Profunctor Optics

Categorical View

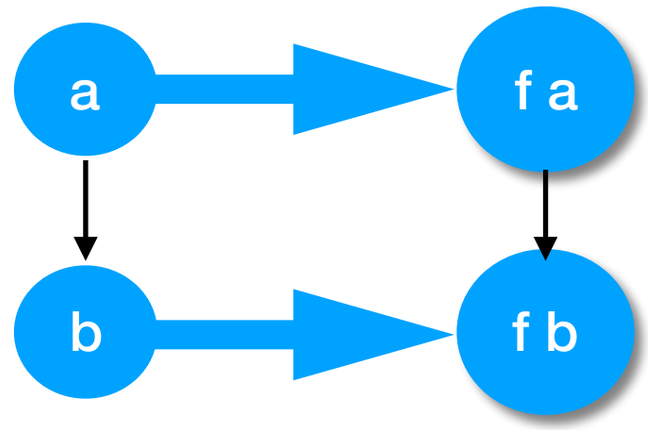
Bartosz Milewski

Optics

```
type Lens s t a b = forall p. Strong p => p a b -> p s t

type Prism s t a b = forall p. Choice p => p a b -> p s t


class Profunctor p => Strong p where
    first' :: p a b  -> p (a, c) (b, c)

class Profunctor p => Choice p where
    left'  :: p a b -> p (Either a c) (Either b c)


class Profunctor p where
    dimap :: (a -> b) -> (c -> d) -> (p b c -> p a d)
```

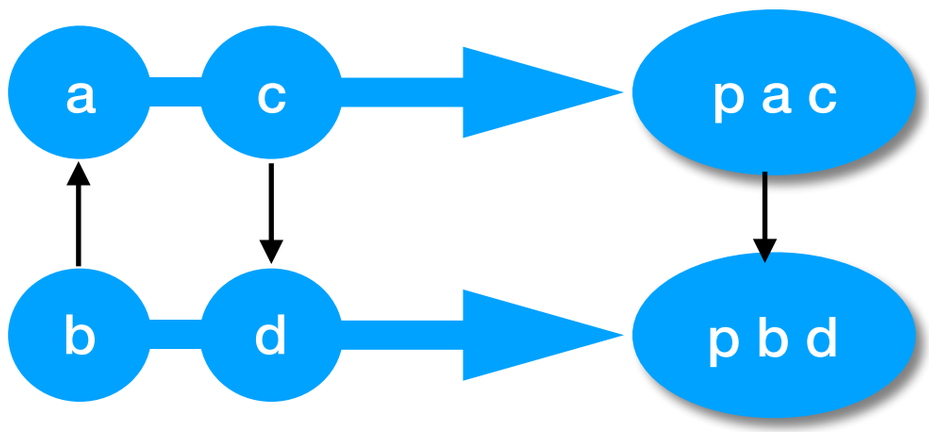
Functors

```
class  Functor f  where
     fmap :: (a -> b) -> (f a -> f b)
```



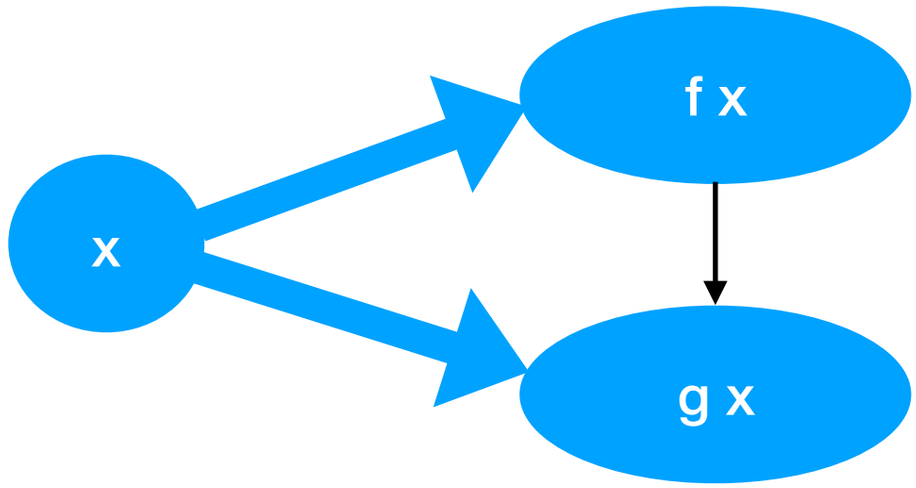

```
class Profunctor p where
    dimap :: (b -> a) -> (c -> d) -> (p a c -> p b d)
```

