1. Initial algebra and catamorphism

The initial algebra can be defined by its mapping out property.

```
newtype Mu f = Mu (forall a. (f a -> a) -> a)
```

Notice that this definition requires the following language pragma

```
{-# language RankNTypes #-}
```

This definition works because for every least fixed point one can define a catamorphism, which can be rewritten as

```
cata :: Functor f => Fix f -> (forall a . (f a -> a) -> a)
cata (Fix x) = \alg -> alg (fmap (flip cata alg) x)
```

(`flip` is a function that reverses the order of arguments of its (function) argument.) What the definition of `Mu` is saying is that it's an object that, for all algebras, has a mapping out to a catamorphism.

It's easy to define a catamorphism in terms of `Mu`, since `Mu` *is* a catamorphism

```
cataMu :: Functor f => Algebra f a -> Mu f -> a
cataMu alg (Mu cata) = cata alg
```

The challenge is to construct terms of type `Mu f`. For instance, let's convert a list of `a` to a term of type `Mu (ListF a)`

```
mkList :: forall a. [a] -> Mu (ListF a)
mkList as = Mu cata
  where cata :: forall x. (ListF a x -> x) -> x
        cata unf = go as
          where
            go [] = unf NilF
            go (n: ns) = unf (ConsF n (go ns))
```

Notice that we use the type `a` defined in the type signature of `mkList` to define the type signature of the helper function `cata`. For the compiler to identify the two, we have to use the pragma

```
{-# language ScopedTypeVariables #-}
```

You can now verify that

```
cataMu sumAlg (mkList [1..10])
```

produces the correct result for the following algebra

```
sumAlg :: Algebra (ListF Int) Int
sumAlg NilF = 0
sumAlg (ConsF a x) = a + x
```

2. Terminal coalgebra and anamorphism

The terminal coalgebra, on the other hand, is defined by its mapping in property. This requires a definition in terms of existential types. If Haskell had an existential quantifier, we could write the following definition for the terminal coalgebra

```
data Nu f =  Nu (exists a. (a -> f a, a))
```

Existential types can be encoded in Haskell using the so called Generalized Algebraic Data Types or GADTs

```
data Nu f where
  Nu :: (a -> f a) -> a -> Nu f
```

The use of GADTs requires the language pragma

```
{-# language GADTs #-}
```

The argument is that, for every greatest fixed point one can define an anamorphism

```
ana :: Functor f => forall a. (a -> f a) -> a -> Fix f
ana coa x = Fix (fmap (ana coa) (coa x))
```

We can uncurry it

```
ana :: Functor f => forall a. (a -> f a, a) -> Fix f
ana (coa, x) = Fix (fmap (curry ana coa) (coa x))
```

A universally quantified mapping out

```
forall a. ((a -> f a, a) -> Fix f)
```

is equivalent to a mapping out of an existential type (in pseudo-Haskell)

```
(exists a. (a -> f a, a)) -> Fix f
```

which is the type signature of the constructor of `Nu f`.

The intuition is that, if you want to implement a function from an existential type—a type which hides some other type `a` to which you have no access—your function has to be prepared to handle any `a`. In other words, it has to be polymorphic in `a`.

Since in an existential type we have no access to the hidden type, it has to provide both the "producer" and the "consumer" for this type. Here we are given a value of type `a` on the produces side, and the function `a -> f a` as the consumer. All we can do is to apply this function to `a` and obtain the term of the type `f a`. Since `f` is a functor, we can lift our function and apply it again, to get something of the type `f (f a)`. Continuing this process, we can obtain arbitrary powers of `f` acting on `a`. We get a recursive data type.

An anamorphism in terms of `Nu` is given by

```
anaNu :: Functor f => Coalgebra f a -> a -> Nu f
anaNu coa a = Nu coa a
```

Notice however that we cannot directly pass the result of `anaNu` to `cataMu` because we are no longer guaranteed that the initial algebra is the same as the terminal coalgebra for a given functor.

