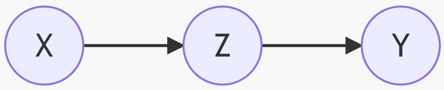
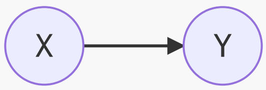
VAEs

Variational autoencoders have the following graph structure. The inputs are encoded into some latent distribution decoded to match the output as closely as possible.

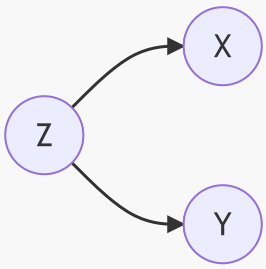


Linear Regression



My Approach

I restructure the graph such that both the inputs and outputs are generated from some latent space.



Ideal Characteristics Linear Regression

- Ability to use an unlimited number of inputs
- Adding variables does not neccisarily improve the likelihood, but can also decrease the likelihood
- Exploit all information in the variables (e.g. do not test on significance)

Review

Process

$$ \begin{aligned} X_i & = AZ_i + \epsilon_i \ Y_i & = BZ_i + \epsilon'_i \ \epsilon & \sim N(\mu, \Sigma) \ \epsilon' & \sim N(\mu', \Sigma') \ \end{aligned}

$$

Parameters

$$ \begin{aligned} X & : \textrm{(M x N) : input variables}\ Z & : \textrm{(F x N) : latent variables}\ Y & : \textrm{(P x N) : output variables}\ A & : \textrm{(M x F) }\ B & : \textrm{(P x F)}\ \mu & : \textrm{(M x 1)}\ \mu' & : \textrm{(P x 1)}\ \Sigma & : \textrm{(M x M)}\ \Sigma' & : \textrm{(P x P)} \ N & : \textrm{number of observations} \ M & : \textrm{number of input features} \ P & : \textrm{number of output variables} \ F & : \textrm{number of latent features} \end{aligned}

$$

Derivation

$$ \begin{aligned} \textrm{Objective : } \argmax_{\theta} p(Y|X;\theta) \end{aligned}

$$

$$ \begin{aligned} p(Y|X) & = \frac{\int p(Y|Z)p(X|Z)p(Z)dZ}{p(X)} \ \end{aligned}

$$

$$ \begin{aligned} p(Z) & = (2\pi)^{-\frac{k}{2}} det(\Sigma)^{-\frac{1}{2}}e^{-\frac{1}{2} Z'Z} \ & \propto e^{Z'Z}\

p(X|Z) & = \prod_{t=1}^T (2 \pi)^{-\frac{M}{2}}det(\Sigma_X)^{-\frac{1}{2}}e^{-\frac{1}{2}(X_t-A Z_t)'\Sigma_X^{-1}(X_t - A Z_t)} \ & \propto \prod_{t=1}^T det(\Sigma_X)^{-\frac{1}{2}}e^{-\frac{1}{2}(X_t-A Z_t)'\Sigma_X^{-1}(X_t - A Z_t)} \ & = \prod_{t=1}^T det(\Sigma_X)^{-\frac{1}{2}}e^{-\frac{1}{2}(X_t-A \hat{Z}_t)'\Sigma_X^{-1}(X_t - A \hat{Z}_t) -\frac{1}{2}(Z_t-\hat{Z}_t)'A' \Sigma_X^{-1}A(Z_t - \hat{Z}_t)}\ \end{aligned}

$$

$$ \begin{aligned} p(Y|Z) & = \prod_{t=1}^T (2 \pi)^{-\frac{M}{2}}det(\Sigma_Y)^{-\frac{1}{2}}e^{-\frac{1}{2}(Y_t-B Z_t)'\Sigma_Y^{-1}(Y_t - B Z_t)} \ & \propto \prod_{t=1}^T det(\Sigma_Y)^{-\frac{1}{2}}e^{-\frac{1}{2}(Y_t - B Z_t)'\Sigma_Y^{-1}(Y_T - B Z_t)} \ & = \prod_{t=1}^T det(\Sigma_Y)^{-\frac{1}{2}}e^{-\frac{1}{2}(Y_t - B \hat{Z}_t)'\Sigma_Y^{-1}(Y_t - B \hat{Z}_t) -\frac{1}{2}(Z_t-\hat{Z}_t)' B' \Sigma_Y^{-1} B (Z_t - \hat{Z}_t)}\ \end{aligned}

