On a class of vector optimization problems

Ariana Pitea[1,a)]

[1]*Department of Mathematics and Informatics, University Politehnica of Bucharest, Splaiul Independentei 313, 060042, Bucharest.*

[a)]Corresponding author: arianapitea@yahoo.com

Abstract.

The aim of this work is to introduce a new class of multiobjective optimization problems. The study of such kind of programs is imposed by practical reasons, since a significant number of real world problems have a multiobjective nature. In this respect, we could mention the shortest path method, which involves the length of the paths and their costs. More than that, multiple criteria may refer to the length of a journey, its price, number of transfers. Also, the timetable information could be considered as a result of multiobjective optimization, if we have in view the unknown delays. Another field which provides real world multiobjective optimization problems is material sciences, where there is required an optimal estimation of the parameters of the materials, or the non-destructive determination of faults is needed. Further more, such kind of optimization problems can be found also in economics, or game theory, see [2], [3] and the references therein.

The class of problems which are to be proposed in the work refers to minimizing a vector of curvilinear integrals, where the integrand depends also on the velocities. This kind of problems are connected, for example, with Mechanical Engineering, considering that curvilinear integral objectives are extensively used because of their physical meaning as mechanical work, and there is a need to minimize simultaneously such kind of quantities, subject to some constraints.

In order to introduce our class of problems and the results regarding our class of vector optimization problems, we need the following well-known convention for equalities and inequalities.

For any $x = \left(x_1, x_2, \dots, x_p\right)$, $y = \left(y_1, y_2, \dots, y_p\right)$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = \overline{1,p}$;

(ii) $x > y$ if and only if $x_i > y_i$ for all $\overline{1,p}$;

(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $\overline{1,p}$;

(iv) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

This product order relation will be used on $\mathbb{R}^p$, more precisely on the hyperparallelepiped Ω_{t_0,t_1} in $\mathbb{R}^p$, with diagonal opposite points $t_0 = (t_0^1, ..., t_0^p)$. Assume that γ_{t_0,t_1} is a piecewise C^1-class curve joining the points t_0 and t_1.

The local coordinates on Ω_{t_0,t_1}, and $\mathbb{R}^n$ will be written $t = (t^\alpha)$, $\alpha = \overline{1,p}$, and $x = (x^i)$, $i = \overline{1,n}$, respectively.

By $C^\infty\left(\Omega_{t_0,t_1}, \mathbb{R}^n\right)$ we denote the space of all functions $x\colon \Omega_{t_0,t_1} \to \mathbb{R}^n$ of C^∞-class, and $\Pi = \left\{(t, (x(t), x_\eta(t)) : x(\cdot) \in C^\infty\left(\Omega_{t_0,t_1}, \mathbb{R}^n\right)\right)\}$, where $x_\eta(t) = \frac{\partial x}{\partial t^\eta}(t)$, $\eta = \overline{1,p}$, are partial velocities.. The closed Lagrange 1-forms densities of C^∞-class

$$k_\alpha = \left(k_\alpha^i\right) : \Pi \to \mathbb{R}^r, q_\alpha = \left(q_\alpha^i\right) : \Pi \to \mathbb{R}^r, i = \overline{1,r}, \alpha = \overline{1,p},$$

determine the following path independent curvilinear functionals (actions)

$$K^i(x(\cdot)) = \int_{\gamma_{t_0,t_1}} k_\alpha^i(\pi_x(t))dt^\alpha, i = \overline{1,r}, \alpha = \overline{1,p}, \tag{1}$$

where $\pi_x(t) = (t, x(t), x_\eta(t))$, and $x_\eta(t) = \frac{\partial x}{\partial t^\eta}(t)$, $\eta = \overline{1,p}$, are partial velocities.

The closedness conditions (complete integrability conditions) are $D_\beta f_\alpha^i = D_\alpha f_\beta^i$, $\alpha, \beta = \overline{1,p}$, $\alpha \neq \beta$, $i = \overline{1,r}$, where D_β is the total derivative.