3. End/Coend formulation

Let's rewrite `Mu` using GADTs

```
data Mu f where
  Mu :: (forall a. (f a -> a) -> a) -> Mu f
```

We use a natural transformation to construct a `Mu`. Categorically, we can write it as an end

$$\mu f = \int_a a^{C(fa,a)}$$

It's an end over the profunctor

$$pab = b^{C(fb,a)}$$

Where the power is defined as

$$C(x, a^s) \cong Set(s, C(x, a))$$

Projection from the end is a catamorphism

$$\pi_a \colon \mu f \to a^{C(fa,a)}$$

It's a morphism from the hom-set

$$C(\mu f, a^{C(fa,a)})$$

or an element of

$$Set(C(fa, a), C(\mu f, a))$$

Similarly, we can rewrite `Nu`

```
data Nu f where
  Nu :: (a -> f a) -> a -> Nu f
```

as a coend

$$\nu f = \int^a C(a, fa) \cdot a$$

over the profunctor

$$pab = C(a, fb) \cdot b$$

where the copower is defined as

$$C(s \cdot a, x) \cong Set(s, C(a, x))$$

Injection into the coend is an anamorphism

$$\iota_a \colon C(a, fa) \cdot a \to \nu f$$

It's a morphism from the hom-set

$$C(C(a, fa) \cdot a, \nu f)$$

or an element of

$$Set(C(a, fa), C(a, \nu f))$$

Because of Lambek's lemma, an initial algebra is also a coalgebra, and a terminal coalgebra is also an algebra. Universality, therefore, tells us that there is a unique algebra morphism (as well as a unique coalgebra morphism)

$$\phi \colon \mu f \to \nu f$$

This is a canonical embedding, but not necessarily an isomorphism.

The two profunctors in the definition of `Mu` and `Nu` can be written as

```
data M f a b = M ((f b -> a) -> b)
```

```
instance Functor f => Profunctor (M f) where
  dimap g g' (M h) = M (\j -> g'( h (g . j . fmap g')))
```

```
data N f a b = N (a -> f b) b
```

```
instance Functor f => Profunctor (N f) where
  dimap g g' (N h b) = N (fmap g' . h . g) (g' b)
```

4. Hylomorphism

If the mapping from from the terminal coalgebra ν to the initial algebra μ exists, it is an element of the following hom-set

$$C\Big(\int^a C(a, fa) \cdot a, \int_b b^{C(fb,b)}\Big)$$

By co-continuity of the hom-functor, this is isomorphic to

$$\int_a C\Big(C(a, fa) \cdot a, \int_b b^{C(fb,b)}\Big)$$

Using continuity we get

$$\int_{a,b} C\Big(C(a, fa) \cdot a, b^{C(fb,b)}\Big)$$

Using the definition of the copower

$$C(s \cdot a, x) \cong Set(s, C(a, x))$$

we get

$$\int_{a,b} Set\Big(C(a, fa), C(a, b^{C(fb,b)})\Big)$$

And using the definition of the power

$$C(x, a^s) \cong Set(s, C(x, a))$$

we get

$$\int_{a,b} Set\Big(C(a, fa), Set\big(C(fb, b), C(a, b)\big)\Big)$$

Finally, applying the currying adjunction we get

$$\int_{a,b} Set\Big(C(a, fa) \times C(fb, b), C(a, b)\Big)$$

in which you may recognize a hylomorphism

```
hylo :: Functor f => Coalgebra f a -> Algebra f b -> a -> b
hylo coa alg = alg . fmap (hylo coa alg) . coa
```

