we define step functions $A(a, \lambda)$ and $B(b, \lambda)$ as follows:

$$\begin{aligned} A(a, \lambda) &= \text{sgn}(1 - H^{\text{loop}}(a, \lambda)) \\ B(b, \lambda) &= \text{sgn}(1 - H^{\text{loop}}(b, \lambda)) \end{aligned} \quad (42)$$

where

$$\text{sgn}(v) = \begin{cases} -1 & \text{if } v < 0 \\ 0 & \text{if } v = 0 \\ +1 & \text{if } v > 0 \end{cases} \quad (43)$$

If we ignore the case where $H^{\text{loop}} = 1$, which has zero probability of occurring, then Eq. (44) is an implication of the hypothesis that the loop is a model of Carol's experiment.

$$g_A = \begin{cases} \text{green} & \text{if } A^{\text{loop}}(a, \lambda) = -1 \\ \text{not-green} & \text{if } A^{\text{loop}}(a, \lambda) = 1 \end{cases} \quad (44)$$

$$g_B = \begin{cases} \text{green} & \text{if } B^{\text{loop}}(b, \lambda) = -1 \\ \text{not-green} & \text{if } B^{\text{loop}}(b, \lambda) = 1 \end{cases}$$

One can check that the earlier Eq. (28) follows from Eqs. (41), (42), (43) and (44).

6.2 Expectation values $P^{\text{loop}}(a, b)$ and $P^{\text{obs}}(s_A, s_B)$

In Fig. 26, we have $A(a, \lambda) = 1$ (*not-green*) since $H^{\text{loop}}(a, \lambda) = 0.74 < 1$ and we have $B(b, \lambda) = -1$ (*green*) because $H^{\text{loop}}(b, \lambda) = 1.78 > 1$. In this trial of Carol's experiment, the product of the outcomes of Alice and Bob's spin component measurements is therefore determined by a, b and the hidden variable λ as in Eq. (45).

$$A(a, \lambda) B(b, \lambda) = 1 \times -1 = -1 \quad (45)$$

Let us consider the average value of $A(a, \lambda) B(b, \lambda)$ if λ is uniformly distributed over the interval $[0, 4]$. This average can take any value in the interval $[-1, 1]$ and is the expectation value $P^{\text{loop}}(a, b)$ of $A(a, \lambda) B(b, \lambda)$, even though we do not expect this product ever to be different from either -1 or 1 . We can compute $P^{\text{loop}}(a, b)$ by taking the average of infinitely many terms while varying λ over all values between 0 and 4 in infinitesimally small but constant steps, as follows:

$$P^{\text{loop}}(a, b) = \lim_{N \rightarrow +\infty} \frac{1}{4N-1} \sum_{k=0}^{4N-1} A\left(a, \frac{k}{N}\right) B\left(b, \frac{k}{N}\right)$$

$$= \frac{1}{4} \int_0^4 A(a, \lambda) B(b, \lambda) d\lambda \quad (46)$$

We can express $P^{\text{loop}}(a, b)$ in terms of the separation $H^{\text{loop}}(a, b)$ as follows. $A(a, \lambda) = 1$ on half of the loop, namely, where $H^{\text{loop}}(a, \lambda) < 1$. On the other half of the loop, where $H^{\text{loop}}(a, \lambda) \geq 1$, $A(a, \lambda) = -1$. Similarly, $B(b, \lambda) = 1$ on half of the loop. On the other half of the loop, $B(b, \lambda) = -1$. The borders between the half loops are not the same for a and b , but offset from each other by the separation $H^{\text{loop}}(a, b) = |b - a|$ between a and b . The two sections where $A(a, \lambda) = -B(b, \lambda)$ (the sections marked blue in Fig. 25) together have a length $2 \times H^{\text{loop}}(a, b)$. The two sections where $A(a, \lambda) = B(b, \lambda)$ (the sections marked red in Fig. 25) together have a length

