Intro to Profunctor Optics

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Profunctors

Definition

Profunctors are **Set**-valued functors of the type:

$$p\colon \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$$

- For each pair of objects, a set:

$$p\langle c, d\rangle$$

- For each pair of morphisms:

$$f : s \to c$$
$$g : d \to t$$

  a function:

$$p\langle f, g\rangle : p\langle c, d\rangle \to p\langle s, t\rangle$$

Profunctor zoo

- Endo-Profunctors

$$p\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$$

- They generalize hom-functors:

$$\mathcal{C}(-, =)\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$$

- Enriched profunctors.
  $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{V}$-enriched categories.

$$p\colon \mathcal{C}^{op} \otimes \mathcal{D} \to \mathcal{V}$$

Profunctors as proof-relevant relations

- Element of $p\langle c, d\rangle$ is a proof that c is related to d
- Relations that are compatible with categorical structure.
  If c is related to d then s is related to t, if the objects are connected by morphisms:

$$f : s \to c$$
$$g : d \to t$$

Profunctors as linear transformations

- Elements of a co-presheaf as vectors:

$$v : \mathcal{C} \to \mathbf{Set}$$

  v_i is a component of v at object i
- Especially if $\mathcal{C}$ is a discrete category.
  $\mathcal{C}(i, j)$ acts like a Kronecker delta.
- A profunctor acts like a linear transformation, a matrix, or a mixed covariance tensor:

$$p_{ij}$$

Composition

$$\circ\colon \mathcal{C}(b, c) \times \mathcal{C}(a, b) \to \mathcal{C}(a, c)$$

- Profunctor composition:

$$\diamond\colon q\langle b, c\rangle \times p\langle a, b\rangle \to (q \diamond p)\langle a, c\rangle$$

- As relations: A proof of $(q \diamond p)\langle a, c\rangle$ is:

$$\exists b.\ q\langle b, c\rangle \wedge p\langle a, b\rangle$$

- As linear transformations:

$$(q \diamond p)_{ik} = \sum_j q_{jk} \times p_{ij} = Tr(q_{-k} \times p_{i-})$$

- The sum (coproduct) works well for discrete categories. Otherwise, it over-counts.

Coend as sum

- As trace

$$\sum_i p_{ii} \qquad\qquad \int^a p\langle a, a\rangle$$

- As sum (only in discrete categories)

$$\begin{tikzcd} p\langle x, x\rangle \arrow[dr, "i_x"'] & p\langle y, y\rangle \arrow[d, "i_y"] & p\langle z, z\rangle \arrow[dl, "i_z"] \\ & \sum_a p\langle a, a\rangle & \end{tikzcd}$$

- Overcounting: if there's a morphism $f : x \to y$:

$$p\langle y, f\rangle : p\langle y, x\rangle \to p\langle y, y\rangle$$
$$p\langle f, x\rangle : p\langle y, x\rangle \to p\langle x, x\rangle$$

Co-wedge condition

- Like sum, modulo identifications:

$$\begin{tikzcd} & p\langle y, x\rangle \arrow[dl, "p\langle y,f\rangle"'] \arrow[dr, "p\langle f,x\rangle"] & \\ p\langle y, y\rangle \arrow[dr, "i_y"'] & & p\langle x, x\rangle \arrow[dl, "i_x"] \\ & \int^a p\langle a, a\rangle & \end{tikzcd}$$

- Universal co-wedge is the coend
- Like a sum, co-end has the mapping out property
- Generalizes to enriched categories

Profunctor composition

- Composition

$$(q \diamond p)\langle a, b\rangle = \int^{c} q\langle c, b\rangle \times p\langle a, c\rangle$$

- Compare with:

$$(q \diamond p)_{ik} = \sum_{j} q_{jk} \times p_{ij} = Tr(q_{-k} \times p_{i-})$$

$$(q \diamond p)\langle a, c\rangle = \exists b.\ q\langle b, c\rangle \wedge p\langle a, b\rangle$$

