Generalized Ideal Points

Andrzej M.J. Skulimowski[1,a)]

[1]*AGH University of Science and Technology, Chair of Automatic Control and Robotics, Decision Sciences Laboratory, al. Mickiewicza 30, 30-059 Kraków, Poland*

[a)]Corresponding author: ams@agh.edu.pl

Abstract. This paper presents an extension of the ideal point concept, which is one of the most widespread ideas in multicriteria optimization. In the simplest case an ideal point of a multicriteria optimization problem is an element of the criteria space with the coordinates equal to the optima of each criterion functions considered as scalar objectives. First, we will provide an overview of properties of ideal points and their earlier generalizations, including the nadir points. Then, we will present an extension of ideal points in case when the dimension of the criteria space is higher than 2. Specifically, scalar optima used in the classical definition of ideal points are replaced by Pareto optima with respect to criteria subsets. It is assumed that the generating criteria subsets belong to a minimal covering of the set of all criteria.We will distinguish several classes of generalized ideal points (GIP) so defined. For example, the proper GIPs are calculated with respect to the criteria subsets that form a partition of the criteria set. The regular GIPs are those which correspond to a covering with criteria subsets of the same cardinality and such that all intersections of these sets do also contain the same number of criteria or are empty. Finally, we will propose an application of the GIP concept to formulating criteria independence conditions, as well as to modeling group decision making with multiple.

INTRODUCTION

The ideal point concept belongs to the most widespread ideas in multicriteria optimization. In the simplest case, for the multicriteria optimization problem with the natural coordinatewise ordering in the criteria space $E:=\mathbb{R}^N$,

$$(F:U\rightarrow\mathbb{R}^N)\rightarrow\min,\ U\subset\mathbb{R}^M,\ F:=(F_1,\ldots,F_N), \tag{1}$$

where all functions $F_1,\ldots,F_N$ are bounded from below, *ideal point* x^* is an element of the criteria space with the coordinates equal to the optima of each criterion function considered as a scalar objective, i.e.

$$x^*(U,F):=(\inf\{F_1(u):\ u\in U\},\ldots,\inf\{F_N(u):\ u\in U\})\in\mathbb{R}^N. \tag{2}$$

Let us recall that the primary solution concept for problem (1) is Pareto optimality, by definition $u\in U$ is *Pareto optimal* iff for any $v\in U$ the coordinatewise inequality $F(v)\leq F(u)$ implies $u=v$. The set of Pareto optimal elements, in short *Pareto set*, will be denoted by $P(U,F)$. The values of F on $P(U,F)$ will be termed nondominated, the set of nondominated points in E will be denoted $FP(U)$. In the next section these notions are extended for arbitrary Banach spaces E with an order generated by a convex, closed, pointed, and nontrivial cone θ.

Ideal points, sometimes termed also utopia points [20], attracted the attention of multicriteria decision making (MCDM) researchers since the inception period of multicriteria decision analysis, becoming one of the most wide-spread concepts. Their usual non-attainability in non-trivial multicriteria optimization problems motivated the re-searchers to look for surrogate decision rules by finding admissible Pareto optimal solutions closest, in a certain sense, to x^*, cf. e.g. [5]. Thus, in so called compromise programming, ideal points plays the role of a target reference points [15]. Lower bounds of the criteria values are also useful when investigating the properties of other reference points in the criteria space [14]. Namely, the situation of a reference point q with respect to the ideal one can be helpful to determine whether minimizing the distance from q to $F(U)$ yields a nondominated element of $F(U)$ [14].

Further applications of ideal points in multicriteria analysis refer to the redundancy of the multicriteria optimization problem formulation, namely to the existence of redundant (termed also non-essential [6]) criteria.

Namely, a straightforward observation that if an ideal point in a multicriteria problem is attainable then this problem is, in fact, scalar and all but one criteria can be eliminated, led to formulating a criteria redundancy condition for multicriteria linear programming [7]. Conditions of this type were later extended [19] and formulated for nonlinear problems as well, cf. [1,10]. An overview of this class of applications is provided in [21].

This paper presents an extension of ideal points which may be defined in case when the dimension of the real criteria space is at least 3, but preferably when it is higher. In the literature devoted to evolutionary multicriteria analysis such problems are sometimes referred to as many criteria optimization [21]. Specifically, scalar optima used in the classical definition of ideal points are replaced by the Pareto optima calculated with respect to certain subsets of the criteria set. The latter are assumed to cover the set of all criteria and fulfill some further properties, for example the covering generating a generalized ideal point (GIP) should be minimal.

