Covering a Square with Six Circles by Deterministic Global Optimization

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Abstract. We consider the problem of covering a square with exactly 6 identical circles of minimal radius. In the literature, a covering is presented by Melissen and Schuur, and conjectured to be optimal. We adress the problem proposing a mathematical programming formulation and solving it to global optimality. We prove that the conjectured optimal covering is indeed the global optimum.

INTRODUCTION

We consider the problem of optimal covering a square with a given number n of identical circles. More precisely, in this paper the case of $n = 6$ circles is considered, and the optimal covering is corresponding to the one obtained by minimizing the radius of the circles. The problem of covering a (compact) polygonal set by circles is adressed in the literature for various polygonal sets and numbers of circles. Some numerical algorithms are proposed, and theoretical results are proved for some special cases. Optimal coverings are known for $n \leq 5$ and $n = 7$ [1]. For $n = 6$, Melissen and Schuur [2, 3] consider the covering of rectangles with various side lengths (including the case of the square), and show the possible configurations for realizing the covering, each configuration corresponding to a different way to place the circles. They give some theoretical results for some of these configurations, and for $n = 6$ and a square to be covered, they conjecture an optimal covering. The solution is indeed not proved to be optimal. The authors use a simulated annealing algorithm, starting from a grid that is gradually refined. Stoyan and Patsuk [4] address the problem of covering compact polygonal sets, and propose a mathematical model based on Voronoi polygons and an algorithm specific for the problem. Kazakov et al. [5] deal with the problem of covering in the context of Logistics, and propose an algorithm based on physical principles due to Fermat and Huygens. Nurmela and Ostergard [6] propose a computational method for finding good coverings of a square with up to 30 equal circles. They use a quasi-Newton method with BFGS secant update to minimize the uncovered area by moving the circles.
Some applications arising in telecommunication-based problems are presented in [7, 8].

To the best of our knowledge, mathematical programming formulations have not been proposed for the considered problem. We adress the problem proposing a mathematical programming formulation and solving it to global optimality using a deterministic global optimization method. We computationally prove that the optimal covering conjectured by Melissen and Schuur is indeed the global optimal one.

The rest of the paper is organized as follows. We first present our mathematical programming formulation, that is a mixed-integer nonlinear program. Then, we present the global solution obtained via a deterministic global optimization method. Finally, we conclude the paper giving some perspectives.

MATHEMATICAL PROGRAMMING FORMULATION FOR COVERING A SQUARE WITH 6 CIRCLES

In this section, we propose a mathematical programming formulation for the problem of covering a square with 6 circles. The problem consists in seeking the minimum radius of the circles, and placing the circles in such a way to entirely cover the considered surface. Decision variables are then represented by the radius, r, of the circles, and by the coordinates (x_i, y_i), $\forall i = 1, \ldots, 6$, of the centers of the circles in an Euclidean space.
The objective function is simply r, to be minimized.

The constraints of the problem have to ensure that the circles are placed in such a way that the square is entirely covered. In the following, we describe how these constraints are formulated. Let us first remark that, given a square and six circles to cover it, only a few configurations are possible. The circles can be organized in two groups of three circles, located, say, on two rows (equivalently, columns). With this organization of the circles, it has been shown by Tarnai and Gaspar [9], then by Melissen and Schuur [2], that the configuration having three axes of symmetry, that one could expect to be optimal, can be improved (other configurations with a smaller radius can be obtained). Melissen and Schuur [2] proposed a configuration which is symmetric only with respect to the center of the square, and that has indeed a smaller radius associated. That was the configuration they conjectured being optimal. Another possibility (see [6]) is to place a circle in the center of the square and the other ones around it, but the corresponding result does not improve the above conjectured optimal solution. We then consider the circles arranged on two rows and do not seek for a completely symmetric configuration. Let us consider the coordinates of the vertices of the square be (clockwise): $(0,0), (1,0), (1,1)$ and $(0,1)$. Without loss of generality, we name the circles C_i, with centers (x_i, y_i), starting from $i = 1$ for the circle on the upper row, and increasing i from left to right. The 4 "external" circles, C_1, C_3, C_4 and C_6, must have centers whose distance from the square vertices is no more than r:

$$(x_1 - 0)^2 + (y_1 - 1)^2 \leq r^2 \tag{1}$$
$$(x_3 - 1)^2 + (y_3 - 1)^2 \leq r^2 \tag{2}$$
$$(x_4 - 0)^2 + (y_4 - 0)^2 \leq r^2 \tag{3}$$
$$(x_6 - 1)^2 + (y_6 - 0)^2 \leq r^2 \tag{4}$$