We also accept that the Lagrange densities matrix

$$g = \left(g_a^j\right)\colon \Pi \to \mathbb{R}^{ms}, a = \overline{1,s}, j = \overline{1,m}, m < n,$$

of C^∞-class defines the partial differential inequalities (PDI) (of evolution)

$$g\left(\pi_x(t)\right) \leqq 0, t \in \Omega_{t_0,t_1},$$

and the Lagrange densities matrix

$$h = \left(h_a^l\right)\colon \Pi \to \mathbb{R}^{ms}, a = \overline{1,s}, l = \overline{1,z}, z < n,$$

defines the partial differential inequalities (PDI) (of evolution)

$$h\left(\pi_x(t)\right) = 0, t \in \Omega_{t_0,t_1}.$$

In the paper, we consider the multitime multiobjective variational problem (MFP) of minimizing a vector of path independent curvilinear functionals defined by

$$\begin{aligned} \min K(x(\cdot)) = \left(K^1\left(x(\cdot)\right), ..., K^r\left(x(\cdot)\right)\right) \\ g(\pi_x(\cdot)) \leqq 0, \\ h(\pi_x(\cdot)) = 0, \\ x(t_0) = x_0, x(t_1) = x_1, \end{aligned} \tag{MFP}$$

Let

$$D = \left\{x \in C^\infty\left(\Omega_{t_0,t_1}, M\right) : t \in \Omega_{t_0,t_1}, x(t_0) = x_0, x(t_1) = x_1, \right. \\ \left. g(\pi_x(t)) \leqq 0, h(\pi_x(t)) = 0\right\}$$

denote the set all feasible solutions of problem (MFP).

Definition 0.1 *A feasible solution $\bar{x}(\cdot) \in D$ is called an efficient solution to the problem (MFP) if there is no other feasible solution $x(\cdot) \in D$ such that*

$$K(x(\cdot)) \leq K(\bar{x}(\cdot)).$$

If, in this relation, we use the strict inequality <, then $\bar{x}(\cdot)$ is called weakly efficient solution to the problem (MFP).

Let $A\colon C^{\infty}\left(\Omega_{t_0,t_1}, M\right) \to \mathbb{R}^r$ be a path independent curvilinear vector functional

$$A\left(x\left(\cdot\right)\right) = \int_{\gamma_{t_0,t_1}} a_{\alpha}(\pi_x(t))dt^{\alpha}.$$

Now, we introduce the definition of the vectorial (Φ, ρ) -convexity for the vectorial functional A, which will be useful to state the results established in the paper. Before we do this, we give the definition of a convex functional.

Definition 0.2 *The functional $F\colon \Pi \times \Pi \times C^{\infty}\Omega_{t_0,t_1}, \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is convex with respect to the third component, if, for all $x(\cdot)$, $\bar{x}(\cdot)$, $y_1(\cdot)$, $y_2(\cdot)$, the following inequality holds*

$$\begin{aligned} & F\Big(\pi_x(t), \pi_{\bar{x}(t)}; (\lambda(y_1(t), q_1) + (1-\lambda_2)(y_2(t), q_2))\Big) \\ \leqq \quad & \lambda F\left(\pi_x(t), \pi_{\bar{x}}(t); (y_1(t), q_1)\right) + (1-\lambda) F\left(\pi_x(t), \pi_{\bar{x}}(t); (y_2(t), q_2)\right), \end{aligned}$$

for q, q_1, $q_2 \in \mathbb{R}^n$, $\lambda \in [0, 1]$.

Let S be a nonempty subset of $C^{\infty}\left(\Omega_{t_0,t_1}, M\right)$, and $\bar{x}(\cdot) \in S$ be given. Following the footsteps of [1], we have the following definition.