Basic properties of ideal points and their earlier generalizations that are also relevant for the new extended class, are given in the next section. It contains also an example of non-unique ideal points that may occur in problems with an ordering cone generated by more than N linearly independent vectors, where N is the dimension of the criteria space. In section 3 we will distinguish several classes of GIPs and discuss their properties. For example, proper GIPs are calculated with respect to the criteria subsets that form a disjoint partition of the criteria set F. The regular GIPs are those which correspond to the subsets of F of the same cardinality and all their intersections do also contain the same number of elements or are empty. In the final section, we will discuss some applications of GIPs, for example to formulating criteria independence and redundancy conditions, as well as to modeling group decision making with multiple criteria and in mixed cooperative-conflicting multicriteria games.

BASIC PROPERTIES OF TOTALLY DOMINATING AND IDEAL POINTS

Let us consider now a more general problem than (1) assuming that the criteria are valued in a Banach space E partially ordered by a closed, convex, nontrivial and pointed cone θ. The order in E is defined as $x \leq_\theta y \overset{\mathrm{df}}{\Leftrightarrow} y-x \in \theta$. Let us recall that a nontrivial cone θ is pointed iff $\theta \cap (-\theta)=\{0\}$. The new problem can now be formulated as

$$(F: U \rightarrow E) \rightarrow \min(\theta),\ U \subset Y, \tag{3}$$

where Y is an arbitrary topological vector space, and the notation $\min(\theta)$ denotes the minimization problem with respect to the order $\leq_\theta$.

The definitions of Pareto optimal and nondominated points for the problem (1) remain valid when the relation "$\leq$" is replaced by "$\leq_\theta$". When the ordering cone may be ambiguous, we will also use the term "θ-optimality" and include θ in the notation, for example as $P(U,F,\theta)$. We will also recall the definition of *totally dominating points*, $TD(F(U),\theta)$, [14],[16] for the problem (3):

$$TD(F(U),\theta) := \{y \in E:\ F(U) \subset y+\theta\}. \tag{4}$$

The corresponding definition of ideal points based on the notion of total dominance can be formulated as follows [14],[15],[16],[18]:

Definition 1. Any $(-\theta)$-optimal element of $TD(F(U),\theta)$ will be termed an *ideal point* for $F(U)$. The set of ideal points will be denoted by $x^*(F(U),\theta)$. ■

In other terms, ideal points are defined as θ-maximal elements of totally dominating points. The Def. 1 may be used to define *nadir points* $\eta^*(F(U),\theta)$ as ideal points of $FP(U,\theta)$ with respect to the negative cone, i.e.

$$\eta^*(F(U),\theta) := x^*((FP(U,\theta), (-\theta)). \tag{5}$$

The existence and uniqueness of ideal points cannot be taken for granted for any subset of E. The following Theorem 1 [15],[16] states a necessary and sufficient condition for the existence of at least one ideal point and establishes another relation between $TD(F(U),\theta)$ and $x^*(F(U),\theta)$.

Theorem 1. If θ is closed and pointed, and $TD(F(U),\theta)$ is non-empty then the latter set can be expressed as

$$TD(F(U),\theta) := x^*(F(U),\theta) - \theta. \tag{6}$$

■

From Thm. 1 it follows that $x^*(F(U),\theta)$ is nonempty iff $F(U)$ is θ-bounded, i.e. iff there exists $y \in E$ such that $F(U) \subset y+\theta$. Theorem 2 given below relates the uniqueness of ideal points to the properties of θ.

Theorem 2 [15],[16]. Suppose that E is a linear space partially ordered by a closed, convex, nontrivial and pointed cone θ. Then the following conditions are equivalent:

a) *If the subset X of E is θ-bounded then the set of ideal points for X, $x^*(X,\theta)$, consists of a single point.*
b) *For every two points $x_1,x_2 \in E$ there exists the unique $z \in E$ such that*

$$(x_1+\theta)\cap(x_2+\theta)=z+\theta \tag{7}$$

c) *(E,θ) is a vector lattice, i.e. $\forall a,b \in E$ the minimal element of a and b, $a\wedge b$, is defined as $a\wedge b := x^*(\{a,b\},\theta)$.* ■

Proof of this theorem was given first in [15]. Let us recall that, by definition, the minimal element of a and b in a lattice E is unique and it should not be confused with the maximal lower bound with respect to the partial order in E.