$4 - 2 \times H^{loop}(a, b)$. The contribution to the integral from the two sections of the loop where $A(a, \lambda) = -B(b, \lambda)$ is $-1 \times [2 \times H^{loop}(a, b)]$. The contribution from the two loop sections for which $A(a, \lambda) = B(b, \lambda)$ is $1 \times [4 - 2 \times H^{loop}(a, b)]$. Adding all contributions together and dividing by the circumference of the loop, we obtain

$$P^{loop}(a, b) = \frac{-1 \times [2 \times H^{loop}(a, b)] + 1 \times [4 - 2 \times H^{loop}(a, b)]}{4} = 1 - H^{loop}(a, b) \quad (47)$$

According to Eq. (40), it then follows that Eq. (48) is true.

$$P^{loop}(a, b) = 1 - H^{obs}(s_A, s_B) \quad (48)$$

From Eqs. (21) and (48) follows Eq. (49).

$$P^{obs}(s_A, s_B) = P^{loop}(a, b) \quad (49)$$

If Alice's and Bob's green fluid levels, $H^{loop}(a, \lambda)$ and $H^{loop}(b, \lambda)$, have uniform distributions in $[0, 2]$ and are also independent, then $H^{loop}(a, b)$ cannot be fixed but must also have a uniform distribution in $[0, 2]$. With Eq. (47), it follows that P^{loop} has values in the interval $[0, 2]$ and that it is uniform in that interval. Both these conditions agree with Fig. 10, which shows the frequencies of the values of $P^{obs}(s_A, s_B)$. There is therefore no evidence as yet that Eq. (49) is false.

6.3 From keys a, b to S-G device settings \hat{a} and \hat{b}

In Carol's experiment, the number of keys is many times larger than the number of settings in the formulation of the experiment by Bell and CHSH, where Alice and Bob each have to choose between two settings. For example, in the description by CHSH, Alice can choose between \hat{a}_1 and \hat{a}_2 , and Bob can choose between \hat{b}_1 and \hat{b}_2 . See Eq. (3). In Bell and CHSH's approach, the number of combinations of settings is the minimum needed to derive the inequality that, according to Bell and CHSH, disqualifies local hidden variable theories. That minimalism has a price: QM only predicts violation of the Bell and CHSH inequalities for *some* combinations of settings, but not for *all* combinations. To test whether the inequalities are violated, the experiment must be set up with carefully chosen possible settings that are shared between Alice and Bob before the experiment starts. In Carol's experiment, preparatory coordination is completely absent. All Carol needs to do is command Alice's and Bob's apparatuses to select a large number, say 30, of isotropically distributed directions. This is done before the experiment starts. Each direction constitutes a possible setting for an S-G device that is steered by an apparatus with keys, one key for each possible setting. When a key is pressed, the S-G device is aligned in the corresponding direction. There is no coordination whatsoever between Alice's and Bob's apparatuses, and nobody, not even Carol, needs to share or even know which random directions were generated inside the apparatuses before the experiment started.

To ensure that Eq. (46) is a viable HV model, we must look for fulfilment of Eq. (47). We must find a separation function $H(\hat{a}, \hat{b})$ of the directions \hat{a} and \hat{b} that, just

like $P(\hat{a}, \hat{b})$, varies uniformly when \hat{a} and \hat{b} are repeatedly chosen at random from the two sets of isotropically distributed directions. In 3-D space, the angle between Alice's S-G device and Bob's S-G device will not be distributed uniformly in the interval $[0, \pi]$, but a linear function of the cosine of that angle will be. The separation function must obey one of the two sets of boundary conditions, as given in Eqs. (50a) and (50b).

$$H^{loop}(a, b) = \begin{cases} 0 & \text{if } \angle_{\hat{a}\hat{b}} = 0 \\ 2 & \text{if } \angle_{\hat{a}\hat{b}} = \pi \end{cases} \quad (50a)$$

$$H^{loop}(a, b) = \begin{cases} 0 & \text{if } \angle_{\hat{a}\hat{b}} = \pi \\ 2 & \text{if } \angle_{\hat{a}\hat{b}} = 0 \end{cases} \quad (50b)$$