Then, in order to cover all sides of the square, we impose the following constraints to ensure that all circles intersect each other (on the horizontal and on the vertical axis):

$$y_1 - \sqrt{r^2 - x_1^2} \leq y_4 + \sqrt{r^2 - x_4^2} \tag{5}$$
$$y_3 - \sqrt{r^2 - (1 - x_3)^2} \leq y_6 + \sqrt{r^2 - (1 - x_6)^2} \tag{6}$$
$$x_2 - \sqrt{r^2 - (1 - y_2)^2} \leq x_1 + \sqrt{r^2 - (1 - y_1)^2} \tag{7}$$
$$x_3 - \sqrt{r^2 - (1 - y_3)^2} \leq x_2 + \sqrt{r^2 - (1 - y_2)^2} \tag{8}$$
$$x_5 - \sqrt{r^2 - y_5^2} \leq x_4 + \sqrt{r^2 - y_4^2} \tag{9}$$
$$x_6 - \sqrt{r^2 - y_6^2} \leq x_5 + \sqrt{r^2 - y_5^2} \tag{10}$$
$$\tag{11}$$

where the coordinates of the considered points are computed by simply applying the Pythagorean theorem. For example, the third of these constraints concerns the two triangles, each having a vertex in the center of C_1 and C_2 respectively, and having sides of length r, $1 - y_1$ and $\sqrt{r^2 - (1 - y_1)^2}$ for the first triangle and r, $1 - y_2$ and $\sqrt{r^2 - (1 - y_2)^2}$ for the second one. The constraint imposes that the vertex of the second triangle, which is at the extremity of the side r and is not coinciding with the center (x_2, y_2), has an abscissa less than or equal to the one of the vertex of the other triangle which is at the extremity of the side r and is not conciding with the center (x_1, y_1).
Auxiliary variables are then introduced to represent the coordinates (x_{jk}, y_{jk}) of intersection points of two circles C_j and C_k, with $(j, k) = (1,4), (2,5), (3,6)$, and (x_{jk}, y_{jk}) having superscript $\{[1], [2]\}$ where there are two intersections.

Their defining constraints are added as follows:

$$(x_{jk} - x_j)^2 + (y_{jk} - y_j)^2 = r^2 \qquad (j,k) = (1,4),(2,5),(3,6) \tag{12}$$

$$(x_{jk} - x_k)^2 + (y_{jk} - y_k)^2 = r^2 \qquad (j,k) = (1,4),(2,5),(3,6) \tag{13}$$

Then, to ensure the covering, the above intersection points are constrained to belong to the neighboring circles: (x_{14}, y_{14}) must belong to C_2 or C_5, $(x_{25}^{[1]}, y_{25}^{[1]})$ must belong to C_1 or C_4, $(x_{25}^{[2]}, y_{25}^{[2]})$ must belong to C_3 or C_6, and (x_{36}, y_{36}) to C_2 or C_5. Binary auxiliary variables $z_{jk}, j = 1,3, k = 4,6$ and $z_{jk}^{[s]}, j = 2, k = 5, s = 1,2$ are introduced to model the disjunction, and corresponding constraints added to the formulation:

$$z_{14}\left((x_{14} - x_2)^2 + (y_{14} - y_2)^2 - r^2\right) \leq 0 \tag{14}$$

$$(1 - z_{14})\left((x_{14} - x_5)^2 + (y_{14} - y_5)^2 - r^2\right) \leq 0 \tag{15}$$

