WHEN IS A FUNCTOR REPRESENTABLE?

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Category theory is full of aha! moments. You open a book, read a sentence containing the words "of course," or "obviously," and have no idea what they are talking about. You start digging into it, go back to definitions of terms, draw a few pictures, and then you get it. It's a very satisfying feeling.

I recently opened Max Kelly's book *Basic Concepts of Enriched Category Theory* in the section about representable functors, in which he says that there is no simple criterion for deciding whether an enriched functor is representable. Then he remarks:

"It is of course otherwise in the classical case $\mathcal{V} = \mathbf{Set}$; there we have the comma category $1/F = el\,F$ of "elements of F", whose objects are pairs (A, x) with $A \in \mathcal{A}$ and $x \in FA$; and α is invertible if and only if (K, η) is initial in $el\,F$, so that F is representable if and only if $el\,F$ has an initial object."

If you understand this one-sentence explanation then you are a better person than me and you can stop reading right now. Otherwise, bear with me.

1. Representable Functor

Let's start with the definition of a representable functor. A functor F from some category $\mathcal{A}$ to $\mathbf{Set}$ is representable if there exists an object K such that

$$\alpha\colon \mathcal{A}(K, -) \to F$$

is a natural isomorphism. In other words, for every object A in $\mathcal{A}$ the set FA is isomorphic to the hom-set $\mathcal{A}(K, A)$, and α is a family of bijections

$$\alpha_A\colon \mathcal{A}(K, A) \to FA$$

The *unit* η of the representation is the image of the identity morphism under α

$$\eta = \alpha_K(id_K) \in FK$$

2. Comma category

A comma category is defined using three categories and two functors

$$\mathcal{B} \xrightarrow{S} \mathcal{C} \xleftarrow{T} \mathcal{A}$$

The name "comma category" is a historical accident. Modern notation for a comma category is either $S \downarrow T$ or S/T (which is what Kelly uses). It is, essentially, a category of arrows in $\mathcal{C}$, whose sources come from the embedding of $\mathcal{B}$ using S, and targets come from the embedding of $\mathcal{A}$ using T. So objects of S/T are triples

$$(B, A, f\colon SB \to TA)$$

where B is an object in $\mathcal{B}$, A is an object in $\mathcal{A}$ and f is a morphism in $\mathcal{C}$.

A morphism from (B, A, f) to (B', A', f') is a pair of morphisms $(h\colon B \to B', h'\colon A \to A')$ that makes the following diagram commute:

$$\begin{tikzcd} SB \arrow[r, "Sh"] \arrow[d, "f"] & SB' \arrow[d, "f'"] \\ TA \arrow[r, "Th'"] & TA' \end{tikzcd}$$

This is a very general construction that can be specialized in many ways. For instance, you can take $\mathcal{B}$ to be the terminal single-object category. The functor S then selects one object in $\mathcal{C}$. Furthermore, if $\mathcal{C}$ is **Set**, you can specialize S to pick the terminal set, the singleton $*$. In that case, the arrows in S/T all have the singleton as the source. Such arrows pick individual elements in the set TA.

If we call 1 the functor that picks the singleton set $*$, and rename T to F, we can say that the objects of the comma category $1/F$ are pairs (A, a) where A is an object in $\mathcal{A}$ and a is an element $a \in FA$.

A morphism from (A, a) to (A', a') is a morphism $h \colon A \to A'$ such that the following diagram commutes

$$\begin{tikzcd} & * \arrow[dl, "a"'] \arrow[dr, "a'"] & \\ FA \arrow[rr, "Fh"] & & FA' \end{tikzcd}$$

In other words, the lifted h must map the point $a \in FA$ to the point $a' \in FA'$.

3. Category of elements

If you are intimately familiar with comma categories then the easiest way to describe the category of elements of a functor F is to say that it's $1/F$. For the rest of us, a more direct definition may be more intuitive.

