NOTES

BARTOSZ MILEWSKI

Definition of a catamorphism. It's a unique algebra morphism κ, from the initial algebra (i, ι) to the algebra (a, α):

$$\begin{tikzcd} Fi \arrow[r, "F\kappa"] \arrow[d, "\iota"] & Fa \arrow[d, "\alpha"] \\ i \arrow[r, dashed, "\kappa"] & a \end{tikzcd}$$

a solution to:

$$\kappa \circ \iota = \alpha \circ F\kappa$$

By Lambek's lemma ι is an isomorphism.

Yoneda lemma is a natural isomorphism:

$$\phi\colon Fa \to \mathrm{Nat}(\mathcal{C}(a, -), F)$$

between F and the functor:

$$Ya = \mathrm{Nat}(\mathcal{C}(a, -), F)$$

$$\phi p = (f\colon a \to x) \mapsto (Ff)p$$

$$\phi^{-1}\nu = \nu_a(id_a)$$

Yoneda applied to a catamorphism:

$$\begin{tikzcd} \mathrm{Nat}(\mathcal{C}(i, -), F) \arrow[r, "Y\kappa"] & \mathrm{Nat}(\mathcal{C}(a, -), F) \arrow[d, "\phi^{-1}"] \\ Fi \arrow[u, "\phi"] \arrow[r, "F\kappa"] \arrow[d, "\iota"] & Fa \arrow[d, "\alpha"] \\ i \arrow[r, dashed, "\kappa"] & a \end{tikzcd}$$

$$\kappa \circ \iota = \alpha \circ \phi^{-1} \circ Y\kappa \circ \phi$$

Hinze defines:

$$\Psi\colon \int_x \mathbf{Set}(\mathcal{C}(x, a), \mathcal{C}(Fx, a))$$

For $f \in \mathcal{C}(x, a)$, $\Psi f \in \mathcal{C}(Fx, a)$ is given by:

$$\Psi f = \alpha \circ \phi^{-1} \circ Yf \circ \phi$$

Algebra morphism f:

$$\begin{tikzcd}
\mathrm{Nat}(\mathcal{C}(x,-),F) \arrow[r, "Yf"] & \mathrm{Nat}(\mathcal{C}(a,-),F) \arrow[d, "\phi^{-1}"] \\
Fx \arrow[u, "\phi"] \arrow[r, "Ff"] \arrow[d, "\xi"] & Fa \arrow[d, "\alpha"] \\
x \arrow[r, "f"] & a
\end{tikzcd}$$

Naturality of Ψ. For any $h\colon y \to x$:

$$\begin{tikzcd}
\mathcal{C}(y,a) \arrow[r, "\Psi_y"] & \mathcal{C}(Fy,a) \\
\mathcal{C}(x,a) \arrow[u, "(-\circ h)"] \arrow[r, "\Psi_x"] & \mathcal{C}(Fx,a) \arrow[u, "(-\circ Fh)"']
\end{tikzcd}$$

For $f\colon x \to a$, this reads:

$$\begin{tikzcd}
f \circ h \arrow[r, mapsto, "\Psi_y"] & \Psi_y(f \circ h) = \Psi_x(f) \circ Fh \\
f \arrow[u, mapsto, "(-\circ h)"] \arrow[r, mapsto, "\Psi_x"] & \Psi_x(f) \arrow[u, mapsto, "(-\circ Fh)"']
\end{tikzcd}$$

$$\Psi_y(f \circ h) = \Psi_x(f) \circ Fh$$

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psi :: forall x. (x -> Nat) -> (StackF x -> Nat)
psi tot EmptyF = Z
psi tot (PushF (n, s)) = n + tot s
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data StackF stack = EmptyF | PushF (Nat, stack)

instance Functor StackF where
  fmap f EmptyF = EmptyF
  fmap f (PushF (n, a)) = PushF (n, f a)
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total :: Mu StackF -> Nat
total (In l) = psi total l
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```
alg :: Algebra StackF Nat
alg EmptyF = Z
alg (PushF (n, a)) = n + a
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1. Optics

$$O\langle s,t\rangle\langle a,b\rangle = \int^{m:\mathcal{M}} \mathcal{C}(s, m \bullet a) \times \mathcal{D}(m \bullet b, t)$$

Adjunctions:

$$\mathcal{C}(s, m \bullet a) \cong \mathcal{M}(L_a s, m)$$
$$O\langle s,t\rangle\langle a,b\rangle \cong \mathcal{D}(L_a s \bullet b, t)$$

Or:

$$\mathcal{C}(s, m \times a) \cong (\mathcal{C} \times \mathcal{C})(\Delta s, \langle m, a\rangle) \cong \mathcal{C}(s, m) \times \mathcal{C}(s, a)$$

Or:

$$\mathcal{D}(m \bullet b, t) \cong \mathcal{M}(m, R_b t)$$
$$O\langle s,t\rangle\langle a,b\rangle \cong \mathcal{C}(s, R_b t \bullet a)$$

2. Deep Learning

$$R_m \colon \mathcal{C} \to \mathcal{C} \times \mathcal{P}^n$$
$$L_m \colon \mathcal{C} \to \mathcal{C} \times \mathcal{P}^n$$

These are two families of functors parameterized by m:

$$L_m^0 \colon \mathcal{C} \to \mathcal{C}$$
$$L_m^k \colon \mathcal{C} \to \mathcal{P}, k = 1 ... n$$

$$\int^m (\mathcal{C} \times \mathcal{P}^n)(\langle s, p...\rangle, L_m a) \times (\mathcal{C} \times \mathcal{P}^n)(R_m b, \langle t, q...\rangle)$$

Or:

$$\int^m \mathcal{C}(s \times p \times ..., L_m a) \times (\mathcal{C} \times \mathcal{P}^n)(\langle R_m^0 b, R_m^1 a\rangle, \langle t, q...\rangle)$$

$$\int^m \mathcal{C}(s \times p \times ..., L_m a) \times \mathcal{C}(R_m^0 b, t) \times \mathcal{P}(R_m^1 b, q^1) \times ...$$

$$get \colon s \times p \times ... \to a$$
$$set \colon s \times p \times ... \to (b \to \langle t, q...\rangle)$$

$$get \colon P \times A \to B$$
$$set \colon P \times A \times B' \to (A' \times P')$$

$$\begin{tikzcd} fx \arrow[r, "fm"] \arrow[d, "\mu_x"] & fy \arrow[d, "\mu_y"] \\ gx \arrow[r, "fn"] & gy \end{tikzcd}$$

$$\begin{tikzcd}
fx \arrow[r, "fm"] \arrow[d, "\mu_x"] & fy \arrow[d, "\mu_y"] \\
gx \arrow[r, "fn"] & gy
\end{tikzcd}$$

$$\begin{tikzcd}
& fy \arrow[dr, "\mu_y"] & \\
fx \arrow[ur, "fm"] \arrow[dr, "\mu_x"] \arrow[dd, "\text{cat}"] & & gy \arrow[dd, "\text{cat'}"] \\
& gx \arrow[ur, "gm"] & \\
x \arrow[rr, "m"] & & y
\end{tikzcd}$$