OPTICS AND DOUBLES OF MONOIDAL CATEGORIES

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Hom-object in the dubles of a monoidal category $\mathscr{A}$ in Pastro-Street:

$$\mathscr{D}_l\mathscr{A}((X, Y), (U, V)) = \int^A \mathscr{A}(U, A \otimes X) \otimes \mathscr{A}(A \otimes Y, V)$$

In a closed category, we can use the internal-hom adjunction:

$$\int^A \mathscr{A}(U, A \otimes X) \otimes \mathscr{A}(A, [Y, V])$$

Co-Yoneda lets us eliminate A:

$$\mathscr{A}(U, [Y, V] \otimes X)$$

This formulation is often interpreted as a generalized coalgebra of the functor (in fact, a comonad):

$$S_{X,Y}(V) = X \otimes [Y, V]$$

(which becomes a slice for X equal to Y).

In particular, for a cartesian product we get:

$$\mathscr{A}(U, [Y, V]) \times \mathscr{A}(U, X)$$

In Set, an element of this hom-set is a pair of functions:

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set :: u -> y -> v
get :: u -> x
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This is a lens that focuses on an x inside a u. It can replace the x with a y to produce a v.

If we replace the tensor product with coproduct, there is no adjunction. In that case the original formula defines a prism.

Another optic, grate, replaces the tensor product with the internal hom:

$$\int^A \mathscr{A}(U, [A, X]) \otimes \mathscr{A}([A, Y], V)$$

hom-adjunction:

$$\int^A \mathscr{A}(U \otimes A, X) \otimes \mathscr{A}([A, Y], V)$$

symmetry:

$$\int^A \mathscr{A}(A \otimes U, X) \otimes \mathscr{A}([A, Y], V)$$

hom-adjunction again:

$$\int^A \mathscr{A}(A, [U, X]) \otimes \mathscr{A}([A, Y], V)$$

co-Yoneda:

$$\mathscr{A}([[U, X], Y], V)$$

Equivalent to Haskell definition:

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type Grate s t a b = (((s -> a) -> b) -> t)
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Iso is a special case:

$$\mathscr{A}(U, X) \otimes \mathscr{A}(Y, V)$$

1. Traversals

Traversals don't have the same form. They can be written as:

$$\mathscr{A}\big(U, F_{Mon}(S_{X,Y})(V)\big)$$

Here, F_{Mon} is a functor that freely generates lax monoidal functors from copresheaves. It is the initial algebra of the following functor (the star denotes Day convolution, and I is its unit):

$$A(F, G) = I + F \star G$$

It is acting on the functor we've seen before:

$$S_{X,Y} = X \otimes [Y, -]$$

(Obviously, this really only works in *Set*.) Symbolically, we have:

$$\mathscr{A}\Big(U, 1 + X \otimes [Y, V] + \big(X \otimes [Y, -] \star X \otimes [Y, -]\big)(V) + ...\Big)$$

It can be shown inductively that, at least in Set, the following is true:

$$F_{Mon}(S_{X,Y})(V) \cong \coprod_{n \in \mathbb{N}} X^n \otimes [Y^n, V]$$

A traversal then becomes:

$$\mathscr{A}\big(U, \coprod_{n \in \mathbb{N}} X^n \otimes [Y^n, V]\big)$$

The intuition is that a traversal focuses on n locations at a time, but n is not known up front. It is determined given a particular U.

There is an equivalent formula:

$$F_{Mon}(S_{X,Y})(V) \cong \int_{F \colon FMon} \big[[X, FY], FV\big]$$

where the end is taken over all lax monoidal functors. The result is a more symmetric definition of traversal as:

$$\int_{F \in FMon} \mathscr{A}([X, FY], [U, FV])$$

2. Profunctor Optics

All these optics, including traversals, have another representation:

$$\int_{P \in \mathscr{E}} Set\big(P(X, Y), P(U, V)\big)$$

where the end is over different subcategories of profunctors. The advantage of these representations is that they compose just like functions.

Let's start with Yoneda embedding in the category of functors from $\mathscr{C}$ to *Set*:

$$\int_F Set\big(Nat(G, F), Nat(H, F)\big) \cong Nat(H, G)$$

and evaluate it for two representable functors:

$$\int_F Set(Nat(C(X, -), F), Nat((C(Y, -), F) \cong Nat(C(Y, -), C(X, -))$$

By Yoneda, we get:

$$\int_F Set(F(X), F(Y)) \cong C(X, Y)$$

Now let's assume that we have a pair of adjoint functors $L \dashv R$ between functor categories. Instead of bare representable functors, we can use their image under L:

$$\int_{F \in \mathscr{E}} Set\Big(\mathscr{E}\big(L(C(X, -)), F\big), \mathscr{E}\big(L(C(Y, -)), F\big)\Big) \cong \mathscr{E}\big(L(C(Y, -)), L(C(X, -))\big)$$

Using the adjunction we get:

$$\int_{F \in \mathscr{E}} Set\Big(Nat\big(C(X, -), RF\big), Nat\big(C(Y, -), RF\big)\Big) \cong Nat\big(C(Y, -), (R \circ L)(C(X, -))\big)$$

Again, using Yoneda, we get:

$$\int_{F \in \mathscr{E}} Set\big((RF)(X)), (RF)(Y)\big) \cong \Phi(C(X, -))(Y)$$

where Φ is the monad $R \circ L$.

