Lekcja: Algebra - 1

#algebra #Liczby_zespolone

2024-10-02

Zadania z ćwiczeń

1.

$$\sin(\alpha) - i\cos(\alpha) = \cos\left(\frac{\pi}{2} - \alpha\right) - i\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\left(-\left(\frac{\pi}{2} - \alpha\right)\right) + i\sin\left(-\left(\frac{\pi}{2} - \right.\right.$$

2.

$$\sin(\alpha) + i\cos(\alpha) = \cos\left(\frac{\pi}{2} - \alpha\right) + i\sin\left(\frac{\pi}{2} - \alpha\right)$$

3.

$$-\sin(\alpha) + i\cos(\alpha) = \cos\left(\frac{\pi}{2} + \alpha\right) + i\sin\left(\frac{\pi}{2} + \alpha\right)$$

Sposób na obliczenie niewygodnego pierwiastka:

$$\sqrt{3-4i} = x + iy \Leftrightarrow (x+iy)^2 = 3 - 4i$$
$$\begin{cases} x^2 - y^2 = 3, wynika\ z\ Re \\ 2xy = -4, wynika\ z\ Im \\ x^2 + y^2 = 5, wynika\ z\ |z| \end{cases}$$

$$\begin{gathered} 2x^2 = 8 \\ x^2 = 4 \\ x = -2 \vee x = 2 \\ y = 1 \vee y = -1 \\ z_1 = -2 + i,\ z_2 = 2 - i \end{gathered}$$

Zadania z karty

1. Wykaż równości i nierówności dla liczb zespolonych:

a)

$$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$$

$$\overline{(x_1, y_1) \cdot (x_2, y_2)} = (x_1 x_2 - y_1 y_2, -x_1 y_2 - x_2 y_1)$$
$$= (x_1, -y_2) \cdot (x_2, -y_2) = \overline{(x_1, y_1} \cdot \overline{(x_2, y_2)}$$

b)

$$|z_1 z_2| = |z_1| \cdot |z_2$$

$$|(x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)| = |(x_1, y_1)| \cdot |(x_2, y_2)|$$
$$\sqrt{(x_1 x_2)^2 + (y_1 y_2)^2 - 2x_1 x_2 y_1 y_2 + (x_1 y_2)^2 + (x_2 y_1)^2 + 2x_1 x_2 y_1 y_2} = \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 +$$
$$\sqrt{(x_1 x_2)^2 + (y_1 y_1^2 + (x_2 y_2)^2 + (x_2 y_1)^2} = \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2}$$
$$L = P$$

c)

$$z \cdot \overline{z} = |z|^2$$

$$(x, y) \cdot (x, -y) = (x^2 + y^2, -xy + xy) = (x^2 + y^2, 0) =$$
$$= x^2 + y^2$$

d)

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{x_1 y_2 - x_2 y_1}{x_2^2 + y_2^2}\right) = \frac{(x_1, -y_1)}{(x_2, -y_2)} = \frac{\overline{z_1}}{\overline{z_2}}$$

e)

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$Z \;\; x \leq \sqrt{x^2 + y^2} \;\; wynika$$
$$Re\; z \leq |z|$$
$$Im\; z \leq |z|$$
$$1 = Re\frac{z_1 + z_2}{z_1 + z_2} =$$
$$= Re\frac{z_1}{z_1 + z_2} + Re\frac{z_2}{z_1 + z_2} \leq \frac{|z_1|}{|z_1 + z_2|} + \frac{|z_2|}{|z_1 + z_2|}$$
$$|z_1 + z_2| \leq |z_1| + |z_2|$$

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2. Oblicz

a)

$$\frac{2+3i}{1+i} = \frac{(2+3i)(1-i)}{(1+i)(1-i)} = \frac{2-2i+3i+3}{2} = \frac{5+i}{2} = \frac{5}{2} + \frac{1}{2} \cdot i$$

b)

$$\frac{(i+\sqrt{3})(-1-i\sqrt{3})}{1+2i} = \frac{-4i}{1+2i} = \frac{(-4i)\cdot(1-2i)}{(1+2i)\cdot(1-2i)} = \frac{-8-4i}{5} = -\frac{8}{5} - \frac{4}{5} \cdot i$$

c)

$$|3-4i| = \sqrt{3^2+4^2} = 5$$

d)

$$arg(-2+2i) = \frac{3\pi}{4}$$

e)

$$\frac{(1+i)^n}{(1-i)^{n-2}} = \frac{(1+i)^n \cdot (1-i)^2}{(1-i)^n} = (\frac{(1+i)^2}{2})^n \cdot (-2i) =$$
$$= (\frac{2i}{2})^n \cdot (-2i) = -2i^{n+1}$$

