### Lekcja: Algebra - 1

#algebra #Liczby zespolone

2024-10-02

### Zadania z ćwiczeń

1.

$$\sin(lpha) - i\cos(lpha) = \cos\left(rac{\pi}{2} - lpha
ight) - i\sin\left(rac{\pi}{2} - lpha
ight) = \cos\left(-\left(rac{\pi}{2} - lpha
ight)
ight) + i\sin\left(-\left(rac{\pi}{2} - lpha
ight)
ight)$$

2.

$$\sin(lpha) + i\cos(lpha) = \cos\left(rac{\pi}{2} - lpha
ight) + i\sin\left(rac{\pi}{2} - lpha
ight)$$

3.

$$-\sin(lpha)+i\cos(lpha)=\cos\left(rac{\pi}{2}+lpha
ight)+i\sin\left(rac{\pi}{2}+lpha
ight)$$

Sposób na obliczenie niewygodnego pierwiastka:

$$\sqrt{3-4i}=x+iy\Leftrightarrow (x+iy)^2=3-4i \ \begin{cases} x^2-y^2=3, wynika\ z\ Re \ 2xy=-4, wynika\ z\ Im \ x^2+y^2=5, wynika\ z\ |z| \end{cases}$$

$$2x^2 = 8$$
 $x^2 = 4$ 
 $x = -2 \lor x = 2$ 
 $y = 1 \lor y = -1$ 
 $z_1 = -2 + i, \ z_2 = 2 - i$ 

#### 1. Wykaż równości i nierówności dla liczb zespolonych:

a)

$$\overline{z_1z_2} = \overline{z_1} \cdot \overline{z_2}$$

$$\overline{(x_1,y_1)\cdot(x_2,y_2)} = (x_1x_2 - y_1y_2, -x_1y_2 - x_2y_1) \ = (x_1,-y_2)\cdot(x_2,-y_2) = \overline{(x_1,y_1)\cdot(x_2,y_2)}$$

b)

$$|z_1z_2|=|z_1|\cdot|z_2$$

$$egin{aligned} |(x_1x_2-y_1y_2,x_1y_2+x_2y_1)| &= |(x_1,y_1)|\cdot |(x_2,y_2)| \ \sqrt{(x_1x_2)^2+(y_1y_2)^2-2x_1x_2y_1y_2+(x_1y_2)^2+(x_2y_1)^2+2x_1x_2y_1y_2} &= \sqrt{x_1^2+y_1^2}\cdot \sqrt{x_2^2+y_2^2} \ \sqrt{(x_1x_2)^2+(y_1y_1^2+(x_2y_2)^2+(x_2y_1)^2} &= \sqrt{x_1^2+y_1^2}\cdot \sqrt{x_2^2+y_2^2} \ L &= P \end{aligned}$$

c)

$$z\cdot \overline{z}=|z|^2$$

$$(x,y)\cdot(x,-y)=(x^2+y^2,-xy+xy)=(x^2+y^2,0)=\ =x^2+y^2$$

d)

$$\overline{\left(rac{z_1}{z_2}
ight)}=rac{\overline{z_1}}{\overline{z_2}}$$

$$\overline{\left(\frac{z_1}{z_2}\right)}=(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2},\frac{x_1y_2-x_2y_1}{x_2^2+y_2^2})=\frac{(x_1,-y_1)}{(x_2,-y_2)}=\frac{\overline{z_1}}{\overline{z_2}}$$

e)

$$|z_1+z_2| \le |z_1|+|z_2|$$

$$egin{aligned} Z \;\; x \leq \sqrt{x^2 + y^2} \;\; wynika \ Re \; z \leq |z| \ Im \; z \leq |z| \ 1 = Re rac{z_1 + z_2}{z_1 + z_2} = \ &= Re rac{z_1}{z_1 + z_2} + Re rac{z_2}{z_1 + z_2} \leq rac{|z_1|}{|z_1 + z_2|} + rac{|z_2|}{|z_1 + z_2|} \ &= |z_1| + |z_2| \end{aligned}$$

#### 2. Oblicz

a)

$$rac{2+3i}{1+i} = rac{(2+3i)(1-i)}{(1+i)(1-i)} = rac{2-2i+3i+3}{2} = rac{5+i}{2} = rac{5}{2} + rac{1}{2} \cdot i$$

b)

$$\frac{(i+\sqrt{3})(-1-i\sqrt{3})}{1+2i} = \frac{-4i}{1+2i} = \frac{(-4i)\cdot(1-2i)}{(1+2i)\cdot(1-2i)} = \frac{-8-4i}{5} = -\frac{8}{5} - \frac{4}{5} \cdot i$$

c)

$$|3-4i|=\sqrt{3^2+4^2}=5$$

d)

$$arg(-2+2i)=rac{3\pi}{4}$$

e)

$$rac{(1+i)^n}{(1-i)^{n-2}} = rac{(1+i)^n \cdot (1-i)^2}{(1-i)^n} = (rac{(1+i)^2}{2})^n \cdot (-2i) = \ = (rac{2i}{2})^n \cdot (-2i) = -2i^{n+1}$$

