Perspective Envelopes for Bilinear Functions

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Abstract. We characterize the convex hull of the set defined by a bilinear function f(x,y) = xy and a linear inequality defined in x and y. The new characterization, based on perspective functions, dominates the standard McCormick convexification approach. In practice, this result can be of great value in global optimization frameworks, helping to tighten the bounds provided by convex relaxations. The new formulation has the potential to improve the pruning process in spatial branch and bound schemes and consequently reduce the search space effort.

Introduction

Bilinear expressions are the most common nonconvex components in mathematical formulations modeling problems arising in chemical engineering, pooling and blending, supply chain, transportation science and energy systems, to name a few. Global optimization solvers ans many convexification approaches [10, 15, 12] are based on the McCormick envelopes [11]. Given the set $\mathcal{S}_0 = \{(x, y, z) \in (\mathcal{B} \times \mathbb{R}) \mid z = xy\}$, where $\mathcal{B} = \left\{[\boldsymbol{x}^l, \boldsymbol{x}^u] \times [\boldsymbol{y}^l, \boldsymbol{y}^u]\right\}$, its convex hull is defined as,

$$\mathsf{conv}(\mathcal{S}_0) = \left\{ (x, y, z) \in \left([\boldsymbol{x}^l, \boldsymbol{x}^u] \times [\boldsymbol{y}^l, \boldsymbol{y}^u] \times \mathbb{R}\right) \left| \begin{array}{l} z \geq \boldsymbol{x}^l y + \boldsymbol{y}^l x - \boldsymbol{x}^l \boldsymbol{y}^l \\ z \geq \boldsymbol{x}^u y + \boldsymbol{y}^u x - \boldsymbol{x}^u \boldsymbol{y}^u \\ z \leq \boldsymbol{x}^l y + \boldsymbol{y}^u x - \boldsymbol{x}^l \boldsymbol{y}^u \\ z \leq \boldsymbol{x}^u y + \boldsymbol{y}^l x - \boldsymbol{x}^u \boldsymbol{y}^l \end{array} \right. \right\}$$

This set of constraints, known as the McCormick envelopes, defines both the convex and the concave envelopes of the bilinear function $f(x, y) = xy$ on the rectangular domain $[\boldsymbol{x}^l, \boldsymbol{x}^u] \times [\boldsymbol{y}^l, \boldsymbol{y}^u]$. The quality of this polyhedral relaxation highly depends on the initial bounds on variables x and y. State-of-the-art global optimization solvers implement bound contraction techniques in order to improve their underlying relaxations. Once bound propagation is completed, domain partitioning becomes necessary. Spatial branch and bound schemes [14] are among the most effective partitioning methods in global optimization. By splitting the domain of a given variable, the solver is able to divide the original domain into two smaller regions, further tightening the convex relaxations of each partition. In general, the variables involved in bilinear expressions are also linked through other constraints in the problem formulation. It is thus possible to tighten the convex relaxation of the feasible region by combining the bilinear term with other constraints.

Related Work. In [13], the convex hull of the bilinear function over D-polytopes is derived in the space of original variables. Thereafter, Linderoth [7] produces analytical characterizations on triangular domains. Concurrently, Benson [2] derives the convex hull on parallelograms and trapezoids. More recently, Anstreicher and Burer [1] study higher dimension characterizations. Locatelli and Schoen [9] propose a different approach for computing convex envelopes, based on solving convex programs. Locatelli [8] then uses this result to derive closed-form solutions for various domains including 2d polytopes.

In this work, we consider the bilinear function xy in conjunction with a linear constraint linking variables x and y, i.e., $x \leq y$, and leading to a family of polyhedral domains including right triangles, right trapezoids, and rectangles. While Locatelli's [8] technique can be applied in this setting, the current proof, based on perspective functions, offers a new angle on deriving such convex hulls, and can be easily extended to handle arbitrary linear constraints. The main result is presented in the next section.

