

Homework Assignment 5

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1. The probability density function of normal distribution is defined as

$$f(\mathbf{x}) = \frac{1}{Z} \exp \left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right),$$

where

$$\begin{aligned} Z &= \int_{\mathbf{x} \in \mathbb{R}^d} \exp \left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right) d\mathbf{x} \\ &= (2\pi)^{d/2} |\boldsymbol{\Sigma}|^{1/2}, \end{aligned}$$

where $|\boldsymbol{\Sigma}|$ is the determinant of the covariance matrix.

Let us assume that the covariance matrix $\boldsymbol{\Sigma}$ is a diagonal matrix, as below:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_d^2 \end{bmatrix}.$$

The probability density function simplifies to

$$f(\mathbf{x}) = \prod_{i=1}^d \frac{1}{\sqrt{2\pi}\sigma_i} \exp \left(-\frac{1}{2} \frac{1}{\sigma_i^2} (x_i - \mu_i)^2 \right).$$

Show that this is indeed true.

2.

(a) Show that the following equation, called Bayes' rule, is true.

$$p(Y|X) = \frac{p(X|Y)p(Y)}{p(X)}.$$

(b) We learned the definition of expectation:

$$\mathbb{E}[X] = \sum_{x \in \Omega} xp(x).$$

Assuming that X and Y are discrete random variables, show that

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

(c) Further assume that $c \in \mathbb{R}$ is a scalar and is not a random variable, show that

$$\mathbb{E}[cX] = c\mathbb{E}[X].$$

(d) We learned the definition of variance:

$$\text{Var}(X) = \sum_{x \in \Omega} (x - \mathbb{E}[X])^2 p(x).$$

Assuming X being a discrete random variable, show that

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

3. An optimal linear regression machine (without any regularization term), that minimizes the empirical cost function given a training set

$$D_{\text{tra}} = \{(\mathbf{x}_1, \mathbf{y}_1^*), \dots, (\mathbf{x}_N, \mathbf{y}_N^*)\},$$

can be found directly without any gradient-based optimization algorithm. Assuming that the distance function is defined as

$$D(M^*(\mathbf{x}), M, \mathbf{x}) = \frac{1}{2} \|M^*(\mathbf{x}) - M(\mathbf{x})\|_2^2 = \frac{1}{2} \sum_{k=1}^q (y_k^* - y_k)^2,$$

derive the optimal weight matrix \mathbf{W} . (Hint: Moore–Penrose pseudoinverse)

4. Suppose that we have a data distribution $Y = f(\mathbf{X}) + \varepsilon$, where \mathbf{X} is a random vector, ε is an independent random variable with zero mean and fixed but unknown variance σ^2 , and f is an unknown deterministic function that maps a vector into a scalar.

Now, we wish to approximate $f(\mathbf{x})$ with our own model $\hat{f}(\mathbf{x}; \Theta)$ with some learnable parameters Θ .

- (a) Show that considering all possible \hat{f} and Θ , the minimum of L2 loss

$$\mathbb{E}_{\mathbf{X}}[(Y - \hat{f}(\mathbf{X}; \Theta))^2]$$

is achieved when for all \mathbf{x} ,

$$\hat{f}(\mathbf{x}; \Theta) = f(\mathbf{x})$$

(Hint: find the minimum of L2 loss for a single example first.)

- (b) If we train the same model varying initializations and examples from the underlying data distribution, we may end up with different Θ . So we can also consider Θ as a random variable if we fix \hat{f} .

Show that for a single unseen input vector \mathbf{x}_0 and a fixed \hat{f} , the expected squared error between the ground truth $f(\mathbf{x}_0)$ and the prediction $\hat{f}(\mathbf{x}_0; \Theta)$ can be decomposed into:

$$\mathbb{E}[(f(\mathbf{x}_0) - \hat{f}(\mathbf{x}_0; \Theta))^2] = (\mathbb{E}[f(\mathbf{x}_0) - \hat{f}(\mathbf{x}_0; \Theta)])^2 + \text{Var}[\hat{f}(\mathbf{x}_0; \Theta)] + \sigma^2$$

(Side note: this is usually known as the *bias-variance decomposition*, closely related to *bias-variance tradeoff*, and other concepts such as underfitting and overfitting.)