Definition 0.3 Let $\rho = (\rho_1, ..., \rho_r) \in \mathbb{R}^r$ and $\Phi\colon \Pi \times \Pi \times \mathbb{R}^r \to \mathbb{R}$ be convex with respect to the third component. The vectorial functional A is called (strictly) (Φ, ρ)-convex at the point $\bar{x}(\cdot)$ on S if, for each i, $i = \overline{1, r}$, the following inequality

$$A^i(x(\cdot)) - A^i(\bar{x}(\cdot)) \quad \geqq \quad \int_{\gamma_{t_0,t_1}} \Phi\left(\pi_x(t), \pi_{\bar{x}}(t); \left(\frac{\partial a^i_{\alpha}}{\partial x}(\pi_{\bar{x}}(t))\right.\right. \tag{2}$$

$$\left.\left. - \; D_{\gamma}\left(\frac{\partial a^i_{\alpha}}{\partial x_{\gamma}}(\pi_{\bar{x}}(t))\right), \rho_i\right)\right) dt^{\alpha} \tag{3}$$

holds for all $x(\cdot) \in S$, $(x\left(\cdot\right) \neq \bar{x}(\cdot))$). If inequalities (2) are satisfied at each $\bar{x}(\cdot) \in S$, then A is called (strictly) (Φ, ρ)-convex on S.

We begin with a theorem which gives conditions in which a critical point of the unconstrained problem related to (MVP) becomes weakly efficient to the unconstrained problem.

Theorem 0.1 Let $\bar{x}(\cdot)$ be a critical point for the unconstrained optimization problem (UMVP)

$$\min K(x(\cdot)) = \left(K^1(x(\cdot)), \ldots K^r(x(\cdot))\right), \qquad \text{(UMVP)}$$

associated with the problem (MVP), and $\lambda \in \mathbb{R}^r$ be the corresponding Lagrange multiplier. If K is (Φ, ρ)-convex at $\bar{x}(\cdot)$, and $\langle \lambda, \rho \rangle \geq 0$, then $\bar{x}(\cdot)$ is a weakly efficient solution to the unconstrained problem (UMVP).

In the following, we study the multiobjective problem (MVP) by the use of the weighting method. In this respect, we introduce the problem

$$\min \langle K(x(\cdot)), \lambda \rangle, \qquad (W_{\lambda})$$

where $x(\cdot) \in c^{\infty}(\Omega_{t_0,t_1}, \mathbb{R}^n)$, and $\lambda \in \mathbb{R}^r$.

In some additional hypotheses, any weakly efficient solution to the unconstrained vector problem (UMVP) is also an optimal solution to an adequate weighted problem (W_{λ}), as the following theorem states.

Theorem 0.2 Let K be (Φ, ρ)-convex on $C^{\infty}(\Omega_{t_0,t_1}, \mathbb{R}^n)$, and $\bar{x}(\cdot)$ be a weakly efficient solution to the unconstrained vector problem (UMVP). If $\lambda \in \mathbb{R}^r$, $\lambda \geqq 0$, and the Lagrange multiplier of $\bar{x}(\cdot)$, is endowed with the property that $\langle \lambda, \rho \rangle \geq 0$, then $\bar{x}(\cdot)$ is also an optimal solution to (W_{λ}).

In [4], there were proved necessary optimality conditions for a problem similar to (MFP); for our case we obtain the next theorem.

Theorem 0.3 Let $\bar{x}(\cdot) \in D$ be a normal efficient solution in multitime multiobjective problem (MVP). Then there exist the vector $\overline{\Lambda} \in \mathbb{R}^r$ and the smooth functions $\overline{\mu} : \Omega_{t_0,t_1} \to \mathbb{R}^{msp}$, $\overline{\xi} : \Omega_{t_0,t_1} \to \mathbb{R}^{rsp}$ such that