The New Convex Envelope

Background on perspective functions

Perspective formulations have been successfully used to model disjunctive constraints in Mixed-Integer Nonlinear Programming (MINLP) [5], dominating standard big-M formulations. Given a convex function $f : \mathbb{R}^n \to \mathbb{R}$ and a real number $u > 0$, the function,

$$f_u(\mathbf{x}) = uf(\mathbf{x}/u)$$

is convex, and represents a *dilated* version of f [6]. The perspective of f, denoted $\tilde{f} : (\mathbb{R}^n \times \mathbb{R}) \to (\mathbb{R} \cup \{+\infty\})$, is defined as the operator considering all dilations of f, i.e.,

$$\tilde{f}(\mathbf{x}, u) = \begin{cases} uf(\mathbf{x}/u) \text{ if } u > 0 \\ +\infty \text{ otherwise.} \end{cases} \tag{1}$$

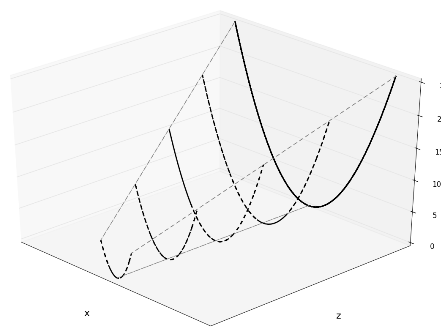

FIGURE 1: Several dilations of the square function using the perspective operator.

Figure 1 illustrates the dilation property of the perspective operator on the square function.

Notations

$\mathbf{x}$ denotes a vector variable in $\mathbb{R}^n$, x a variable in $\mathbb{R}$, and $\boldsymbol{x}$ a constant in $\mathbb{R}$. Given a convex domain $\mathcal{D} \subseteq \mathbb{R}^n$, the epigraph of a continuous function f over $\mathcal{D}$, denoted $\mathsf{epi}_{\mathcal{D}}\ f$, is defined as, $\mathsf{epi}_{\mathcal{D}}\ f = \{(\mathbf{x}, z) \in \mathcal{D} \times \mathbb{R} \mid f(\mathbf{x}) \leq z\}$. The *convex envelope* of f over $\mathcal{D}$, denoted $\mathsf{conv}_{\mathcal{D}}(f)$, represents the convex hull of its epigraph, $\mathsf{conv}_{\mathcal{D}}(f) = \mathsf{conv}\left(\{(\mathbf{x}, z) \in \mathcal{D} \times \mathbb{R} \mid f(\mathbf{x}) \leq z\}\right)$.

The Convex Envelope

Our main result is the following,

Theorem 1. *Consider the bilinear function $f(x, y) = xy$ on the domain $\mathcal{D} \subset \mathbb{R}^2 = \left\{(x, y) \in [\boldsymbol{x}^l, \boldsymbol{x}^u] \times [\boldsymbol{y}^l, \boldsymbol{y}^u] \mid x \leq y\right\}$, s.t. $\boldsymbol{x}^l < \boldsymbol{y}^u$,*

$$\textit{then } \mathsf{conv}_{\mathcal{D}}(f) = \left\{ (x, y, z) \in \mathcal{D} \times \mathbb{R} \;\middle|\; \begin{array}{l} \left(x - \boldsymbol{x}^l \dfrac{x - y}{\boldsymbol{\delta}}\right)^2 \leq \left(1 - \dfrac{x - y}{\boldsymbol{\delta}}\right)\left(z - \boldsymbol{x}^l \boldsymbol{y}^u \dfrac{x - y}{\boldsymbol{\delta}}\right) \\ z \geq \boldsymbol{x}^u y + \boldsymbol{y}^u x - \boldsymbol{x}^u \boldsymbol{y}^u \\ z \geq \boldsymbol{x}^l y + \boldsymbol{y}^l x - \boldsymbol{x}^l \boldsymbol{y}^l \end{array} \right\},$$

and $\boldsymbol{\delta} = (\boldsymbol{x}^l - \boldsymbol{y}^u)$.