Eq. (51a) defines the separation function that has a uniform distribution in the interval $[0, 2]$ and fulfils the boundary conditions in Eq. (50a). The boundary conditions Eq. (50b) are fulfilled by Eq. (51b)

$$H^{loop}(a, b) = 1 - \cos(\angle_{\hat{a}\hat{b}}) \quad (51a)$$

$$H^{loop}(a, b) = 1 + \cos(\angle_{\hat{a}\hat{b}}) \quad (51b)$$

The boundary conditions Eq. (50a) and separation function Eq. (51a) pertain to experiments where Alice's and Bob's particles are the same in every aspect. That is not the case in Bell-type experiments, where the particles received by Alice and by Bob have opposite spins. If Alice and Bob measure the spin components while the magnetic fields are oriented in the same direction, then Alice's and Bob's results - colour observations, in Carol's experiment - are always different. Although Alice and Bob share the hidden variable λ , Alice's copy λ_A of the hidden variable must neutralise Bob's copy λ_B . In the loop model, this can be done by offsetting λ_B from λ_A by half of the circumference of the loop, as in Eq. (52).

$$\lambda_B = (\lambda_A + 2) \pmod{4} \quad (52)$$

We define the step functions A and B as the same step function A , operating on Alice's λ_A and Bob's λ_B , respectively. See Eq. (53)

$$\begin{aligned} A(x, \lambda) &\stackrel{\triangle}{=} A(x, \lambda_A) \\ B(x, \lambda) &\stackrel{\triangle}{=} A(x, \lambda_B) \end{aligned} \quad (53)$$

Combining Eqs. (52) and (53) we obtain Eq. (54).

$$\begin{aligned} B(b, \lambda) &= A(b, (\lambda_A + 2) \pmod{4}) \\ &= -A(b, \lambda_A) \\ &= -A(b, \lambda) \end{aligned} \quad (54)$$

Inserting Eq. (54) into Eq. (46) gives Eq. (55):

$$P^{loop}(a, b) = -\frac{1}{4} \int_0^4 A(a, \lambda) A(b, \lambda) d\lambda \quad (55)$$

Eq. (55) is of the form Eq. (1), which is Bell's general local hidden variable expression.

Finally, Eqs. (51b) and (47) lead to Eq. (56).

$$P^{loop}(a, b) = -\cos(\angle_{\hat{a}\hat{b}}) \quad (56)$$

7 Contradiction

Eq. (55) has the same form as Eq. (1). According to Bell, any local hidden variable expression for the expectation value P must be expressible in that form. In Eq. (55), the role of the hidden variable λ is played by one of the three points in Fig. 25, which, with foresight, was already called λ . The density ϱ is constant, and points a and b take the roles of settings \hat{a} and \hat{b} . Bell and CHSH argued that any Local Hidden Variable theory of the form of Eq. (1) obeys the CHSH inequality Eq. (2), which the predictions of QM sometimes violate. This proves, according to Bell, that Eq. (1) cannot reproduce Eq. (4).

$$P^{LHV}(\hat{a}, \hat{b}) \neq P^{QM}(\hat{a}, \hat{b}) \quad (\text{according to Bell}) \quad (57)$$

In Sec. 4, we have seen that the outcomes produced by Alice and Bob's trials are similar to what an ideal Bell-type experiment might generate, even to the point that the CHSH inequality is violated to the same degree as predicted by QM. In Sec. 5, we demonstrated how hidden causes explained the outcomes in each trial. Sec. 6 presented a theory that incorporates the causes that were introduced in Sec. 5. That theory was modelled along the lines of Bell's template. In other words, we have shown that Eq. (58) is true.

$$P^{obs}(a, b) = P^{loop}(a, b) \cong P^{LHV}(\hat{a}, \hat{b}) \quad (58)$$

We have also shown that Eq. (59) is false for some values of a_1 , a_2 , b_1 , and b_2 , contradicting the conclusions of Bell and CHSH.

$$|P^{obs}(a_1, b_1) + P^{obs}(a_2, b_1) + P^{obs}(a_2, b_2) - P^{obs}(a_1, b_2)| \leq 2 \quad (\text{false}) \quad (59)$$

Who is right? Can Eq. (55) reproduce the predictions of QM and hence violate the CHSH inequality, or is Bell right?

