

ORBITAL FUNCTORS

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1. CATEGORY OF ORBITAL FUNCTORS

Action of a monoidal category \mathcal{M} on \mathcal{C} given by an endofunctor:

$$m \bullet : \mathcal{C} \rightarrow \mathcal{C}$$

satisfying the laws:

$$\begin{aligned} 1 \bullet a &= a \\ (m \cdot n) \bullet a &= m \bullet (n \bullet a) \end{aligned}$$

Lax orbital functor $F : \mathcal{C} \rightarrow \mathbf{Set}$ is equipped with the structure:

$$\alpha_{m,a} : Fa \rightarrow F(m \bullet a)$$

which is a natural transformations that let's us extend the functor along the orbits of $m \bullet$. We have the following coherence conditions:

$$\begin{aligned} \alpha_{1,a} &= id_{Fa} \\ \alpha_{n \cdot m, a} &= \alpha_{n,a} \circ \alpha_{m,a} \end{aligned}$$

These functors form a category **Orb**. A morphisms from (F, α) to (G, β) is a natural transformation that maps the two structures:

$$\begin{array}{ccc} Fa & \xrightarrow{\mu_a} & Ga \\ \downarrow \alpha & & \downarrow \beta \\ F(m \bullet a) & \xrightarrow{\mu_{m \bullet a}} & G(m \bullet a) \end{array}$$

2. CATEGORY OF ORBITS

Category \mathcal{O} of orbits has the same objects as \mathcal{C} , but its hom-sets that are given by the colimit formula. We use the coend notation for the colimit (a colimit of F is a coend of a profunctor $P \langle x, y \rangle = Fy$):

$$\mathcal{O}(a, b) = \int^m \mathcal{C}(a, m \bullet b)$$

An element of the hom-set can be constructed by injecting a pair $(m, f : a \rightarrow m \bullet b)$ into the colimit.

Composition:

$$\circ : \mathcal{O}(b, c) \times \mathcal{O}(a, b) \rightarrow \mathcal{O}(a, c)$$

is a member of the set:

$$\mathbf{Set}(\int^m \mathcal{C}(b, m \bullet c) \times \int^n \mathcal{C}(a, n \bullet b), \int^k \mathcal{C}(a, k \bullet c))$$

By cocontinuity of the hom-functor, we can replace the colimit with the limit (using the end notation):

$$\int_{m,n} \mathbf{Set}(\mathcal{C}(b, m \bullet c) \times \mathcal{C}(a, n \bullet b), \int^n \mathcal{C}(a, n \bullet c))$$

Given two morphisms $f: b \rightarrow m \bullet c$ and $g: a \rightarrow n \bullet b$, we can produce the composition (using the functoriality of $m \bullet$):

$$(n \bullet f) \circ g: a \rightarrow (n \cdot m) \bullet c$$

and inject it into the colimit with $k = n \cdot m$. The identity morphism at a is given by the pair $(1, id_a)$

3. PRESHEAVES ON ORBITS

The category of presheaves $[\mathcal{O}^{op}, \mathbf{Set}]$ is isomorphic to the category **Orb** of lax orbital functors.

$$\begin{array}{ccc} \mathbf{Set} & & \mathbf{Set} \\ \hat{F} \uparrow & (F, \alpha) \uparrow & \\ \mathcal{O}^{op} & & \mathcal{C} \end{array}$$

Indeed, given a presheaf $\hat{F}: \mathcal{O}^{op} \rightarrow \mathbf{Set}$ we construct a functor $F: \mathcal{C} \rightarrow \mathbf{Set}$ which on objects is the same as \hat{F} .

Let's define the action of F on morphisms. Given a morphism $f: a \rightarrow b$ in \mathcal{C} , we first construct a pair $(f, 1)$ with the unit monoidal action, inject it into the colimit to get an element of $\mathcal{O}(a, b)$, and apply \hat{F} to it.