As an easy corollary we see that a sufficient uniqueness condition the intersection of any two translations of an ordering cone θ must be non-empty and congruent to θ. In $\mathbb{R}^N$ this property is satisfied by the polyhedral cones generated by exactly N linearly independent vectors, so called *simplicial cones* (cf. e.g. [3]). They can be equivalently characterized as isomorphic transformations of the natural positive cone $\mathbb{R}_+^N$. Observe that if a vector function F is to be minimized on a set U with respect to a simplicial cone θ in $\mathbb{R}^N$ generated by the base vectors $\{s_1,\ldots,s_N\}$, it is sufficient to calculate the global infima on U of the function $G:=S^{-1}\circ F$, where S is the automorphism of $\mathbb{R}^N$ defined by the matrix $[s_1,\ldots,s_N]$. This observation provides a constructive algorithm to compute the ideal point in virtually all situations where its uniqueness is guaranteed.

As the second observation, we conclude that in $\mathbb{R}^2$ the ideal points with respect to any nontrivial convex cone are either unique or do not exist. However, the class of convex cones that do not satisfy the assumptions of Thm. 2 is large even in $\mathbb{R}^3$. This is exemplified below.

Example 1. We will show that the ideal and nadir points for the set $F(U)=[x_1,x_2]\subset\mathbb{R}^3$ are not unique if the order in $\mathbb{R}^3$ is defined by the "ice-cream cone" $\theta:=\{(x,y,z)\in\mathbb{R}^3: x^2+y^2\le az\}$, with $a>0$, and $[x_1,x_2]$ is not contained in the main axis of this cone. It is illustrated in Figure 1 with a=1, $x_1:=A=(1, 0.5,1)$, $x_2:=B=(1,-0.5,1)$. For clarity's sake, the set $F(U)$ is inherently nondominated, i.e. $F(U)=FP(U,\theta)$ and is situated in a plane perpendicular to the axis of the cone. By the construction, $x^*(F(U),\theta)$ consists of mutually non-comparable points dominating [A,B]. An analogous argument applies to and $\eta^*(F(U),\theta)$, and the order introduced by and $(-\theta)$.

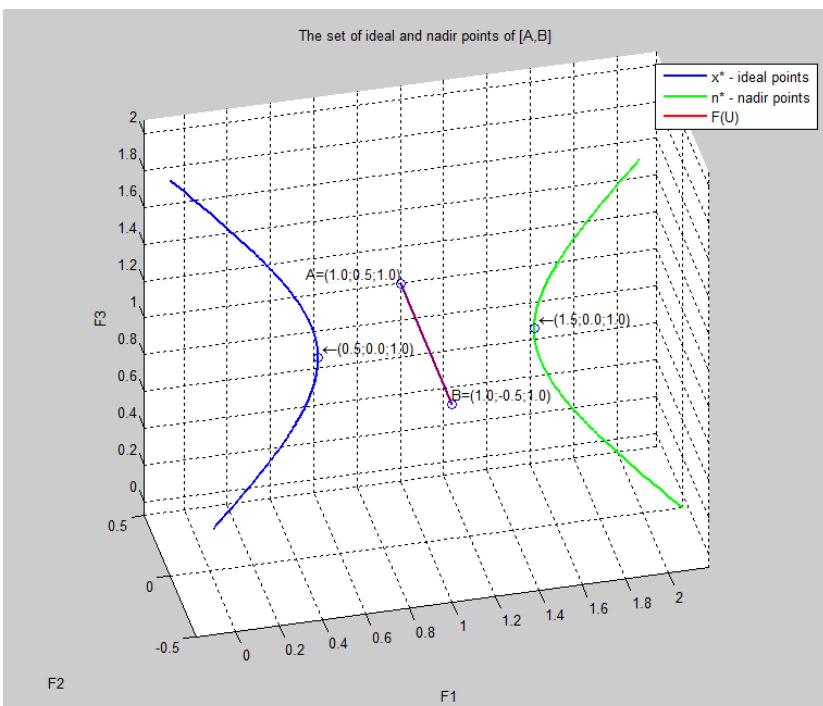

FIGURE 1. An example of situation, where the sets of ideal points (right) and nadir points (left) are not unique

