Lower and Upper Bounds For The General Multiobjective Optimization Problem

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Abstract. Searching over the Pareto front for the most preferred decision requires providing multiple Pareto optimal solutions to an instance of the general multiobjective optimization problem. This process, especially for large-scale problems can be time consuming and even can go over the resource (time, memory required) budget. In that case, lower and upper bounds are necessary to judge if the solutions derived thus far are acceptable approximations of the Pareto optimal ones.

To this aim we provide lower and upper bounds for the general multiobjective optimization problem. This work generalizes our similar development valid for a special case.

INTRODUCTION

The ubiquitous access to more and more powerful computing platforms stimulates the appetite to solve more and more larger optimization problems. However, various barriers (time and/or memory limits, ”the curse of dimensionality” in combinatorial problems) are the cause that in practice one has to contend himself with approximate solutions and the existence of lower and upper bounds on the exact solution is essential. The multiobjective optimization literature is rich with various approaches to this issue, cf. Kaliszewski, Miroforidis 2014,2017 for brief overviews.

In this work, we present a method to establish lower and upper bounds for a general formulation of the multiobjective optimization problem.

PRELIMINARIES

Let x denote a solution, X a space of solutions, $X \subseteq \mathcal{R}^n$, X_0 a set of feasible solutions, $X_0 \subseteq X$. Then the general multiobjective optimization problem is defined as:

$$\begin{aligned} &vmax f(x) \\ &x \in X_0 , \end{aligned} \tag{1}$$

where $f : X \to \mathcal{R}^k$, $f = (f_1, ..., f_k)$, $f_l :\to \mathcal{R},\ l = 1, ..., k,\ k \geq 2$, are objective functions, and $vmax$ denotes the operator of deriving all Pareto optimal solutions in X_0. $\mathcal{R}^k$ is called the objective space.

Solution $\bar{x}$ is Pareto optimal (or: efficient) if $f_l(x) \geq f_l(\bar{x}),\ l = 1, ..., k$, implies $f(x) = f(\bar{x})$. If $f_l(x) \geq f_l(\bar{x}),\ l = 1, ..., k$ and $f(x) \neq f(\bar{x})$, then we say that x dominates $\bar{x}$. Below, we shall denote the set of Pareto optimal solutions to (1) by N. Set $f(N)$ is called the Pareto front (PF).

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It is a well-established result (cf. Kaliszewski 2006, Ehrgott 2005, Miettinen 1999) that solution x is Pareto optimal (actually, this solution is properly Pareto optimal, for a formal treatment of this issue cf., e.g., Kaliszewski 2006, Kaliszewski et al. 2016, Ehrgott 2005, Miettinen 1999) if and only if it solves the Chebyschev weighted optimization problem

$$\min \max_l \lambda_l((y^*_l - f_l(x)) + \rho e^k(y^* - f(x))) \qquad (2)$$

$$x \in X_0\,,$$

where weights $\lambda_l > 0,\ l = 1,...,k,\ e^k = (1,1,...,1),\ y^*_l = \hat{y}_l + \varepsilon, \hat{y}_l = \max_{x\in X_0} f_l(x),\ l = 1,...,k,\ \varepsilon > 0$, and ρ is a positive "sufficiently small" number.

By the "only if" part of this result, no Pareto optimal solution is a priori excluded from being derived by solving an instance of optimization problem (2). In contrast to that, maximization of a weighted sum of objective functions over X_0 does not possess, in general (and especially in the case of problems with discrete variables), this property [1].

On the first glance, the objective function in (2) seems to be difficult to handle. However, problem (2) is equivalent to

$$\min s\,,$$

$$s \geq \lambda_l((y^*_l - f_l(x)) + \rho e^k(y^* - f(x))),\ \ l = 1,...,k\,, \qquad (3)$$

$$x \in X_0\,.$$

In the sequel, we will assume that Pareto optimal solutions are derived by solving problem (2) with varying λ.

DEVELOPMENT OF GENERAL LOWER AND UPPER BOUNDS

Given λ, let $x^{P_{opt}}(\lambda)$ denote the Pareto optimal solution to problem (2), which would be derived if this problem were solved to optimality. By the definition of $\hat{y}$, the locus of $f(x^{P_{opt}}(\lambda))$ is in $Y = \{y \in \mathcal{R}^k \,|\, y_l \leq \hat{y}_l, l = 1,...,k\}$.

Lower bounds

The lower bound we developed in our earlier works (Kaliszewski 2006, Kaliszewski et al. 2012) is general, i.e. it is valid for any problem of the form (1). Here, for completeness, we present the underlying arguments.