Functor from $C^{op} \times C$ -> Set



Natural Transformations

```
type f ~> g = forall x. f x -> g x
```




```
safeHead :: [] ~> Maybe
safeHead []     = Nothing
safeHead (a:as) = Just a
```

```
type Lens s t a b = forall p. Strong p => p a b -> p s t
```

Yoneda Lemma

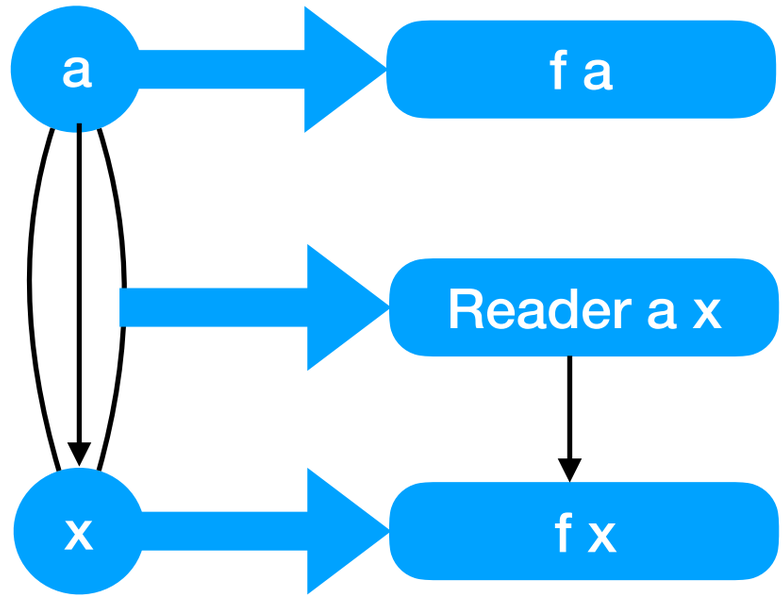
```
type Reader a x = a -> x
```

```
instance Functor (Reader a) where
  fmap f atox = f . atox
```

```
type Yo f a = Functor f => Reader a ~> f
```

```
Yo f a ~ f a
```