$$

$$ \begin{aligned} p(X) & = \int p(X|Z) p(Z) dZ \ & \propto \int e^{-\frac{1}{2} Z'Z} \prod_{t=1}^T det(\Sigma_X)^{-\frac{1}{2}} e^{-\frac{1}{2}(X_t-A \hat{Z}_t)'\Sigma_X^{-1}(X_t - A \hat{Z}_t) -\frac{1}{2}(Z_t-\hat{Z}_t)'A' \Sigma_X^{-1}A(Z_t - \hat{Z}*t)} dZ \ & = \prod*{t=1}^T \int det(\Sigma_X)^{-\frac{1}{2}}e^{-\frac{1}{2}(X_t-A \hat{Z}_t)'\Sigma_X^{-1}(X_t - A \hat{Z}_t) -\frac{1}{2}(Z_t-\hat{Z}_t)'A' \Sigma_X^{-1}A(Z_t - \hat{Z}*t)-\frac{1}{2} Z_t'Z_t} dZ_t \ & = det(\Sigma_X)^{-\frac{T}{2}} det(\Sigma')^{\frac{T}{2}} \prod*{t=1}^T \Big(e^{-\frac{1}{2}(X_t-A \hat{Z}_t)'\Sigma_X^{-1}(X_t - A \hat{Z}_t)} \Big) \ \Sigma' & = (I + A' \Sigma_X^{-1} A)^{-1} \end{aligned}

$$

$$ \begin{aligned} p(Y|Z)p(Z|X)p(Z) & \propto det(\Sigma_Y)^{-\frac{1}{2}}e^{-\frac{1}{2}(Y - B \hat{Z})'\Sigma_Y^{-1}(Y - B \hat{Z}) -\frac{1}{2}(Z-\hat{Z})' B' \Sigma_Y^{-1} B (Z - \hat{Z})} det(\Sigma_X)^{-

\frac{1}{2}}e^{-\frac{1}{2}(X-A \hat{Z})'\Sigma_X^{-1}(X - A \hat{Z}) -\frac{1}{2}(Z-\hat{Z})'A' \Sigma_X^{-1} A(Z - \hat{Z})} e^{-\frac{1}{2} Z'Z} \end{aligned}

$$

$$ \begin{aligned} \int p(Y|Z)p(Z|X)p(Z) dZ & \propto \int e^{-\frac{1}{2} Z'Z} \prod_{t=1}^T det(\Sigma_X)^{-\frac{1}{2}} det(\Sigma_Y)^{-\frac{1}{2}}
e^{-\frac{1}{2}(Y - B \hat{Z})'\Sigma_Y^{-1}(Y - B \hat{Z})-\frac{1}{2}(X-A \hat{Z})'\Sigma_X^{-1}(X - A \hat{Z})} e^{ -\frac{1}{2}(Z-\hat{Z}_t)' B' \Sigma_Y^{-1} B (Z_t - \hat{Z}_t)-\frac{1}{2}(Z_t-\hat{Z}_t)'A' \Sigma_X^{-1} A(Z_t - \hat{Z}*t)} dZ\ & = det(\Sigma_X)^{-\frac{T}{2}} det(\Sigma_Y)^{-\frac{T}{2}} \prod*{t=1}^T \Big(e^{-\frac{1}{2}(Y_t - B \hat{Z}_t)'\Sigma_Y^{-1}(Y_t - B \hat{Z}_t)-\frac{1}{2}(X_t-A \hat{Z}_t)'\Sigma_X^{-1}(X_t - A \hat{Z}*t)} \Big) \int \prod*{t=1}^T e^{ -\frac{1}{2}(Z_t-\hat{Z}_t)' B' \Sigma_Y^{-1} B (Z_t - \hat{Z}_t)-\frac{1}{2}(Z_t-\hat{Z}_t)'A' \Sigma_X^{-1} A(Z_t - \hat{Z}_t)- \frac{1}{2}Z_t'Z_t} dZ \

& = det(\Sigma_X)^{-\frac{T}{2}} det(\Sigma_Y)^{-\frac{T}{2}} \prod_{t=1}^T \Big(e^{-\frac{1}{2}(Y_t - B \hat{Z}_t)'\Sigma_Y^{-1}(Y_t - B \hat{Z}_t)-\frac{1}{2}(X_t-A \hat{Z}_t)'\Sigma_X^{-1}(X_t - A \hat{Z}*t)} \Big) \prod*{t=1}^T \int e^{ -\frac{1}{2}(Z_t-\hat{Z}_t)' B' \Sigma_Y^{-1} B (Z_t - \hat{Z}_t)-\frac{1}{2}(Z_t-\hat{Z}_t)'A' \Sigma_X^{-1} A(Z_t - \hat{Z}_t)- \frac{1}{2}Z_t'Z_t} dZ_t \

