A CATEGORY OF GENERALIZED TAMBARA MODULES

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We start with a monoidal category $\mathcal{M}$ and its action in a pair of categories $\mathcal{C}$ and $\mathcal{D}$ (we use the same infix symbol for both):

$$\bullet\colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$$
$$\bullet\colon \mathcal{M} \times \mathcal{D} \to \mathcal{D}$$

A Tambara module is a profunctor $p\colon \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$ equipped with a family of transformations:

$$\alpha_{m,\langle a,b\rangle}\colon p\langle a,b\rangle \to p\langle m \bullet a, m \bullet b\rangle$$

natural in m and dinatural in $\langle a,b\rangle$, satisfying the unit and associativity conditions.

Tambara modules form a category. A morphism between Tambara modules (p,α) and (q,α') is a natural transformation:

$$\rho_{\langle a,b\rangle}\colon p\langle a,b\rangle \to q\langle a,b\rangle$$

satisfying the coherence conditions:

$$\begin{tikzcd}
p\langle a,b\rangle \arrow[r, "\alpha_{m,\langle a,b\rangle}"] \arrow[d, "\rho_{\langle a,b\rangle}"'] & p\langle m \bullet a, m \bullet b\rangle \arrow[d, "\rho_{\langle m\bullet a, m\bullet b\rangle}"] \\
q\langle a,b\rangle \arrow[r, "\alpha'_{m,\langle a,b\rangle}"] & q\langle m \bullet a, m \bullet b\rangle
\end{tikzcd}$$

In what follows I'll consider, for the sake of simplicity, Tambara endomodules of the type $p\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$

1. Generalized Tambara Category

We start with two monoidal categories $\mathcal{M}$ and $\mathcal{N}$ with two separate actions on two categories:

$$\bullet\colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$$
$$\bullet\colon \mathcal{N} \times \mathcal{D} \to \mathcal{D}$$

We have a Tambara module p in $\mathcal{C}^{op} \times \mathcal{C}$ with the structure:

$$\alpha_{m,\langle a,b\rangle}\colon p\langle a,b\rangle \to p\langle m \bullet a, m \bullet b\rangle$$

and another one q in $\mathcal{D}^{op} \times \mathcal{D}$ with the structure:

$$\beta_{n,\langle s,t\rangle}\colon q\langle s,t\rangle \to q\langle n \bullet s, n \bullet t\rangle$$

A morphism between Tambara modules is a functor $c\colon \mathcal{C} \to \mathcal{D}$ and a family of functions:

$$\rho_{c,\langle a,b\rangle}\colon p\langle a,b\rangle \to q\langle c \bullet a, c \bullet b\rangle$$

satisfying the following coherence conditions:

$$\begin{tikzcd}
p\langle a,b\rangle \arrow[r, "\alpha_{m,\langle a,b\rangle}"] \arrow[d, "\rho_{c\cdot m,\langle a,b\rangle}"'] & p\langle m \bullet a, m \bullet b\rangle \arrow[d, "\rho_{c,\langle m\bullet a, m\bullet b\rangle}"] \\
q\langle (c\cdot m) \bullet a, (c \cdot m) \bullet b\rangle \arrow[r] & q\langle c \bullet (m \bullet a), c \bullet (m \bullet b)\rangle
\end{tikzcd}$$

$$\begin{tikzcd}
p\langle a,b\rangle \arrow[r, "\rho_{c,\langle a,b\rangle}"] \arrow[d, "\rho_{n\cdot c,\langle a,b\rangle}"'] & q\langle c \bullet a, c \bullet b\rangle \arrow[d, "\beta_{n,\langle c\bullet a, c\bullet b\rangle}"] \\
q\langle (n\cdot c) \bullet a, (n \cdot c) \bullet b\rangle \arrow[r] & q\langle n \bullet (c \bullet a), n \bullet (c \bullet b)\rangle
\end{tikzcd}$$

2. References

- D. Tambara, *Distributors on a tensor category.* Hokkaido Math. J. 35(2): 379-425 (May 2006)
- Craig Pastro and Ross Street. *Doubles for monoidal categories.* Theory and applications of categories, 21(4):61–75, 200