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3. Przedstaw podane liczby zespolone w postaci trygonometrycznej:

a)

$$7+7i, \quad |z| = \sqrt{7^2 \cdot 2} = 7\sqrt{2}, \quad \varphi = \frac{\pi}{4}$$
$$z = 7\sqrt{2}(cos(\frac{\pi}{4}) + i \cdot sin(\frac{\pi}{4}))$$

b)

$$\sqrt{3} - i = 2(cos(\frac{11\pi}{6}) + i \cdot sin(\frac{11\pi}{6}))$$

c)

$$\frac{1}{i} \cdot \frac{1}{1+i} = \frac{1}{i-1} = \frac{1+i}{2} = \frac{1}{\sqrt{2}}(cos(\frac{\pi}{4}) + i \cdot sin(\frac{\pi}{4}))$$

d)

$$1 + \tan(\alpha) = \frac{1}{\cos(\alpha)}(\cos(\alpha) + i \cdot \sin(\alpha))$$

e)

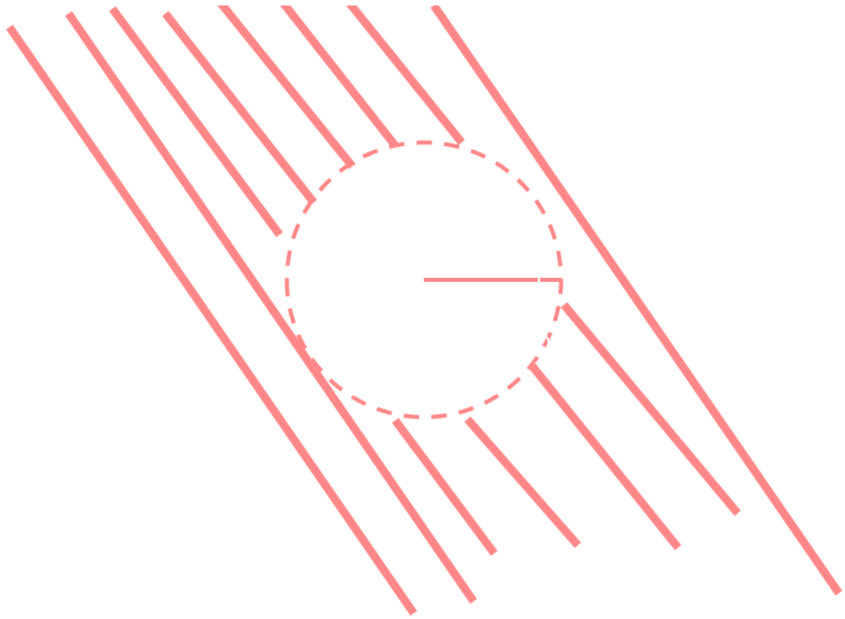
$$1 + \cos(\alpha) + i\sin(\alpha) = 1 + 2i\cos(\frac{\alpha}{2})^2 - 1 + 2\sin(\frac{\alpha}{2})\cos(\frac{\alpha}{2}) =$$
$$= 2\cos(\frac{\alpha}{2})(\cos(\frac{\alpha}{2}) + i \cdot \sin(\frac{\alpha}{2}))$$

4. Zilustruj na płaszczyźnie zespolonej następujące zbiory:

a)

$$\{z \in \mathbb{C} : |z - i + 3| > 3\}$$

$$\sqrt{(x+3)^2 + (y-1)^2} > 3 \ /^2$$
$$(x+3)^2 + (y-1)^2 > 9$$
$$S(-3, 1), \quad r = 3$$



b)