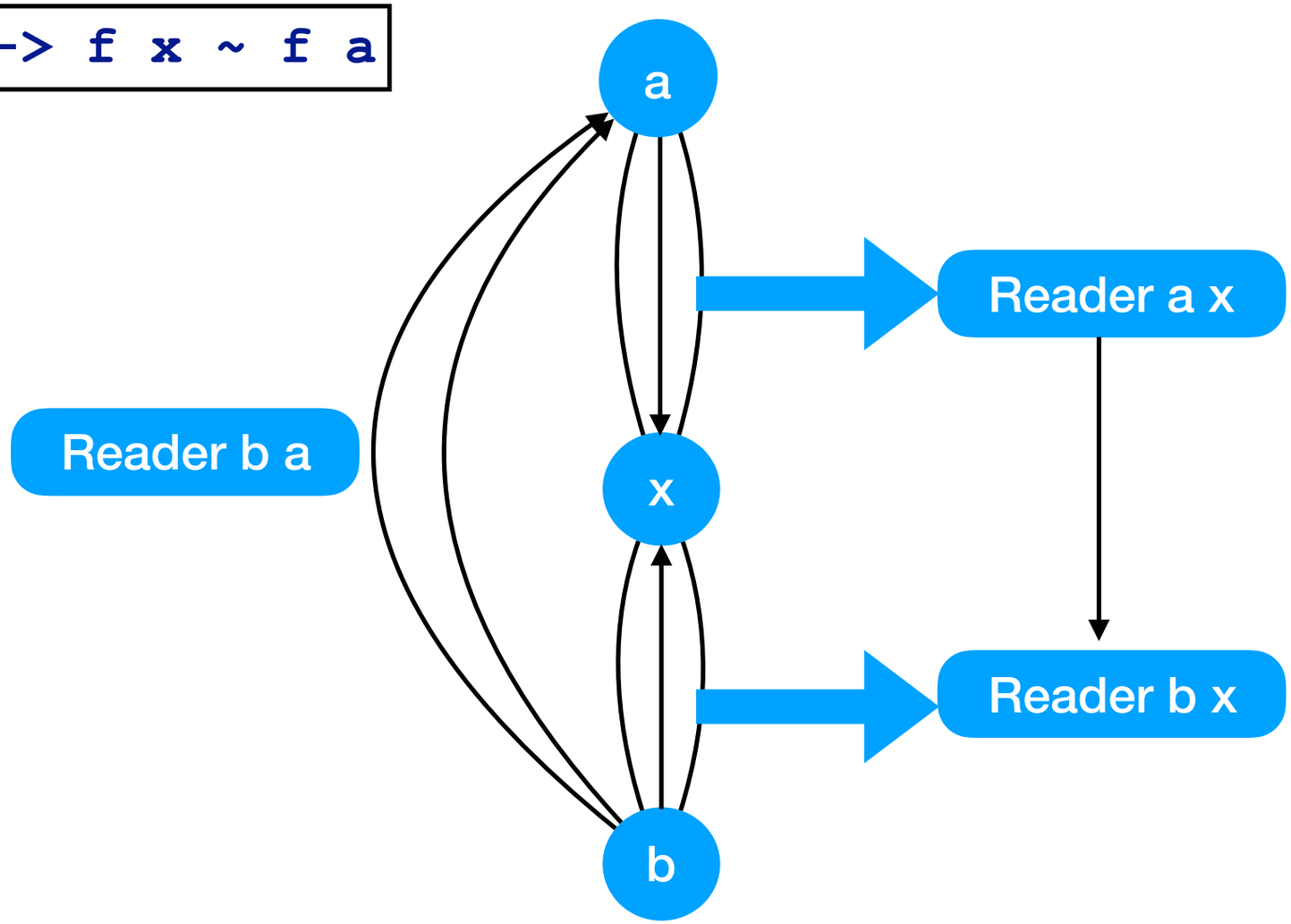
```
forall x. (a -> x) -> f x ~ f a
```




```
toYo :: Functor f => f a -> Yo f a
toYo fa = \atox -> fmap atox fa

fromYo :: Functor f => Yo f a -> f a
fromYo alpha = alpha id
```

Yoneda Embedding

```
forall x. (a -> x) -> f x ~ f a
```

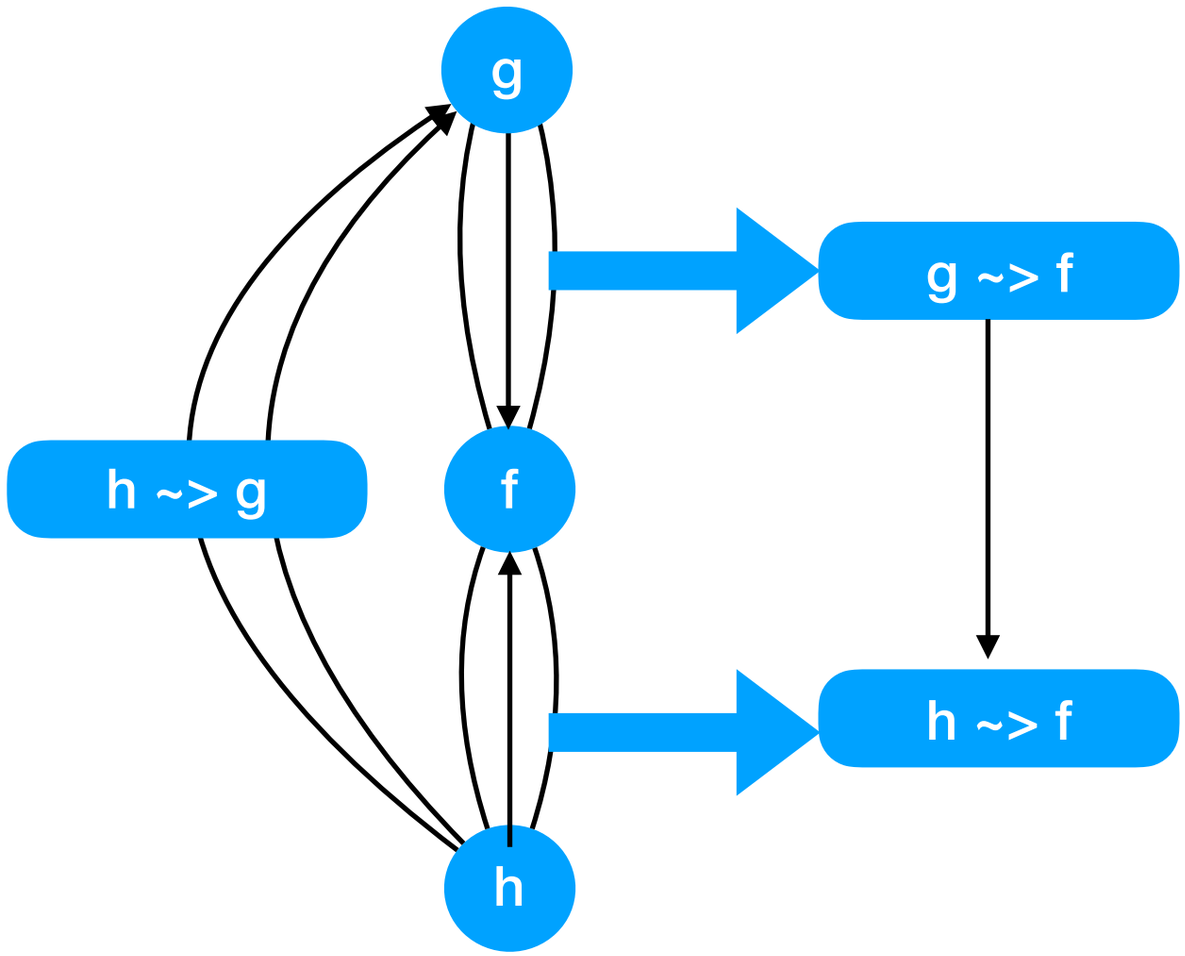



```
f x = (b -> x)
```

```
forall x. (a -> x) -> (b -> x) ~ (b -> a)
```