In contrast to singleobjective problems, where any feasible solution yields a lower bound for the optimal value of the objective function, in MO, where the notion of Pareto optimal solutions (elements of N) replace the notion of the optimal solutions and set $f(N)$ is not a singleton, this is not the case. Suppose for a while $k = 2$. Since $x^{P_{opt}}(\lambda)$ is a solution to (2) and thus it is efficient, for a given element $\bar{x}$ of X_0, $f(x^{P_{opt}}(\lambda))$ is neither in the set $\{f(x) \in Y \,|\, x \in X_0,\ f_1(x) \leq f_1(\bar{x})$ AND $f_2(x) < f_2(\bar{x})\}$ nor in the set $\{f(x) \in Y \,|\, x \in X_0,\ f_1(x) < f_1(\bar{x})$ AND $f_2(x) \leq f_2(\bar{x})\}$ but it is in the set $\{f(x) \in Y \,|\, x \in X_0,\ f_1(x) \geq f_1(\bar{x})$ OR $f_2(x) \geq f_2(\bar{x})\}$. The same observation is valid for any $k \geq 2$. Lower bounds provided by elements of X_0 are of disjunctive type and therefore not constructive.

To propose constructive, i.e. conjunctive bounds, one has to exploit the fact that Pareto optimal solutions are derived by solving problem (2). To simplify presentation, we assume $\rho = 0$ (cf. Kaliszewski et al. 2011 for the treatment of the case $\rho > 0$).

The contours of the objective function in (2) are borders of shifted cones $\{y \,|\, y_l \geq 0,\ l = 1,...,k\}$ with cone apexes on the *compromise half line* $\{y \,|\, y_l = \hat{y}_l - \frac{1}{\lambda_l}t, t \geq 0,\ l = 1,...,k\}$ (cf. Kaliszewski 2006, Kaliszewski et al. 2012). Given λ, elements of $\{f(x) \in Y \,|\, x \in X_0,\ f_1(x) < L_1(\lambda,\{\bar{x}\})$ OR $f_2(x) < L_2(\lambda,\{\bar{x}\})$ ... OR $f_k(x) < L_k(\lambda,\{\bar{x}\})\}$, where $L_l(\lambda,\{\bar{x}\}),\ l = 1,...,k$, are lower bounds as is illustrated in Figure 1 for $k = 2$, cannot be solutions to (2) since any such element yields a smaller value of the objective function of (2) than $f(\bar{x})$. Thus, $L(\lambda,\{\bar{x}\})$ is a lower bound for $f(x^{Popt}(\lambda))$.

Given multiple elements of $\bar{x} \in X_0$, the maximal lower bound is selected. Elements which are dominated by another elements of X_0 clearly provide lower bounds lower than lower bounds provided by dominating elements. Therefore, to avoid redundancy, it is reasonable to work with elements which do not dominate one another and this is formalized by the notion of lower shell.

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[1] ibidem.

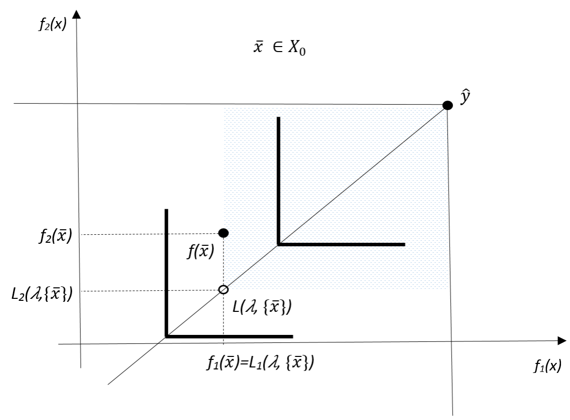

FIGURE 1. Lower bounds $L_l(\lambda, \{\bar{x}\})$ on $f_l(x^{Popt}(\lambda))$ provided by element $\bar{x} \in X_0$ are constructive: $L_1(\lambda, \{\bar{x}\}) \leq f_1(x^{Popt}(\lambda))$ AND $L_2(\lambda, \{\bar{x}\}) \leq f_2(x^{Popt}(\lambda))$. With this lower bound, the dotted area is the locus of $f(x^{Popt}(\lambda))$.

Lower shell to problem (2) is a finite nonempty set $S_L \subseteq X_0$, elements of which satisfy

$$\forall\, x \in S_L \;\; \nexists x' \in S_L \;\; x \prec x' \,. \tag{4}$$

The formula for lower bounds is

$$f_l(x^{P_{opt}}(\lambda)) \geq L_l(\lambda, S_L) = \max_{x \in S_L}(y_l^* - \tfrac{1}{\lambda_l}[\max_j \lambda_j(y_j^* - f_j(x))]\,),\;\; l = 1, 2. \tag{5}$$

For details, see the works cited above.

Upper bounds

Upper shell (In contrast to our earlier works (Kaliszewski et al. 2012, Kaliszewski and Miroforidis 2014,2017), here we omit the requirement that elements of a lower shell should satisfy $f(x) \geq y^{nad}$, where y^{nad} is the nadir point. This specific and rather technical requirement was related to multiobjective evolutionary optimization context in which the idea of bounds was first proposed.) is a finite nonempty set $S_U \subseteq \mathcal{R}^n$, elements of which satisfy

$$\forall\, x \in S_U \;\; \nexists x' \in S_U \;\; x' \prec x \,, \tag{6}$$

$$\forall\, x \in S_U \;\; \nexists x' \in N \;\; x \prec x' \,, \tag{7}$$

Not every element of an upper shell provides a constructive upper bound. To do so, elements of upper shell have to be appropriately located with respect to a lower bound $L(\lambda, S_L)$.