The resulting functor is automatically lax orbital, with the structure $\alpha_{m,a}$ given by the action of \hat{F} on the colimit constructed from $(id_{m \bullet a}, m)$, which is a member of $\mathcal{O}(m \bullet a, a)$

$$\begin{array}{ccc} \mathbf{Set} & \hat{F}a \xrightarrow{\alpha_{m,a}} \hat{F}(m \bullet a) & \\ \hat{F} \uparrow & & \\ \mathcal{O}^{op} & a \xleftarrow[\mathcal{O}(m \bullet a, a)]{} m \bullet a & \end{array}$$

$$\alpha_{m,a} \in \hat{F}(\mathcal{O}(m \bullet a, a)) = \hat{F}(\int^m \mathcal{C}(m \bullet a, m \bullet a))$$

Conversely, given a lax orbital functor $F: \mathcal{C} \rightarrow \mathbf{Set}$, we can build a presheaf $\hat{F}: \mathcal{O}^{op} \rightarrow \mathbf{Set}$ that is identity on objects. Its action on morphisms is a mapping of hom-sets:

$$\mathcal{O}(b, a) \rightarrow \mathbf{Set}(Fa, Fb)$$

Such a mapping is a member of:

$$\mathbf{Set}\left(\int^m \mathcal{C}(b, m \bullet a), \mathbf{Set}(Fa, Fb)\right)$$

Using continuity and the currying adjunction in \mathbf{Set} we get:

$$\int_m \mathbf{Set}(\mathcal{C}(b, m \bullet a) \times Fa, Fb)$$

Given a morphism $f: b \rightarrow m \bullet a$ and the set Fa , we first apply $\alpha_{m,a}$ to Fa to get $F(m \bullet a)$ and follow it by Ff to get Fb .

The mapping between these two functor categories is functorial. Consider a natural transformation $\mu: \hat{F} \rightarrow \hat{G}$. For any morphisms $g \in \mathcal{O}(b, a)$, the naturality square reads:

$$\begin{array}{ccc} \hat{F}b & \xrightarrow{\mu_b} & \hat{G}b \\ \hat{F}g \uparrow & & \uparrow \hat{G}g \\ \hat{F}a & \xrightarrow{\mu_a} & \hat{G}a \end{array}$$

We map μ to a natural transformation between two orbital functors: $(F, \alpha) \rightarrow (G, \beta)$. Since \hat{F} and F are the same on objects, the same μ can be used to map F to G .

Naturality condition, however, involves the mapping of morphisms. We have to show that, for any $f: a \rightarrow b$ the following square commutes:

$$\begin{array}{ccc} Fb & \xrightarrow{\mu_b} & Gb \\ Ff \uparrow & & \uparrow Gf \\ Fa & \xrightarrow{\mu_a} & Ga \end{array}$$

The lifting of f is done by injecting $(f, 1)$ into $\mathcal{O}(a, b)$ and applying \hat{F} (or \hat{G}) to it. So the naturality of μ with respect to F follows from the naturality with respect to \hat{F} .

Similarly, the induced structure maps are preserved, since they are given by the lifting of morphisms in $\mathcal{O}(m \bullet a, a)$:

$$\begin{array}{ccc} Fa & \xrightarrow{\mu_a} & Ga \\ \downarrow \alpha & & \downarrow \beta \\ F(m \bullet a) & \xrightarrow{\mu_{m \bullet a}} & G(m \bullet a) \end{array}$$

4. TANNAKIAN RECONSTRUCTION

Tannakian reconstruction for categories expresses a hom-set in a category as an end over all co-presheaves on that category:

$$\int_{\hat{F}: [\mathcal{C}, \mathbf{Set}]} \mathbf{Set}(\hat{F}a, \hat{F}b) \cong \mathcal{C}(a, b)$$

In particular, if we choose our category as the (opposite of the) category of orbits, we get:

$$\int_{\hat{F}: [\mathcal{O}^{op}, \mathbf{Set}]} \mathbf{Set}(\hat{F}a, \hat{F}b) \cong \mathcal{O}(b, a)$$

As we've seen, this category is equivalent to the category of orbital functors, so we get:

$$\int_{F: \mathbf{Orb}} \mathbf{Set}(Fa, Fb) \cong \int^{m: \mathcal{M}} \mathcal{C}(b, m \bullet a)$$

The left-hand side is a set of natural transformations between two fiber functors. A fiber functor maps a functor F to its value (a set) at a fixed object a :

$$\Phi_a: F \mapsto Fa$$

Conceptually, a fiber functor “probes” the environment around a fixed object.

The advantage of this representation is that it replaces a somewhat messy composition of morphisms in \mathcal{O} (the right hand-side) with simple composition of functions in \mathbf{Set} (the left-hand side).