#### 3. Przedstaw podane liczby zespolone w postaci trygonometrycznej:

a)

$$egin{aligned} 7+7i, & |z|=\sqrt{7^2\cdot 2}=7\sqrt{2}, & arphi=rac{\pi}{4}\ z=7\sqrt{2}(cos(rac{\pi}{4})+i\cdot sin(rac{\pi}{4})) \end{aligned}$$

b)

$$\sqrt{3}-i=2(cos(rac{11\pi}{6})+i\cdot sin(rac{11\pi}{6}))$$

c)

$$rac{1}{i} \cdot rac{1}{1+i} = rac{1}{i-1} = rac{1+i}{2} = rac{1}{\sqrt{2}}(cos(rac{\pi}{4}) + i \cdot sin(rac{\pi}{4}))$$

d)

$$1+ an(lpha)=rac{1}{\cos(lpha)}(\cos(lpha)+i\cdot\sin(lpha))$$

e)

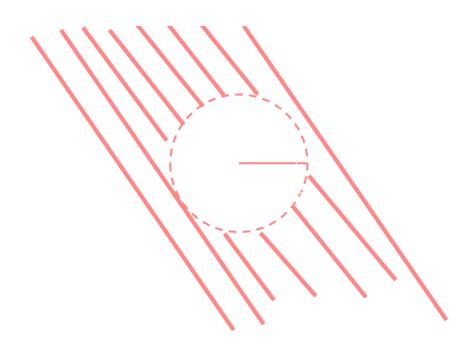
$$1+\cos(lpha)+i\sin(lpha)=1+2i\cos(rac{lpha}{2})^2-1+2\sin(rac{lpha}{2})\cos(rac{lpha}{2})= \ =2\cos(rac{lpha}{2})(\cos(rac{lpha}{2})+i\cdot\sin(rac{lpha}{2}))$$

## 4. Zilustruj na płaszczyźnie zespolonej następujące zbiory:

a)

$$\{z\in\mathbb{C}:|z-i+3|>3\}$$

$$\sqrt{(x+3)^2+(y-1)^2} > 3 \ /^2 \ (x+3)^2+(y-1)^2 > 9 \ S(-3,1), \ \ r=3$$

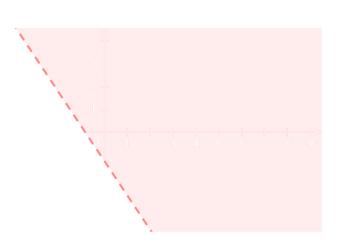


b)

$$\{z\in\mathbb{C}:|z-1|=|z+1|\}$$

$$\sqrt{(x-1)^2+y^2}=\sqrt{(x+1)^2+y^2}\ /^2 \ (x-1)^2=(x+1)^2 \ x^2-2x+1=x^2+2x+1 \ 4x=0 \ x=0$$

$$egin{aligned} \{z\in\mathbb{C}:rac{|z-2i|}{|z+3|}<1\} \ &rac{|z-2i|}{|z+3|}<1 \ &|z-2i|<|z+3| \ &\sqrt{x^2+(y-2)^2}<\sqrt{(x+3)^2+y^2}\ /^2 \ &x^2+y^2-4y+4< x^2+6x+9+y^2\ /-x^2-y^2 \ &-4y+4<6x+9\ /-4 \ &-4y<6x+5\ /:-4 \ &y>-rac{3}{2}x-rac{5}{4} \end{aligned}$$



d)

$$Re(x+iy-i)^2 \leq 0 \ Re(x+i(y-1))^2 \leq 0 \ Re(x^2+2ix(y-1)-(y-1)^2) \leq 0 \ x^2-(y-1)^2 \leq 0 \ x^2 \leq (y-1)^2 \ |x| \leq |y-1|$$

 $\{z \in \mathbb{C} : Re(z-i)^2 < 0\}$ 

$$\{z\in\mathbb{C}: arg(z-3+i)=rac{2\pi}{3}\}$$

$$arg((x-3)+i(y+1)) = rac{2\pi}{3}$$
 $Re(z) = x-3, \ Im(z) = y+1$ 
 $rac{Im}{Re} = rac{y+1}{x-3}$ 
 $an^{-1} rac{y+1}{x-3} = arg(z)$ 
 $rac{y+1}{x-3} = anrac{2\pi}{3}$ 
 $y+1 = -\sqrt{3}(x-3)$ 
 $y = -\sqrt{3}x + 3\sqrt{3} - 1$ 
 $P = (-3, 1)$ 