8 Discussion

The Loop of Four is not a physical theory. It is a mathematical proposition that states that Bell's theorem does not apply to an HV theory that aims to explain how a common cause can result in the outcomes of a pair of actual spin component measurements. The Loop of Four proposition does not state whether that common cause is physical or the product of post-hoc rationalisation, analogous to the application of realistic-looking colours to a Daguerreotype. The Loop of Four can violate the CHSH inequality, but

dimension	$ S _{max}$	dimension	$ S _{max}$	dimension	$ S _{max}$
2	2.000000	6	3.697653	10	3.940175
3	2.828427	7	3.800699	11	3.959522
4	3.273240	8	3.867418	12	3.972511
5	3.535534	9	3.911184	20	3.998648

Table 1: The maximum value of the CHSH expression in n -dimensional space for some values of n , $2 \leq n \leq 20$

it does not predict the degree to which that inequality is violated. For that, a physical theory must be developed that explains the mechanism by which the values of the hidden variable ('green fluid levels') at Alice's and Bob's places are determined.

The Loop of Four requires that for each key Alice chooses, there is a uniform distribution of expectation values over the interval between -1 and $+1$ as Bob varies his key choices. And vice versa, Bob will see a uniform distribution of expectation values for any key he chooses. There are infinitely many ways in which such uniform distributions between a minimum value of -1 and a maximum value of $+1$ can be realised. The most straightforward way is as follows. Create the Cartesian product T of the set of Alice's keys $\{a_1, a_2, \dots, K | K \gg 2\}$ and Bob's keys $\{b_1, b_2, \dots, M | M \gg 2\}$. The cardinality of T is $|T| = K \times M$. Picking one random element (a_i, b_j) of T at a time, assign a unique separation to each element of T as in Eq. (60).

$$H(a_i, b_j) \in \left\{ \frac{2n - 1}{|T|} \mid n \in 1, 2, \dots, |T| \right\} \quad (60)$$

Use Eq. (48) to derive the expectation values from the separation values $H(a_i, b_j)$.

Although most sets T that are constructed in the way described in Eq. (60) will violate the CHSH inequality, few will do so to the same degree as QM. Probably most T will violate the CHSH inequality close to the maximum possible, 4.

There are natural selection criteria for T to reproduce the predictions by QM. The first condition is that T must violate the CHSH inequality by a value close to, and not exceeding, the Tsirelson boundary. In addition, T must allow that each key that Alice or Bob can press as preparation for the measurement that follows is associated with a vector in 3D-space such that Eq. (56) is fulfilled for all pairs of keys (a_i, b_j) . It is not the aim of this paper to speculate why these conditions exist.

Both conditions can be simultaneously met by considering isotropic distributions of vectors in spaces with D dimensions and uniform separation functions between pairs of such vectors. If space has only two spatial dimensions, the angles between pairs of randomly chosen vectors are uniformly distributed. The separation $H(\hat{a}, \hat{b})$ and the expectation value $P(\hat{a}, \hat{b})$ would vary linearly with the angle between \hat{a} and \hat{b} . Due to the triangle inequality, the CHSH inequality would not be violated in Flatland. In worlds with more than two dimensions, the angle between randomly chosen vectors is not uniformly distributed, and the CHSH inequality is violated. In three spatial dimensions, $|S|_{max}$ is Tsirelson's bound, $2\sqrt{2}$. In higher dimensions, $|S|_{max}$ approaches the super-quantum correlation [8] limit of 4; see Table 1 and Appendix B.

9 Conclusion

The original purpose of hidden variables theories was to describe Einstein's *elements of reality* that QM is unable to account for. The Loop of Four proposition does not satisfy Einstein's wish to portray every element of reality, because it does not accommodate the simultaneous determination of properties that are incommensurable according to QM. Rather, the Loop of Four is a way to attribute the collapse of the quantum wave function to a shared hidden variable.

Elements of reality can be treated as *properties* that have *values*. The purpose of measurements, in that view, is to bring such pre-existing values to light. However, some properties do not have pre-existing values but acquire those in the presence of a reference system.

A reference system is an abstract, theory-laden construct. However, in the context of Bell's thought experiment, which takes place far away from masses that are large enough to exert an appreciable influence on the structure of spacetime, it is almost a heresy to doubt the absolute truth of a Lorentzian system of coordinates pervading both Alice's and Bob's laboratories. This epistemological complication may seem innocuous, but if we circumvent any dependence on a system of reference in the description of Bell's thought experiment, we arrive at a conclusion that contradicts Bell's.