5. Bibliography

- Michael Barr, Terminal coalgebras for endofunctors on sets

6. Random things

6.1. **Initial algebra structure map.**

$$j \colon f(\mu f) \to \mu f$$

$$C\Big(f(\mu f), \int_b b^{C(fb,b)}\Big)$$

is a member of

$$\int_b C\Big(f(\mu f), b^{C(fb,b)}\Big)$$

Using the definition of the power

$$C(x, a^s) \cong Set(s, C(x, a))$$

we get

$$\int_b Set\Big(C(fb, b), C\big(f(\mu f), b\big)\Big)$$

Using Yoneda lemma we replace b with $f(\mu f)$

$$C(f(f(\mu f)), f(\mu f))$$

6.2. **Terminal coalgebra structure map.**

$$k \colon \nu f \to f(\nu f)$$

is a member of

$$C\Big(\int^a C(a, fa) \cdot a, f(\nu f)\Big)$$

$$\int_a C\Big(C(a, fa) \cdot a, f(\nu f)\Big)$$

Using the definition of the copower

$$C(s \cdot a, x) \cong Set(s, C(a, x))$$

we get

$$\int_a Set\Big(C(a, fa), C\big(a, f(\nu f)\big)\Big)$$

Yoneda lemma

$$C(f(\nu f), f(f(\nu f)))$$

6.3. **Kan extensions.**

$$\mu f = \int_a a^{C(fa,a)}$$

$$\nu f = \int^a C(a, fa) \cdot a$$

These formulas are reminiscent of Kan extensions. For comparison, the right Kan extension of g along f is given by

$$(Ran_f g)c = \int_a (ga)^{C(c,fa)}$$

The left Kan extension is

$$(Lan_f g)c = \int^a C(fa, c) \cdot ga$$

If f has left and right adjoints, they are given by

$$Ran_f Id \dashv f \dashv Lan_f Id$$

In particular, using the adjunction

$$(Lan_f Id)c = \int^a C(a, (Lan_f Id)c) \cdot a$$

This shows that $(Lan_f Id)c$ is a fixed point of the functor

$$\Phi(x) = \int^a C(a, x) \cdot a$$

6.4. **Ends as limits.** Twisted arrow category on $Tw(\mathbf{C})$ has, as objects, morphisms in $\mathbf{C}$ (or, strictly speaking, triples $(a, b, f\colon a \to b)$). A morphism from $f\colon a \to b$ to $g\colon a' \to b'$ is a pair of morphisms

$$(h\colon a' \to a, h'\colon b \to b')$$

For every profunctor $p\colon C^{op} \times C \to \mathbf{Set}$ define a functor $\bar{p}\colon Tw(\mathbf{C}) \to Set$. On objects

$$\bar{p}(a, b, f) = pab$$

and on morphisms, it's just profunctor lifting.

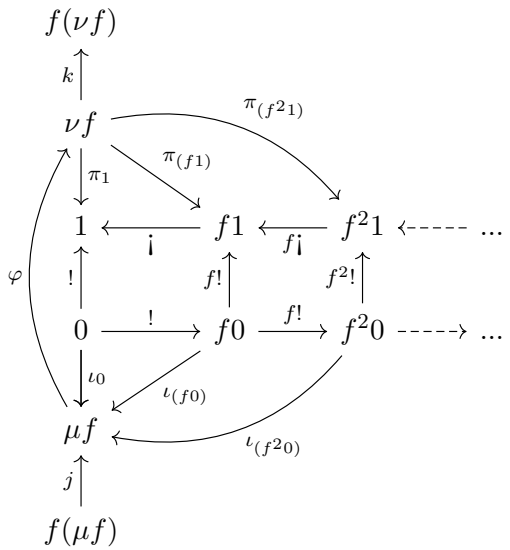
It can be shown that the end is just a limit over the twisted arrow category

$$\int_c pcc \cong \lim_{Tw(C)} \bar{p}$$

Similarly, the coend is a colimit over $Tw(C^{op})^{op}$

$$\int^c pcc \cong \operatorname*{colim}_{Tw(C^{op})^{op}} \bar{p}$$

6.5. **Iterative solution.** Terminal coalgebra is a limit, and initial algebra is a colimit of these two chains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