$$\{z\in\mathbb{C}: argrac{i}{i-z}=rac{4\pi}{3}\}$$
  $rgrac{i}{i-z}=rgi-rgi-rgi-z=rac{\pi}{2}-rgi-z=rac{4\pi}{3}$   $-rac{5\pi}{6}=rgi-z$ 

$$rg z - i = rac{\pi}{6}$$

## ZAPAMIĘTAJ $\operatorname{arg} -z = \operatorname{arg} z + \pi$

h)

$$\{z\in\mathbb{C}: rgrac{i}{z}\leqrac{3\pi}{4}\} \ rgrac{i}{z}=rgi-rgz+2k\pi \ 0\leqrac{\pi}{2}-rgz\leqrac{3\pi}{4}\ argumenty\ naleza\ do\ [0,2\pi] \ 2k\pi-rac{\pi}{2}\leq-rgz\leqrac{\pi}{4}+2k\pi \ 2k\pi+rac{\pi}{2}\geqrgz\geq-rac{\pi}{4}+2k\pi \ 2k\pi+rac{\pi}{2}\geqrgz\geqrac{7\pi}{4}$$



i)

$$\{z\in\mathbb{C}: Im((z-2i)^4)>0\}$$

$$Im((z-2i)^4) = Im()$$

# 5. Oblicz wartości podanych wyrażeń (wyniki podaj w postaci algebraicznej):

a)

$$(1-i)^6 = \sqrt{2}^6 (cos \frac{7\pi \cdot 6}{4} + i \cdot sin \frac{7\pi \cdot 6}{4}) = 8(cos \frac{2\pi}{4} + i \cdot sin \frac{2\pi}{4}) = 8 \cdot (0+i \cdot 1) = 8i$$

b)

$$\left(rac{1+i\cdot\sqrt{3}}{1-i}
ight)^{20} = \left(rac{2\cdot(\cosrac{\pi}{3}+i\cdot\sinrac{\pi}{3})}{\sqrt{2}\cdot(\cos-rac{\pi}{4}+i\cdot\sin-rac{\pi}{4})}
ight)^{20} = 2^{10}\cdot(\cos{(2\pi-rac{\pi}{3})}+i\cdot\sin{(2\pi-i\pi)})^{20} = 2^{10}\cdot(\cos{(2\pi-rac{\pi}{3})}+i\cdot\sin{(2\pi-i\pi)})^{20}$$

c)

$$rac{(1+i)^{22}}{(1-i\sqrt{3})^6} = rac{2^{11} \cdot (\cos rac{\sqrt{2}}{2} + i \cdot \sin rac{\sqrt{2}}{2})}{\cdots} = cos \ tam$$

d)

$$(-\cos{\pi\over 7}+i\sin{\pi\over 7})^{14}=(\cos{6\pi\over 7}+i\sin{6\pi\over 7})^{14}=1$$

## 6. Znajdź funkcję rzeczywistą f taką że:

a)

$$\cos 3x = f(\cos x) \ t = \cos x \ \cos 3x = \cos^3 x - 3 \cdot \cos x \cdot \sin^2 x = \cos^3 x - 3 \cdot \cos x \cdot (1 - \cos^2 x) = \ = 4\cos^3 x - 3\cos x \ f(t) = 4t^3 - 3t$$

b)

$$\sin 5x = f(\sin x) \ \sin 5x = 5\cos^4 x \sin^1 x - 10\cos^2 x \sin^3 x + \sin^5 x = \ = \sin^5 -10 \cdot (1 - \sin^2 x) \sin^3 x + 5(1 - \sin^2 x)^2 \sin x = \ = \sin^5 x - 10\sin^3 x + 10\sin^5 x + 5(\sin x - 2\sin^3 x + \sin^5 x) = \ = 16\sin^5 x - 20\sin^3 x + 5\sin x$$

c)

$$\cot 4x = f(\cot x) \ \cot 4x = rac{\cos 4x}{\sin 4x} \ \cos 4x = \cos^4 x - 6\cos^2 x \sin^2 x + \sin^4 x = \ = \cos^4 x - 6(1 - \cos^2 x)\cos^2 x + (1 - \cos^2 x)^2 = \ = \cos^4 x - 6\cos^2 x + 6\cos^4 x + 1 - 2\cos^2 x + \cos^4 x = \ = 8\cos^4 x - 8\cos^2 x + 1$$

albo dwa razy wzor podwojonego cosinusa

$$\sin 4x = \dots$$

### 7. Oblicz pierwiastki z liczb zespolonych:

a)

$$\sqrt{-1+\sqrt{3}i}:$$

$$-1+\sqrt{3}i=2\cdot(\cosrac{2\pi}{3}+i\cdot\sinrac{2\pi}{3})$$
 $\sqrt{=\sqrt{2}\cdot(\cosrac{2\pi}{3}+2k\pi}{2}+i\cdot\sinrac{2\pi}{3}+2k\pi}{2})$ 
 $k=0:$ 
 $\sqrt{2}\cdot(\cosrac{2\pi}{6}+i\cdot\sinrac{2\pi}{6})$ 
 $k=1:$ 
 $\sqrt{2}\cdot(\cosrac{8\pi}{6}+i\cdot\sinrac{8\pi}{6})$ 