Yoneda on Functors

```
forall x. (a -> x) -> (b -> x)
~ (b -> a)
```




```
forall f. Functor f => (g ~> f) -> (h ~> f)
~ (h ~> g)
```

Toy Optic

```
forall f. Functor f. (g ~> f) -> (h ~> f)
~ (h ~> g)
```

```
forall f.(forall x. g x -> f x) -> (forall x. h x -> f x)
~ (forall x. h x -> g x)
```

```
                    g x = a -> x
                    h x = s -> x
```

```
forall f.(forall x. (a->x) -> f x) -> (forall x. (s->x) -> f x)
~ (forall x. (s->x) -> (a->x))
```

```
forall f. f a -> f s ~ (a -> s)
```

```
type Toy s a = forall f. Functor f => f a -> f s
Toy s a ~ (a -> s)
```

Yoneda for Profunctors

```
type ProReader a b x y = (x -> a, b -> y)

instance Profunctor (ProReader a b) where
  dimap from to (f, g) = (f.from, to.g)
```

```
type ProYo p a b =
   Profunctor p => forall x y. (x -> a, b -> y) -> p x y
```

```
ProYo p a b ~ p a b
```

```
forall x y.(x -> a) -> (b -> y) -> p x y ~ p a b
```

Yoneda on Profunctors

```
forall p. Profunctor p => (q ~> p) -> (r ~> p) ~ (r ~> q)
```

```
forall p. Profunctor p => (forall x y. q x y -> p x y) ->
                          (forall x y. r x y -> p x y)
 ~ (forall x y. r x y -> q x y)
```

```
q x y = (a->x, y->b)
r x y = (s->x, y->t)

forall p.(forall x y. (a->x, y->b) -> p x y) ->
          (forall x y. (s->x, y->t) -> p x y)
 ~ (forall x y. (s->x, y->t) -> q x y)
```

```
forall p. Profunctor p => p a b -> p s t
~ q s t ~ (a -> s, t -> b) ~ Iso s t a b
```

