Well-Posedness for a Class of Variational Inequalities

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Abstract. By using the concept of normal subdefferential for set-valued maps we consider a generalization of well-posedness for variational inequalities and give some metric characterizations for them. Also, we show that the well-posedness for a multi-valued variational inequality is equivalent to the existence and uniqueness of its solution.

INTRODUCTION

In the last decades, various concepts of well-posedness such as Levitin-Polyak well-posedness, α-well-posedness and well-posedness by perturbations have been introduced and studied for optimization problems, see [1, 2, 3, 4] and references therein. The concept of well-posedness for hemivariational inequality was first introduced by Goeleven and Mentagui [5] to provide some conditions guaranteing the existence and uniqueness of a solution for a hemivariational inequality. Very recently, Xiao et al. [2] established two kinds of conditions under which the strong and weak well-posedness for the hemivariational inequality are equivalent to the existence and uniqueness of its solutions, respectively.

In this paper, motivated by the works mentioned above, we generalize the concept of well-posedness to a class of multi-valued variational inequality which include as a special case the classical variational inequalities. By using the normal subdifferential for set-valued maps, we establish some equivalence results for the well-posedness of the multi-valued variational inequalities. The paper is organized as follows: Section 2 prepares briefly some preliminary notions and results used in sequel. Section 3 is devoted to obtain a metric characterization for the well-posedness of multi-valued variational inequality. We also show that the strong well-posedness for the multi-valued variational inequality is equivalent to the existence and uniqueness of its solution.

PRELIMINARIES

Let X be a Banach space and X^* its topological dual space. The norm in X and X^* will be denoted by $\|.\|$. We denote $\langle .,. \rangle$, $[x, y]$ and $]x, y[$ the dual pair between X and X^*, the line segment for $x, y \in X$, and the interior of $[x, y]$, respectively. Also, suppose that B_X and S_X to be the closed unit ball and unit sphere of X, respectively. Now, we recall some concepts of subdifferentials and coderivatives that we need in the next section.

Definition 1:[6] Let X be a Banach space, Ω be a nonempty subset of X, $x \in \Omega$ and $\varepsilon \geq 0$. The set of ε-normals to Ω at x is defined by

$$\widehat{N}_\varepsilon(x;\Omega) := \{x^* \in X^* | \limsup_{u \to^{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \leq \varepsilon\}.$$

If $\varepsilon = 0$, the above set is denoted by $\widehat{N}(x;\Omega)$ and called regular normal cone to Ω at x. Let $\bar{x} \in \Omega$, the basic normal cone to Ω at $\bar{x}$ is

$$N(\bar{x};\Omega) := \mathrm{Limsup}_{x \to \bar{x}, \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x;\Omega).$$

Given a set-valued mapping $F : X \rightrightarrows Y$ between Banach spaces with the range space Y partially ordered by a nonempty, closed and convex cone K. Denoting the ordering relation on Y by "$\leq$", we have

$$y_1 \leq y_2 \quad \text{if and only if} \quad y_2 - y_1 \in K.$$

The normal coderivative of F at $(\bar{x}, \bar{y}) \in \mathrm{gr}F$ is the set-valued mapping $D_N^* F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ given by

$$D_N^* F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* | (x^*, -y^*) \in N((\bar{x}, \bar{y}); \mathrm{gr}F)\}.$$

The epigraphical multifunction $\mathcal{E}_F : X \rightrightarrows Y$ is defined by

$$\mathcal{E}_F(x) := \{y \in Y | y \in F(x) + K\}.$$

The normal subdifferentials of F at the point $(\bar{x}, \bar{y}) \in \mathrm{epi}F$ in the direction $y^* \in Y^*$ is defined by $\partial F(\bar{x}, \bar{y})(y^*) := D_N^* \mathcal{E}_F(\bar{x}, \bar{y})(y^*)$.

Definition 2:[6] Let $F : \Omega \subset X \rightrightarrows Y$ with $\mathrm{dom}F \neq \emptyset$.

1. F is said to be Lipschitz-like around $(\bar{x}, \bar{y}) \in \mathrm{gr}F$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ and number $\ell \geq 0$ such that
$$F(x) \cap V \subset F(u) + \ell \|x - u\| B_Y, \quad \text{for all } x, u \in \Omega \cap U.$$
2. F is said to be epi-Lipschitz-like around $(\bar{x}, \bar{y}) \in \mathrm{epi}F$ if $\mathcal{E}_F$ is Lipschitz-like around this point.

Definition 3: Let $T : X \to X^*$ be an operator. T is said to be relaxed monotone, if there exists a real constant c such that for any $x_1, x_2 \in X$, one has

$$\langle T(x_1) - T(x_2), x_1 - x_2 \rangle \geq c \|x_1 - x_2\|^2.$$

When $c = 0$, we obtain the definition of an monotone operator.
Let $F : X \rightrightarrows Y$. The set-valued map $\partial F : X \times Y \times Y^* \rightrightarrows X^*$ is said to be relaxed monotone if there exists a constant α such that for any $y^* \in K^+ \cap S_{Y^*}$, $x_i \in X$, there exists $y_i \in F(x_i)$ such that for any $\xi_i \in \partial F(x_i, y_i)(y^*)$, (i=1,2), one has

$$\langle \xi_1 - \xi_2, x_1 - x_2 \rangle \geq \alpha \parallel x_1 - x_2 \parallel^2 .$$

Definition 4:[7] A mapping $T : X \to X^*$ is said to be hemicontinuous if for any $x_1, x_2 \in X$, the function $t \mapsto \langle T(x_1 + tx_2), x_2 \rangle$ from $[0, 1]$ into $]-\infty, +\infty[$ is continuous at 0_+.