& = det(\Sigma_X)^{-\frac{T}{2}} det(\Sigma_Y)^{-\frac{T}{2}} \prod_{t=1}^T \Big(e^{-\frac{1}{2}(Y_t - B \hat{Z}_t)'\Sigma_Y^{-1}(Y_t - B \hat{Z}_t)-\frac{1}{2}(X_t-A \hat{Z}_t)'\Sigma_X^{-1}(X_t - A \hat{Z}*t)} \Big) \prod*{t=1}^T \int e^{ -\frac{1}{2}(Z_t-\hat{Z}_t)' B' \Sigma_Y^{-1} B (Z_t - \hat{Z}_t)-\frac{1}{2}(Z_t-\hat{Z}_t)'A' \Sigma_X^{-1} A(Z_t - \hat{Z}_t)- \frac{1}{2}Z_t'Z_t} dZ_t \

& \propto det(\Sigma_X)^{-\frac{T}{2}} det(\Sigma_Y)^{-\frac{T}{2}} det(\Sigma)^{\frac{T}{2}} \prod_{t=1}^T \Big(e^{-\frac{1}{2}(Y_t - B \hat{Z}_t)'\Sigma_Y^{-1}(Y_t - B \hat{Z}_t)-\frac{1}{2}(X_t-A \hat{Z}_t)'\Sigma_X^{-1}(X_t - A \hat{Z}_t)} \Big) \ \Sigma & = (I + A'\Sigma_X^{-1}A + B'\Sigma_Y^{-1}B)^{-1} \end{aligned}

$$

$$ \begin{aligned} p(Y|X;\Sigma_X, \Sigma_Y, \hat{Z}) & = \frac{\int p(Y|Z)p(X|Z)p(Z)dZ}{p(X)} \ & \propto det(\Sigma_Y)^{-\frac{T}{2}} det(\Sigma)^{\frac{T}{2}} det(\Sigma')^{-\frac{T}{2}} \prod_{t=1}^T \Big(e^{-\frac{1}{2}(Y_t - B \hat{Z}_t)'\Sigma_Y^{-1}(Y_t - B \hat{Z}_t)} \Big) \ \Sigma & = (I + A'\Sigma_X^{-1}A + B'\Sigma_Y^{-1}B)^{-1} \ \Sigma' & = (I + A' \Sigma_X^{-1} A)^{-1} \end{aligned}

$$

$$ \begin{aligned} \log{p(Y|X)} & = \frac{T}{2} \log{|\Sigma_Y^{-1}|} - \frac{T}{2}\log{|I + A' \Sigma_X^{-1} A + B' \Sigma_Y^{-1} B|} + \frac{T}{2}\log{|I + A'\Sigma_X^{-1}A|} - \frac{1}{2} \sum_{t=1}^T (Y_t - B \hat{Z}_t)'\Sigma_Y^{-1}(Y_t - B \hat{Z}_t) \end{aligned}

$$

link

first order deriviative

example with trace

derivatives in a trace (first order approximation)

$$ \begin{aligned} \frac{\partial p(Y|X)}{\partial B} & = ... + \frac{1}{2} \sum_{t=1}^T 2 \Sigma_Y^{-1} (Y_t - B \hat{Z}_t)B' \end{aligned}

$$

$$ \begin{aligned} \frac{\partial p(Y|X)}{\partial A} & = ... + \frac{1}{2} \sum_{t=1}^T 2 \Sigma_Y^{-1} (Y_t - B \hat{Z}_t)B' \end{aligned}

$$

$$ \begin{aligned} \frac{\log{p(Y|X)}}{\partial \Sigma_Y^{-1}} & = \frac{T}{2} \Sigma_Y + ... - \frac{1}{2} \sum_{t=1}^T (Y_t - B \hat{Z}_t)(Y_t - B \hat{Z}_t)' \end{aligned}

$$

$$ \begin{aligned} \frac{\partial \log{p(Y|X)}}{\partial \hat{Z}} & = \end{aligned}

$$

Decomposition Rule

$$ \begin{aligned} (Y-XB)'(Y-XB) & = (Y-X \hat{B})'(Y-X \hat{B}) + (B - \hat{B})'X'X (B - \hat{B}) \end{aligned}

$$

$$ \begin{aligned} X & : \textrm{M x N}\ A & : \textrm{M x F}\ Z & : \textrm{F x N}\ \end{aligned}