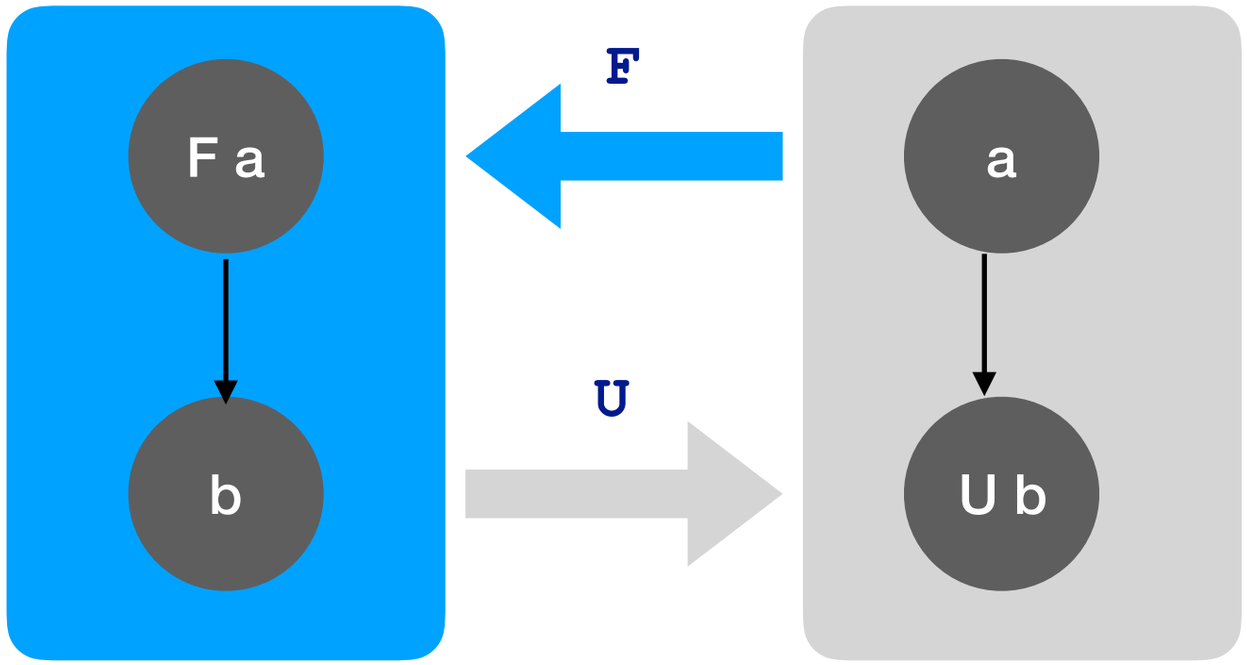
Optics

```
type Iso s t a b = forall p. Profunctor p => p a b -> p s t

type Lens s t a b = forall p. Strong p => p a b -> p s t

type Prism s t a b = forall p. Choice p => p a b -> p s t
```

```
class Profunctor p => Strong p where
    first' :: p a b  -> p (a, c) (b, c)

class Profunctor p => Choice p where
    left'  :: p a b -> p (Either a c) (Either b c)

class Profunctor p => Tambara p where
    left'  :: p a b -> p (a ⊗ c) (b ⊗ c)
```

Adjoint Functors

```
F left adjoint to U

F a -> b ~ a -> U b
```




```
F a = (a, x)
U b = (x -> b)
```

```
(a, x) -> b   ~   a -> (x -> b)
```

Yoneda

```
exists x. (f x, x -> a) ~ f a
```

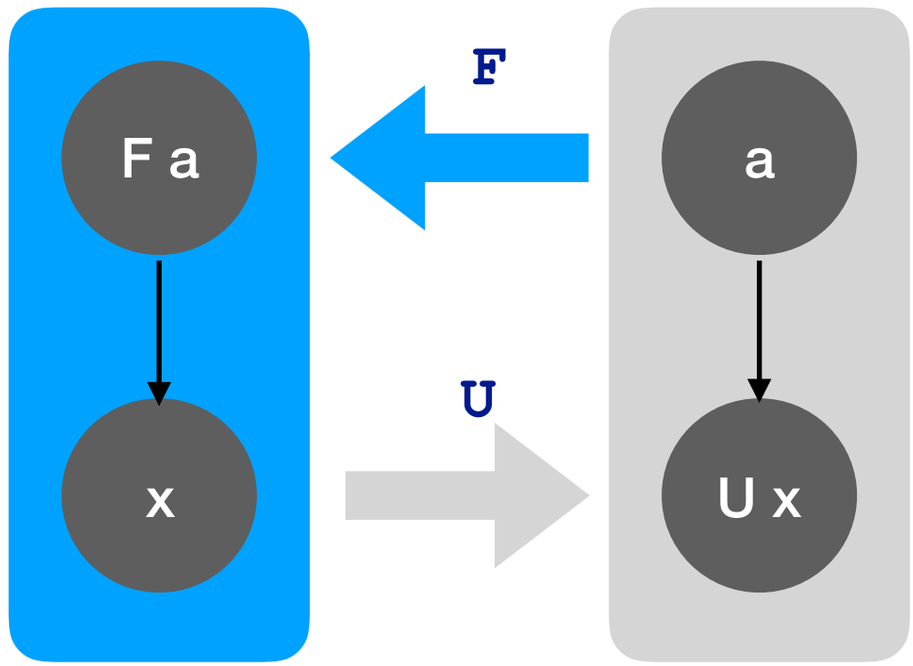
Yoneda + Adjunction

```
forall x. (a -> x) -> (b -> x) ~ (b -> a)
```

```
forall x. (F a -> x) -> (F b -> x)
~ (F b -> F a)
```

```
forall x. (a -> U x) -> (b -> U x)
~ (b -> (U ∘ F) a)
```