 $alternatywny\ sposob$ 

$$egin{aligned} w_k \cdot (\cos rac{2\pi}{n} + i \sin rac{2\pi}{n}) &= w_{k+1} \ w_{k+1} &= (-1 + \sqrt{3}i) \cdot (\cos rac{2\pi}{n} + i \sin rac{2\pi}{n}) \end{aligned}$$

d)

$$\sqrt[4]{(-2+3i)^4}$$
 $w^4=(-2+3i)^4$ 
 $w^4-(-2+3i)^4=0$ 
 $(w-(-2+3i))(w+(-2+3i))(w^2+(-2+3i)^2)=0$ 
 $w_1=-2+3i,\ w_2=2-3i$ 
 $w_4,w_5:\sqrt{-(-2+3i)^2}=m\Leftrightarrow m^2=-(-2+3i)^2=5+12i$ 
 $x^2-y^2=5$ 
 $2xy=12$ 
 $x^2+y^2=169$ 
 $2x^2=174$ 
 $x^2=87$ 

. . .

## 10. Wykorzystując postać wykładniczą liczby zespolonej, rozwiąż równania:

a)

$$|z|^3=iz^3 \ r^3=i(re^{iarphi})^3 \ r^3=i\cdot r^3e^{i\cdot 3\cdot arphi} \ r^3(1-i\cdot e^{i\cdot 3\cdot arphi})=0,\ r
eq 0 \ 1=i\cdot e^{i\cdot 3\cdot arphi} \ 1=e^{i\cdot 3\cdot arphi+rac{\pi}{2}} \ r=1,\ 3arphi+rac{\pi}{2}=2k\pi \ arphi=rac{2k\pi}{3}-rac{\pi}{6}$$

b)

$$egin{aligned} (\overline{z})^6 &= 4|z^2| \ r^6 e^{-i \cdot 6 arphi} &= 4 r^2 \ r^4 e^{-i \cdot 6 arphi} &= 4 \ r &= 0 \lor r = \sqrt{2}, \quad -6 arphi = 2 k \pi \ arphi &= -rac{k \pi}{3} \ z_0 &= 0 \ k &= 1, \ z_1 &= rac{\sqrt{2}}{2} + i \end{aligned}$$

c)

$$(\overline{z})^6 = -8z|z|\overline{z}$$
 $r^6e^{-i\cdot 6arphi} = -8re^{iarphi} \cdot r \cdot re^{-iarphi}$ 
 $r^6e^{-i\cdot 6arphi} = -8r^3e^{2k\pi}$ 
 $r^6 + 8r^3 = 0$ 
 $r^3(r^3 + 8) = 0$ 
 $r = 0 \ \lor r = 2$ 
 $-6arphi = 2k\pi$ 
 $arphi = -\frac{k\pi}{3}$ 
 $dla\ r = 0, \ arphi \in \mathbb{R} \Leftrightarrow z_0 = 0$ 
 $dla\ r = 2, \ \ arphi_k = -\frac{k\pi}{3}$ 
 $k = 0, \ \ arphi = 0 \ ext{d} = 1$ 
 $k = 1, \ \ arphi = -\frac{\pi}{3} \ ext{d} = 1$ 
 $k = 2, \ \ arphi = -\frac{2\pi}{3} \ ext{d} = 1$ 
 $k = 3, \ \ arphi = -\pi \ ext{d} = -2$ 
 $k = 4, \ \ arphi = -\frac{4\pi}{3} \ ext{d} = 2$ 
 $k = 5, \ \ arphi = -\frac{5\pi}{3} \ ext{d} = 1$ 

### 11. Punkty

## 12. Ile wynosi suma wszystkich pierwiastków zespolonych stopnia n z1?

$$w=\sqrt[n]{1}\Leftrightarrow w^n-1=0 \ suma: \ a_n=1,\ a_{n-1}=0 \ z\ uogolnionych\ wzorow\ Viete'a \ \sum_{k=0}^{n-1}w_k=rac{-a_n-1}{a_n}=0/1=0$$

## 1. Przedstaw w postaci trygonometrycznej

$$-\cos(lpha)+i\cdot\sin(lpha)=cos(\pi-lpha)+i\cdot\sin(\pi-lpha)$$

## 2. Zilustruj

$$A=\{z\in\mathbb{C}:1\leq|z|\leq2,arg\;z\in[-rac{\pi}{2},rac{3\pi}{4}]\}$$