- Enriched version uses tensor product in $\mathcal{V}$

End

- Like a product of diagonal elements (in a discrete category)

$$\prod_a p\langle a, a\rangle$$

- Like a universal quantifier $\forall_a \; p\langle a, a\rangle$
- Wedge condition

$$\begin{tikzcd}
 & \int_a p\langle a, a\rangle \arrow[dl, "\pi_x"'] \arrow[dr, "\pi_y"] & \\
p\langle x, x\rangle \arrow[dr, "p\langle x,f\rangle"'] & & p\langle y, y\rangle \arrow[dl, "p\langle f,y\rangle"] \\
 & p\langle x, y\rangle &
\end{tikzcd}$$

- Universal wedge is the end
- Like a product, end has a mapping in property

Natural transformations

- Two functors F and G from $\mathcal{C}$ to $\mathcal{D}$

$$[\mathcal{C}, \mathcal{D}](F, G) \cong \int_{x\colon \mathcal{C}} \mathcal{D}(Fx, Gx)$$

- Element of this end: a giant tuple of morphisms. Wedge contition = naturality square
- Can be used to define natural transformations between enriched functors
- (Ninja) Yoneda lemma

$$\int_{x\colon \mathcal{C}} \mathbf{Set}(\mathcal{C}(a, x), Fx) \cong Fa$$

- (Ninja) co-Yoneda

$$\int^{x\colon \mathcal{C}} \mathcal{C}(x, a) \times Fx \cong Fa$$

- Colimit of representables. $a = \sum_i e_i a_i$

Continuity of hom-functor

- Mapping out of a coend

$$\mathbf{Set}(\int^a p\langle a, a\rangle, C) \cong \int_a \mathbf{Set}(p\langle a, a\rangle, C)$$

- Mapping into an end

$$\mathbf{Set}(C, \int_a p\langle a, a\rangle) \cong \int_a \mathbf{Set}(C, p\langle a, a\rangle)$$

Actegories

- The action of a monoidal category $\mathcal{M}$ on $\mathcal{C}$

$$\bullet \colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$$

- Preserving monoidal structure

$$m \bullet (n \bullet a) \cong (m \otimes n) \bullet a$$

- Action of $\mathcal{C}$ on itself through:
  - product: $c \bullet a = c \times a$
  - sum (coproduct): $c \bullet a = c + a$
- Polynomial action (power series): streams of $\mathcal{C}$ acting on $\mathcal{C}$:

$$\bullet \colon [\mathbb{N}, \mathcal{C}] \times \mathcal{C} \to \mathcal{C}$$

$$c \bullet a = \sum_k c_k \times a^k$$

- Dependent lens: fibrations with pullbacks (locally closed categories)
- Action of profunctors on co-presheaf categories

Existential optics

- Mixed optics:

$$\mathcal{O}\langle s, t\rangle\langle a, b\rangle = \int^{m \colon \mathcal{M}} \mathcal{C}(s, m \bullet a) \times \mathcal{D}(m \bullet b, t)$$

- Lens

$$\mathcal{L}\langle s, t\rangle\langle a, b\rangle = \int^{c \colon \mathcal{C}} \mathcal{C}(s, c \times a) \times \mathcal{C}(c \times b, t)$$

- Currying adjunction:

$$\int^{c \colon \mathcal{C}} \mathcal{C}(s, c \times a) \times \mathcal{C}(c, [b, t])$$

- co-Yoneda $\to$ set/get:

$$C(s, [b, t] \times a) \cong C(s, [b, t]) \times C(s, a)$$

Tannakian reconstruction

- Reconstruct a monoid from all its representations
- Monoid $\mathcal{M}$: single object $*$, hom-set $\mathcal{M}(*, *)$
- Representation $F \colon \mathcal{M} \to \mathbf{Set}$
- Category of representations $[\mathcal{M}, \mathbf{Set}]$ with natural transformations as equivariant functions
- Tannakian reconstruction:

$$\int_{F \colon [\mathcal{M}, \mathbf{Set}]} \mathbf{Set}(F*, F*) \cong \mathcal{M}(*, *)$$

- More general

$$\int_{F \colon [\mathcal{C}, \mathbf{Set}]} \mathbf{Set}(Fa, Fb) \cong \mathcal{C}(a, b)$$