Let S_U be an upper shell to problem (2).

Lemma 1 *$x \in S_U$ provides an upper bound for some $f_l(x^{Popt}(\lambda))$ only if for at least one l, $L_l(\lambda, S_L) \geq f_l(x)$.*

Proof. $x \in S_U$ provides no upper bound for $f_l(x^{Popt}(\lambda))$ if for all l, $l = 1, ..., k$, $L_l(\lambda, S_L)) < f_l(x)$. □

Remark 1 *It is worth observing that also some Pareto optimal solutions can satisfy conditions of Lemma 1. This relates to the development of Kaliszewski (2006) where lower and upper bounds were proposed with shells (subsets of Pareto optimal solutions), the concept in which lower and upper shells are the same. Later on, with applications to space sampling algorithms (e.g. evolutionary) in mind (Kaliszewski et al. 2012, Kaliszewski, Miroforidis 2014,2017), to clearly delineate feasible and infeasible sampling spaces, we have adopted the condition $S_U \subseteq \mathcal{R}^n \setminus X_0$. However, for the sake of generality, here we stick to the definition of upper shell as that given above.*

Lemma 2 *Suppose $x \in S_U$ and $L_{\bar{l}}(\lambda, S_L) \leq f_{\bar{l}}(x)$ for some $\bar{l}$ and $L_l(\lambda, S_L) \geq f_l(x)$ for all $l = 1, ..., k,\; l \neq \bar{l}$. Then x provides an upper bound for $f_{\bar{l}}(x^{Popt}(\lambda))$, namely $f_{\bar{l}}(x^{Popt}(\lambda)) \leq f_{\bar{l}}(x)$.*

Proof. By assumption, $f_l(x^{Popt}(\lambda)) \geq L_l(\lambda, S_L) \geq f_l(x)$ for all $l = 1, ..., k,\; l \neq \bar{l}$. Since $x \in S_U$ and hence x is not dominated by $f(x^{Popt}(\lambda))$, the condition $f_{\bar{l}}(x^{Popt}(\lambda)) \leq f_{\bar{l}}(x)$ must hold. □

A special case

Lower and upper bounds were originally developed (Kaliszewski 2006, Kaliszewski et al. 2011,2012) for the special case when at $x^{Popt}(\lambda)$, the optimal solution to (3) (and hence, also to (2)),

$$s = \lambda_l((y_l^* - f_l(x^{Popt}(\lambda)) + \rho e^k(y^* - f(x^{Popt}(\lambda)))), \quad l = 1, ..., k\,. \tag{8}$$

In other words, $f(x^{Popt}(\lambda))$ lies at the apex of the contour of the objective function of (2).

As said, lower bounds developed there are valid for the general case, and only upper bounds are specific for this special case.

Given $\bar{x} \in S_U$, by the definition of S_U, the part of the half line emanating from $\hat{y}$ (the locus of apexes of the objective function of (2)) belonging to $\{f(x) \,|\, f(x) \geq f(\bar{x})\}$ cannot contain $f(x^{Popt}(\lambda))$. Thus, the intercept point of this half line with $\{f(x) \,|\, f(x) \geq f(\bar{x})\}$ defines upper bound for some components of $f(x^{Popt}(\lambda))$. For $k = 2$, this is illustrated in Figure 2. The formula for upper bounds for this case can be found in three references cited above.

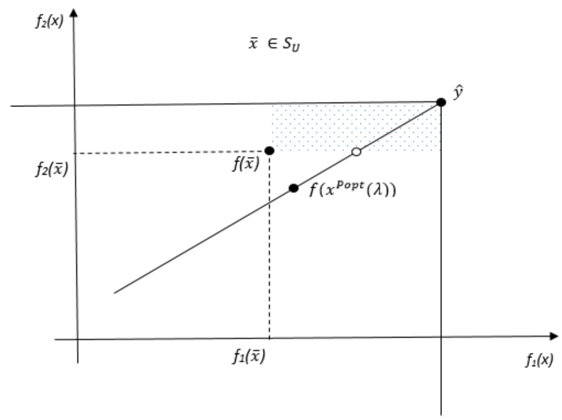

FIGURE 2. $\bar{x} \in S_U$ provides a constructive upper bound for $f(x^{Popt}(\lambda))$ (hypothetical location), namely $f_2(\bar{x})$, as indicated by $\circ$. Since $\bar{x} \in S_U$, $f(x^{Popt}(\lambda))$ cannot belong to the dotted area.

FINAL REMARKS

When it comes to solve large-scale optimization problems, the concept of deriving the whole Pareto front looses its appeal and becomes just impractical. At most, the Pareto front can be navigated by solving instances of problem (2) with various λ (the method to interpret λ in decision making terms was given in Kaliszewski 2006, Kaliszewski et al. 2012). Then, as suggested by Lemma 2, a number of upper shell elements can mutually provide upper bonds, and these bounds can be strengthened by deriving new upper shell elements.

The strength of the proposed development, as well as its weakness, lies it its generality (no assumption about the problem (1) has been adopted).

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