Moreover, the set of ideal points is a curve unbounded from below, which means that $x^*(x^*(F(U),\theta),\theta)$ is empty. ■

The definition of ideal points used above is not the only one that can be found in the literature. For example, in [2] and in the subsequent papers referring to the above cited paper (cf. e.g. [4],[9]), an ideal point with respect to a closed and pointed cone θ is defined as an ideal point in the sense of Def.1 with respect to another closed and pointed cone ψ such that $\theta\subset\psi$. So defined ideal points are likely to be elements of $F(U)$ and are related to the extreme points of the set $F(U)+\theta$. Their density in $F(U)$ and other properties have been studied e.g. in [2] and [9].

HIGHER-ORDER IDEAL POINTS

According to the definitions presented in the above section, the coordinates of ideal points in multicriteria optimization problem (1) with $\theta=\mathbb{R}_+^N$ are determined as the best values of the corresponding criteria. They are calculated separately on the same set of admissible decisions with a scalar optimization algorithm. In this section, we present a generalization of this concept, namely, we will look for vectors in the criteria space such that selected subsets of their coordinates are nondominated with respect to appropriate subsets $\{F_{i1},\ldots,F_{ik}\}$ of the criteria set $\{F_1,\ldots,F_N\}$ ($k<N$). It is assumed that for a given problem (1) a family of criteria subsets covers the whole set $\{F_1,\ldots,F_N\}$ and no element of such a family is contained in a union of the others. Every such covering will generate a set of generalized ideal points. We will distinguish several classes of GIPs which are defined below. To provide the most general form of GIP, we will define first the *inner product* of sets spanned by a given Cartesian product of n spaces.

Definition 3. Let $X_1,X_2,\ldots,X_n$ be arbitrary non-empty sets and let $A\subset X_{i1}\times\ldots\times X_{ik}$, $1\le i1<i2<\ldots<ik\le n$, $B\subset X_{j1}\times\ldots\times X_{jm}$, $1\le j1<j2<\ldots<jm\le n$. Denote $I:=\{i1,i2,\ldots,ik\}$, $J:=\{j1,j2,\ldots,jm\}$, $X_I:=\{X_{i1},\ldots,X_{ik}\}$, $X_J:=\{X_{j1},\ldots,X_{jm}\}$. The projection from $X_{i1}\times\ldots\times X_{ik}$ onto $X_{ip(1)}\times\ldots\times X_{ip(q)}$, where $I':=\{ip(1),\ldots,ip(q)\}\subset I$, will be denoted $\mathrm{pr}_{I,I'}$. Then, the *inner product* of A and B, denoted $A\#B$, will be defined as

$$A\#B:=\mathrm{pr}_{I,I\setminus J}(A)\times(\mathrm{pr}_{I,I\cap J}(A)\cup\mathrm{pr}_{J,I\cap J}(B))\times\mathrm{pr}_{J,J\setminus I}(B) \qquad \text{iff } I\cap J\neq\varnothing \tag{8}$$

■

From the above definition it follows that the inner product of two sets spanned by X_I and X_J is spanned by $X_{I\cup J}$. Admitting additionally the convention $\mathrm{pr}_{I,\varnothing}:=\varnothing$, $X_\varnothing:=\varnothing$, and $X\times\varnothing=X=\varnothing\times X$, it is easy to see that if $I\cap J=\varnothing$ then $A\#B=A\times B$ and if $I=J$ then $A\#B=A\cup B$. The assumed ordering of indices in I and J is motivated by the anticipated application of the inner product to the nondominated sets. Since the Pareto sets are invariant to a permutation of criteria, so in a problem (1) one can order them arbitrarily to obtain the desired order of the criteria space coordinates, which span the images of nondominated sets.