Proof. Due to lack of space, we only outline the intuition behind the proof here, a detailed version can be found in [4]. Observe that the intersection of the constraint $x = y$ with the graph of $f(x, y) = xy$ defines a parabola. The proof is based on showing that the convex hull is obtained by applying the perspective operator on the function underlying this parabola, dilating it to the appropriate corner feasible point, as can be seen in Figure 2 (right-side). $\square$

Remark 1. *Observe that for $\boldsymbol{x}_l = \boldsymbol{y}_l = 0$ and $\boldsymbol{x}_u = \boldsymbol{y}_u = 1$, we get the triangular domains studied in [13, 7], and their convex envelope characterization coincide with this result.*

FIGURE 2: Left: Original polyhedral McCormick relaxation for $f(x, y) = xy$. The intersection between f and the constraint $x = y$ is represented in dashed lines.
Right: The new perspective hull, a cone pointed at $\left(\boldsymbol{x}^l, \boldsymbol{y}^u, \boldsymbol{x}^l\boldsymbol{y}^u\right)$.

Computational Impact

To illustrate the potential impact of this result in practice, we design the following computational experiment, based on the nonlinear program,

$$\min \sum_{i=1}^{2} 2z_i^2 + \left(x_i - \frac{\boldsymbol{x}_i^u - \boldsymbol{x}_i^l}{2}\right)^2 + \left(y_i - \frac{\boldsymbol{y}_i^u - \boldsymbol{y}_i^l}{2}\right)^2$$
$$\text{s.t.} \begin{cases} z_i = x_i y_i,\ x_i \le y_i,\ i \in \{1, 2\} \\ \boldsymbol{x}_i^l \le x_i \le \boldsymbol{x}_i^u,\ \boldsymbol{y}_i^l \le y_i \le \boldsymbol{y}_i^u, i \in \{1, 2\} \end{cases} \tag{NLP}$$

The variable bounds are randomly generated based on the following two schemes,

1. $\boldsymbol{x}_i^l = \boldsymbol{y}_i^l = \mathsf{uniform}_{[-2,0]}$, $\boldsymbol{x}_i^u = \boldsymbol{x}_i^l + \mathsf{uniform}_{[0,5]}$, and $\boldsymbol{y}_i^u = \max\left(\boldsymbol{x}_i^u, \boldsymbol{y}_i^l + \mathsf{uniform}_{[0,5]}\right)$.

2. $\boldsymbol{x}_i^l = \mathsf{uniform}_{[-10,10]}$, $\boldsymbol{x}_i^u = \boldsymbol{x}_i^l + \mathsf{uniform}_{[0,10]}$, $\boldsymbol{y}_i^l = \boldsymbol{x}_i^l + \mathsf{uniform}_{[0,2]}$, and $\boldsymbol{y}_i^u = \max\left(\boldsymbol{x}_i^u, \boldsymbol{y}_i^l + \mathsf{uniform}_{[0,10]}\right)$.

$\mathsf{uniform}_{[\mathbf{l},\mathbf{u}]}$ returns a random number following the uniform distribution on the interval $[\mathbf{l}, \mathbf{u}]$.

Observe that the objective function is designed to drive the optimal solution away from the boundaries of the variables' domain, where both McCormick envelopes and the new perspective hull are tight.

(NLP) is solved using the nonlinear solver Ipopt [16] as a heuristic, the resulting primal solution is then evaluated using the standard McCormick relaxation, and the new perspective envelopes. Observe that the nonlinear component of the perspective formulation is a rotated second-order cone constraint ($uv \ge w^2$), and thus can be handled by commercial solvers such as Gurobi [3].

Table 1 reports the average and the maximum gap reduction comparing between the standard and the new approach, on 200 randomly generated instances for each scenario. The first two columns report the average optimality gap produced by, respectively, the McCormick envelopes, and the perspective hull. The last column reports the maximum gap reduction obtained by using the new envelope.

TABLE 1: Optimality gap reduction

	McCormick av.	Perspective av.	Reduction max
Set 1	59%	29%	71%
Set 2	6%	3%	55%

Conclusion

Given the bilinear function $f(x, y) = xy$, and the constraint $x \leq y$, we characterize the convex and the concave envelopes of f in the space of original variables. The new characterization, based on perspective functions, dominates the standard McCormick approach, with promising optimality gap reductions. This result can have a strong impact in global optimization frameworks, potentially improving the pruning process by providing better lower bounds in spatial branch and bound algorithms.

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