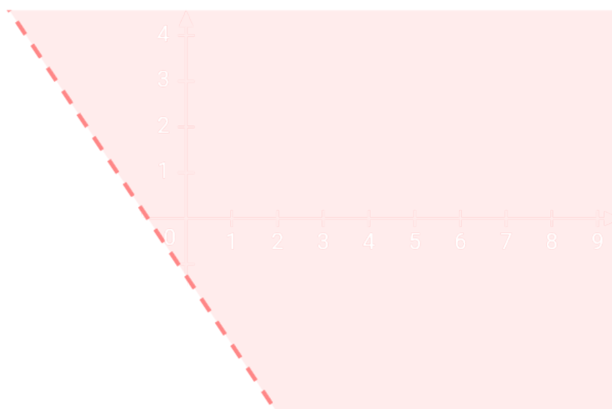
$$\{z \in \mathbb{C} : |z-1| = |z+1|\}$$

$$\sqrt{(x-1)^2+y^2} = \sqrt{(x+1)^2+y^2} \;/^2$$
$$(x-1)^2 = (x+1)^2$$
$$x^2 - 2x + 1 = x^2 + 2x + 1$$
$$4x = 0$$
$$x = 0$$

c)

$$\{z \in \mathbb{C} : \frac{|z-2i|}{|z+3|} < 1\}$$

$$\frac{|z-2i|}{|z+3|} < 1$$
$$|z-2i| < |z+3|$$
$$\sqrt{x^2+(y-2)^2} < \sqrt{(x+3)^2+y^2} \; /^2$$
$$x^2+y^2-4y+4 < x^2+6x+9+y^2 \; / -x^2-y^2$$
$$-4y+4 < 6x+9 \; / -4$$
$$-4y < 6x+5 \; / : -4$$
$$y > -\frac{3}{2}x - \frac{5}{4}$$



d)

$$\{z \in \mathbb{C} : Re(z-i)^2 \leq 0\}$$

$$Re(x+iy-i)^2 \leq 0$$
$$Re(x+i(y-1))^2 \leq 0$$
$$Re(x^2+2ix(y-1)-(y-1)^2) \leq 0$$
$$x^2-(y-1)^2 \leq 0$$
$$x^2 \leq (y-1)^2$$
$$|x| \leq |y-1|$$

f)

$$\{z \in \mathbb{C} : arg(z - 3 + i) = \frac{2\pi}{3}\}$$

$$arg((x-3) + i(y+1)) = \frac{2\pi}{3}$$
$$Re(z) = x - 3,\ Im(z) = y + 1$$
$$\frac{Im}{Re} = \frac{y+1}{x-3}$$
$$\tan^{-1}\frac{y+1}{x-3} = arg(z)$$
$$\frac{y+1}{x-3} = \tan\frac{2\pi}{3}$$
$$y + 1 = -\sqrt{3}(x-3)$$
$$y = -\sqrt{3}x + 3\sqrt{3} - 1$$
$$P = (-3, 1)$$

g)

$$\{z \in \mathbb{C} : arg\frac{i}{i-z} = \frac{4\pi}{3}\}$$

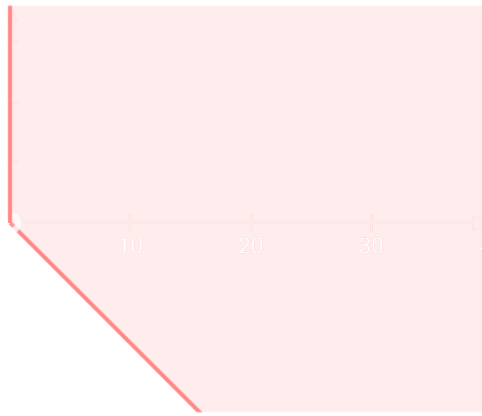
$$\arg\frac{i}{i-z} = \arg i - \arg i - z = \frac{\pi}{2} - \arg i - z = \frac{4\pi}{3}$$
$$-\frac{5\pi}{6} = \arg i - z$$
$$\arg z - i = \frac{\pi}{6}$$

ZAPAMIĘTAJ $\arg -z = \arg z + \pi$

h)

$$\{z \in \mathbb{C} : \arg \frac{i}{z} \leq \frac{3\pi}{4}\}$$

$$\arg \frac{i}{z} = \arg i - \arg z + 2k\pi$$
$$0 \leq \frac{\pi}{2} - \arg z \leq \frac{3\pi}{4} \; \textit{argumenty naleza do } [0, 2\pi]$$
$$2k\pi - \frac{\pi}{2} \leq - \arg z \leq \frac{\pi}{4} + 2k\pi$$
$$2k\pi + \frac{\pi}{2} \geq \arg z \geq -\frac{\pi}{4} + 2k\pi$$
$$2k\pi + \frac{\pi}{2} \geq \arg z \geq \frac{7\pi}{4}$$