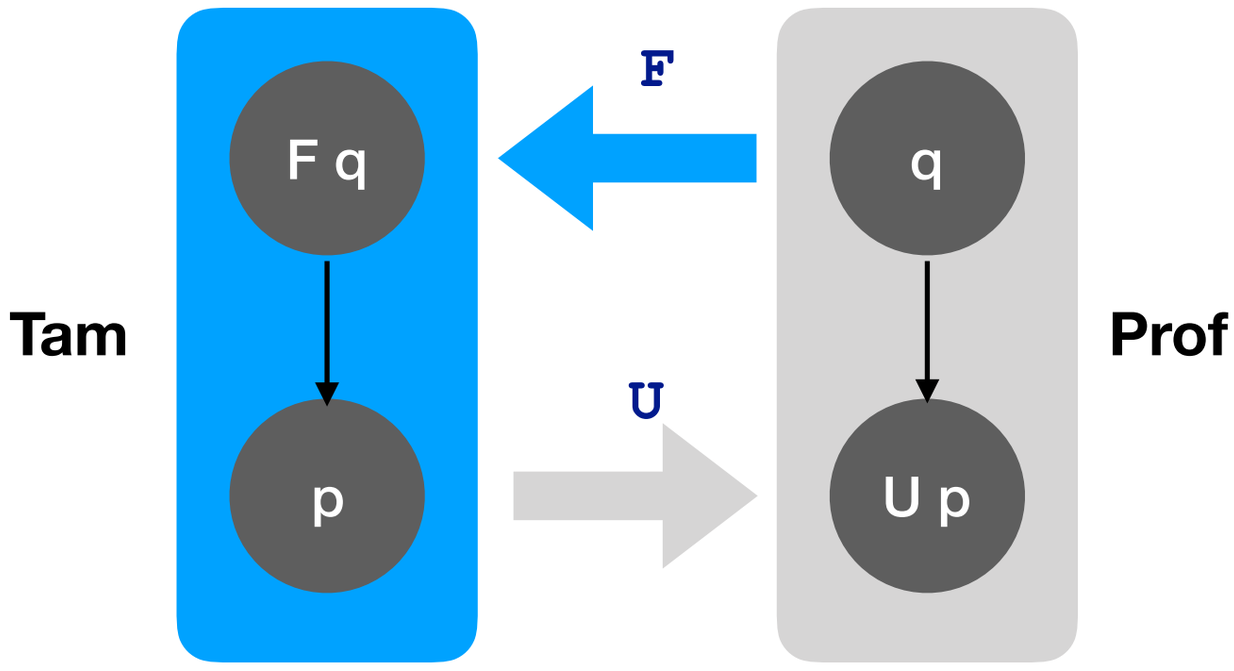
U ∘ F is a monad




```
F a -> x ~ a -> U x
```

Profunctor Adjunction

```
forall p. Profunctor p => (forall x y. q x y -> p x y) ->
                          (forall x y. r x y -> p x y)
 ~ (forall x y. r x y -> q x y)
```




```
forall p. Tambara p => (forall x y. q x y -> (U p) x y) ->
                       (forall x y. r x y -> (U p) x y)
 ~ (forall x y. r x y -> ((U ∘ F) q) x y)
```

Optics + Adjunction

```
forall p. Tambara p => (forall x y. q x y -> (U p) x y) ->
                       (forall x y. r x y -> (U p) x y)
 ~ (forall x y. r x y -> ((U ∘ F) q) x y)
```

```
q x y = (a->x, y->b)
r x y = (s->x, y->t)
```