$$\begin{gathered}\left\langle \overline{\Lambda}, \tfrac{\partial k_\alpha}{\partial x}(\pi_{\bar{x}}(t)) \right\rangle + \left\langle \overline{\mu}_\alpha(t), \tfrac{\partial g}{\partial x}(\pi_{\bar{x}}(t)) \right\rangle + \left\langle \overline{\xi}_\alpha(t), \tfrac{\partial h}{\partial x}(\pi_{\bar{x}}(t)) \right\rangle \\ -D_\gamma \left(\left\langle \overline{\Lambda}, \tfrac{\partial k_\alpha}{\partial x_\gamma}(\pi_{\bar{x}}(t)) \right\rangle + \left\langle \overline{\mu}_\alpha(t), \tfrac{\partial g}{\partial x_\gamma}(\pi_{\bar{x}}(t)) \right\rangle + \left\langle \overline{\xi}_\alpha(t), \tfrac{\partial h}{\partial x_\gamma}(\pi_{\bar{x}}(t)) \right\rangle \right) = 0, \\ t \in \Omega_{t_0,t_1}, \alpha = \overline{1,p}, \text{(Euler-Lagrange PDEs)}\end{gathered} \tag{4}$$

$$\langle \overline{\mu}_\alpha(t), g(\pi_{\bar{x}}(t)) \rangle = 0, \in \Omega_{t_0,t_1}, \alpha = \overline{1,p}, \tag{5}$$

$$\overline{\Lambda} \geq 0, \left\langle \overline{\Lambda}, e \right\rangle = 1, \overline{\mu}_\alpha(t) \geqq 0, t \in \Omega_{t_0,t_1}, \alpha = \overline{1,p}. \tag{6}$$

The following theorem establishes sufficient conditions of weakly efficiency for the problem (MVP).

Theorem 0.4 Suppose that the following conditions are fulfilled:
1) $\bar{x}(\cdot) \in D$, $\bar{\Lambda}$, $\mu(\cdot)$ and $\nu(\cdot)$ satisfy the necessary conditions of efficiency (4)-(6).
2) The objective functional K is (Φ, ρ_K)-convex.
3) $\left\langle \bar{\mu}_{\alpha j}(\cdot), g^j(\pi_x(\cdot)) \right\rangle$, $j = \overline{1,m}$, are (Φ, ρ_{g_j})-convex at $\bar{x}(\cdot)$ on D;
4) $\left\langle \bar{\xi}_{\alpha l}(\cdot), h^l(\pi_x(\cdot)) \right\rangle$, $l = \overline{1,z}$, are (Φ, ρ_{h_l})-convex at $\bar{x}(\cdot)$ on D;
5) $\langle \lambda, \rho_K \rangle + \sum_{j=1}^m \rho_{g_j} + \sum_{l=1}^z \rho_{h_l} \geq 0$.
Then $\bar{x}(\cdot)$ is a weakly efficient solution to the problem (MVP).

Analogously to Theorem 0.2 there can be established connection between weakly efficient solutions to the problem (MVP) and appropriate weighting problems.

Consider the dual problem (to (MVP)) in the sense of Mond-Weir

$$\begin{aligned}&\max K(x(\cdot)) \\ &\left\langle \lambda, \frac{\partial k_\alpha}{\partial x}(\pi_{\bar{x}}(t)) \right\rangle + \left\langle \overline{\mu}_\alpha(t), \frac{\partial g}{\partial x}(\pi_{\bar{x}}(t)) \right\rangle + \left\langle \overline{\xi}_\alpha(t), \frac{\partial h}{\partial x}(\pi_{\bar{x}}(t)) \right\rangle \qquad \text{(DMVP)} \\ &-D_\gamma \left(\left\langle \lambda, \frac{\partial k_\alpha}{\partial x_\gamma}(\pi_{\bar{x}}(t)) \right\rangle + \left\langle \overline{\mu}_\alpha(t), \frac{\partial g}{\partial x_\gamma}(\pi_{\bar{x}}(t)) \right\rangle + \left\langle \overline{\xi}_\alpha(t), \frac{\partial h}{\partial x_\gamma}(\pi_{\bar{x}}(t)) \right\rangle \right) = 0, \\ &\qquad t \in \Omega_{t_0,t_1}, \alpha = \overline{1,p}, \text{(Euler} - \text{LagrangePDEs)} \\ &\left\langle \overline{\mu_\alpha(t)}, g(\pi_{\bar{x}}(t)) \right\rangle = 0, t \in \Omega_{t_0,t_1}, \\ &\overline{\mu}_\alpha(t) \geqq 0, t \in \Omega_{t_0,t_1}, \alpha = \overline{1,p}.\end{aligned}$$

Let ΔD be the set of the feasible solutions to the dual problem (DMVP), and $\Delta = \{y(\cdot) : (y(\cdot), \lambda, \mu(\cdot), \nu(\cdot)) \in \Delta D\}$. By using (Φ, ρ)-convexity hypothesis, weak, strong, and converse duality results may be stated and proved, as in the sequel.

