Intro to Profunctor Optics

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Profunctors

Definition

Profunctors are **Set**-valued functors of the type:

$$p \colon \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$$

• For each pair of objects, a set:

$$p\langle c, d \rangle$$

For each pair of morphisms:

$$f: s \rightarrow c$$

$$g: d \rightarrow t$$

a function:

$$p\langle f,g\rangle \colon p\langle c,d\rangle \to p\langle s,t\rangle$$



Profunctor zoo

Endo-Profunctors

$$p \colon \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$$

They generalize hom-functors:

$$\mathcal{C}(-,=) \colon \mathcal{C}^{op} \times \mathcal{C} \to \mathsf{Set}$$

Enriched profunctors.
 C and D, V-enriched categories.

$$p \colon \mathcal{C}^{op} \otimes \mathcal{D} \to \mathcal{V}$$



Profunctors as proof-relevant relations

- Element of $p\langle c, d \rangle$ is a proof that c is related to d
- Relations that are compatible with categorical structure.
 If c is related to d then s is related to t, if the objects are connected by morphisms:

$$f: s \to c$$

 $g: d \to t$

Profunctors as linear transformations

• Elements of a co-presheaf as vectors:

$$v \colon \mathcal{C} \to \mathbf{Set}$$

 v_i is a component of v at object i

- Especially if C is a discrete category. C(i,j) acts like a Kronecker delta.
- A profunctor acts like a linear transformation, a matrix, or a mixed covariance tensor:

 p_{ij}

Composition

$$\circ \colon \mathcal{C}(b,c) \times \mathcal{C}(a,b) \to \mathcal{C}(a,c)$$

Profunctor composition:

$$\diamond \colon q\langle b,c\rangle \times p\langle a,b\rangle \to (q\diamond p)\langle a,c\rangle$$

• As relations: A proof of $(q \diamond p)\langle a, c \rangle$ is:

$$\exists b. \ q\langle b, c \rangle \land p\langle a, b \rangle$$

As linear transformations:

$$(q \diamond p)_{ik} = \sum_{j} q_{jk} \times p_{ij} = Tr(q_{-k} \times p_{i-})$$

The sum (coproduct) works well for discrete categories.
 Otherwise, it over-counts.

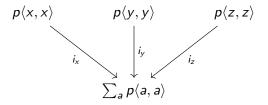


Coend as sum

As trace

$$\sum_{i} p_{ii} \qquad \qquad \int^{a} p\langle a, a \rangle$$

As sum (only in discrete categories)



• Overcounting: if there's a morphism $f: x \to y$:

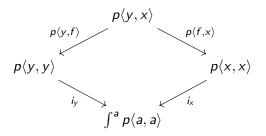
$$p\langle y, f \rangle \colon p\langle y, x \rangle \to p\langle y, y \rangle$$

 $p\langle f, x \rangle \colon p\langle y, x \rangle \to p\langle x, x \rangle$



Co-wedge condition

Like sum, modulo identifications:



- Universal co-wedge is the coend
- Like a sum, co-end has the mapping out property
- Generalizes to enriched categories

Profunctor composition

Composition

$$(q \diamond p)\langle a, b \rangle = \int^c q\langle c, b \rangle \times p\langle a, c \rangle$$

Compare with:

$$(q \diamond p)_{ik} = \sum_{j} \ q_{jk} \times p_{ij} = Tr(q_{-k} \times p_{i-})$$

 $(q \diamond p)\langle a, c \rangle = \exists b. \ q\langle b, c \rangle \wedge p\langle a, b \rangle$

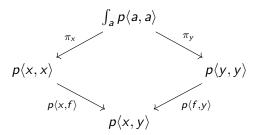
ullet Enriched version uses tensor product in ${\cal V}$

End

Like a product of diagonal elements (in a discrete category)

$$\prod_{a} p\langle a, a \rangle$$

- Like a universal quantifier $\forall_a \ p\langle a, a \rangle$
- Wedge condition