i)

$$\{z \in \mathbb{C} : Im((z-2i)^4) > 0\}$$

$$Im((z-2i)^4) = Im()$$

5. Oblicz wartości podanych wyrażeń (wyniki podaj w postaci algebraicznej):

a)

$$(1-i)^6 = \sqrt{2}^6 (cos\frac{7\pi \cdot 6}{4} + i \cdot sin\frac{7\pi \cdot 6}{4}) = 8(cos\frac{2\pi}{4} + i \cdot \sin\frac{2\pi}{4}) = 8 \cdot (0 + i \cdot 1) = 8i$$

b)

$$\left(\frac{1+i\cdot\sqrt{3}}{1-i}\right)^{20} = \left(\frac{2\cdot(\cos\frac{\pi}{3} + i\cdot\sin\frac{\pi}{3})}{\sqrt{2}\cdot(\cos-\frac{\pi}{4} + i\cdot\sin-\frac{\pi}{4})}\right)^{20} = 2^{10}\cdot(\cos(2\pi - \frac{\pi}{3}) + i\cdot\sin(2\pi -$$
$$= 2^9 - i\cdot\sqrt{3}\cdot 2^9$$

c)

$$\frac{(1+i)^{22}}{(1-i\sqrt{3})^6} = \frac{2^{11}\cdot(\cos\frac{\sqrt{2}}{2} + i\cdot\sin\frac{\sqrt{2}}{2})}{\cdots} = cos\ tam$$

d)

$$(-\cos\frac{\pi}{7}+i\sin\frac{\pi}{7})^{14}=(\cos\frac{6\pi}{7}+i\sin\frac{6\pi}{7})^{14}=1$$

6. Znajdź funkcję rzeczywistą f taką że:

a)

$$\cos 3x = f(\cos x)$$
$$t=\cos x$$
$$\cos 3x=\cos^3 x-3\cdot\cos x\cdot\sin^2 x=\cos^3 x-3\cdot\cos x\cdot(1-cos^2x)=$$
$$=4cos^3x-3cosx$$
$$f(t)=4t^3-3t$$

b)

$$\sin 5x=f(\sin x)$$
$$\sin 5x=5\cos^4 x\sin^1 x-10\cos^2 x\sin^3 x+\sin^5 x=$$
$$=\sin^5-10\cdot(1-\sin^2 x)\sin^3 x+5(1-\sin^2 x)^2\sin x=$$
$$=\sin^5 x-10\sin^3 x+10\sin^5 x+5(\sin x-2\sin^3 x+\sin^5 x)=$$
$$=16\sin^5 x-20\sin^3 x+5\sin x$$

c)

$$\cot 4x=f(\cot x)$$
$$\cot 4x=\frac{\cos 4x}{\sin 4x}$$
$$\cos 4x=\cos^4 x-6\cos^2 x\sin^2 x+\sin^4 x=$$
$$=\cos^4 x-6(1-\cos^2 x)\cos^2 x+(1-cos^2x)^2=$$
$$=\cos^4 x-6\cos^2 x+6\cos^4 x+1-2\cos^2 x+\cos^4 x=$$
$$=8\cos^4 x-8\cos^2 x+1$$

$$albo\ dwa\ razy\ wzor\ podwojonego\ cosinusa$$

$$\sin 4x=\dots$$

7. Oblicz pierwiastki z liczb zespolonych:

a)

$$\sqrt{-1+\sqrt{3}i}:$$

$$-1+\sqrt{3}i = 2\cdot(\cos\frac{2\pi}{3} + i\cdot\sin\frac{2\pi}{3})$$
$$\sqrt{\ } = \sqrt{2}\cdot(\cos\frac{\frac{2\pi}{3}+2k\pi}{2} + i\cdot\sin\frac{\frac{2\pi}{3}+2k\pi}{2})$$

$$k=0:$$
$$\sqrt{2}\cdot(\cos\frac{2\pi}{6} + i\cdot\sin\frac{2\pi}{6})$$
$$k=1:$$
$$\sqrt{2}\cdot(\cos\frac{8\pi}{6} + i\cdot\sin\frac{8\pi}{6})$$

$$alternatywny\ sposob$$

$$w_k\cdot(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}) = w_{k+1}$$
$$w_{k+1} = (-1+\sqrt{3}i)\cdot(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n})$$

d)

