1. Random things

1.1. Initial algebra structure map.

$$j\colon f(\mu f) \to \mu f$$

$$C\Big(f(\mu f), \int_b b^{C(fb,b)}\Big)$$

is a member of

$$\int_b C\Big(f(\mu f), b^{C(fb,b)}\Big)$$

Using the definition of the power

$$C(x, a^s) \cong Set(s, C(x,a))$$

we get

$$\int_b Set\Big(C(fb,b), C\big(f(\mu f), b\big)\Big)$$

Using Yoneda lemma we replace b with $f(\mu f)$ (not really!)

$$C(f(f(\mu f)), f(\mu f))$$

1.2. Terminal coalgebra structure map.

$$k\colon \nu f \to f(\nu f)$$

is a member of

$$C\Big(\int^a C(a, fa)\cdot a, f(\nu f)\Big)$$

$$\int_a C\Big(C(a, fa)\cdot a, f(\nu f)\Big)$$

Using the definition of the copower

$$C(s\cdot a, x) \cong Set(s, C(a,x))$$

we get

$$\int_a Set\Big(C(a, fa), C\big(a, f(\nu f)\big)\Big)$$

Yoneda lemma (not really!)

$$C(f(\nu f), f(f(\nu f)))$$

1.3. Kan extensions.

$$\mu f = \int_a a^{C(fa,a)}$$

$$\nu f = \int^a C(a, fa)\cdot a$$

These formulas are reminiscent of Kan extensions. For comparison, the right Kan extension of g along f is given by

$$(Ran_f g)c = \int_a (ga)^{C(c,fa)}$$

The left Kan extension is

$$(Lan_f g)c = \int^a C(fa, c)\cdot ga$$

If f has left and right adjoints, they are given by

$$Ran_f Id \dashv f \dashv Lan_f Id$$

In particular, using the adjunction

$$(Lan_f Id)c = \int^a C(a, (Lan_f Id)c)\cdot a$$

This shows that $(Lan_f Id)c$ is a fixed point of the functor

$$\Phi(x) = \int^a C(a, x) \cdot a$$

1.4. **Ends as limits.** Twisted arrow category on $Tw(\mathbf{C})$ has, as objects, morphisms in $\mathbf{C}$ (or, strictly speaking, triples $(a, b, f : a \to b)$). A morphism from $f : a \to b$ to $g : a' \to b'$ is a pair of morphisms

$$(h : a' \to a, h' : b \to b')$$

For every profunctor $p : C^{op} \times C \to \mathbf{Set}$ define a functor $\bar{p} : Tw(\mathbf{C}) \to Set$. On objects

$$\bar{p}(a, b, f) = pab$$

and on morphisms, it's just profunctor lifting.

It can be shown that the end is just a limit over the twisted arrow category

$$\int_c pcc \cong \lim_{Tw(C)} \bar{p}$$

Similarly, the coend is a colimit over $Tw(C^{op})^{op}$

$$\int^c pcc \cong \operatorname*{colim}_{Tw(C^{op})^{op}} \bar{p}$$

1.5. **Iterative solution.** Terminal coalgebra is a limit, and initial algebra is a colimit of these two chains

$$\begin{tikzcd}
f(\nu f) & & & \\
\nu f \arrow[u, "k"] \arrow[d, "\pi_1"] \arrow[dr, "\pi_{(f1)}"] \arrow[drr, bend left, "\pi_{(f^2 1)}"] & & & \\
1 & f1 \arrow[l, "¡"'] & f^2 1 \arrow[l, "f¡"'] & \dots \arrow[l, dashed] \\
0 \arrow[u, "!"] \arrow[r, "!"] \arrow[d, "\iota_0"] & f0 \arrow[u, "f!"] \arrow[r, "f!"] \arrow[dl, "\iota_{(f0)}"] & f^2 0 \arrow[u, "f^2!"] \arrow[r, dashed] \arrow[dll, bend left, "\iota_{(f^2 0)}"] & \dots \\
\mu f \arrow[uuu, bend left, "\varphi"] & & & \\
f(\mu f) \arrow[u, "j"] & & &
\end{tikzcd}$$