The inner product of p sets $A_1,\ldots,A_p$, spanned by $X_{I_1},\ldots\times X_{I_p}$, respectively will be defined in the natural way as $A_1\#\ldots\#A_p:=(A_1\#\ldots\#A_{p-1})\#A_p$. Since the operation "#" is well-defined, we can formulate the following

Definition 4. Let us consider the multicriteria problem (1) with the criteria set $\{F_1,\ldots,F_N\}$, and let Ω be a minimal covering of $\{F_1,\ldots,F_N\}$, i.e. $\forall\omega\in\Omega: \Omega\setminus\{\omega\}$ is not a covering of $\{F_1,\ldots,F_N\}$. For an $\omega:=\{F_{i1},\ldots,F_{ik(\omega)}\}$ let us admit the notation $P(U,\omega):=P(U,(F_{i1},\ldots,F_{ik(\omega)}))$ and, analogously, $F_\omega P(U):=(F_{i1},\ldots,F_{ik(\omega)})P(U)$. Then the set of Ω-generalized ideal points, Ω-GIP, is defined as

$$\Omega\text{-}GIP(U,F):=\#_{\omega\in\Omega}F_\omega P(U)\subset I\!R^N. \tag{9}$$

The criteria space $E:=I\!R^N$ is regarded as the Cartesian product of spaces $E_1,\ldots E_N$, each of them is isomorphic to $I\!R$ but it corresponds to the values of a different criterion F_i. ■

The correctness of the above definition follows from the minimality of the covering Ω. However, in real life applications, a major role will be played by the GIPs related to a simpler case of the above general concept, namely the situation, where Ω is disjoint, i.e. it is a partition with $\forall\omega_1,\omega_2\in\Omega\ \omega_1\cap\omega_2=\varnothing$. GIPs calculated with respect to such a disjoint partition will be termed *proper*. Below, we will provide further useful notions.

Regular GIPs are those which correspond to the partitions Ω consisting of subsets of the same cardinality k and such that all pairwise intersections of $\omega_1,\omega_2\in\Omega$ do also contain the same number of criteria $m<k$ or are empty. They are denoted (k,m)-GIPs. Its relevant subclass are *regular proper* GIPs, denoted (k,0)-GIP, which - by definition - are GIPs based on a partition of $\{F_1,\ldots,F_N\}$ that consist of k-element subsets. *Quasi-regular proper* (k,0,r)-GIP is a GIP based on a partition of the criteria set, where all but one subsets consist of k elements and one contains $r<k$ criteria with $N=kp+r$, and $p+1$ - the cardinality of Ω. From the combinatorics of partitions [11] it follows the following fact.

Proposition 1. There are exactly B_N-1 proper GIPs for (1), where B_N is the N-th Bell number [11, Thm.1.12]. ■

Observe that in the above terminology, the classical ideal point (2) can be characterized as (1,0)-GIP and corresponds to the partition Ω consisting of singletons. Generally, Ω-GIPS may be useful to describe the consensus seeking procedures among K decision makers whose preferences are described by the vector criteria $\{F_\omega\}_{\omega\in\Omega}$. A relevant subclass of GIPs are (2,0)- or (2,0,1)-GIP that can be characterized and visualized with their 3D projections. Figure 2 presents an example of situation, where three different regular (2,0) GIPs can be constructed in a four-criteria optimization problem. They correspond to all two-element proper partitions of the set $\{F_1,F_2,F_3,F_4\}$.