$$\sqrt[4]{(-2+3i)^4}$$
$$w^4 = (-2+3i)^4$$
$$w^4 - (-2+3i)^4 = 0$$
$$(w-(-2+3i))(w+(-2+3i))(w^2+(-2+3i)^2) = 0$$
$$w_1 = -2+3i,\ w_2 = 2-3i$$
$$w_4, w_5 : \sqrt{-(-2+3i)^2} = m \Leftrightarrow m^2 = -(-2+3i)^2 = 5+12i$$
$$x^2 - y^2 = 5$$
$$2xy = 12$$
$$x^2 + y^2 = 169$$
$$2x^2 = 174$$
$$x^2 = 87$$
$$\dots$$

10. Wykorzystując postać wykładniczą liczby zespolonej, rozwiąż równania:

a)

$$|z|^3 = iz^3$$
$$r^3 = i(re^{i\varphi})^3$$
$$r^3 = i \cdot r^3 e^{i \cdot 3 \cdot \varphi}$$
$$r^3(1 - i \cdot e^{i \cdot 3 \cdot \varphi}) = 0,\ r \neq 0$$
$$1 = i \cdot e^{i \cdot 3 \cdot \varphi}$$
$$1 = e^{i \cdot 3 \cdot \varphi + \frac{\pi}{2}}$$
$$r = 1,\ 3\varphi + \frac{\pi}{2} = 2k\pi$$
$$\varphi = \frac{2k\pi}{3} - \frac{\pi}{6}$$

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b)

$$(\overline{z})^6 = 4|z^2|$$
$$r^6 e^{-i \cdot 6\varphi} = 4r^2$$
$$r^4 e^{-i \cdot 6\varphi} = 4$$
$$r = 0 \vee r = \sqrt{2},\ -6\varphi = 2k\pi$$
$$\varphi = -\frac{k\pi}{3}$$
$$z_0 = 0$$
$$k = 1,\ z_1 = \frac{\sqrt{2}}{2} + i$$

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c)

$$(\overline{z})^6 = -8z|z|\overline{z}$$
$$r^6 e^{-i\cdot 6\varphi} = -8re^{i\varphi} \cdot r \cdot re^{-i\varphi}$$
$$r^6 e^{-i\cdot 6\varphi} = -8r^3 e^{2k\pi}$$
$$r^6 + 8r^3 = 0$$
$$r^3(r^3 + 8) = 0$$
$$r = 0 \ \vee r = 2$$
$$-6\varphi = 2k\pi$$
$$\varphi = -\frac{2k\pi}{6} = -\frac{k\pi}{3}$$
$$dla\ r = 0, \varphi \in \mathbb{R} \Leftrightarrow z_0 = 0$$
$$dla\ r = 2, \ \ \varphi_k = -\frac{k\pi}{3}$$
$$k = 0, \ \varphi = 0 \to z_1 = 2$$
$$k = 1, \ \varphi = -\frac{\pi}{3} \to z_2 = 1 - i\sqrt{3}$$
$$k = 2, \ \varphi = -\frac{2\pi}{3} \to z_3 = -1 - i\sqrt{3}$$
$$k = 3, \ \varphi = -\pi \to z_4 = -2$$
$$k = 4, \ \varphi = -\frac{4\pi}{3} \to z_5 = -1 + i\sqrt{3}$$
$$k = 5, \ \varphi = -\frac{5\pi}{3} \to z_6 = 1 + i\sqrt{3}$$

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11. Punkty

12. Ile wynosi suma wszystkich pierwiastków zespolonych stopnia n z 1?

$$w = \sqrt[n]{1} \Leftrightarrow w^n - 1 = 0$$
$$suma:$$
$$a_n = 1, \ a_{n-1} = 0$$
$$z\ uogolnionych\ wzorow\ Viete'a$$
$$\sum_{k=0}^{n-1} w_k = \frac{-a_n - 1}{a_n} = 0/1 = 0$$

Zadania z ćwiczeń 2

1. Przedstaw w postaci trygonometrycznej

$$-\cos(\alpha) + i \cdot \sin(\alpha) = cos(\pi - \alpha) + i \cdot \sin(\pi - \alpha)$$

2. Zilustruj

$$A = \{z \in \mathbb{C} : 1 \leq |z| \leq 2, arg\, z \in [-\frac{\pi}{2}, \frac{3\pi}{4}]\}$$