Tannakian reconstruction with adjunction

- Tannaka
$$\int_{F \colon [\mathcal{C}, \mathbf{Set}]} \mathbf{Set}(Fa, Fb) \cong \mathcal{C}(a, b)$$
- More general:
  - A category $\mathcal{T}$ of functors with additional structure
  - forgetful functor $U \colon \mathcal{T} \to [\mathcal{C}, \mathbf{Set}]$
  - Free functor F adjoint to U
$$\int_{P \colon \mathcal{T}} \mathbf{Set}\big((UP)a, (UP)b\big) \cong \big((U \circ F)\mathcal{C}(a, -)\big)\, b$$
- Fiber functor $\mathcal{T} \to \mathbf{Set}$ ("stalk" over a)
$$P \mapsto (UP)a$$
- The monad $\Phi = U \circ F$ in the functor category $[\mathcal{C}, \mathbf{Set}]$ acting on the representable $\mathcal{C}(a, -)$

Profunctor optics

- Replace $\mathcal{C}$ with $\mathcal{C}^{op} \times \mathcal{C}$ to get Tannakian reconstruction for profunctors:

$$\mathcal{O}\langle s, t\rangle\langle a, b\rangle = \int_{P : \mathcal{T}} \mathbf{Set}\big((UP)\langle a, b\rangle, (UP)\langle s, t\rangle\big)$$

$$\cong \big(\Phi(\mathcal{C}^{op} \times \mathcal{C})(\langle a, b\rangle, -)\big)\langle s, t\rangle$$

- $\Phi = U \circ F$ is a profunctor functor.
- We get optics when $\mathcal{T}$ is a category of Tambara modules

Tambara modules

- A profunctor equipped with a family of transformations

$$\alpha_{\langle a,b\rangle,m} \colon p\langle a,b\rangle \to p\langle m \bullet a, m \bullet b\rangle$$

- suitably natural and compatible with the monoidal action
- Mixed optics: two separate actions of the same monoidal category in two categories
- Pastro and Street calculated the monad Φ

Pastro-Street monad

- The monad in $[\mathcal{C}^{op} \times \mathcal{D}, \mathbf{Set}]$

$$(\Phi P)\langle s, t\rangle = \int^{\langle u,v\rangle,m} (\mathcal{C}^{op} \times \mathcal{D})\big(m \bullet \langle u, v\rangle, \langle s, t\rangle\big) \times P\langle u, v\rangle$$

- where

$$m \bullet \langle u, v\rangle = \langle m \bullet a, m \bullet b\rangle$$

- Substituting $\big(\Phi(\mathcal{C}^{op} \times \mathcal{D})(\langle a, b\rangle, -)\big)\langle s, t\rangle$

$$\int^{\langle u,v\rangle,m} (\mathcal{C}^{op} \times \mathcal{D})\big(m \bullet \langle u, v\rangle, \langle s, t\rangle\big) \times (\mathcal{C}^{op} \times \mathcal{D})\big(\langle a, b\rangle, \langle u, v\rangle\big)$$

$$\int^{m} (\mathcal{C}^{op} \times \mathcal{D})\big(m \bullet \langle a, b\rangle, \langle s, t\rangle\big)$$

$$\int^{m} \mathcal{C}(s, m \bullet a) \times \mathcal{D}(m \bullet b, t)$$

Co-presheaf optics

- Polynomial functor

$$P(y) = \sum_{n \in N} s_n \times [t_n, y]$$

- Natural transformations between two polynomials:

$$\int^{c_{ki}} \prod_{k \in K} \mathbf{Set}\left(s_k, \sum_{n \in N} a_n \times c_{nk}\right) \times \prod_{i \in K} \mathbf{Set}\left(\sum_{m \in N} b_m \times c_{mi}, t_i\right)$$

- c_{nk} are not "square". Not a monoidal action!
- If a_n, s_k interpreted as co-presheaves, then c_{nk} are profunctors:

$$c \colon [\mathcal{N}, \mathbf{Set}] \to [\mathcal{K}, \mathbf{Set}]$$

of actegory we have the action of **Prof** (with profunctor composition) on co-presheaves,

$$(c \bullet a)k = \int^{n} a(n) \times p\langle n, k \rangle$$

Bibliography

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