$$

$$ \begin{aligned} p(Z|X) & \propto p(X|Z) p(Z)\ p(Z) & \sim N(0, I) \ & = (2 \pi)^{-\frac{M}{2}}e^{-\frac{1}{2}Z'Z}\ p(X|Z) & = (2 \pi)^{-\frac{M}{2}}det(\Sigma)^{-\frac{1}{2}}e^{-\frac{1}{2}(X-A Z)'\Sigma^{-1}(X - A Z)} \ p(Z|X) & \propto (2 \pi)^{-\frac{M+F}{2}} det(\Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2}(X-AZ)'\Sigma^{-1}(X-AZ)-\frac{1}{2}Z'Z}\ & \propto e^{-\frac{1}{2} (X-A\hat{Z})\Sigma^{-1}(X-A\hat{Z}) - \frac{1}{2}(Z-\hat{Z})A'\Sigma^{-1}A(Z-\hat{Z}) -\frac{1}{2} Z'Z} \ p(Y|Z) & \propto e^{-\frac{1}{2}(Y-BZ)\Sigma^{-1}(Y-BZ)} \ & = e^{-\frac{1}{2} (Y-B\hat{Z})\Sigma^{-1}(Y-B\hat{Z})-\frac{1}{2}(Z-\hat{Z})B'\Sigma^{-1}B(Z-\hat{Z}) } \ p(Y|X) & = \int p(Y,Z|X) dZ \ & = \int p(Y|Z) p(Z|X) dZ\ & = \int (2 \pi)^{-\frac{P}{2}}det(\Sigma')^{-\frac{1}{2}}e^{-\frac{1}{2}(Y-B Z)'\Sigma'^{-1}(Y-B Z)} (2 \pi)^{-\frac{M}{2}}det(\Sigma)^{-\frac{1}{2}}e^{-\frac{1}{2}(X-A Z)'\Sigma^{-1}(X - A Z)} dZ \ & = \int (2 \pi)^{-\frac{P}{2}}det(\Sigma')^{-\frac{1}{2}}e^{-\frac{1}{2}(Y-B Z)'\Sigma'^{-1}(Y-B Z)} (2 \pi)^{-\frac{M}{2}}det(\Sigma)^{-\frac{1}{2}}e^{-\frac{1}{2}(A Q-A Z)'\Sigma^{-1}(A Q - A Z)} dZ \ & = \int (2 \pi)^{-\frac{P}{2}}det(\Sigma')^{-\frac{1}{2}}e^{-\frac{1}{2}(A A^{-1} Y - A B Z)'\Sigma'^{-1}(A A^{-1} Y - A B Z)} (2 \pi)^{-\frac{M}{2}}det(\Sigma)^{-\frac{1}{2}}e^{-\frac{1}{2}(A Q-A Z)'\Sigma^{-1}(A Q - A Z)} dZ \ p(Z|X) & = \frac{p(X|Z)p(Z)}{p(X)} = \frac{p(X|Z)p(Z)}{\int p(X|Z) p(Z) dZ} \ p(Y|X) & = \frac{\int p(Y|Z)p(X|Z)p(Z)dZ}{\int p(X|Z) p(Z) dZ} \ p(Z) & \sim N(0, \sigma^2 I) \end{aligned}

$$

Parameters:

$$ A, B, \Sigma, \Sigma', \mu, \mu'

$$

Multivariate Gaussian Distribution

$$ \begin{aligned} p(x) & = (2 \pi)^{-\frac{k}{2}}det(\Sigma)^{-\frac{1}{2}}e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)} \end{aligned}

$$

Bayesian Regression

Univariate

Process

$$ \begin{aligned} y &= X \beta + \epsilon \end{aligned}

$$

Assume flat priors

$$ \begin{aligned} p(\beta) & \propto 1 \ p(\sigma^2) & \propto \sigma^{-2} \end{aligned}

$$

Bayesian prediction

$$ \begin{aligned} y & = X \beta + \epsilon\ p(y|X) & = \int p(Y, \beta, \sigma^2|X) d\beta d\sigma^2 \ & = \int p(Y|X, \beta, \sigma^2) p(\beta, \sigma^2 | X) d\beta d\sigma^2 \end{aligned}

$$

$$ \begin{aligned} p(\sigma^2|y) & \propto \sigma^{-(N+2-k)} e^{-\frac{1}{2 \sigma^2}(y - X \hat{\beta})'(y - X\hat{\beta})} \end{aligned}

$$

Singular Value Decomposition

$$ \begin{aligned}

M & = U \Sigma V*\ \textrm{where} \ M & : \textrm{M x N} \ U & : \textrm{M x M} \ \Sigma & : \textrm{M x N} \ V^* & : \textrm{N x N} \ \end{aligned}

$$

Estimating Covariance matrix

link