- Universal wedge is the end
- Like a product, end has a mapping in property



Natural transformations

• Two functors F and G from C to D

$$[\mathcal{C},\mathcal{D}](F,G)\cong\int_{x\colon\mathcal{C}}\mathcal{D}(Fx,Gx)$$

- Element of this end: a giant tuple of morphisms. Wedge contition = naturality square
- Can be used to define natural transformations between enriched functors
- (Ninja) Yoneda lemma

$$\int_{x \in \mathcal{C}} \mathbf{Set}(\mathcal{C}(a, x), Fx) \cong Fa$$

(Ninja) co-Yoneda

$$\int_{-\infty}^{\infty} \mathcal{C}(x,a) \times Fx \cong Fa$$

• Colimit of representables. $a = \sum_{i} e_{i} a_{i}$



Continuity of hom-functor

Mapping out of a coend

$$\mathsf{Set}(\int^a p\langle a,a\rangle,C)\cong \int_a \mathsf{Set}(p\langle a,a\rangle,C)$$

Mapping into an end

$$\mathsf{Set}(C, \int_{\mathsf{a}} p\langle \mathsf{a}, \mathsf{a} \rangle) \cong \int_{\mathsf{a}} \mathsf{Set}(C, p\langle \mathsf{a}, \mathsf{a} \rangle)$$

Actegories

ullet The action of a monoidal category ${\mathcal M}$ on ${\mathcal C}$

$$\bullet : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$$

Preserving monoidal structure

$$m \bullet (n \bullet a) \cong (m \otimes n) \bullet a$$

- Action of C on itself through:
 - product: $c \bullet a = c \times a$
 - sum (coproduct): $c \bullet a = c + a$
- Polynomial action (power series): streams of C acting on C:

$$ullet$$
: $[\mathbb{N}, \mathcal{C}] \times \mathcal{C} \to \mathcal{C}$

$$c \bullet a = \sum_{k} c_{k} \times a^{k}$$

- Dependent lens: fibrations with pullbacks (locally closed categories)
- Action of profunctors on co-presheaf categories



Existential optics

• Mixed optics:

$$\mathcal{O}\langle s,t \rangle \langle a,b \rangle = \int^{m: \mathcal{M}} \mathcal{C}(s,m \bullet a) \times \mathcal{D}(m \bullet b,t)$$

Lens

$$\mathcal{L}\langle s,t
angle\langle a,b
angle = \int^{c:\mathcal{C}} \mathcal{C}(s,c imes a) imes \mathcal{C}(c imes b,t)$$

Currying adjunction:

$$\int^{c:C} \mathcal{C}(s,c\times a)\times \mathcal{C}(c,[b,t])$$

• co-Yoneda \rightarrow set/get:

$$C(s,[b,t]\times a)\cong C(s,[b,t])\times C(s,a)$$



Tannakian reconstruction

- Reconstruct a monoid from all its representations
- Monoid \mathcal{M} : single object *, hom-set $\mathcal{M}(*,*)$
- Representation $F: \mathcal{M} \to \mathbf{Set}$
- Category of representations [M, Set] with natural transformations as equivariant functions
- Tannakian reconstruction:

$$\int_{F \colon [\mathcal{M},\mathsf{Set}]} \mathsf{Set}(F*,F*) \cong \mathcal{M}(*,*)$$

More general

$$\int_{F: [\mathcal{C}, \mathbf{Set}]} \mathbf{Set}(Fa, Fb) \cong \mathcal{C}(a, b)$$



Tannakian reconstruction with adjunction

Tannaka

$$\int_{F \colon [\mathcal{C},\mathsf{Set}]} \mathsf{Set}(\mathit{Fa},\mathit{Fb}) \cong \mathcal{C}(\mathsf{a},\mathsf{b})$$