$$z_{25}^{[1]}\left((x_{25}^{[1]} - x_1)^2 + (y_{25}^{[1]} - y_1)^2 - r^2\right) \leq 0 \tag{16}$$

$$(1 - z_{25}^{[1]})\left((x_{25}^{[1]} - x_4)^2 + (y_{25}^{[1]} - y_4)^2 - r^2\right) \leq 0 \tag{17}$$

$$z_{25}^{[2]}\left((x_{25}^{[2]} - x_3)^2 + (y_{25}^{[2]} - y_3)^2 - r^2\right) \leq 0 \tag{18}$$

$$(1 - z_{25}^{[2]})\left((x_{25}^{[2]} - x_6)^2 + (y_{25}^{[2]} - y_6)^2 - r^2\right) \leq 0 \tag{19}$$

$$z_{36}\left((x_{36} - x_2)^2 + (y_{36} - y_2)^2 - r^2\right) \leq 0 \tag{20}$$

$$(1 - z_{36})\left((x_{36} - x_5)^2 + (y_{36} - y_5)^2 - r^2\right) \leq 0 \tag{21}$$

This completes the proposed formulation, that is mixed-integer and nonlinear (MINLP), due to the presence of continuous as well as binary variables and to nonlinearities arising in the constraints: quadratic terms, square roots, and products of binary and continuous variables. Note that the bilinear products involving binary and continuous variables can easily be reformulated using the Fortet's linearization [10].

GLOBALLY SOLVING THE PROBLEM

The proposed model is implemented using the AMPL modeling Language [11]. We are interested in the global exact solution of the problem at hand, and we solve it by the deterministic MINLP solver COUENNE [12], which implements a spatial-Branch-and-Bound. This solver is used with its default setting.

We obtain a global solution which is guaranteed to be optimal and corresponds to a radius $r = 0.298727$.

The global optimal solution is illustrated in Figure 1, and the corresponding center coordinates (x_i, y_i) of the circles $C_i, i = 1, \ldots, 6$ are given in Table 1. Note that this solution is symmetric only with respect to the center point of the square.

TABLE 1. Coordinates of the centers of the circles in the global optimal solution.

	C_1	C_2	C_3	C_4	C_5	C_6
x_i	0.141625	0.459783	0.818138	0.181862	0.540217	0.858379
y_i	0.737	0.758985	0.763011	0.236989	0.241015	0.262998

CONCLUSIONS

In this paper, we prove, using mathematical programming and deterministic global optimization, that the Melissen and Schuur conjecture for the covering of a square with 6 circles of minimal radius is indeed optimal. We propose a MINLP formulation that is solved to global optimality by a deterministic global optimization solver. Future work will adress the case of a rectangle to be covered by six circles. In this case, a larger number of configurations are possible for the circles, and the best configuration changes according to to the rectangle side-length value range. We will investigate mathematical programming formulations for identifying the optimal way to place the circles and their minimum radius.

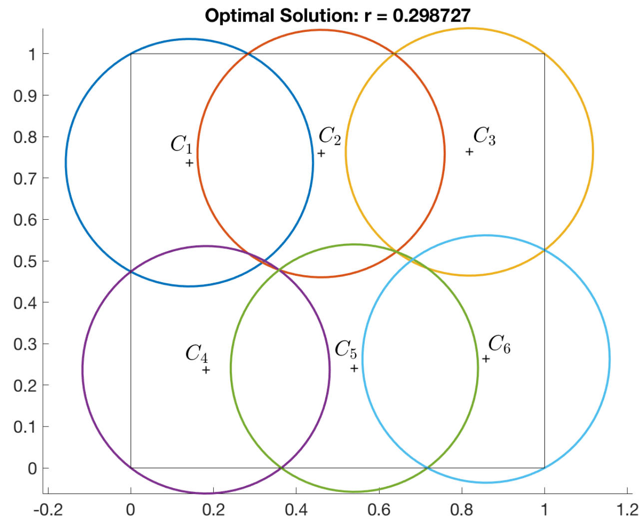

FIGURE 1. Covering a square with 6 circles: global optimal solution.

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