ORBITAL FUNCTORS

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1. Category of Orbital Functors

Action of a monoidal category $\mathcal{M}$ on $\mathcal{C}$ given by an endofunctor:

$$m\bullet \colon \mathcal{C} \to \mathcal{C}$$

satisfying the laws:

$$1 \bullet a = a$$
$$(m \cdot n) \bullet a = m \bullet (n \bullet a)$$

Lax orbital functor $F \colon \mathcal{C} \to \mathbf{Set}$ is equipped with the structure:

$$\alpha_{m,a} \colon Fa \to F(m \bullet a)$$

which is a natural transformations that let's us extend the functor along the orbits of $m\bullet$. We have the following coherence conditions:

$$\alpha_{1,a} = id_{Fa}$$
$$\alpha_{n \cdot m,a} = \alpha_{n,a} \circ \alpha_{m,a}$$

These functors form a category $\mathbf{Orb}$. A morphisms from (F, α) to (G, β) is a natural transformation that maps the two structures:

$$\begin{tikzcd}
Fa \arrow[r, "\mu_a"] \arrow[d, "\alpha"] & Ga \arrow[d, "\beta"] \\
F(m \bullet a) \arrow[r, "\mu_{m\bullet a}"] & G(m \bullet a)
\end{tikzcd}$$

2. Category of Orbits

Category $\mathcal{O}$ of orbits has the same objects as $\mathcal{C}$, but its hom-sets that are given by the colimit formula. We use the coend notation for the colimit (a colimit of F is a coend of a profunctor $P\langle x, y\rangle = Fy$):

$$\mathcal{O}(a, b) = \int^m \mathcal{C}(a, m \bullet b)$$

(By analogy with optics, I call this hom-set a monocle.) An element of the hom-set can be constructed by injecting a pair $(m, f \colon a \to m \bullet b)$ into the colimit.

Composition:

$$\circ \colon \mathcal{O}(b, c) \times \mathcal{O}(a, b) \to \mathcal{O}(a, c)$$

is a member of the set:

$$\mathbf{Set}\Big(\int^m \mathcal{C}(b, m \bullet c) \times \int^n \mathcal{C}(a, n \bullet b), \int^k \mathcal{C}(a, k \bullet c)\Big)$$

By cocontuity of the hom-functor, we can replace the colimit with the limit (using the end notation):

$$\int_{m,n} \mathbf{Set}\big(\mathcal{C}(b, m \bullet c) \times \mathcal{C}(a, n \bullet b), \int^n \mathcal{C}(a, n \bullet c)\big)$$

Given two morphisms $f\colon b \to m \bullet c$ and $g\colon a \to n \bullet b$, we can produce the composition (using the functoriality of $m\bullet$):

$$(n \bullet f) \circ g\colon a \to (n \cdot m) \bullet c$$

and inject it into the colimit with $k = n \cdot m$. The identity morphism at a is given by the pair $(1, id_a)$

3. Presheaves on Orbits

The category of presheaves $[\mathcal{O}^{op}, \mathbf{Set}]$ is isomorphic to the category **Orb** of lax orbital functors.

$$\begin{tikzcd} \mathbf{Set} & & \mathbf{Set} \\ \mathcal{O}^{op} \arrow[u, "\hat{F}"] & & \mathcal{C} \arrow[u, "(F,\alpha)"'] \end{tikzcd}$$

Indeed, given a presheaf $\hat{F} : \mathcal{O}^{op} \to \mathbf{Set}$ we construct a functor $F\colon \mathcal{C} \to \mathbf{Set}$ which on objects is the same as $\hat{F}$.

Let's define the action of F on morphisms. Given a morphism $f\colon a \to b$ in $\mathcal{C}$, we first construct a pair $(f, 1)$ with the unit monoidal action, inject it into the colimit to get an element of $\mathcal{O}(a, b)$, and apply $\hat{F}$ to it.