We start with a functor $F \colon \mathcal{A} \to \mathbf{Set}$. The objects of $el\,F$ are pairs $(A, a \in FA)$, where A is an object of $\mathcal{A}$. A morphism from (A, a) to (A', a') is a morphism $h \colon A \to A'$ such that $(Fh)a = a'$.

Let's dissect this definition a little. The category of elements explodes every object A of $\mathcal{A}$ into as many separate objects of $el\,F$ as there are elements in the set FA. It's a giant disjoint union of sets FA indexed by objects of $\mathcal{A}$. This is reflected in the coend notation

$$el\,F = \int^{A:\mathcal{A}} FA$$

Every morphism h of $\mathcal{A}$ explodes into as many morphisms of $el\,F$ as there are elements in the domain of Fh. So a morphism is $el\,F$ is uniquely determined by a pair

$$(h \colon \mathcal{A}(A, A'), a \in FA)$$

The domain of this morphism is (A, a) and the codomain is $(A', (Fh)a)$.

Moreover, for a fixed pair (A, a), we get a natural transformation

$$\phi^{(A,a)} \colon \mathcal{A}(A, -) \to F$$

whose component at A' is implemented by lifting $h \in \mathcal{A}(A, A')$ and applying it to a

$$\phi^{(A,a)}_{A'} h = (Fh)a$$

This mapping is natural in A'. To show this, suppose that we have a morphism $f \colon A' \to A''$. The naturality square is

$$
\begin{tikzcd}
\mathcal{A}(A, A') \arrow[r, "f \circ -"] \arrow[d, "\phi_{A'}"] & \mathcal{A}(A, A'') \arrow[d, "\phi_{A''}"] \\
FA' \arrow[r, "Ff"] & FA''
\end{tikzcd}
$$

Indeed, acting on $h \in \mathcal{A}(A, A')$, the composition with f produces $f \circ h$ and then $\phi_{A''}$ produces $(F(f \circ h))a$. By functoriality, this is equal to $(Ff)((Fh)a)$. The other path produces first $(Fh)a$ and then acts on it with Ff.

Notice that, in order to show that F is representable, all we have to do is to find a pair (A, a) for which $\phi^{(A,a)}$ is an isomorphism. In other words, every component of $\phi^{(A,a)}$ must be a bijection of sets.

There are two ways in which this condition may break down. First, it's possible that two morphisms h and h' from the same hom-set $\mathcal{A}(A, A')$ produce the same element a' in FA'. In other words, it's possible to have two different morphisms in $\mathit{el}\, F$ from (A, a) to (A', a').

The other way of not being bijective is if there is an element a' in A' that is not covered by our mapping. In other words, there may be no such h in $\mathcal{A}(A, A')$ for which $(Fh)a$ is equal to a'. That would mean that there is no morphism in $\mathit{el}\, F$ from (A, a) to (A', a').

What we need is to have exactly one morphism from (A, a) to (A', a'). That is only possible if (A, a) is the initial object.

4. Initial object in the category of elements

Suppose that there is an initial object in the category of elements $\mathit{el}\, F$. It's a pair (K, η), where η is an element of FK. There is a unique morphism from it to any other object (A, a) of $\mathit{el}\, F$. Such a morphism is a morphism $h \in \mathcal{A}(K, A)$, such that Fh maps η to a.

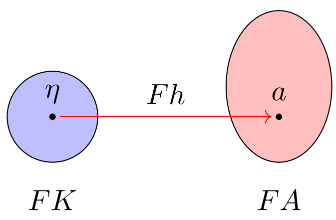
Let's dissect this statement. For every h we can produce an element a of FA

$$a = (Fh)\eta$$

That gives us a mapping

$$\alpha_A \colon \mathcal{A}(K, A) \to FA$$

Initiality of (K, η) means that this mapping is a bijection. We can reach every $a \in FA$ and every h maps η to a different a.



This proves that F is representable if and only if $\mathit{el}\, F$ has an initial object. So it was obvious after all!