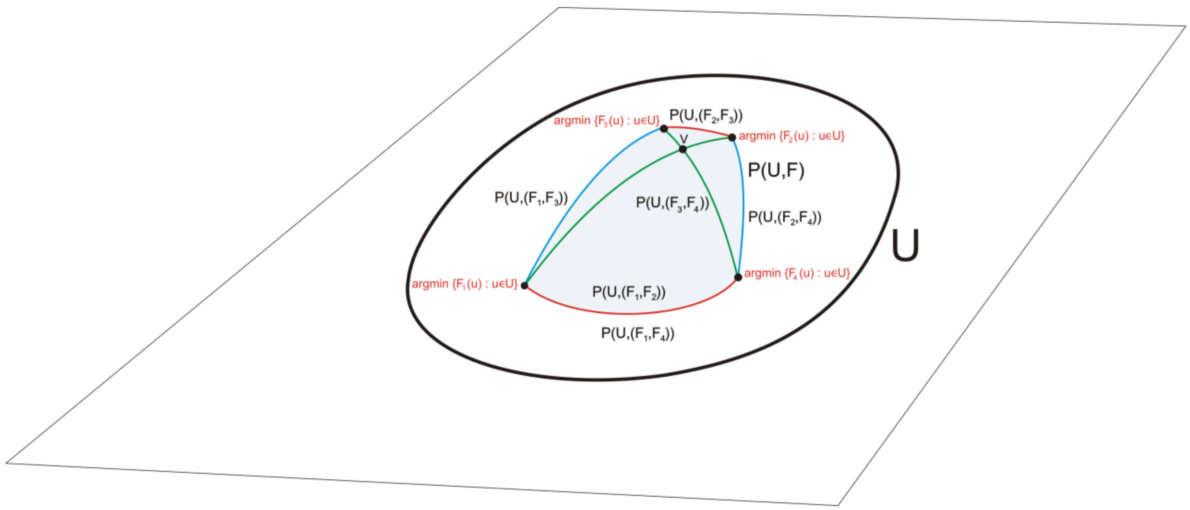

FIGURE 2. An example of Pareto sets in $I\!R^2$ generating regular GIPs in 4-dimensional criteria space

Observe that the partition {1,2} and {3,4} corresponds to the situation where the (2,0)-GIP contains the point $((F_1(v),F_2(v)),(F_3(v),F_4(v))=F(v)\in F(U)$. Generally, a non-empty intersection of Pareto sets corresponding to the partition subsets implies the existence of attainable points in such a GIP. Hence, one can derive the following corollary.

Proposition 2. If the dimension of the criteria space in problem (1) is higher than 4, the decision set $U\subset I\!R^2$ is convex as well as all the criteria, then in (2,0)-GIP there exists at least one attainable point. ■

The proof of this Proposition follows from the Kuratowski's planar graph theorem and the connectivity of P(U,(F_i,F_j)) for convex problems.

Similarly as in case of classical ideal point, the GIP of any class from those defined above may be a base of a distance minimization procedure, when the decision maker's preferences are expressed as a proximity measure to a target set [17]. The θ-optimality conditions of such distance minimization procedures are subject of ongoing research.

DISCUSSION AND CONCLUSIONS

The fact that the uniqueness of ideal points for all θ-bounded subsets of the criteria space E in a multicriteria optimization problem (1) can only be guaranteed if E is ordered by a cone isomorphic to $\mathbb{R}^N_+$ has some far-reaching consequences. First of all, in problems with non-standard preference structures, lacking the uniqueness of $x^*(F(U),\theta)$, a compromise-seeking procedure based in distance minimization with respect to the ideal point must be replaced by finding a least-distance solution to a subset of the criteria space [17],[18]. In group decision making, where each decision maker's preferences are expressed with several independent criteria, a least distance solution to a GIP constructed with Pareto optimal points of each decision maker corresponds to finding a compromise between individual goals. Thus GIPs may serve to extend consensus-reaching procedures [8].

The GIPs introduced in the previous section open new horizons for investigating the independence and redundancy of criteria in so-called many criteria ($N\geq 4$) optimization problems. This became recently a field of intensive research, in particular in the literature devoted to evolutionary multicriteria optimization [13],[21],[12]. Specifically, corners considered in [13] as ideal point with respect to essential subsets of the criteria set inherit thus relevant properties of ideal points, yet in a different way than GIPs proposed in this paper. Using GIPs, this concept can be generalized in a similar way as the ideal points, by considering corners consisting of GIPs with respect to a subset of criteria. These could be used further to provide criteria independence conditions and eliminate one of the limitations of corners, which by definition are always elements of *F(U)*.

Another useful concept that can be further generalized with GIPs are local ideal points (LIPs). Their basic properties were investigated in [15]. An application of LIPs to construct a class of scalarization methods for non-convex vector optimization problems proposed in [15] can be extended to the case where LIPs are replaced by local GIPs. These ideas will again involve finding compromise solutions with the additional preference information given in form of reference sets and proximity measures, yet in the non-convex case.