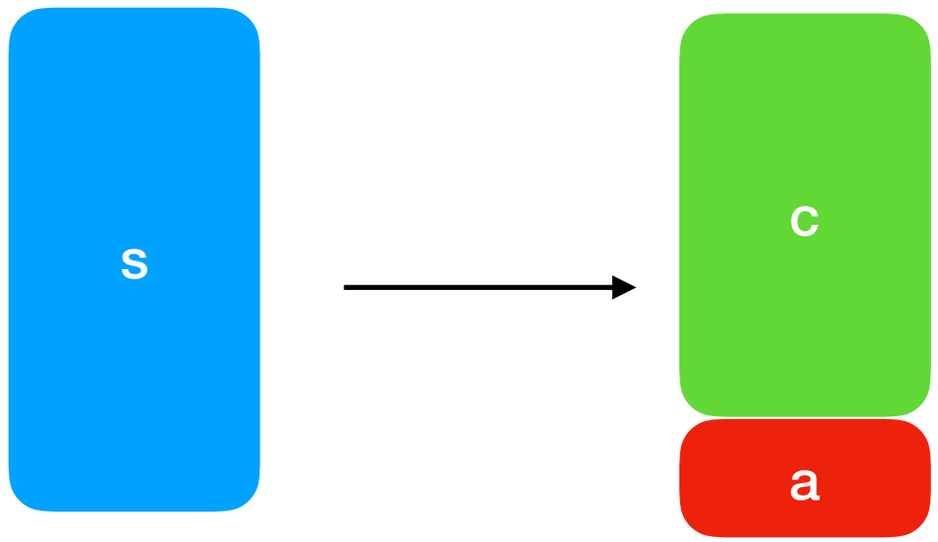
```
forall p. Tambara p => (U p) a b -> (U p) s t
 ~((U ∘ F) q) s t
```

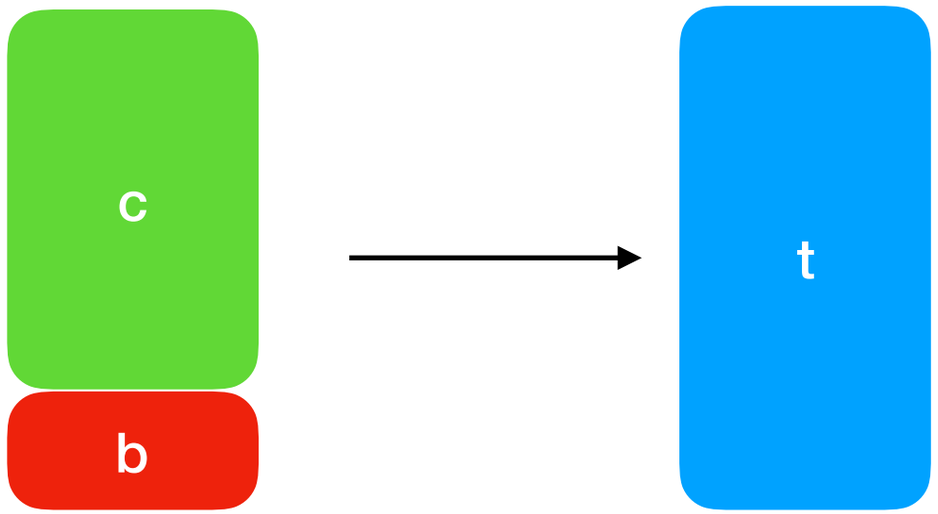
```
forall p. Tambara p => p a b -> p s t
 ~(Φ q) s t
```

Free Tambara

```
(Φ q) s t = ∫ c x y A(s, c⊗x) ⊗ A(c⊗y, t) ⊗ q x y

(Phi q) s t = exists c x y. ((s -> c⊗x, c⊗y -> t), q x y)

q x y = (a->x, y->b)

(Phi q) s t = exists c. (s -> c⊗a, c⊗b -> t)

type Lens s t a b = forall p. Strong p => p a b -> p s t

Lens s t a b = exists c. (s -> (c, a), (c, b) -> t)

type Prism s t a b = forall p. Choice p => p a b -> p s t

Prism s t a b = exists c. (s -> Either c a, Either c b -> t)
```

```
Lens s t a b = exists c. (s -> (c, a), (c, b) -> t)
```






```
Prism s t a b = exists c. (s -> Either c a, Either c b -> t)
```