# A CATEGORY OF GENERALIZED TAMBARA MODULES

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We start with a monoidal category $\mathcal{M}$ and its action in a pair of categories $\mathcal{C}$ and $\mathcal{D}$ (we use the same infix symbol for both):

$$\bullet \colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$$
$$\bullet \colon \mathcal{M} \times \mathcal{D} \to \mathcal{D}$$

A Tambara module is a profunctor $p \colon \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$ equipped with a family of transformations:

$$\alpha_{m,\langle a,b\rangle} \colon p\langle a, b\rangle \to p\langle m \bullet a, m \bullet b\rangle$$

natural in $m$ and dinatural in $\langle a, b\rangle$, satisfying the unit and associativity conditions.

Tambara modules form a category. A morphism between Tambara modules $(p, \alpha)$ and $(q, \alpha')$ is a natural transformation:

$$\rho_{\langle a,b\rangle} \colon p\langle a, b\rangle \to q\langle a, b\rangle$$

satisfying the coherence conditions:

$$\begin{tikzcd}
p\langle a, b\rangle \arrow[r, "\alpha_{m,\langle a,b\rangle}"] \arrow[d, "\rho_{\langle a,b\rangle}"'] & p\langle m \bullet a, m \bullet b\rangle \arrow[d, "\rho_{\langle m\bullet a, m\bullet b\rangle}"] \\
q\langle a, b\rangle \arrow[r, "\alpha'_{m,\langle a,b\rangle}"] & q\langle m \bullet a, m \bullet b\rangle
\end{tikzcd}$$

In what follows I'll consider, for the sake of simplicity, Tambara endomodules of the type $p \colon \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$

## 1. Generalized Tambara Category

We start with two monoidal categories $\mathcal{M}$ and $\mathcal{N}$ with two separate actions on two categories:

$$\bullet \colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$$
$$\bullet \colon \mathcal{N} \times \mathcal{D} \to \mathcal{D}$$

The two monoidal actions can also be seen, after currying, as strict monoidal functors:

$$\mathcal{M} \to [\mathcal{C}, \mathcal{C}]$$
$$\mathcal{N} \to [\mathcal{D}, \mathcal{D}]$$

Next we define a category of monoidal functors $\mathcal{M} \to \mathcal{N}$. Given a functor $c$, we'll write its action on $m \colon \mathcal{M}$ as $c \cdot m$. This produces an object of $\mathcal{N}$. This action satisfies the usual associativity and unit conditions of a monoidal functor.

We lift this action to a functor from $\mathcal{C}$ to $\mathcal{D}$, or the action:

$$\bullet \colon \mathcal{M} \times \mathcal{C} \to \mathcal{D}$$

satisfying the coherence conditions:

$$\begin{tikzcd}
a \arrow[r, "m\bullet"] \arrow[d, "(c\cdot m)\bullet"'] & m \bullet a \arrow[d, "c\bullet"] \\
(c \cdot m) \bullet a \arrow[r, "\cong"] & c \bullet (m \bullet a)
\end{tikzcd}$$

$$\begin{tikzcd}
a \arrow[r, "c\bullet"] \arrow[d, "(n\cdot c)\bullet"'] & c \bullet a \arrow[d, "n\bullet"] \\
(n \cdot c) \bullet a \arrow[r, "\cong"] & n \bullet (c \bullet a)
\end{tikzcd}$$

We have a Tambara module $p$ in $\mathcal{C}^{op} \times \mathcal{C}$ with the structure:

$$\alpha_{m,\langle a,b\rangle} \colon p\langle a, b\rangle \to p\langle m \bullet a, m \bullet b\rangle$$

and another one $q$ in $\mathcal{D}^{op} \times \mathcal{D}$ with the structure:

$$\beta_{n,\langle s,t\rangle} \colon q\langle s, t\rangle \to q\langle n \bullet s, n \bullet t\rangle$$

A morphism between Tambara modules is a functor $c \colon \mathcal{C} \to \mathcal{D}$ and a family of functions:

$$\rho_{c,\langle a,b\rangle} \colon p\langle a, b\rangle \to q\langle c \bullet a, c \bullet b\rangle$$

satisfying the following coherence conditions:

$$\begin{tikzcd}
p\langle a, b\rangle \arrow[r, "\alpha_{m,\langle a,b\rangle}"] \arrow[d, "\rho_{c\cdot m,\langle a,b\rangle}"'] & p\langle m \bullet a, m \bullet b\rangle \arrow[d, "\rho_{c,\langle m\bullet a, m\bullet b\rangle}"] \\
q\langle (c \cdot m) \bullet a, (c \cdot m) \bullet b\rangle \arrow[r] & q\langle c \bullet (m \bullet a), c \bullet (m \bullet b)\rangle
\end{tikzcd}$$

$$\begin{tikzcd}
p\langle a, b\rangle \arrow[r, "\rho_{c,\langle a,b\rangle}"] \arrow[d, "\rho_{n\cdot c,\langle a,b\rangle}"'] & q\langle c \bullet a, c \bullet b\rangle \arrow[d, "\beta_{n,\langle c\bullet a, c\bullet b\rangle}"] \\
q\langle (n \cdot c) \bullet a, (n \cdot c) \bullet b\rangle \arrow[r] & q\langle n \bullet (c \bullet a), n \bullet (c \bullet b)\rangle
\end{tikzcd}$$

## 2. References


- D. Tambara, *Distributors on a tensor category.* Hokkaido Math. J. 35(2): 379-425 (May 2006)
- Craig Pastro and Ross Street. *Doubles for monoidal categories.* Theory and applications of categories, 21(4):61–75, 200