REFERENCES

1. P.J. Agrell, On redundancy in multi criteria decision making, Eur. J. Oper. Res. **98**, 571-586 (1997).
2. A. Balbás, M.E. Ballvé, and P. Jiménez Guerra, Density theorems for ideal points in vector optimization. Eur. J. Oper. Res. **133**, 260–266 (2001).
3. G.P. Barker, Theory of Cones, Linear Algebra and its Applications **39**, 263-291(1981).
4. Y.-F. Chai, Y.J. Cho, and J. Li, Some Characterizations of Ideal Points in Vector Optimization Problems, J. of Inequalities and Appl. **2008**, Article ID 231845, 1-8 (2008), doi:10.1155/2008/231845.
5. R. Estrella, D. Cattrysse, and J. Van Orshoven, Comparison of Three Ideal Point-Based Multi-Criteria Decision Methods for Afforestation Planning, Forests **5**, 3222-3240 (2014), doi:10.3390/f5123222.
6. T. Gal, T. Hanne, Nonessential objectives within network approaches for MCDM. Eur. J. Oper. Res. **168**(2), 584–592 (2006).
7. T. Gal and H. Leberling, Redundant objective functions in linear vector maximum problems and their determination, Eur. J. Oper. Res. **1**(3), 176-184, (1977).
8. D. Gołuńska, J. Kacprzyk, and S. Zadrożny, "A Consensus Reaching Support System Based on Concepts of Ideal and Anti-ideal Point", in: *2014 IEEE Conference on Norbert Wiener in the 21st Century (21CW)*, Boston, Mass., 24-26 June 2014, pp.1-6, DOI: 10.1109/NORBERT.2014.6893874.
9. P. Jiménez Guerra, M.A. Melguizo, and M.J. Muñoz-Bouzo, Conic set-valued maps in vector optimization. Set-Valued Anal. **15**, 47–59 (2007).
10. A.B. Malinowska, and D.F.M. Torres, Computational Approach to Essential and Nonessential Objective Functions in Linear Multicriteria Optimization, J. Optim. Theory Appl. **139**, 577–590 (2008) DOI 10.1007/s10957-008-9397-z.
11. T. Mansour, *Combinatorics of Set Partitions*, (CRC Press, Taylor & Francis Group, Boca Raton, 2013), p.587.
12. P. Lindroth, M. Patriksson, and A.B. Stromberg, Approximating the Pareto optimal set using a reduced set of objective functions, Eur. J. Oper. Res. **207**(3), 1519-1534 (2010), DOI: 10.1016/j.ejor.2010.07.004.
13. H.K. Singh, A. Isaacs, and T. Ray, A Pareto Corner Search Evolutionary Algorithm and Dimensionality Reduction in Many-Objective Optimization Problems, IEEE Trans. Evol. Comp. **15**(4), 539–556 (2011).
14. A.M.J. Skulimowski, Classification and Properties of Dominating Points in Vector Optimization Problems, Meth. Oper. Res. **58**, 99-112 (1990).
15. A.M.J. Skulimowski, "Applicability of ideal points in multicriteria decision-making", in: *Multiple Criteria Decision-Making, Proceedings of the Ninth International Conference: Theory and Applications in Business, Industry, and Government*, Fairfax (VA), 5-8.08.1990, edited by A. Goicoechea, L. Duckstein, S. Zionts, (Springer-Verlag, New York, 1992), pp. 387-400 (available from www.researchgate.net).
16. A.M.J. Skulimowski, *Decision Support Systems Based on Reference Sets*, Monographs No.40, (AGH Sci. Publ., Kraków, 1996), p.167.
17. A.M.J. Skulimowski, "Methods of Multicriteria Decision Support Based on Reference Sets", in *Advances in Multiple Objective and Goal Programming*, edited by R. Caballero, F. Ruiz, R.E. Steuer, Lecture Notes in Economics and Mathematical Systems 455, Springer, Berlin-Heidelberg-New York, 1997, pp. 282-290.
18. A.M.J. Skulimowski, *Selected methods, applications, and challenges of multicriteria opti¬mization*. (Committee for Automation and Robotics of Pol. Acad. Sci., Vol.19, AGH Publishers, Kraków, 2018), p.380.
19. N.V. Thoai, Criteria and dimension reduction of linear multiple criteria optimization problems, J. Glob. Optim. **52**, 499–508 (2012), DOI 10.1007/s10898-011-9764-4.
20. P.L. Yu, A Class of Solutions for Group Decision Problems. Management Science **19**(8), 936-946 (1973).
21. Y. Yuan, Y.-S. Ong, A. Gupta, and H. Xu, Objective Reduction in Many-Objective Optimization: Evolutionary Multiobjective Approaches and Comprehensive Analysis", IEEE Trans. Evol. Comp. **22**, 189-210 (2018).