Theorem 0.5 Suppose that $x(\cdot)$ and $(y(\cdot), \lambda, \mu(\cdot), \nu(\cdot))$ are feasible solutions to the problems (MVP), and (DMVP), respectively. Additionally, presume that hypotheses 2)-5) from Theorem 0.4 are satisfied. Then $K(x(\cdot)) \nleq K(y(\cdot))$.

Theorem 0.6 Consider that $x(\cdot)$ is an efficient solution to the primal problem (MVP). Then there exists λ, $\mu(\cdot)$, $\nu(\cdot)$ so that $(x(\cdot), \lambda, \mu(\cdot), \nu(\cdot)) \in \Delta D$. More than that, if assumptions 2)-5) from Theorem 0.4 are fulfilled, then $(x(\cdot), \lambda, \mu(\cdot), \nu(\cdot))$ is an efficient solution to the dual problem (DMVP).

Theorem 0.7 Let $(y(\cdot), \lambda, \mu(\cdot), \nu(\cdot))$ be an efficient solution to the dual problem (DMVP). Assume that conditions 2)-5) from Theorem 0.4 are satisfied. Then $y(\cdot)$ is an efficient solution to the primal problem (MVP).

In a similar manner, a dual problem in the sense of Wolfe can be associated to our vector problem (MVP). First, we introduce the objective of this problem.

$$\varphi(y(\cdot), \mu(\cdot), \xi(\cdot)) = \int_{\gamma_{t_0,t_1}} \left\{ k_\alpha\left(\pi_y(t)\right) + \left[\left\langle \mu_\alpha(t), g\left(\pi_y(t)\right) \right\rangle + \left\langle \xi_\alpha(t), h\left(\pi_y(t)\right) \right\rangle \right] e \right\} dt^\alpha,$$

where $e = (1, \ldots, 1)^T \in \mathbb{R}^r$.

The associated multitime multiobjective dual problem of (MVP) in the sense of Wolfe is (WDMVP), as in the following.

$$\begin{aligned}
&\min \varphi(y(\cdot), \mu(\cdot), \xi(\cdot)) \\
&\left\langle \lambda, \frac{\partial f_\alpha}{\partial y}\left(\pi_y(t)\right)\right\rangle + \left\langle \mu_\alpha(t), \frac{\partial g}{\partial y}\left(\pi_y(t)\right)\right\rangle + \left\langle \xi_\alpha(t), \frac{\partial h}{\partial y}\left(\pi_y(t)\right)\right\rangle \\
&-D_\gamma\left(\left\langle \lambda, \frac{\partial f_\alpha}{\partial y_\gamma}\left(\pi_y(t)\right)\right\rangle + \left\langle \mu_\alpha(t), \frac{\partial g}{\partial y_\gamma}\left(\pi_y(t)\right)\right\rangle + \left\langle \xi_\alpha(t), \frac{\partial h}{\partial y_\gamma}\left(\pi_y(t)\right)\right\rangle\right) = 0, \qquad \text{(WDMVP)} \\
&t \in \Omega_{t_0,t_1}, \quad y(t_0) = y_0, \quad y(t_1) = y_1, \\
&\lambda \geq 0, \quad \langle \lambda, e\rangle = 1, \quad \mu_\alpha(t) \geqq 0, \quad t \in \Omega_{t_0,t_1}, \quad \alpha = 1, \ldots, p,
\end{aligned}$$

Again, by the use of the notion of (Φ, ρ)-convexity, some weak, strong and converse duality results can be stated and proved.

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