The presented version of the Bohm-Bell thought experiment is different from the usual exposition in two ways. Firstly, the experiment is described as an administrative task that involves columns, rows, numbers, and identifiers. I avoided mentioning *vectors* and *space*. Secondly, I applied statistics to the sample spaces available to the two assistants, Alice and Bob. Those sample spaces must be much larger than the few instrument settings available in the original formulation of the thought experiment.

I have shown that a hidden variable may exist with these properties:

- Alice and Bob share the hidden variable. The equation that expresses how the expectation value of the product of a pair of spin component measurements on a quantum mechanical singlet state depends on a hidden variable is isomorphic with the formula that, according to Bell, applies to any hidden variable theory that tries to reproduce the predictions made by QM.
- The hidden variable has a uniform distribution for each of Alice and Bob separately. Therefore, if Alice does not know Bob's setting, but somehow could observe the hidden variable, that knowledge would not enable her to predict the result of Bob's observation of the hidden variable, and vice versa. This property complies with locality and separability.
- The hidden variable can violate the CHSH inequality to the same degree as QM. For that, space must have three dimensions. The maximum degree to which the CHSH inequality is violated is shown to be indicative of the number of spatial dimensions.

A Computer algorithm

All statistics in this paper are the result of the analysis of real data. These data are not the result of a real experiment but were generated by a computer program¹. The program takes four parameters and does the following:

A.1 Represent Alice's and Bob's apparatuses as data structures

Each of these data structures is an array of settings. Command-line parameters set the number of array elements in each array. Throughout this text, it is assumed that each array has thirty elements, apart from a few explicit exceptions. Each setting has a name. For Alice's apparatus, the names run from $a1$ to $a30$ and for Bob's, from $b1$ to $b30$. By choosing a key, Alice or Bob decides on a setting. A setting has, apart from a name, also a data structure that represents a randomly generated vector in a d -dimensional space. The number of dimensions, d , is specified on the command line. Alice's and Bob's vector spaces have the same number of dimensions. The algorithm must generate vectors as though they were sampled at random from an infinite set of vectors with an isotropic distribution in all available dimensions. It is, for example, wrong to choose vectors that are evenly spread out in a plane if the number of dimensions is more than 2.

A.2 Compute separations between Alice's and Bob's settings

The separation H between one of Alice's vectors and one of Bob's vectors is a function of the angle between the vectors. This H must have the property that for an arbitrary vector of Alice, the number of Bob's vectors that are closer than a separation H is proportional to H . The same must be the case when Alice's and Bob's roles are swapped. If the number of dimensions is 2, then the angle itself is the correct function, but for all higher dimensions, the separation is a more complex function of the angle. See Appendix B.

The separation function must ensure that separation values are uniformly distributed over the interval $[0, 2]$ and that parallel vectors have a separation $H = 0$. Vectors that are each other's opposites have a separation $H = 2$. For example, if γ is the angle between Alice's vector and Bob's vector, then in the 2-dimensional case, the separation function is $H_2 = \frac{2\gamma}{\pi}$. If space is set to have 3 dimensions, then the correct function is $H_3 = 1 - \cos(\gamma)$.

A.3 Simulate trials using random hidden variables

Repeat 30,000,000 times:

- Draw a setting a from Alice's settings and, independently of the choice of Alice's setting, also draw a setting b from Bob's settings.
- Compute (or look up) the separation $H^{loop}(a, b)$ between the two setting vectors.
- Assign an arbitrary value in the interval $[0, 4]$ to Alice's setting.
- Assign an arbitrary value in the interval $[0, 4]$ to Bob's setting, with the restriction that the absolute value of the difference between Alice's setting and Bob's setting

¹<https://github.com/BartJongejan/simulated-Bell-data>

must be equal (modulo 4) to the separation between the vectors representing those two settings. Thus, if the separation $H^{\text{loop}}(a, b)$ is 0.4 and Alice's setting was assigned the arbitrary value 0.3, then Bob's setting is chosen at random from $\{0.7, 3.9\}$. Alternatively, one can set Alice's setting value to 0 and set Bob's setting value equal to the value $H^{\text{loop}}(a, b)$.