- More general:
 - ullet A category ${\mathcal T}$ of functors with additional structure
 - forgetful functor $U \colon \mathcal{T} \to [\mathcal{C}, \mathbf{Set}]$
 - ullet Free functor F adjoint to U

$$\int_{P \colon \mathcal{T}} \mathsf{Set} \big((\mathit{UP}) \mathsf{a}, (\mathit{UP}) \mathsf{b} \big) \cong \big((\mathit{U} \circ \mathit{F}) \mathcal{C} (\mathsf{a}, -) \big) \mathsf{b}$$

ullet Fiber functor $\mathcal{T} o \mathbf{Set}$ ("stalk" over a)

$$P \mapsto (UP)a$$

• The monad $\Phi = U \circ F$ in the functor category $[C, \mathbf{Set}]$ acting on the representable C(a, -)



Profunctor optics

• Replace $\mathcal C$ with $\mathcal C^{op} \times \mathcal C$ to get Tannakian reconstruction for profunctors:

$$\mathcal{O}\langle s,t \rangle \langle a,b \rangle = \int_{P \colon \mathcal{T}} \mathbf{Set} \big((\mathit{UP})\langle a,b \rangle, (\mathit{UP})\langle s,t \rangle \big)$$

$$\cong (\Phi(\mathcal{C}^{op} \times \mathcal{C})(\langle a, b \rangle, -))\langle s, t \rangle$$

- $\Phi = U \circ F$ is a profunctor functor.
- ullet We get optics when ${\mathcal T}$ is a category of Tambara modules

Tambara modules

A profunctor equipped with a family of transformations

$$\alpha_{\langle a,b\rangle,m} \colon p\langle a,b\rangle \to p\langle m \bullet a, m \bullet b\rangle$$

- suitably natural and compatible with the monoidal action
- Mixed optics: two separate actions of the same monoidal category in two categories
- ullet Pastro and Street calculated the monad Φ

Pastro-Street monad

ullet The monad in $[\mathcal{C}^{op} imes \mathcal{D}, \mathbf{Set}]$

$$(\Phi P)\langle s,t
angle = \int^{\langle u,v
angle,m} (\mathcal{C}^{op} imes \mathcal{D}) ig(m ullet \langle u,v
angle, \langle s,t
angle ig) imes P\langle u,v
angle$$

where

$$m \bullet \langle u, v \rangle = \langle m \bullet a, m \bullet b \rangle$$

• Substituting $(\Phi(\mathcal{C}^{op} \times \mathcal{D})(\langle a, b \rangle, -))\langle s, t \rangle$

$$\int^{\langle u,v\rangle,m} (\mathcal{C}^{op} \times \mathcal{D}) \big(m \bullet \langle u,v\rangle, \langle s,t\rangle \big) \times \big(\mathcal{C}^{op} \times \mathcal{D} \big) \big(\langle a,b\rangle, \langle u,v\rangle \big)$$

$$\int^{m} (\mathcal{C}^{op} \times \mathcal{D}) \big(m \bullet \langle a,b\rangle, \langle s,t\rangle \big)$$

$$\int^{m} \mathcal{C}(s,m \bullet a) \times \mathcal{D}(m \bullet b,t)$$

Co-presheaf optics

Polynomial functor

$$P(y) = \sum_{n \in N} s_n \times [t_n, y]$$

Natural transformations between two polynomials:

$$\int^{c_{ki}} \prod_{k \in K} \mathbf{Set} \left(s_k, \sum_{n \in N} a_n \times c_{nk} \right) \times \prod_{i \in K} \mathbf{Set} \left(\sum_{m \in N} b_m \times c_{mi}, t_i \right)$$

- ullet c_{nk} are not "square". Not a monoidal action!
- If a_n , s_k interpreted as co-presheaves, then c_{nk} are profunctors:

$$c \colon [\mathcal{N}, \mathsf{Set}] \to [\mathcal{K}, \mathsf{Set}]$$

of actegory we have the action of **Prof** (with profunctor composition) on co-presheaves,

$$(c \bullet a)k = \int_{-\infty}^{n} a(n) \times p\langle n, k \rangle$$



Bibliography

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