Now consider a free/forgetful adjunction. We start with a functor category with some additional structure and we get:

$$\int_{F \in \mathscr{E}} Set\big((UF)(X)), (UF)(Y)\big) \cong \Phi(C(X, -))(Y)$$

Notice the resemblance to Tannakian reconstruction.

This construction also works in the profunctor category:

$$\int_{P \in \mathscr{E}} Set\big((UP)(X, Y)), (UP)(U, V)\big) \cong \Phi(C(-, X) \otimes C(Y, -))(U, V)$$

The simplest case reproduces the trivial *Iso*:

$$\int_P Set\big(P(X, Y)), P(U, V)\big) \cong C(U, X) \otimes C(Y, V)$$

The lens/prism is obtained when we restrict the profunctors to Tambara modules. A (left) Tambara module is a profunctor with a natural transformation:

$$\alpha(A, X, Y) \colon P(X, Y) \to P(A \otimes X, A \otimes Y)$$

It turns out (Pastro/Street) that Tambara modules are free algebras for the following monad:

$$\Phi(P)(U, V) = \int^{A, X, Y} \mathscr{A}(U, A \otimes X) \otimes \mathscr{A}(A \otimes Y, V) \otimes P(X, Y)$$

We can evaluate:

$$\Phi(C(-, X) \otimes C(Y, -))(U, V) \cong \int^A \mathscr{A}(U, A \otimes X) \otimes \mathscr{A}(A \otimes Y, V)$$

This is exactly the hom-object in the double of monoidal category, which becomes a lens/prism when instantiated with product/coproduct. The reconstruction then reads:

$$\int_{P \in Tamb} Set\big((UP)(X, Y)), (UP)(U, V)\big) \cong \int^A \mathscr{A}(U, A \otimes X) \otimes \mathscr{A}(A \otimes Y, V)$$

3. Other Optics

Grate:

$$\int^{A} \mathscr{A}(U,[A,X]) \otimes \mathscr{A}([A,Y],V)$$

turns out to be generated by *closed* profunctors eqipped with:

$$\alpha(A,X,Y) \colon P(X,Y) \to P([A,X],[A,Y])$$

This is related to the following monad:

$$\Phi(P)(U,V) = \int^{A,X,Y} \mathscr{A}(U,[A,X]) \otimes \mathscr{A}([A,Y],V) \otimes P(X,Y)$$

A pattern emerges, where we have a family of transformations of the underlying category, parameterized by A. The simplest is the scaling using categorical product:

$$X \to A \times X$$

The coproduct, similarly, produces translations:

$$X \to A + X$$

In a general monoidal category we have:

$$X \to A \otimes X$$

and in a closed monoidal category:

$$X \to [A,X]$$

In general, there is a family of (associative) transformations:

$$X \to F(X)$$

which includes identity. There is a corresponding profunctor category equipped with a set of natural transformations:

$$\alpha(A,X,Y) \colon P(X,Y) \to P(F(X),F(Y))$$

If these profunctors can be freely generated, they induce a monad Φ and the corresponding reconstruction theorem.

This pattern doesn't work for traversals, although it is known that they are generated by so called *monoidal* profunctors, which transform both under product and coproduct (so they are Tambara modules under both), as well as under this family of non-linear transformations that laxly preserve the product:

$$P(X,Y) \times P(U,V) \to P(X \times U, Y \times V)$$

and unit:

$$1 \to P(1,1)$$

Traversals are thus generated by:

$$\int_{P \in \mathscr{E}} Set\big((UP)(X,Y)),(UP)(U,V)\big)$$

with $\mathscr{E}$ being the category of profunctor with the sturcture defined above.

Another formulation (due to Russell O'Connor?) is to consider transformations using *traversable* functors:

$$\alpha_T(X,Y) \colon P(X,Y) \to P(T(X),T(Y))$$

for all T's that distribute across monoidal functors, that is:

$$T \circ F \to F \circ T$$

for all lax monoidal functors F.

If traversable functors could be generated freely, we could use the same derivation, but that's unlikely.