- Draw a random value in the interval $[0, 4)$ and assign that value to λ . It makes no difference whether the values of λ form a set of 30,000,000 equidistant fractional numbers $\{0/7500000, 1/7500000, 2/7500000, \dots, 29999999/7500000\}$ shuffled in a random order, or whether they are a random set of real numbers in the interval $[0, 4)$ with a uniform distribution in that interval.
- Compute the green fluid levels $H^{\text{loop}}(a, \lambda)$ and $H^{\text{loop}}(b, \lambda)$ for Alice and Bob: compute the absolute value of the smallest difference (modulo 4) between the value of λ and the value assigned to Alice's or Bob's setting. The result of this computation is always a number in the interval $[0, 2]$. That value is the green fluid level.
- Compute the outcomes $A(a, \lambda)$ and $A(b, \lambda)$ of the observations of the colours seen in the narrow slits of Alice's and Bob's apparatuses. If the green fluid level is less than 1.0, the observation is *not-green*, and otherwise it is *green*.

After all iterations have been performed, analyse the generated data as described in the previous sections.

B The uniformly distributed separation between isotropically distributed vectors in D dimensions

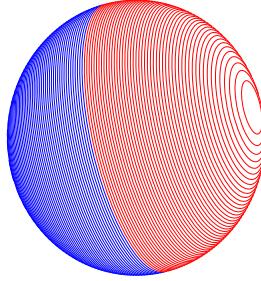


Fig. 27: A 2-sphere cut into rings having the same outward area

Consider a hyper-spherical cap of a d -sphere with unit radius. Let h be the height of the cap, and let γ be the polar angle between the unit vector from the centre of the sphere to the apex of the cap and the unit vectors from the centre of the sphere to the edge of the disk forming the base of the cap, $\gamma = \arccos(1 - h)$. For a positive constant number C , the area $O_d(\gamma)$ of this hyper-spherical cap is

$$O_d(\gamma) = C \int_0^\gamma \sin^{d-1}(\vartheta) \, d\vartheta = CI(\gamma, d) \quad (61)$$

Using integration by parts, we have, for $0 \leq \gamma \leq \pi$ and $d \geq 1$:

$$I(\gamma, d) = \int_0^\gamma \sin^{d-1}(\vartheta) d\vartheta = \begin{cases} \gamma & \text{if } d = 1 \\ 1 - \cos(\gamma) & \text{if } d = 2 \\ -\frac{1}{d-1} \sin^{d-2}(\gamma) \cos(\gamma) + \frac{d-2}{d-1} I(\gamma, d-2) & \text{if } d \geq 3 \end{cases} \quad (62)$$

Eq. (62) computes a number that is proportional to the area of the d -hyper-spherical cap that contains all points that are less than an angle γ from the apex point $\gamma = 0$. If N points are uniformly distributed over the d -sphere, then a fraction $\frac{I(\gamma, d)}{I(\pi, d)} N$ is expected to lie within the d -hyper-spherical cap.

Define the separation function $H_d(\gamma)$ as follows:

$$H_d(\gamma) = \frac{2I(\gamma, d)}{I(\pi, d)} \quad (63)$$

$H_d(\gamma)$ is proportional to the number of random unit vectors at angles smaller than γ from the apex. $H_d(\gamma)$ varies uniformly over the interval $[0, 2]$ if γ is the angle between unit vectors that are drawn at random. Eqs. (64), (65), (66), and (67) are the separation functions in 2, 3, 4 and 5 spatial dimensions. (The subscripts 1...4 refer to the dimensionality of the hypersphere, which is 1 lower than the number of dimensions of the embedding space.)

$$H_1(\gamma) = \frac{2\gamma}{\pi} \quad (64)$$

$$H_2(\gamma) = 1 - \cos(\gamma) \quad (65)$$

$$H_3(\gamma) = \frac{2\gamma}{\pi} - \frac{8}{\pi} \cos(\gamma) \sin^2(\gamma) \quad (66)$$

$$H_4(\gamma) = 1 - \cos(\gamma) - \frac{9}{2} \cos(\gamma) \sin^3(\gamma) \quad (67)$$

Formulate as follows a correlation $P_d(\gamma)$, $-1 \leq P_d(\gamma) \leq 1$ that is uniformly distributed if γ is the angle between two vectors \hat{x} and \hat{y} that both are chosen randomly from a uniform distribution over a d -sphere:

$$P_d(\gamma) = 1 - H_d(\gamma) \quad (68)$$

In the case of a circle ($d = 1$), P varies linearly with the length γ of the arc separating the vectors \hat{x} and \hat{y} . In the case of a 2-sphere ($d = 2$), P varies as $\cos(\gamma)$.