Now, suppose that $F : X \rightrightarrows Y$ is a set-valued mapping, $A : X \to X^*$ is a mapping and $g \in X^*$ is some given element. Consider the following multi-valued variational inequality associated with (A, g, F):
$MVI(A, g, F)$: Find $\bar{x} \in X$ such that for any $x \in X$ and $y^* \in K^+ \cap S_{Y^*}$, there exist $y \in F(x)$ and $\xi \in \partial F(\bar{x}, \bar{y})(y^*)$, such that

$$\langle A\bar{x} - g + \xi, x - \bar{x} \rangle \geq 0.$$

Definition 5: A sequence $\{x_n\} \subset X$ is said to be an approximating sequence for the $MVI(A, g, F)$ if there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ such that for any $x \in X$ and $y^* \in K^+ \cap S_{Y^*}$ there exist $y_n \in F(x_n)$ and $x_n^* \in \partial F(x_n, y_n)(y^*)$ such that

$$\langle Ax_n - g + x_n^*, x - x_n \rangle \geq -\epsilon_n \|x - x_n\|.$$

The multi-valued variational inequality $MVI(A, g, F)$ is said to be strongly well-posed if it has a unique solution $\bar{x}$ on X and for every approximating sequence $\{x_n\}$, it converges strongly to $\bar{x}$.

MAIN RESULTS

In this section, we investigate some equivalence formulations of multi-valued variational inequality considered under different monotonicity assumptions. Also, we establish some conditions under which the well-posedness for the multi-valued variational inequality is equivalent to the existence and uniqueness of its solution.

Theorem 1: Assume that operator $A : X \to X^*$ is hemicontinuous and relaxed monotone with constant c and $F : X \rightrightarrows Y$ is epi-Lipschitz-like such that ∂F satisfies relaxed monotonicity condition with constant α. Consider the following assertions:

(i) $\bar{x}$ is a solution of the $MVI(A, g, F)$.

(ii) $\bar{x}$ is a solution of the following associated multi-valued variational inequality: $AMVI(A, g, F)$: Find $\bar{x} \in X$ such that for any $x \in X$ and $y^* \in K^+ \cap S_{Y^*}$ there exist $y \in F(x)$ and $x^* \in \partial F(x, y)(y^*)$ such that $\langle Ax - g + x^*, x - \bar{x} \rangle \geq 0$.

Then $(ii) \Rightarrow (i)$. If $c + \alpha \geq 0$, then $(i) \Rightarrow (ii)$

For any $\epsilon > 0$, we define the following two sets:

$$\Omega(\epsilon) = \{\bar{x} : \forall x \in X, y^* \in K^+ \cap S_{Y^*}, \exists \bar{y} \in F(\bar{x}), x^* \in \partial F(\bar{x}, \bar{y})(y^*), s.t.\ \langle A\bar{x} - g + x^*, x - \bar{x} \rangle \geq -\epsilon\|x - \bar{x}\|\},$$

and

$$\Psi(\epsilon) = \{\bar{x} : \forall x \in X, y^* \in K^+ \cap S_{Y^*}, \exists \bar{y} \in F(\bar{x}), x^* \in \partial F(\bar{x}, \bar{y})(y^*), s.t.\ \langle Ax - g + x^*, x - \bar{x} \rangle \geq -\epsilon\|x - \bar{x}\|\}.$$

Lemma 2: Suppose that $A : X \to X^*$ is monotone and hemicontinuous. Then $\Omega(\epsilon) = \Psi(\epsilon)$ for all $\epsilon > 0$.

Lemma 3: Suppose that $A : X \to X^*$ is a hemicontinuous mapping. If F is a epi-Lipschitz-like set-valued map, then $\Omega(\epsilon)$ is closed in X for all $\epsilon > 0$.

Theorem 4: Suppose that $A : X \to X^*$ is hemicontinuous and monotone. Then $MVI(A, g, F)$ is strongly well-posed if and only if

$$\Omega(\epsilon) \neq \emptyset, \forall \epsilon > 0, \quad diam\Omega(\epsilon)) \to 0, \quad as \quad \epsilon \to 0.$$

Proposition 5:[8] Let $C^* \subset X^*$ be nonempty, closed, convex and bounded, $\varphi : X \to \mathbb{R}$ be proper, convex and lower semi-continuous and $y \in X$ be arbitrary. Assume that for each $x \in X$, there exists $x^*(x) \in C^*$ such that

$$\langle x^*(x), x - y \rangle \geq \varphi(y) - \varphi(x).$$

Then, there exists $y^* \in C^*$ such that

$$\langle y^*, x - y \rangle \geq \varphi(y) - \varphi(x), \quad \forall x \in X.$$

Theorem 6: Let $A : X \to X^*$ be relaxed monotone with constant c and $F : X \rightrightarrows Y$ an epi-Lipschitz-like set-valued map that its normal subdifferential satisfies relaxed monotonicity condition with constant α. If $\alpha + c > 0$, then $MVI(A, g, F)$ is strongly well-posed if and only if it has a unique solution on X.

Remark: Theorem 6 extends Theorem 4.1 in [2] from Clarke subdifferential for real-valued functions which is convex to normal subdifferential for set-valued maps which is not generally convex.

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