The resulting functor is automatically lax orbital, with the structure $\alpha_{m,a}$ given by the action of $\hat{F}$ on the colimit constructed from $(id_{m\bullet a}, m)$, which is a member of $\mathcal{O}(m \bullet a, a)$

$$\begin{tikzcd} \mathbf{Set} & & \hat{F}a \arrow[r, "\alpha_{m,a}"] & \hat{F}(m \bullet a) \\ \mathcal{O}^{op} \arrow[u, "\hat{F}"] & & a & m \bullet a \arrow[l, "\mathcal{O}(m\bullet a,a)"] \end{tikzcd}$$

$$\alpha_{m,a} \in \hat{F}\big(\mathcal{O}(m \bullet a, a)\big) = \hat{F}\Big(\int^m \mathcal{C}(m \bullet a, m \bullet a)\Big)$$

Conversely, given a lax orbital functor $F\colon \mathcal{C} \to \mathbf{Set}$, we can build a presheaf $\hat{F}\colon \mathcal{O}^{op} \to \mathbf{Set}$ that is identity on objects. Its action on morphisms is a mapping of hom-sets:

$$\mathcal{O}(b, a) \to \mathbf{Set}(Fa, Fb)$$

Such a mapping is a member of:

$$\mathbf{Set}\big(\int^m \mathcal{C}(b, m \bullet a), \mathbf{Set}(Fa, Fb)\big)$$

Using continuity and the currying adjunction in **Set** we get:

$$\int_m \mathbf{Set}\big(\mathcal{C}(b, m \bullet a) \times Fa, Fb\big)$$

Given a morphism $f \colon b \to m \bullet a$ and the set Fa, we first apply $\alpha_{m,a}$ to Fa to get $F(m \bullet a)$ and follow it by Ff to get Fb.

The mapping between these two functor categories is functorial. Consider a natural transformation $\mu \colon \hat{F} \to \hat{G}$. For any morphisms $g \in \mathcal{O}(b, a)$, the naturality square reads:

$$\begin{array}{ccc} \hat{F}b & \xrightarrow{\mu_b} & \hat{G}b \\ {\scriptstyle \hat{F}g}\uparrow & & {\scriptstyle \hat{G}g}\uparrow \\ \hat{F}a & \xrightarrow{\mu_a} & \hat{G}a \end{array}$$

We map μ to a natural transformation between two orbital functors: $(F, \alpha) \to (G, \beta)$. Since $\hat{F}$ and F are the same on objects, the same μ can be used to map F to G.

Naturality condition, however, involves the mapping of morphisms. We have to show that, for any $f \colon a \to b$ the following square commutes:

$$\begin{array}{ccc} Fb & \xrightarrow{\mu_b} & Gb \\ {\scriptstyle Ff}\uparrow & & {\scriptstyle Gf}\uparrow \\ Fa & \xrightarrow{\mu_a} & Ga \end{array}$$

The lifting of f is done by injecting $(f, 1)$ into $\mathcal{O}(a, b)$ and applying $\hat{F}$ (or $\hat{G}$) to it. So the naturality of μ with respect to F follows from the naturality with respect to $\hat{F}$.

Similarly, the induced structure maps are preserved, since they are given by the lifting of morphisms in $\mathcal{O}(m \bullet a, a)$:

$$\begin{array}{ccc} Fa & \xrightarrow{\quad \mu_a \quad} & Ga \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ F(m \bullet a) & \xrightarrow{\mu_{m \bullet a}} & G(m \bullet a) \end{array}$$

4. Tannakian Reconstruction

Tannakian reconstruction for categories expresses a hom-set in a category as an end over all co-presheaves on that category:

$$\int_{\hat{F} \colon [\mathcal{C}, \mathbf{Set}]} \mathbf{Set}(\hat{F}a, \hat{F}b) \cong \mathcal{C}(a, b)$$

In particular, if we choose our category as the (opposite of the) category of orbits, we get:

$$\int_{\hat{F}\colon [\mathcal{O}^{op},\mathbf{Set}]} \mathbf{Set}(\hat{F}a, \hat{F}b) \cong \mathcal{O}(b, a)$$

As we've seen, this category is equivalent to the category of orbital functors, so we get:

$$\int_{F\colon \mathbf{Orb}} \mathbf{Set}(Fa, Fb) \cong \int^{m\colon \mathcal{M}} \mathcal{C}(b, m \bullet a)$$

The left-hand side is a set of natural transformations between two fiber functors. A fiber functor maps a functor F to its value (a set) at a fixed object a:

$$\Phi_a \colon F \mapsto Fa$$

Conceptually, a fiber functor "probes" the environment around a fixed object.

The advantage of this representation is that it replaces a somewhat messy composition of morphisms in $\mathcal{O}$ (the right hand-side) with simple composition of functions in **Set** (the left-hand side).

This construction extends naturally to general optics, as shown by Pietro Vertechi.