Fig. 27 shows one hundred spherical caps with a common apex. Each ring has the same area. Random unit vectors are uniformly distributed over the rings. The number of cuts between a unit vector and the vector pointing to the apex point is proportional to the separation H between those two vectors. Only on a 2-sphere is it the case that the rings have a fixed height. Therefore, H and P vary as $\cos(\gamma)$ on a 2-sphere. For fewer or more dimensions, the slices do still have the same outward area, and H is still proportional to the number of cuts, but the slices do not have the same height.

The expression Eq. (69)

$$3 * P_d(\gamma) - P_d(3\gamma) \quad (69)$$

reaches maxima for $\gamma = \frac{\pi}{4}$ and for $\gamma = \frac{3\pi}{4}$ for all values $d > 1$ because

$$\begin{aligned} \frac{d}{d\gamma} [3 * P_d(\gamma) - P_d(3\gamma)] &\propto \frac{d}{d\gamma} [3 * I(\gamma, d) - I(3\gamma, d)] \\ &\propto 3 \sin^{d-1}(\gamma) - \frac{d3\gamma}{d\gamma} \sin^{d-1}(3\gamma) \propto \sin^{d-1}(\gamma) - \sin^{d-1}(3\gamma) \end{aligned} \quad (70)$$

and

$$\begin{aligned} \sin^{d-1}\left(\frac{\pi}{4}\right) - \sin^{d-1}\left(\frac{3\pi}{4}\right) &= \sin^{d-1}\left(\frac{3\pi}{4}\right) - \sin^{d-1}\left(\frac{9\pi}{4}\right) = \left(\sqrt{\frac{1}{2}}\right)^{d-1} - \left(\sqrt{\frac{1}{2}}\right)^{d-1} \\ &= 0 \end{aligned} \quad (71)$$

References

- [1] Einstein, A., Podolsky, B., Rosen, N.: Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? *Phys. Rev.* **47**, 777–780 (1935) <https://doi.org/10.1103/PhysRev.47.777>
- [2] Bohr, N.: Can Quantum-Mechanical Description of Physical Reality be Considered Complete? *Phys. Rev.* **48**, 696–702 (1935) <https://doi.org/10.1103/PhysRev.48.696>
- [3] Bohm, D.: A Suggested Interpretation of the Quantum Theory in Terms of “Hidden” Variables. I. *Physical Review* **85**(2), 166–179 (1952) <https://doi.org/10.1103/PhysRev.85.166>
- [4] Bell, J.S.: On the Einstein Podolsky Rosen paradox. *Physics Physique Fizika* **1**, 195–200 (1964) <https://doi.org/10.1103/PhysicsPhysiqueFizika.1.195>
- [5] Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A.: Proposed Experiment to Test Local Hidden-Variable Theories. *Phys. Rev. Lett.* **23**, 880–884 (1969) <https://doi.org/10.1103/PhysRevLett.23.880>
- [6] Cirel'son, B.S.: Quantum generalizations of Bell's inequality. *Letters in Mathematical Physics* **4**, 93–100 (1980) <https://doi.org/10.1007/BF00417500>
- [7] Howard, D.: Einstein on locality and separability. *Studies in History and Philosophy of Science Part A* **16**(3), 171–201 (1985) [https://doi.org/10.1016/0039-3681\(85\)90001-9](https://doi.org/10.1016/0039-3681(85)90001-9)
- [8] Popescu, S., Rohrlich, D.: Quantum nonlocality as an axiom. *Foundations of Physics* **24**, 379–385 (1994) <https://doi.org/10.1007/BF02058098>